THE DYNAMICAL MANIN-MUMFORD CONJECTURE AND THE DYNAMICAL BOGOMOLOV CONJECTURE FOR ENDOMORPHISMS OF $(\mathbb{P}^1)^n$

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ABSTRACT. We prove Zhang's Dynamical Manin-Mumford Conjecture and Dynamical Bogomolov Conjecture for dominant endomorphisms Φ of $(\mathbb{P}^1)^n$. We use the equidistribution theorem for points of small height with respect to an algebraic dynamical system, combined with an analysis of the symmetries of the Julia set for a rational function.

1. Introduction

1.1. **Notation.** As always in algebraic dynamics, given a self-map f on a variety X, we denote by f^n its n-th iterate (for any non-negative integer n, where f^0 denotes the identity map). We say that $x \in X$ is periodic if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$; we call x preperiodic if there exists $m \in \mathbb{N}$ such that $f^m(x)$ is periodic. Also, for a subvariety $V \subset X$, we say that V is periodic if $f^n(V)$ equals V for some $n \in \mathbb{N}$; similarly, we say that V is preperiodic if $f^m(V)$ is periodic.

1.2. **The Dynamical Manin-Mumford Conjecture.** Motivated by the classical Manin-Mumford conjecture (proved by Laurent [Lau84] in the case of tori, by Raynaud [Ray83] in the case of abelian varieties and by McQuillan [McQ95] in the general case of semiabelian varieties), Zhang formulated a dynamical analogue of this conjecture (see [Zha06, Conjecture 1.2.1]) for polarizable endomorphisms of any projective variety. We say that an endomorphism Φ of a projective variety X is *polarizable* if there exists an ample line bundle $\mathcal{L}$ on X such that $\Phi^*\mathcal{L}$ is linearly equivalent to $\mathcal{L}^{\otimes d}$ for some integer $d \geq 2$. As initially conjectured by Zhang, one might expect that if X is defined over a field K of characteristic 0 and Φ is a polarizable endomorphism of X, and the subvariety $V \subseteq X$ contains a Zariski dense set of preperiodic points, then V is preperiodic. We prove that Zhang's conjecture holds for dynamical systems $\left((\mathbb{P}^1)^n, \Phi\right)$, where Φ is given by the coordinatewise action

$$(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n)), \tag{1.2.1}$$

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where each rational function f_i is not a Lattés map. A *Lattés map* $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is a rational function coming from the quotient of an affine map $L(z) = az + b$ on a torus $\mathcal{T}$ (elliptic curve), i.e. $f = \Theta \circ L \circ \Theta^{-1}$ with $\Theta : \mathcal{T} \to \mathbb{P}^1$ a finite-to-one holomorphic map; see [Mil04] by Milnor. We prove the following result.

Theorem 1.1. *Let $f_1, \dots, f_n \in \mathbb{C}(x)$ be rational functions of degree $d \geq 2$, and let $\Phi : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$ be given by their coordinatewise action as in (1.2.1). Assume that none of the rational functions f_i is a Lattés map. If a subvariety $V \subseteq (\mathbb{P}^1)^n$ contains a Zariski dense set of preperiodic points under the action of Φ, then V is preperiodic.*

We will prove Theorem 1.1 as a consequence of a more general statement, which we will state in Section 1.3.

1.3. **Statement of our main results.** We first need to introduce the notion of *exceptional* rational functions; they are rational functions which commute with more functions of degree larger than one than a generic rational function does (note that generic rational functions commute only with their iterates). The first examples of such exceptional functions are the monomials x^d, and then related to them we have the Chebyshev polynomials. The Chebyshev polynomial of degree d is the unique polynomial T_d with the property that for each $z \in \mathbb{C}$, we have $T_d(z + 1/z) = z^d + 1/z^d$. For two rational functions f and g, we say they are (linearly) conjugate if there exists an automorphism η of $\mathbb{P}^1$ such that $f = \eta^{-1} \circ g \circ \eta$. We call *exceptional* any rational map of degree $d > 1$ which is conjugate either to $z^{\pm d}$, or to $\pm T_d(z)$, or to a Lattés map.

We prove the following result.

