LINEAR SYSTEM OF GEOMETRICALLY IRREDUCIBLE PLANE CUBICS OVER FINITE FIELDS

SHAMIL ASGARLI AND DRAGOS GHIOCA

Abstract. We examine the maximum dimension of a linear system of plane cubic curves whose $\mathbb{F}_q$-members are all geometrically irreducible. Computational evidence suggests that such a system has a maximum (projective) dimension of 3. As a step towards the conjecture, we prove that there exists a 3-dimensional linear system $\mathcal{L}$ with at most one geometrically reducible $\mathbb{F}_q$-member.

1. Introduction

Let $\mathcal{P}$ describe a property of a degree d algebraic hypersurface in $\mathbb{P}^n$. In algebraic geometry and adjacent fields, we are often interested in measuring the likelihood of the property $\mathcal{P}$ for a "randomly chosen" hypersurface. When working over an infinite field, we can use Zariski dense open sets to show that property $\mathcal{P}$ holds generically. However, the situation over finite fields is more subtle since open sets in the relevant parameter space may not have any $\mathbb{F}_q$-points (despite being Zariski dense over $\overline{\mathbb{F}_q}$).

There are alternative methods to quantify how widespread a property $\mathcal{P}$ holds for hypersurfaces over finite fields. One method is to count the proportion of degree d hypersurfaces over $\mathbb{F}_q$ satisfying $\mathcal{P}$, and consider the asymptotic proportion (either as $d \to \infty$ or $q \to \infty$). As another natural metric, we can ask for the maximum size of a linear family $\mathcal{P}$ can carry. More precisely, we can pose the following question for each finite field $\mathbb{F}_q$, and positive integers d and n.

Question 1. What is the maximum value of $t \in \mathbb{N}$ for which there exist $\{F_i = 0\}$ for $i = 0, 1, ..., t$ such that $X_{[a_0:\cdots:a_t]} = \{a_0F_0 + \cdot + a_tF_t = 0\}$ satisfies the property $\mathcal{P}$ for all choices $[a_0 : a_1 : \ldots : a_t] \in \mathbb{P}^t(\mathbb{F}_q)$?

The question can be rephrased in the language of linear systems: what is the largest (projective) dimension of a linear system $\mathcal{L} \cong \mathbb{P}^t$ of degree d hypersurfaces in $\mathbb{P}^n$ such that each $\mathbb{F}_q$-member of $\mathcal{L}$ satisfies $\mathcal{P}$? An answer to Question 1 measures the extent to which the property $\mathcal{P}$ linearly propagates in the parameter space of all degree d hypersurfaces in $\mathbb{P}^n$. Larger values of t indicate higher levels of prevalence for $\mathcal{P}$.

Question 1 has been addressed in recent works for several specific choices of $\mathcal{P}$. For instance, $\mathcal{P}$ may correspond to the property of being smooth [AGR23], irreducible over $\mathbb{F}_q$ [AGR24], reducible over $\mathbb{F}_q$ [AGR24], or nonblocking with respect to $\mathbb{F}_q$-lines [AGY23]. To illustrate some of these results, we specialize to the setting of cubic plane curves, that is, $d = 3$ and $n = 2$. Below are two concrete examples from the recent literature where Question 1 has a known answer.

Theorem 2 ([AGR23]). *Let $\mathbb{F}_q$ be a finite field with characteristic $p \neq 3$. Then there is a 2-dimensional linear system $\mathcal{L}_{smo} = \langle F_0, F_1, F_2\rangle \cong \mathbb{P}^2$ of cubic plane curves such that every $\mathbb{F}_q$-member of $\mathcal{L}_{smo}$ is smooth. Moreover, no such 3-dimensional system exists.*

