LINEAR SYSTEM OF GEOMETRICALLY IRREDUCIBLE PLANE CUBICS OVER FINITE FIELDS

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Abstract. We examine the maximum dimension of a linear system of plane cubic curves whose $\mathbb{F}_q$-members are all geometrically irreducible. Computational evidence suggests that such a system has a maximum (projective) dimension of 3. As a step towards the conjecture, we prove that there exists a 3-dimensional linear system $\mathcal{L}$ with at most one geometrically reducible $\mathbb{F}_q$-member.

1. Introduction

Let $\mathcal{P}$ describe a property of a degree d algebraic hypersurface in $\mathbb{P}^n$. In algebraic geometry and adjacent fields, we are often interested in measuring the likelihood of the property $\mathcal{P}$ for a "randomly chosen" hypersurface. When working over an infinite field, we can use Zariski dense open sets to show that property $\mathcal{P}$ holds generically. However, the situation over finite fields is more subtle since open sets in the relevant parameter space may not have any $\mathbb{F}_q$-points (despite being Zariski dense over $\overline{\mathbb{F}_q}$).

There are alternative methods to quantify how widespread a property $\mathcal{P}$ holds for hypersurfaces over finite fields. One method is to count the proportion of degree d hypersurfaces over $\mathbb{F}_q$ satisfying $\mathcal{P}$, and consider the asymptotic proportion (either as $d \to \infty$ or $q \to \infty$). As another natural metric, we can ask for the maximum size of a linear family $\mathcal{P}$ can carry. More precisely, we can pose the following question for each finite field $\mathbb{F}_q$, and positive integers d and n.

Question 1. What is the maximum value of $t \in \mathbb{N}$ for which there exist $\{F_i = 0\}$ for $i = 0, 1, ..., t$ such that $X_{[a_0:\cdots:a_t]} = \{a_0F_0 + \cdot + a_tF_t = 0\}$ satisfies the property $\mathcal{P}$ for all choices $[a_0 : a_1 : \ldots : a_t] \in \mathbb{P}^t(\mathbb{F}_q)$?

The question can be rephrased in the language of linear systems: what is the largest (projective) dimension of a linear system $\mathcal{L} \cong \mathbb{P}^t$ of degree d hypersurfaces in $\mathbb{P}^n$ such that each $\mathbb{F}_q$-member of $\mathcal{L}$ satisfies $\mathcal{P}$? An answer to Question 1 measures how much the property $\mathcal{P}$ linearly propagates in the parameter space of all degree d hypersurfaces in $\mathbb{P}^n$. Larger values of t indicate higher levels of prevalence for $\mathcal{P}$. Question 1 has been addressed in recent papers when $\mathcal{P}$ stands for "is smooth" [AGR23], "is irreducible over $\mathbb{F}_q$" [AGR24], "is reducible over $\mathbb{F}_q$" [AGR24], or "nonblocking with respect to $\mathbb{F}_q$-lines" [AGY23].

In this paper, we address Question 1 when $\mathcal{P}$ stands for "is geometrically irreducible" (that is, irreducible over $\overline{\mathbb{F}_q}$) for cubic plane curves: $d = 3$ and $n = 2$. In this special case, every linear system $\mathcal{L}$ of (projective) dimension 4 has an $\mathbb{F}_q$-member that is a reducible plane cubic over $\mathbb{F}_q$ by [AGR24, Theorem 1.3(d)]. Hence, the answer to Question 1 in this setting is at most 3. We predict that the answer is exactly 3.

Conjecture 2. There exists a linear system $\mathcal{L} = \langle F_0, F_1, F_2, F_3 \rangle$ of cubic plane curves where each $\mathbb{F}_q$-member of $\mathcal{L}$ is geometrically irreducible.

As partial progress, we establish the following result.

Theorem 3. *There exists a linear system $\mathcal{L} = \langle F_0, F_1, F_2, F_3 \rangle$ of cubic plane curves where each $\mathbb{F}_q$-member of $\mathcal{L}$ is irreducible over $\mathbb{F}_q$ and there is at most one geometrically reducible $\mathbb{F}_q$-member of $\mathcal{L}$.*

While we focus on the case of cubic plane curves in the present paper, the same question applies to hypersurfaces of degree d in $\mathbb{P}^n$ for any d and n.