Theorem 1.2. *Let n be a positive integer, let $f_i \in \mathbb{C}(x)$ (for $i = 1, \dots, n$) be non-exceptional rational functions of degree $d_i \geq 2$, and let $V \subset (\mathbb{P}^1)^n$ be an irreducible subvariety defined over $\mathbb{C}$. Assume:*

(1) *either that V contains a Zariski dense set of preperiodic points under the action of $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$ and that $d_1 = d_2 = \cdots = d_n$;*
(2) *or that $f_1, \dots, f_n \in \bar{\mathbb{Q}}(x)$, that V is defined over $\bar{\mathbb{Q}}$, and that there exists a Zariski dense sequence of points $(x_{1,i}, \dots, x_{n,i}) \in V(\bar{\mathbb{Q}})$ such that $\lim_{i\to\infty} \sum_{j=1}^n \widehat{h}_{f_j}(x_{j,i}) = 0$, where $\widehat{h}_{f_j}$ is the canonical height with respect to the rational function f_j.*

Then there exists a finite set S of tuples

$$(i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}$$

along with $(\ell_i, \ell_j) \in \mathbb{N} \times \mathbb{N}$ and curves $C_{i,j} \subset \mathbb{P}^1 \times \mathbb{P}^1$ which are preperiodic under the coordinatewise action $(x_i, x_j) \mapsto \left(f_i^{\ell_i}(x_i), f_j^{\ell_j}(x_j)\right)$ such that:

(i) $\deg\left(f_i^{\ell_i}\right) = \deg\left(f_j^{\ell_j}\right)$*; and*

(ii) *V is an irreducible component of*

$$\bigcap_{(i,j)\in S} \pi_{i,j}^{-1}\left(C_{i,j}\right), \tag{1.3.1}$$

where $\pi_{i,j} : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^2$ is the projection on the (i,j)-th coordinate axes for each $(i,j) \in S$.

Our Theorem 1.2 answers Zhang's Dynamical Manin-Mumford Conjecture (over $\mathbb{C}$) and a slightly more general form of Zhang's Dynamical Bogomolov Conjecture (over $\bar{\mathbb{Q}}$) for endomorphisms of $(\mathbb{P}^1)^n$ (see [Zha06, Conjectures 1.2.1 and 4.1.7]). Note that any dominant (regular) endomorphism of $(\mathbb{P}^1)^n$ has an iterate which is of the form

$$\Phi := (f_1, \dots, f_n) : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n;$$

see also [GNY, Remark 1.2]. Our result is slightly stronger than the one conjectured in [Zha06] since in part (2) of Theorem 1.2 we do not assume the endomorphism $\Phi = (f_1, \dots, f_n)$ is necessarily polarizable (i.e., the rational maps f_i might have different degrees). On the other hand, we exclude the case when the f_i's are conjugate to monomials, ±Chebyshev polynomials, or Lattés maps since in those cases there are counterexamples to a formulation when Φ is not polarizable (see [GTZ11] and [GNY, Remark 1.2]). Moreover, if at least two of the maps f_i are Lattés, then even assuming Φ is polarizable, one would still have to impose an additional condition in order to get that the subvariety V is preperiodic (see [GTZ11, Theorem 1.2]). In our next result (see Theorem 1.3) we prove the appropriately modified statement of the Dynamical Manin-Mumford Conjecture (as formulated in [GTZ11, Conjecture 2.4]) for all polarizable endomorphisms of $(\mathbb{P}^1)^n$.

Theorem 1.3. *Let $n \in \mathbb{N}$, let $f_i \in \mathbb{C}(x)$ (for $i = 1, \dots, n$) be rational functions of degree $d > 1$, let $\Phi : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$ be defined by*

$$\Phi(x_1, \dots, x_n) = (f_1(x_1), \dots, f_n(x_n))$$

and let $V \subset (\mathbb{P}^1)^n$ be an irreducible subvariety. Assume there exists a Zariski dense set of smooth points $P = (a_1, \dots, a_n) \in V(\mathbb{C})$ which are preperiodic under Φ and moreover such that the tangent space $T_{V,P}$ of V at P is preperiodic under the induced action of Φ on $\mathrm{Gr}_{\dim(V)}\left(T_{(\mathbb{P}^1)^n,P}\right)$, where $\mathrm{Gr}_{\dim(V)}\left(T_{(\mathbb{P}^1)^n,P}\right)$ is the corresponding Grassmannian. Then the subvariety V must be preperiodic under the action of Φ.