Theorem 3 ([AGR24]). *Let $\mathbb{F}_q$ be a finite field. Then there is a 3-dimensional linear system $\mathcal{L}_{irr} = \langle F_0, F_1, F_2, F_3\rangle \cong \mathbb{P}^3$ of cubic plane curves such that every $\mathbb{F}_q$-member of $\mathcal{L}_{irr}$ is irreducible over $\mathbb{F}_q$. Moreover, no such 4-dimensional system exists.*

Let us compare the two properties in Theorems 2 and 3. The smoothness condition refers to geometric smoothness, while irreducibility over $\mathbb{F}_q$ does not necessarily imply geometric irreducibility (i.e., irreducibility over $\overline{\mathbb{F}_q}$). This distinction naturally raises the goal of establishing a version of Theorem 3 where the conclusion is strengthened from irreducibility over $\mathbb{F}_q$ to irreducibility over $\overline{\mathbb{F}_q}$. The purpose of the present paper is to address this objective.

2020 *Mathematics Subject Classification.* Primary 14N05; Secondary 14C21, 14H50, 14G15.

Key words and phrases. linear system, cubic curves, geometric irreducibility, finite fields.

To summarize, we address Question 1 when $\mathcal{P}$ stands for "is geometrically irreducible" (that is, irreducible over $\overline{\mathbb{F}_q}$) for cubic plane curves: $d = 3$ and $n = 2$. In this special case, every linear system $\mathcal{L}$ of (projective) dimension 4 has an $\mathbb{F}_q$-member that is a reducible plane cubic over $\mathbb{F}_q$ by Theorem 3; see [AGR24, Theorem 1.3(d)] for further details. Hence, the answer to Question 1 in this setting is at most 3. We predict that the answer is exactly 3.

Conjecture 4. There exists a linear system $\mathcal{L} = \langle F_0, F_1, F_2, F_3 \rangle \cong \mathbb{P}^3$ of cubic plane curves where each $\mathbb{F}_q$-member of $\mathcal{L}$ is geometrically irreducible.

As mentioned earlier, Conjecture 4 strengthens Theorem 3. As partial progress, we establish the following.

Theorem 5. *There exists a linear system $\mathcal{L} = \langle F_0, F_1, F_2, F_3 \rangle$ of cubic plane curves where each $\mathbb{F}_q$-member of $\mathcal{L}$ is irreducible over $\mathbb{F}_q$ and there is at most one geometrically reducible $\mathbb{F}_q$-member of $\mathcal{L}$.*

To further motivate the problem and explain its arithmetic origin, we explain why the analogue of Conjecture 4 fails when the base field $\mathbb{F}_q$ is replaced by an algebraically closed field $\mathbb{K}$. Since a cubic form in three variables is described by 10 coefficients, the parameter space of cubic plane curves is $\mathbb{P}^9(\mathbb{K})$. Let $\mathcal{L} = \langle F_0, F_1, F_2, F_3 \rangle$ be a linear system where each $F_i \in \mathbb{K}[x, y, z]$ is a homogeneous polynomial of degree 3.

The set of reducible cubic curves forms a 7-dimensional subvariety Y of $\mathbb{P}^9(\mathbb{K})$. This can be seen as follows: any reducible cubic polynomial can be factored as

$$(a_0x + a_1y + a_2z)(b_0x^2 + b_1y^2 + b_2z^2 + b_3xy + b_4yz + b_5zx). \tag{1.1}$$

The variety Y is the image of the natural map $\mathbb{P}^2 \times \mathbb{P}^5 \to \mathbb{P}^9$ induced by the multiplication from (1.1). Since $\dim(Y) + \dim(\mathcal{L}) = 7 + 3 = 10$, the intersection $C := Y \cap \mathcal{L}$ has dimension at least 1 in $\mathbb{P}^9$. Because $\mathbb{K}$ is algebraically closed, we have $C(\mathbb{K}) \neq \emptyset$, meaning that $\mathcal{L}$ has at least one reducible $\mathbb{K}$-member. In particular, Conjecture 4 does *not* hold over $\mathbb{K}$.