Problem 4. Determine the maximum (projective) dimension of a linear system $\mathcal{L}$ of degree d hypersurfaces in $\mathbb{P}^n$ such that each $\mathbb{F}_q$-member is geometrically irreducible.

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By [AGR24, Theorem 1.3(d)], every linear system of dimension $\binom{n+d-1}{n-1}$ has an $\mathbb{F}_q$-member that is reducible over $\mathbb{F}_q$. Hence, the answer to Problem 4 is at most $\binom{n+d-1}{n-1}-1$. On the other hand, by [AGR23, Theorem 2], when $p=\mathrm{char}(\mathbb{F}_q)\nmid \gcd(d,n+1)$, there exists an n-dimensional linear system whose $\mathbb{F}_q$-members are all smooth, hence geometrically irreducible. Hence, the answer to Problem 4 is at least n. We expect the true answer to Problem 4 to be closer to the upper bound $\binom{n+d-1}{n-1}-1$. After all, we expect most geometrically reducible hypersurfaces defined over $\mathbb{F}_q$ to be reducible over $\mathbb{F}_q$.

We also have a related open problem with the condition "geometrically irreducible" relaxed to "not containing a linear component over $\overline{\mathbb{F}_q}$."

Problem 5. Determine the maximum (projective) dimension of a linear system $\mathcal{L}$ of degree d hypersurfaces in $\mathbb{P}^n$ such that each $\mathbb{F}_q$-member has no linear factor.

By definition, the answer to Problem 4 is less than or equal to the answer to Problem 5. It is reasonable to expect that the two answers agree, at least for all sufficiently large q (as a function of n and d). The heuristic is that most reducible hypersurfaces (over $\overline{\mathbb{F}_q}$) have a linear factor. Note that Conjecture 2 concerns the case $n=2$ and $d=3$ for which Problems 4 and 5 coincide.

While these open problems are new, we note that the study of reducible members in a linear system of algebraic hypersurfaces is rich in literature. One case that has been investigated thoroughly is the number of reducible (or totally reducible) hypersurfaces in a pencil of hypersurfaces [Lor93, Vis93, PY08]. The setting between the cited work and the present work differs in a few places. We only consider $\mathbb{F}_q$-members while the previous work is about controlling reducibility over $\overline{\mathbb{F}_q}$-members. On the other hand, we do not restrict our attention to pencils and allow large-dimensional linear systems.

Structure of the paper. We provide two proofs for Theorem 3. In Section 2, we provide a non-constructive proof in the spirit of the work done in [AGR24], while in Section 3 we provide an explicit construction of a 3-dimensional linear system as desired for the conclusion of Theorem 3. Appendix A provides numerical evidence (computed using SageMath) that supports Conjecture 2 for all $q \leq 11$.

2. Proof 1: Galois orbits

In this section, we discuss the construction in our previous paper [AGR24] joint with Reichstein in the special case of plane cubic curves. Note that [AGR24, Theorem 1.3(c)] provides a linear system of cubics $\mathcal{L}\cong\mathbb{P}^3$ where each $\mathbb{F}_q$-member of $\mathcal{L}$ is irreducible over $\mathbb{F}_q$. We will show that the same linear system $\mathcal{L}$ has at most one geometrically reducible $\mathbb{F}_q$-members, establishing a proof of Theorem 3. We begin with reviewing the construction of $\mathcal{L}$.

The proof of [AGR24, Theorem 1.3(c)] is based on the existence of a point $P\in\mathbb{P}^2(\mathbb{F}_{q^6})$ such that P is not contained in any degree 2 curve C over $\mathbb{F}_q$ [AGR24, Theorem 1.1]. Equivalently, no conic defined over $\mathbb{F}_q$ contains the Galois orbit $S=\{P,P^{\sigma},...,P^{\sigma^5}\}$. Here, P^{σ} denotes the image of the point P under the Frobenius map $[x:y:z]\mapsto[x^q:y^q:z^q]$. For simplicity, let us write $P_i=P^{\sigma^i}$ so that $S=\{P_0,...,P_5\}$.