1.4. **Brief history of previous results towards the Dynamical Manin-Mumford Conjecture and the Dynamical Bogomolov Conjecture.** Motivated by the classical Bogomolov conjecture (proved by Ullmo [Ull98] in the case of curves embedded in their Jacobian and by Zhang [Zha98] in the general case of abelian varieties), Zhang formulated a dynamical analogue also for this conjecture (see [Zha06, Conjecture 4.1.7]) for polarizable endomorphisms Φ of any projective variety X. So, if X is defined over a number field K then one can construct the canonical height $\widehat{h}_\Phi$ for all points in $X(\bar{\mathbb{Q}})$

with respect to the action of Φ (see [CS93] and also our Subsection 3.4) and then Zhang's dynamical version of the Bogomolov Conjecture asks that if a subvariety $V \subseteq X$ is not preperiodic, then there exists $\epsilon > 0$ with the property that the set of points $x \in V(\bar{\mathbb{Q}})$ such that $\widehat{h}_\Phi(x) < \epsilon$ is not Zariski dense in V. Since all preperiodic points have canonical height equal to 0, the Dynamical Bogomolov Conjecture is a generalization of the Dynamical Manin-Mumford Conjecture when the algebraic dynamical system (X, Φ) is defined over a number field.

Besides the case of abelian varieties X endowed with the multiplication-by-2 map Φ (which motivated Zhang's conjectures), there are known only a handful of special cases of the Dynamical Manin-Mumford or the Dynamical Bogomolov conjectures. All of these partial results are for curves contained in $\mathbb{P}^1 \times \mathbb{P}^1$—see [BH05, GT10, GTZ11, GNY]. We also mention here the paper of Dujardin and Favre [DF] who prove a result for plane polynomial automorphisms motivated by Zhang's Dynamical Manin-Mumford Conjecture. Our Theorem 1.2 is the first result towards the Dynamical Manin-Mumford and the Dynamical Bogomolov conjectures for higher dimensional subvarieties of $(\mathbb{P}^1)^n$.

The case $n = 2$ in Theorems 1.2 and 1.3 (i.e., V is a curve in $\mathbb{P}^1 \times \mathbb{P}^1$) was established in [GNY, Theorem 1.1 and 1.3]. Even though the general strategy in our present proof follows the one we employed in [GNY], there are significant new obstacles that we need to overcome; for more details, see Subsection 2.4.

1.5. **Preperiodic subvarieties.** The conclusion from Theorem 1.2 covers the main result of Medvedev's PhD thesis [Med07] (whose main findings were published in [MS14, Proposition 2.21]) who showed that any invariant subvariety $V \subset (\mathbb{P}^1)^n$ under the coordinatewise action of n non-exceptional rational functions must have the form (1.3.1). Our result is stronger than the results from [Med07, MS14] since we only assume that a subvariety $V \subset (\mathbb{P}^1)^n$ contains a Zariski dense set of preperiodic points under the action of $\Phi := (f_1, \dots, f_n)$ and then we derive that V must have the form (1.3.1) (see also our Theorem 1.4). Medvedev and Scanlon assume that V is invariant by Φ (or more generally, preperiodic under the action of Φ) and then using the model theory of difference fields, they conclude that V must have the form (1.3.1). We do not use model theory; instead, we use algebraic geometry (including the powerful Arithmetic Hodge Index Theorem of Yuan and Zhang [YZ16]) coupled with a careful analysis for the local symmetries of the Julia set of a rational function. We state below our formal result which covers the main result of [Med07] thus providing the form of any preperiodic subvariety in $(\mathbb{P}^1)^n$ under the split action of n non-exceptional rational functions.

Theorem 1.4. *Let $n \in \mathbb{N}$, let $f_1, \dots, f_n \in \mathbb{C}(x)$ be non-exceptional rational functions of degrees > 1, and let Φ be their coordinatewise action on $(\mathbb{P}^1)^n$. If $V \subset (\mathbb{P}^1)^n$ is a preperiodic subvariety under the action of Φ, then there exists* *a finite set S of pairs $(i,j) \in \{1,\dots,n\}\times\{1,\dots,n\}$ along with curves $C_{i,j} \subset \mathbb{P}^1 \times \mathbb{P}^1$ which are preperiodic under the coordinate wise action $(x_i,x_j) \mapsto (f_i(x_i), f_j(x_j))$ such that V is an irreducible component of $\bigcap_{(i,j)\in S} \pi_{i,j}^{-1}(C_{i,j})$, where $\pi_{i,j} : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^2$ is the projection on the (i,j)-th coordinate axes.*

1.6. **The Dynamical Pink-Zilber Conjecture.** Analogous to asking the Dynamical Manin-Mumford Conjecture as a dynamical variant of the classical Manin-Mumford conjecture, one could formulate a Dynamical Pink-Zilber Conjecture, at least in the case of split endomorphisms. The following statement is implicitly raised in [GN16].