For example, if we take $\mathbb{K} = \overline{\mathbb{F}_q}$, then any 3-dimensional linear system $\mathcal{L} = \langle F_0, F_1, F_2, F_3 \rangle \cong \mathbb{P}^3$ contains a reducible $\overline{\mathbb{F}_q}$-member. The subtlety of Conjecture 4 lies in the fact that, when $\mathcal{L}$ is defined over $\mathbb{F}_q$, the variety C, which is generically a curve, may lack $\mathbb{F}_q$-points. In fact, Conjecture 4 is equivalent to the following statement: there exists an $\mathbb{F}_q$-linear subspace $\mathcal{L} \cong \mathbb{P}^3$ in the parameter space $\mathbb{P}^9$ such that $Y \cap \mathcal{L}$ has no $\mathbb{F}_q$-points, where Y is the locus of reducible cubics. Viewed through this lens, the difficulty of Conjecture 4 is tied to finding a specific "pointless" curve inside a large-dimensional projective space. Furthermore, the above analysis also shows that the proportion of geometricaly irreducible cubic plane curves defined over $\mathbb{F}_q$ tends to 1 as $q \to \infty$ (since the number of geometrically *reducible* plane cubics defined over $\mathbb{F}_q$ is $O(q^7)$, while we have $O(q^9)$ plane cubics defined over $\mathbb{F}_q$).

While we focus on the case of cubic plane curves in the present paper, the same question applies to hypersurfaces of degree d in $\mathbb{P}^n$ for any d and n.

Problem 6. Determine the maximum (projective) dimension of a linear system $\mathcal{L}$ of degree d hypersurfaces in $\mathbb{P}^n$ such that each $\mathbb{F}_q$-member is geometrically irreducible.

By [AGR24, Theorem 1.3(d)], every linear system of dimension $\binom{n+d-1}{n-1}$ has an $\mathbb{F}_q$-member that is reducible over $\mathbb{F}_q$. Hence, the answer to Problem 6 is at most $\binom{n+d-1}{n-1} - 1$. On the other hand, by [AGR23, Theorem 2], when $p = \mathrm{char}(\mathbb{F}_q) \nmid \gcd(d, n+1)$, there exists an n-dimensional linear system whose $\mathbb{F}_q$-members are all smooth, hence geometrically irreducible. Hence, the answer to Problem 6 is at least n. We expect the true answer to Problem 6 to be closer to the upper bound $\binom{n+d-1}{n-1} - 1$. After all, we expect most geometrically reducible hypersurfaces defined over $\mathbb{F}_q$ to be reducible over $\mathbb{F}_q$.

We also have a related open problem with the condition "geometrically irreducible" relaxed to "not containing a linear component over $\overline{\mathbb{F}_q}$."

Problem 7. Determine the maximum (projective) dimension of a linear system $\mathcal{L}$ of degree d hypersurfaces in $\mathbb{P}^n$ such that each $\mathbb{F}_q$-member has no linear factor.

By definition, the answer to Problem 6 is less than or equal to the answer to Problem 7. It is reasonable to expect that the two answers agree, at least for all sufficiently large q (as a function of n and d). The heuristic is that most reducible hypersurfaces (over $\overline{\mathbb{F}_q}$) have a linear factor. Note that Conjecture 4 concerns the case $n = 2$ and $d = 3$ for which Problems 6 and 7 coincide.

While these open problems are new, we note that the study of reducible members in a linear system of algebraic hypersurfaces is rich in literature. One case that has been investigated thoroughly is the number of reducible (or totally reducible) hypersurfaces in a pencil of hypersurfaces [Lor93, Vis93, PY08]. The setting between the cited work and the present work differs in a few places. We only consider $\mathbb{F}_q$-members while the previous work is about controlling reducibility over $\overline{\mathbb{F}_q}$-members. On the other hand, we do not restrict our attention to pencils and allow large-dimensional linear systems.