Recall that the dimension of the $\mathbb{F}_q$-vector space of cubic forms in 3 variables is 10. Imposing the condition that a cubic passes through a specific point imposes at most 1 linear condition on the coefficients. Since $S=\{P_0,...,P_5\}$ has 6 points and S is defined over $\mathbb{F}_q$ (despite P_i not individually defined over $\mathbb{F}_q$), the $\mathbb{F}_q$-vector subspace of all cubics passing through S has dimension at least $10-6=4$. Let F_0,F_1,F_2,F_3 denote four linearly independent cubic forms in $\mathbb{F}_q[x,y,z]$ each passing through all points of S.

Let $\mathcal{L}=\langle F_0,F_1,F_2,F_3\rangle\cong\mathbb{P}^3$ denote the 3-dimensional linear system of cubic curves passing through S. Let C be a reducible cubic curve (over $\overline{\mathbb{F}_q}$) which is an $\mathbb{F}_q$-member of $\mathcal{L}$. There are two ways in which a reducible cubic $C=L\cup Q$ can pass through the set S:

(a) Let L_{ij} be the line joining P_i and P_j and Q can vary in $\mathbb{P}^1$-worth of conics passing through the remaining 4 points.
(b) Let Q_i be the conic passing through 5 points in the set $S\setminus\{P_i\}$. Then L can vary in $\mathbb{P}^1$-worth of lines passing through the remaining point P_i.

However, if C is defined over $\mathbb{F}_q$, it must be the case that C is a union of three $\mathbb{F}_{q^3}$-lines, Galois conjugated by $\mathrm{Gal}(\mathbb{F}_{q^3}/\mathbb{F}_q)$. It is straightforward to see that *exactly one* one of these curves, namely $\overline{P_0P_3}\cup\overline{P_1P_4}\cup\overline{P_2P_5}$, is defined over $\mathbb{F}_q$. Hence, all $\mathbb{F}_q$-members of $\mathcal{L}$ are irreducible over $\mathbb{F}_q$ and exactly one $\mathbb{F}_q$-member of $\mathcal{L}$ fails to be geometrically irreducible.

3. Proof 2: Explicit Construction

The first proof relies on the existence of a point $P \in \mathbb{P}^2(\mathbb{F}_{q^6})$ which does not lie on any conic defined over $\mathbb{F}_q$. The proof of this assertion in [AGR24, Theorem 1.1] was obtained by an intricate counting argument and hence is nonconstructive by its nature. In this section, we offer an alternative proof of Theorem 3 which has the advantage of providing an explicit construction.

We start with a lemma on reducible cubic curves containing only the monomials x^2y, y^2z, z^2x, xyz.

Lemma 6. *Suppose $ax^2y + by^2z + cz^2x + dxyz = 0$ is a geometrically reducible cubic curve. Then $abc = 0$.*

Proof. The reducible cubic has a linear factor L. Without loss of generality, $L = x + \beta y + \gamma z$ for some scalars β, γ. If $a = 0$, then we are done. Hence, we may assume $a = 1$ after scaling. We have:

$$x^2y + by^2z + cz^2x + dxyz = LQ \tag{3.1}$$

for some quadratic factor Q. We match the coefficients on both sides of (3.1) to prove that $b = 0$. We proceed in five steps:

1. The cubic has no x^3 term, so Q has no x^2 term. The term x^2y can only be constructed from multiplying x from L with a term in xy from Q; thus, the coefficient of xy in Q must be 1.
2. If $\beta \neq 0$, then Q has no y^2 term; in that case, LQ has the term $(\beta y) \cdot xy$ which leads to the term xy^2 in the cubic that cannot be canceled, a contradiction. Therefore, $\beta = 0$.
3. The cubic has no xy^2 term and $\beta = 0$, so Q has no y^2 term. The cubic has no x^2z term and Q has no x^2 term, so Q has no xz term.
4. So, $Q = xy + \delta_1 yz + \delta_2 z^2$ and $L = x + \gamma z$. From (3.1), we see $\gamma\delta_1 = 0$. If $\gamma = 0$, then x divides the cubic, implying that $b = 0$, as desired.
5. If $\gamma \neq 0$, then we have $\delta_1 = 0$. In this case, (3.1) reads:
$$(x + \gamma z)(xy + \delta_2 z^2) = x^2y + by^2z + cz^2x + dxyz.$$
We obtain $b = 0$, as desired.

Thus, any geometrically reducible cubic of the form $ax^2y + by^2z + cz^2x + dxyz = 0$ satisfies $abc = 0$. $\square$

We will now present the second proof of our main theorem.

Proof of Theorem 3. Consider the linear system $\mathcal{L}_1 = \langle x^2y, y^2z, z^2x, xyz \rangle$. By the Normal Basis Theorem, there exists an element $\alpha \in \mathbb{F}_{q^3}$ such that $\alpha, \alpha^q, \alpha^{q^2}$ forms a basis of $\mathbb{F}_{q^3}$ as an $\mathbb{F}_q$-vector space. We construct a new linear system from $\mathcal{L}_1$ where x, y, and z are replaced by appropriate linear forms. Let

$$\begin{aligned}
F &= (\alpha x + \alpha^q y + \alpha^{q^2} z)^2(\alpha^q x + \alpha^{q^2} y + \alpha z), \\
G &= (\alpha^q x + \alpha^{q^2} y + \alpha z)^2(\alpha^{q^2} x + \alpha y + \alpha^q z), \\
H &= (\alpha^{q^2} x + \alpha y + \alpha^q z)^2(\alpha x + \alpha^q y + \alpha^{q^2} z), \\
T &= (\alpha x + \alpha^q y + \alpha^{q^2} z)(\alpha^q x + \alpha^{q^2} y + \alpha z)(\alpha^{q^2} x + \alpha y + \alpha^q z).
\end{aligned}$$

Consider the linear system $\mathcal{L}_2 = \langle F, G, H, T \rangle$. The Frobenius map $t \mapsto t^q$ sends $F \mapsto G \mapsto H \mapsto F$ and fixes T. Thus, the linear system $\mathcal{L}_2$ is defined over $\mathbb{F}_q$, meaning that we can find new generators $R_0, R_1, R_2, R_3 \in \mathbb{F}_q[x, y, z]$ with $\deg(R_i) = 3$ such that $\mathcal{L}_2 = \langle R_0, R_1, R_2, R_3 \rangle$. We claim that each $\mathbb{F}_q$-member of $\mathcal{L}_2$ is geometrically irreducible except the member $T \in \mathcal{L}_2$ which is a union of three lines conjugated by $\mathrm{Gal}(\mathbb{F}_{q^3}/\mathbb{F}_q)$. Indeed, we have a new coordinate system induced by the linear transformation:

$$\begin{aligned}
x' &= \alpha x + \alpha^q y + \alpha^{q^2} z \\
y' &= \alpha^q x + \alpha^{q^2} y + \alpha z \\
z' &= \alpha^{q^2} x + \alpha y + \alpha^q z
\end{aligned}$$

Applying Lemma 6 in the new coordinate system, we see that any geometrically reducible $\mathbb{F}_q$-member of $\mathcal{L}_2$ given by

$$aF + bG + cH + dT = 0,$$

satisfies $abc = 0$. After applying the Frobenius map $t \mapsto t^q$ twice and using the fact that T is defined over $\mathbb{F}_q$, we get two additional equations:

$$aG + bH + cF + dT = 0,$$
$$aH + bF + cG + dT = 0$$

Since $abc = 0$, at least one of a, b, c is zero. The three equations above and the linear independence of F, G, H, T imply $a = b = c = 0$. Hence, the only geometrically reducible $\mathbb{F}_q$-member of $\mathcal{L}_2$ is $\{T = 0\}$. Note that $\{T = 0\}$ is irreducible over $\mathbb{F}_q$. Thus, the linear system $\mathcal{L}_2$ satisfies the desired properties. $\square$