Conjecture 1.5. *Let $n \in \mathbb{N}$, let $f_1,\dots,f_n \in \mathbb{C}(x)$ be non-exceptional rational functions of degrees > 1, and let $\Phi : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$ be their coordinatewise action $(x_1,\dots,x_n) \mapsto (f_1(x_1),\dots,f_n(x_n))$. For each $m \in \{0,\dots,n\}$, we let $\mathrm{Per}^{[m]}$ be the union of all irreducible subvarieties of $(\mathbb{P}^1)^n$ of codimension m, which are periodic under the action of Φ. If $V \subset (\mathbb{P}^1)^n$ is an irreducible subvariety which is not contained in a proper periodic subvariety of $(\mathbb{P}^1)^n$, then $V \cap \mathrm{Per}^{[\dim(V)+1]}$ is not Zariski dense in V.*

We exclude exceptional rational functions in Conjecture 1.5 since in those cases we rediscover the classical Pink-Zilber Conjecture; for more details on the Pink-Zilber Conjecture, see [Zan12].

In Conjecture 1.5, if $V \subset (\mathbb{P}^1)^n$ is an irreducible hypersurface, then we recover essentially the Dynamical Manin-Mumford Conjecture we proved in Theorem 1.2. Quite interestingly, the same Theorem 1.2 can be used (along with other results) in order to solve Conjecture 1.5 if $V \subset (\mathbb{P}^1)^n$ has dimension 1 or codimension 2 and each f_i is a polynomial defined over $\bar{\mathbb{Q}}$; see [GN].

1.7. **Plan for our paper.** In Section 2 we show that in order to prove Theorems 1.1, 1.2 and 1.3, it suffices to assume that $V \subset (\mathbb{P}^1)^n$ is a hypersurface which projects dominantly onto any subset of $(n-1)$ coordinate axes of $(\mathbb{P}^1)^n$. Thus we are left to prove our results for hypersurfaces H (see Theorem 2.2), which will be done over the remaining sections of our paper; the conclusion in Theorem 1.4 will follow from the ingredients we develop for proving Theorem 1.2.

In Sections 3 and 4 we set up our notation, state basic properties for the Julia set of a rational function, construct the heights associated to an algebraic dynamical system and define adelic metrized line bundles which are employed in the main equidistribution result (Theorem 4.1). We note that Theorem 4.1 (of Yuan [Yua08]) is a crucial ingredient in our proof. In Section 5 we prove that under the hypotheses of Theorem 2.2, the measures induced on the hypersuface $H \subset (\mathbb{P}^1)^n$ from the dynamical systems

$$\left((\mathbb{P}^1)^{n-1}, f_1 \times \cdots f_{i-1} \times f_{i+1} \times \cdots f_n\right)$$

are all equal (for $i = 1, \dots, n$). Also, in Section 5 we prove Proposition 5.2, which is a crucial step in our proof of our main results (for more details on this step and also on our overall proof strategy, see Subsection 2.4).

In Section 6 we show how to use the equality of the above measures to infer the preperiodicity of H, assuming also that H satisfies an additional technical hypothesis (see Theorem 6.1). In Section 7 we finalize the proof of Theorem 2.2 (and thus finish our proof for Theorems 1.1, 1.2 and 1.3). We conclude our paper by proving Theorem 1.4.

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2. Reduction to the case of hypersurfaces

In this section we present various reductions which we will employ in proving Theorems 1.1, 1.2 and 1.3. We also provide additional details regarding the overall strategy for our proof.

2.1. **Some reductions.** We start with the following important reduction.