Structure of the paper. We provide two proofs for Theorem 5. In Section 2, we provide a non-constructive proof in the spirit of the work done in [AGR24], while in Section 3 we provide an explicit construction of a 3-dimensional linear system as desired for the conclusion of Theorem 5. We believe both proofs (which are quite different in their approach) could be useful for pursuing Conjecture 4. Appendix A provides numerical evidence (computed using SageMath) that supports Conjecture 4 for all $q \leq 11$.

Acknowledgments. We thank Dino Lorenzini and the referee for valuable comments on the paper.

2. A non-constructive proof

In this section, we discuss the construction in our previous paper [AGR24] joint with Reichstein in the special case of plane cubic curves; this allows us to provide a first (non-constructive) proof of our Theorem 5.

We begin by recalling the motivation behind the main result of [AGR24] specialized to our context. Recall that the parameter space of plane conics corresponds to the projective space $\mathbb{P}^5$, as each conic can be represented by the equation

$$b_0x^2 + b_1y^2 + b_2z^2 + b_3xy + b_4yz + b_5zx = 0$$

for some coefficients $b_0, \ldots, b_5$. Imposing the additional condition that the conic passes through a specific point $Q_1 = [x_1 : y_1 : z_1]$ introduces one linear constraint on the coefficients $b_0, \ldots, b_5$. As a result, the space of conics passing through Q_1 forms a projective subspace of dimension 4, namely $\mathbb{P}^4$. Each time we require the conic to pass through a new point Q, we add another linear constraint on the coefficients. However, it is not guaranteed that these constraints will be linearly independent. We say that the points $Q_1, \ldots, Q_s$ (with $s < 6$) are in *general position* with respect to conics if the $\mathbb{F}_q$-vector space dimension of the space of conics passing through $Q_1, \ldots, Q_s$ is exactly $6 - s$. Over the algebraic closure, the set of such s-tuples forms a Zariski-dense open subset. In particular, when $s = 6$, if the six points are chosen from $\mathbb{P}^2(\overline{\mathbb{F}_q})$ in general position, then no conic passes through all of them. The main contribution of [AGR24] is to show that points in general position can be modeled as Galois orbits in a suitable sense. In the specific case of conics, it is possible to construct a set of six points in general position as the Galois orbit of a single point of degree 6, that is, a point with coordinates in $\mathbb{F}_{q^6}$.

More precisely, [AGR24, Theorem 1.1] asserts the existence of a point $P \in \mathbb{P}^2(\mathbb{F}_{q^6})$ such that P is not contained in any degree 2 curve C over $\mathbb{F}_q$ [AGR24, Theorem 1.1]. Equivalently, no conic defined over $\mathbb{F}_q$ contains the Galois orbit $S = \{P, P^\sigma, \ldots, P^{\sigma^5}\}$. Here, P^σ denotes the image of the point P under the Frobenius map $[x : y : z] \mapsto [x^q : y^q : z^q]$. For simplicity, let us write $P_i = P^{\sigma^i}$ so that $S = \{P_0, ..., P_5\}$.

Next, we follow [AGR24, Theorem 1.3(c)] to construct a linear system of cubics $\mathcal{L} \cong \mathbb{P}^3$ where each $\mathbb{F}_q$-member of $\mathcal{L}$ is irreducible over $\mathbb{F}_q$. We will show that the same linear system $\mathcal{L}$ has at most one geometrically reducible $\mathbb{F}_q$-members, establishing a proof of Theorem 5. To construct $\mathcal{L}$, recall that the dimension of the $\mathbb{F}_q$-vector space of cubic forms in 3 variables is 10. Imposing the condition that a cubic passes through a specific point imposes at most 1 linear condition on the coefficients. Since $S = \{P_0, ..., P_5\}$ has 6 points and S is defined over $\mathbb{F}_q$ (despite the fact that each P_i is not individually defined over $\mathbb{F}_q$), the $\mathbb{F}_q$-vector subspace of all cubics passing through S has dimension at least $10 - 6 = 4$. Let F_0, F_1, F_2, F_3 denote four linearly independent cubic forms in $\mathbb{F}_q[x, y, z]$ each passing through all points of S. Let $\mathcal{L} = \langle F_0, F_1, F_2, F_3 \rangle \cong \mathbb{P}^3$ denote the 3-dimensional linear system of cubic curves passing through S.