Appendix A: computational evidence for the conjecture

We verified Conjecture 2 for all $q \leq 11$ using SageMath [Sage21]. It suffices to randomly generate a cubic linear system $\mathcal{L} = \langle F_0, F_1, F_2, F_3 \rangle$ until all $\mathbb{F}_q$-members of $\mathcal{L}$ are geometrically irreducible. The table below lists the successful linear systems for $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$.

$q = 2$	
$F_0 = x^2y + x^2z + y^2z$	$F_2 = xy^2 + y^3 + xyz + xz^2$
$F_1 = x^3 + yz^2$	$F_3 = x^2y + xy^2 + xz^2 + z^3$
$q = 3$	
$F_0 = y^3 + x^2z + y^2z + yz^2 + z^3$	$F_2 = x^3 - x^2y - xy^2 + xz^2 - yz^2$
$F_1 = x^3 - xy^2 + y^2z - xz^2 + yz^2 - z^3$	$F_3 = -x^3 - x^2y + y^3 + x^2z - xz^2$
$q = 4$	
$F_0 = x^2y + y^3 + x^2z + xyz + yz^2$	$F_2 = x^3 + xy^2 + y^2z + xz^2 + yz^2$
$F_1 = x^2y + xyz + y^2z + z^3$	$F_3 = x^3 + yz^2$
$q = 5$	
$F_0 = 2x^2y + xy^2 + y^3 + xz^2 + yz^2$	$F_2 = 2x^3 + x^2y + xy^2 + y^3 - 2x^2z - xyz - y^2z + xz^2 + 2yz^2$
$F_1 = x^2y + 2xy^2 - 2y^3 - 2x^2z + 2y^2z - 2xz^2 - yz^2$	$F_3 = -2x^2y - 2xy^2 - x^2z - 2xyz + y^2z - xz^2 + 2z^3$
$q = 7$	
$F_0 = -x^3 - 3xy^2 + y^3 + 3y^2z + xz^2 - 2yz^2 + 3z^3$	$F_2 = x^3 - 2x^2y + y^3 - x^2z - 3xyz - 2y^2z + xz^2 - 3z^3$
$F_1 = 3x^3 - 3x^2y - 3xy^2 - 3y^3 + xyz - 2y^2z - 2z^3$	$F_3 = -3x^3 - 2x^2y + 2xy^2 + 2y^3 - 2x^2z - 2y^2z - xz^2 + 3z^3$
$q = 8$	
$F_0 = x^2y + y^2z + xz^2 + yz^2$	$F_2 = x^3 + x^2y + y^2z + xz^2 + z^3$
$F_1 = x^2y + xy^2 + xz^2 + z^3$	$F_3 = x^2y + y^3 + x^2z + xyz + xz^2 + yz^2 + z^3$
$q = 9$	
$F_0 = -x^3 + x^2y + y^3 + x^2z + xyz - y^2z + xz^2 - yz^2$	$F_2 = x^2y + xy^2 + x^2z + xz^2 + yz^2 + z^3$
$F_1 = xy^2 - x^2z - xyz - y^2z - z^3$	$F_3 = xy^2 - y^3 - x^2z + y^2z - yz^2$
$q = 11$	
$F_0 = -3x^3 - 5xy^2 + 2x^2z + 4y^2z - 2xz^2 - 4z^3$	$F_2 = 5x^3 + 3x^2y + y^3 - 2x^2z - 5xyz - y^2z - 5xz^2 - 3yz^2 - 4z^3$
$F_1 = x^3 + xy^2 + 2y^3 + 3x^2z + 4xyz - y^2z - 3xz^2 + 2yz^2 - z^3$	$F_3 = 2x^3 - 3x^2y + 4xy^2 + 2y^3 - 5x^2z + y^2z - 2xz^2 - yz^2 + z^3$

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