Proposition 2.1. *It suffices to prove Theorems 1.1, 1.2 and 1.3 under the additional hypothesis that $V \subset (\mathbb{P}^1)^n$ is a hypersurface which projects dominantly onto any subset of $n-1$ coordinate axes.*

Proof. First we prove that it suffices to assume in each of the Theorems 1.1, 1.2 and 1.3 that $V \subset (\mathbb{P}^1)^n$ is a hypersurface. Indeed, we assume Theorems 1.1, 1.2 and 1.3 hold for all hypersurfaces and we derive the same conclusion for all subvarieties of $(\mathbb{P}^1)^n$. So, let $V \subset (\mathbb{P}^1)^n$ be an irreducible subvariety of dimension $D < n-1$ satisfying the hypotheses of either Theorem 1.1, or of Theorem 1.3, or hypothesis (1) (or (2)) of Theorem 1.2. Then there exist D coordinate axes (without loss of generality, we assume they are $x_1, \dots, x_D$) so that the projection π of $(\mathbb{P}^1)^n$ onto its first D coordinate axes remains dominant when restricted to V. For each $j = D+1, \dots, n$, we let π_j be the natural projection map of $(\mathbb{P}^1)^n$ on coordinates $x_1, \dots, x_D, x_j$, and we let $H_j := \pi_j(V)$. Then $H_j \subset (\mathbb{P}^1)^{D+1}$ is a hypersurface satisfying the hypotheses of Theorem 1.1, or of Theorem 1.3, or hypothesis (1) (or (2)) of Theorem 1.2 with respect to the coordinatewise action of the rational functions $f_1, \dots, f_D, f_j$. Furthermore, for each $j = D+1, \dots, n$, we let $\tilde{H}_j \subset (\mathbb{P}^1)^n$ be the hypersurface $H_j \times (\mathbb{P}^1)^{n-D-1} \subset (\mathbb{P}^1)^n$ (i.e., we insert a copy of $\mathbb{P}^1$ on each coordinate axis not included in the set $\{1, \dots, D, j\}$). Then also $\tilde{H}_j \subset (\mathbb{P}^1)^n$ is a hypersurface satisfying the hypotheses of either one of the three Theorems 1.1, 1.2 or 1.3. Let

$$\tilde{H} := \bigcap_{j=D+1}^{n} \tilde{H}_j; \tag{2.1.1}$$

clearly, $V \subset \tilde{H}$ and so, $D = \dim(V) \le \dim\left(\tilde{H}\right)$.

Since $\dim(V) = D$ and $\pi|_V : V \longrightarrow (\mathbb{P}^1)^D$ is a dominant morphism, then we conclude that there exists a Zariski open subset $U \subset (\mathbb{P}^1)^D$ such that for each $\alpha \in U$, the fiber $\pi^{-1}(\alpha)$ is finite. Therefore for each $\alpha \in U$ and for each $j = D+1, \dots, n$, we have that there exists a finite set $S_{\alpha,j}$ with the property that if $(a_1, \dots, a_n) \in \tilde{H}_j$ and $(a_1, \dots, a_D) = \alpha$, then $a_j \in S_{\alpha,j}$. Thus for each $\alpha \in U$, we have that there exist finitely many points $(a_1, \dots, a_n) \in \tilde{H}$ such that $(a_1, \dots, a_D) = \alpha$. Hence V is an irreducible component of $\tilde{H}$; moreover, any irreducible component W of $\tilde{H}$ for which $\pi|_W : W \longrightarrow (\mathbb{P}^1)^D$ is a dominant morphism has dimension D.

If Theorem 1.2 holds for hypersurfaces, then each hypersurface $\tilde{H}_j \subset (\mathbb{P}^1)^n$ must have the form (1.3.1) since each one of these hypersurfaces satisfies the hypotheses of Theorem 1.2. Actually, since each $\tilde{H}_j$ is a hypersurface, then we must have:

$$\tilde{H}_j = \pi_{i,j}^{-1}(C_{i,j})$$

for some curve $C_{i,j} \subset \mathbb{P}^1 \times \mathbb{P}^1$, which is preperiodic under the action of $(x_i, x_j) \mapsto \left(f_i^{\ell_i}(x_i), f_j^{\ell_j}(x_j)\right)$ for some $\ell_i, \ell_j \in \mathbb{N}$ with the property that $\deg\left(f_i^{\ell_i}\right) = \deg\left(f_j^{\ell_j}\right)$. Because V is an irreducible component of $\tilde{H}$ (see (2.1.1)), we obtain the desired conclusion in Theorem 1.2.