Let C be a reducible cubic curve (over $\overline{\mathbb{F}_q}$) which is an $\mathbb{F}_q$-member of $\mathcal{L}$. There are two ways in which a reducible cubic $C = L \cup Q$ can pass through the set S:

(a) Let L_{ij} be the line joining P_i and P_j and Q can vary in $\mathbb{P}^1$-worth of conics passing through the remaining 4 points.

(b) Let Q_i be the conic passing through 5 points in the set $S \setminus \{P_i\}$. Then L can vary in $\mathbb{P}^1$-worth of lines passing through the remaining point P_i.

However, if C is defined over $\mathbb{F}_q$, it must be the case that C is a union of three $\mathbb{F}_{q^3}$-lines, Galois conjugated by $\mathrm{Gal}(\mathbb{F}_{q^3}/\mathbb{F}_q)$; note that C is assumed to be geometrically reducible, while on the other hand, by [AGR24, Theorem 1.3(c)], C is irreducible over $\mathbb{F}_q$ since the entire $\mathbb{F}_q$-linear space $\mathcal{L}$ consists of $\mathbb{F}_q$-irreducible plane cubics. It is straightforward to see that *exactly one* one of these curves, namely $\overline{P_0P_3} \cup \overline{P_1P_4} \cup \overline{P_2P_5}$, is defined over $\mathbb{F}_q$. Hence, all $\mathbb{F}_q$-members of $\mathcal{L}$ are irreducible over $\mathbb{F}_q$ and exactly one $\mathbb{F}_q$-member of $\mathcal{L}$ fails to be geometrically irreducible. This completes the proof of Theorem 5.

3. An explicit construction

The first proof of Theorem 5 relies on the existence of a point $P \in \mathbb{P}^2(\mathbb{F}_{q^6})$ which does not lie on any conic defined over $\mathbb{F}_q$. The proof of this assertion in [AGR24, Theorem 1.1] was obtained by an intricate counting argument and hence is nonconstructive by its nature. In this section, we offer an alternative proof of Theorem 5 which has the advantage of providing an explicit construction. This latter method offers a new perspective on the problem and suggests future avenues for addressing higher-degree systems (as in Problems 6 and 7) through explicit methods.

We start with a lemma on reducible cubic curves containing only the monomials x^2y, y^2z, z^2x, xyz.

Lemma 8. *Suppose $ax^2y + by^2z + cz^2x + dxyz = 0$ is a geometrically reducible cubic curve. Then $abc = 0$.*

Proof. The reducible cubic has a linear factor L. Without loss of generality, $L = x + \beta y + \gamma z$ for some scalars β, γ. If $a = 0$, then we are done. Hence, we may assume $a = 1$ after scaling. We have:

$$x^2y + by^2z + cz^2x + dxyz = LQ \tag{3.1}$$

for some quadratic factor Q. We match the coefficients on both sides of (3.1) to prove that $b = 0$. We proceed in five steps:

1. The cubic has no x^3 term, so Q has no x^2 term. The term x^2y can only be constructed from multiplying x from L with a term in xy from Q; thus, the coefficient of xy in Q must be 1.
2. If $\beta \neq 0$, then Q has no y^2 term; in that case, LQ has the term $(\beta y) \cdot xy$ which leads to the term xy^2 in the cubic that cannot be canceled, a contradiction. Therefore, $\beta = 0$.
3. The cubic has no xy^2 term and $\beta = 0$, so Q has no y^2 term. The cubic has no x^2z term and Q has no x^2 term, so Q has no xz term.
4. So, $Q = xy + \delta_1 yz + \delta_2 z^2$ and $L = x + \gamma z$. From (3.1), we see $\gamma\delta_1 = 0$. If $\gamma = 0$, then x divides the cubic, implying that $b = 0$, as desired.
5. If $\gamma \neq 0$, then we have $\delta_1 = 0$. In this case, (3.1) reads:
$$(x + \gamma z)(xy + \delta_2 z^2) = x^2y + by^2z + cz^2x + dxyz.$$
We obtain $b = 0$, as desired.

Thus, any geometrically reducible cubic of the form $ax^2y + by^2z + cz^2x + dxyz = 0$ satisfies $abc = 0$. ☐

We will now present the second proof of our main theorem.

Proof of Theorem 5. Consider the linear system $\mathcal{L}_1 = \langle x^2y, y^2z, z^2x, xyz \rangle$. By the Normal Basis Theorem, there exists an element $\alpha \in \mathbb{F}_{q^3}$ such that $\alpha, \alpha^q, \alpha^{q^2}$ forms a basis of $\mathbb{F}_{q^3}$ as an $\mathbb{F}_q$-vector space. We construct a new linear system from $\mathcal{L}_1$ where x, y, and z are replaced by appropriate linear forms. Let

$$\begin{aligned}
F &= (\alpha x + \alpha^q y + \alpha^{q^2} z)^2(\alpha^q x + \alpha^{q^2} y + \alpha z),\\
G &= (\alpha^q x + \alpha^{q^2} y + \alpha z)^2(\alpha^{q^2} x + \alpha y + \alpha^q z),\\
H &= (\alpha^{q^2} x + \alpha y + \alpha^q z)^2(\alpha x + \alpha^q y + \alpha^{q^2} z),\\
T &= (\alpha x + \alpha^q y + \alpha^{q^2} z)(\alpha^q x + \alpha^{q^2} y + \alpha z)(\alpha^{q^2} x + \alpha y + \alpha^q z).
\end{aligned}$$

Consider the linear system $\mathcal{L}_2 = \langle F, G, H, T \rangle$. The Frobenius map $t \mapsto t^q$ sends $F \mapsto G \mapsto H \mapsto F$ and fixes T. Thus, the linear system $\mathcal{L}_2$ is defined over $\mathbb{F}_q$, meaning that we can find new generators

$R_0, R_1, R_2, R_3 \in \mathbb{F}_q[x,y,z]$ with $\deg(R_i) = 3$ such that $\mathcal{L}_2 = \langle R_0, R_1, R_2, R_3\rangle$. We claim that each $\mathbb{F}_q$-member of $\mathcal{L}_2$ is geometrically irreducible except the member $T \in \mathcal{L}_2$ which is a union of three lines conjugated by $\mathrm{Gal}(\mathbb{F}_{q^3}/\mathbb{F}_q)$. Indeed, we have a new coordinate system induced by the linear transformation:

$$x' = \alpha x + \alpha^q y + \alpha^{q^2} z$$
$$y' = \alpha^q x + \alpha^{q^2} y + \alpha z$$
$$z' = \alpha^{q^2} x + \alpha y + \alpha^q z$$

Applying Lemma 8 in the new coordinate system, we see that any geometrically reducible $\mathbb{F}_q$-member of $\mathcal{L}_2$ given by

$$aF + bG + cH + dT = 0,$$

satisfies $abc = 0$. After applying the Frobenius map $t \mapsto t^q$ twice and using the fact that T is defined over $\mathbb{F}_q$, we get two additional equations:

$$aG + bH + cF + dT = 0,$$
$$aH + bF + cG + dT = 0$$

Since $abc = 0$, at least one of a, b, c is zero. The three equations above and the linear independence of F, G, H, T imply $a = b = c = 0$. Hence, the only geometrically reducible $\mathbb{F}_q$-member of $\mathcal{L}_2$ is $\{T = 0\}$. Note that $\{T = 0\}$ is irreducible over $\mathbb{F}_q$. Thus, the linear system $\mathcal{L}_2$ satisfies the desired properties. ☐

Appendix A: computational evidence for the conjecture

We verified Conjecture 4 for all $q \leq 11$ using SageMath [Sage21]. It suffices to randomly generate a cubic linear system $\mathcal{L} = \langle F_0, F_1, F_2, F_3\rangle$ until all $\mathbb{F}_q$-members of $\mathcal{L}$ are geometrically irreducible. The following algorithm formalizes this procedure.

Algorithm 1: Verifying Conjecture 4 for $q \leq 11$

1 **Input:** Prime power q, the base field $\mathbb{F}_q$.
2 **Output:** A cubic linear system $\mathcal{L} = \langle F_0, F_1, F_2, F_3\rangle$ such that all $\mathbb{F}_q$-members are geometrically irreducible.
3 Repeat until all $\mathbb{F}_q$-members of $\mathcal{L}$ are geometrically irreducible:
4 Randomly generate coefficients $c_0, \ldots, c_9 \in \mathbb{F}_q$ to define a cubic form

$$F = c_0x^3 + c_1y^3 + c_2z^3 + c_3x^2y + c_4xy^2 + c_5y^2z + c_6yz^2 + c_7z^2x + c_8zx^2 + c_9xyz.$$

Construct four independent forms F_0, F_1, F_2, F_3 as above.
5 Define $\mathcal{L} = \langle F_0, F_1, F_2, F_3\rangle$.
6 For each $\mathbb{F}_q$-member of $\mathcal{L}$ parametrized by $\vec{a} = (a_0, a_1, a_2, a_3)$ with $a_i \in \mathbb{F}_q$, set:

$$F_{\vec{a}} = a_0F_0 + a_1F_1 + a_2F_2 + a_3F_3 \quad \text{where} \quad a_0, a_1, a_2, a_3 \in \mathbb{F}_q:$$

If $\{F_{\vec{a}} = 0\}$ is geometrically reducible, discard $\mathcal{L}$ and return to Step 4.
7 Return $\mathcal{L}$.

The following table lists the successful linear systems for $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$.

$q = 2$	
$F_0 = x^2y + x^2z + y^2z$	$F_2 = xy^2 + y^3 + xyz + xz^2$
$F_1 = x^3 + yz^2$	$F_3 = x^2y + xy^2 + xz^2 + z^3$

$q = 3$	
$F_0 = y^3 + x^2z + y^2z + yz^2 + z^3$	$F_2 = x^3 - x^2y - xy^2 + xz^2 - yz^2$
$F_1 = x^3 - xy^2 + y^2z - xz^2 + yz^2 - z^3$	$F_3 = -x^3 - x^2y + y^3 + x^2z - xz^2$

$q=4$	
$F_0 = x^2y + y^3 + x^2z + xyz + yz^2$	$F_2 = x^3 + xy^2 + y^2z + xz^2 + yz^2$
$F_1 = x^2y + xyz + y^2z + z^3$	$F_3 = x^3 + yz^2$

$q=5$	
$F_0 = 2x^2y + xy^2 + y^3 + xz^2 + yz^2$	$F_2 = 2x^3 + x^2y + xy^2 + y^3 - 2x^2z - xyz - y^2z + xz^2 + 2yz^2$
$F_1 = x^2y + 2xy^2 - 2y^3 - 2x^2z + 2y^2z - 2xz^2 - yz^2$	$F_3 = -2x^2y - 2xy^2 - x^2z - 2xyz + y^2z - xz^2 + 2z^3$