Now, if Theorems 1.1 or 1.3 hold for hypersurfaces, then each hypersurface $\tilde{H}_j \subset (\mathbb{P}^1)^n$ is preperiodic under the action of $\Phi := (f_1, \dots, f_n)$. Thus, also $\tilde{H}$ is preperiodic under the action of Φ (see (2.1.1)). Combining the following facts:

- $\tilde{H}$ is preperiodic;
- V is an irreducible component of $\tilde{H}$;
- each irreducible component W of $\tilde{H}$ for which $\pi|_W : W \longrightarrow (\mathbb{P}^1)^D$ is a dominant morphism has dimension D; and
- each variety $\Phi^m(V)$ (for $m \in \mathbb{N}$) projects dominantly onto $(\mathbb{P}^1)^D$,

we obtain that V itself must be preperiodic under the action of Φ, as desired.

Now, once we reduced proving Theorems 1.1, 1.2 and 1.3 to the case $V \subset (\mathbb{P}^1)^n$ is a hypersurface, we can reduce further to the special case when V projects dominantly onto each subset of $(n-1)$ coordinate axes. Indeed, assuming otherwise, then (without loss of generality) we may assume $V = \mathbb{P}^1 \times V_0$ for some hypersurface $V_0 \subset (\mathbb{P}^1)^{n-1}$. Therefore, it suffices to prove Theorems 1.2 and 1.3 for the subvariety $V_0 \subset (\mathbb{P}^1)^{n-1}$ under the coordinatewise action of the rational functions $f_2, \dots, f_n$. A simple induction on n finishes our proof. (Finally, as an aside, we observe that in light of [GNY, Theorem 1.1], then due to the reduction proved in Proposition 2.1, we have that Theorem 1.2 is equivalent with proving that if $n > 2$ and also if each f_i is non-exceptional, then there is no hypersurface $H \subset (\mathbb{P}^1)^n$ projecting dominantly onto each subset of $(n-1)$ coordinate axes of $(\mathbb{P}^1)^n$ such that H contains a Zariski dense set of preperiodic points; this is exactly what we will be proving in Theorems 2.2 and 6.1.) ☐

2.2. **A technical result.** The next result (proven in Section 7) in conjunction with [GNY, Theorem 1.1 and 1.3] yields the conclusions of each of the three Theorems 1.1, 1.2 and 1.3.

Theorem 2.2. *Let $n > 2$ be an integer, let $f_i \in \mathbb{C}(x)$ of degree $d_i \geq 2$ (for $i = 1, \dots, n$) and let $H \subset (\mathbb{P}^1)^n$ be an irreducible hypersurface projecting dominantly onto each subset of $(n-1)$ coordinate axes. If there is a Zariski dense sequence of points $(x_{1,i}, \dots, x_{n,i}) \in V(\mathbb{C})$ such that:*

(1) *either each $(x_{1,i}, \dots, x_{n,i})$ is preperiodic under the coordinatewise action of $(f_1, \dots, f_n)$ and also $d_1 = d_2 = \cdots = d_n$,*
(2) *or each $f_i \in \bar{\mathbb{Q}}(x)$ (for $i = 1, \dots, n$), V is defined over $\bar{\mathbb{Q}}$ and $\lim_{i\to\infty} \sum_{j=1}^{n} \widehat{h}_{f_j}(x_{j,i}) = 0$,*

then the following must hold:

(i) *either each $f_i(x)$ is conjugate to $x^{\pm d_i}$ or to $\pm T_{d_i}(x)$,*
(ii) *or each f_i is a Lattès map (for $i = 1, \dots, n$).*

2.3. **Our main results as consequences of the technical result.** We show next how to derive Theorems 1.1, 1.2 and 1.3 from Theorem 2.2.

Proof of Theorem 1.2. As shown in Proposition 2.1, it suffices to prove Theorem 1.2 for irreducible hypersurfaces V, which project dominantly onto each subset of $(n-1)$ coordinate axes of $(\mathbb{P}^1)^n$. Since no f_i is exceptional, then Theorem 2.2 yields that the case of such hypersurfaces is vacuously true when $n > 2$. The case of curves $V \subset (\mathbb{P}^1)^2$ is proven in [GNY, Theorem 1.1], which concludes our proof. ☐

Since both Theorems 1.1 and 1.3 have similar statements and proofs, we will show next in parallel how to derive these two results from Theorem 2.2.