$q=7$	
$F_0 = -x^3 - 3xy^2 + y^3 + 3y^2z + xz^2 - 2yz^2 + 3z^3$	$F_2 = x^3 - 2x^2y + y^3 - x^2z - 3xyz - 2y^2z + xz^2 - 3z^3$
$F_1 = 3x^3 - 3x^2y - 3xy^2 - 3y^3 + xyz - 2y^2z - 2z^3$	$F_3 = -3x^3 - 2x^2y + 2xy^2 + 2y^3 - 2x^2z - 2y^2z - xz^2 + 3z^3$

$q=8$	
$F_0 = x^2y + y^2z + xz^2 + yz^2$	$F_2 = x^3 + x^2y + y^2z + xz^2 + z^3$
$F_1 = x^2y + xy^2 + xz^2 + z^3$	$F_3 = x^2y + y^3 + x^2z + xyz + xz^2 + yz^2 + z^3$

$q=9$	
$F_0 = -x^3 + x^2y + y^3 + x^2z + xyz - y^2z + xz^2 - yz^2$	$F_2 = x^2y + xy^2 + x^2z + xz^2 + yz^2 + z^3$
$F_1 = xy^2 - x^2z - xyz - y^2z - z^3$	$F_3 = xy^2 - y^3 - x^2z + y^2z - yz^2$

$q=11$	
$F_0 = -3x^3 - 5xy^2 + 2x^2z + 4y^2z - 2xz^2 - 4z^3$	$F_2 = 5x^3 + 3x^2y + y^3 - 2x^2z - 5xyz - y^2z - 5xz^2 - 3yz^2 - 4z^3$
$F_1 = x^3 + xy^2 + 2y^3 + 3x^2z + 4xyz - y^2z - 3xz^2 + 2yz^2 - z^3$	$F_3 = 2x^3 - 3x^2y + 4xy^2 + 2y^3 - 5x^2z + y^2z - 2xz^2 - yz^2 + z^3$

Interestingly, the linear system we found for $\mathbb{F}_8$ has coefficients in $\mathbb{F}_2 = \{0,1\}$, which means that the table entry corresponding to $q = 8$ also supports Conjecture 4 for $q = 2, 4, 8$. An intriguing question arises: for how large values of k can we find a linear system over $\mathbb{F}_q$ whose $\mathbb{F}_{q^k}$-members (not just $\mathbb{F}_q$-members) are geometrically irreducible? Such a result would provide an even stronger conclusion than Conjecture 4.

References

[AGR23] Shamil Asgarli, Dragos Ghioca, and Zinovy Reichstein, *Linear families of smooth hypersurfaces over finitely generated fields*, Finite Fields Appl. **87** (2023), Paper No. 102169, 10.

[AGR24] ———, *Linear system of hypersurfaces passing through a Galois orbit*, Res. Number Theory **10** (2024), no. 4, Paper No. 84, 16.

[AGY23] Shamil Asgarli, Dragos Ghioca, and Chi Hoi Yip, *Existence of pencils with nonblocking hypersurfaces*, Finite Fields Appl. **92** (2023), Paper No. 102283, 11.

[Lor93] Dino Lorenzini, *Reducibility of polynomials in two variables*, J. Algebra **156** (1993), no. 1, 65–75.

[PY08] J. V. Pereira and S. Yuzvinsky, *Completely reducible hypersurfaces in a pencil*, Adv. Math. **219** (2008), no. 2, 672–688.

[Sage21] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 9.4)*, 2021. `https://www.sagemath.org`.

[Vis93] Angelo Vistoli, *The number of reducible hypersurfaces in a pencil*, Invent. Math. **112** (1993), no. 2, 247–262.

Department of Mathematics and Computer Science, Santa Clara University, 500 El Camino Real, USA 95053
Email address: `sasgarli@scu.edu`

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2
Email address: `dghioca@math.ubc.ca`