Proof of both Theorems 1.1 and 1.3. Again using Proposition 2.1, it suffices to prove Theorems 1.1 and 1.3 for irreducible hypersurfaces V, which project dominantly onto each subset of $(n-1)$ coordinate axes of $(\mathbb{P}^1)^n$. The case $n = 2$ in Theorem 1.3 was already proven in [GNY, Theorem 1.3]. On the other hand, note that [GNY, Equation (11)] yields that for a plane curve (neither horizontal, nor vertical) which contains infinitely many preperiodic points under the coordinatewise action of two rational functions f_1 and f_2, if f_1 is conjugate to a monomial or ±Chebyshev polynomial, then also f_2 must be conjugated to a monomial or ±Chebyshev polynomial. Furthermore, if we assume $n > 2$, then Theorem 2.2 yields that

(i) either for each $i = 1, \dots, n$ we have that $f_i(x) = \nu_i^{-1}(x) \circ x^{\pm d} \circ \nu_i(x)$ or $f_i(x) = \nu_i^{-1}(x) \circ (\pm T_d(x)) \circ \nu_i(x)$ for some automorphisms $\nu_i : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$,

(ii) or each f_i is a Lattès map corresponding to some elliptic curve E_i (for $i = 1, \dots, n$).

Therefore, from now on, we work in both Theorems 1.1 and 1.3 under the assumption that either hypotheses (i) or (ii) above are met.

If condition (i) is satisfied, then at the expense of replacing V by $\tilde{\nu}(V)$, where $\tilde{\nu}$ is the automorphism of $(\mathbb{P}^1)^n$ given by

$$\tilde{\nu}(x_1, \dots, x_n) := (\nu_1(x_1), \dots, \nu_n(x_n)),$$

we may assume that each $f_i(x)$ is either $x^{\pm d}$ or $\pm T_d(x)$. Next, let $\mu : \mathbb{G}_m^n \longrightarrow (\mathbb{P}^1)^n$ be the morphism given by

$$\mu(x_1, \dots, x_n) =: (\mu_1(x_1), \dots, \mu_n(x_n)),$$

for rational functions μ_i which are:

- $\mu_i(x) = x$ if $f_i(x) = x^{\pm d}$; and
- $\mu_i(x) = x + \frac{1}{x}$ if $f_i(x) = \pm T_d(x)$.

Then there exists an irreducible subvariety W of $\mu^{-1}(V) \subset \mathbb{G}_m^n$ (projecting dominantly onto V through the map μ), which contains a Zariski dense set of preperiodic points under the action of $\Phi : \mathbb{G}_m^n \longrightarrow \mathbb{G}_m^n$ given by

$$(x_1, \dots, x_n) \mapsto \left(\pm x_1^{\pm d}, \dots, \pm x_n^{\pm d}\right).$$

Hence, W contains a Zariski dense set of torsion points of $\mathbb{G}_m^n$. Laurent's theorem [Lau84] (the original Manin-Mumford conjecture for tori) yields that W is a subtorus, thus preperiodic under the action of Φ. This proves that $V = \mu(W)$ is preperiodic under the action of

$$(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n)),$$

as desired in the conclusion of Theorem 1.1. Furthermore, we note that in this case, the conclusion of Theorem 1.3 holds without the extra hypothesis regarding the preperiodicity of the tangent subspaces under the corresponding induced action; we will only need this extra assumption when dealing with hypothesis (ii) above, i.e., when the maps are Lattés.

Now, we assume condition (ii) is verified and so, each f_i is a Lattès map which satisfies $p_i \circ \psi_i = f_i \circ p_i$ where $p_i : E_i \longrightarrow \mathbb{P}^1$ and $\psi_i : E_i \longrightarrow E_i$ are morphisms satisfying $\deg(\psi_1) = \deg(\psi_2) = \cdots = \deg(\psi_n)$ because the Lattès maps f_i have the same degree. Then there exists an irreducible component W of $p^{-1}(V) \subset \tilde{E} := \prod_{i=1}^n E_i$ (where $p : \tilde{E} \longrightarrow (\mathbb{P}^1)^n$ is the morphism given by $p_1 \times \cdots \times p_n$) with the property that it contains a Zariski dense set of (smooth) points P which are preperiodic under the action of the endomorphism $\tilde{\psi}$ of $\tilde{E}$ given by $\psi_1 \times \cdots \psi_n$, and moreover, the tangent space of W at P is preperiodic under the induced action of $\tilde{\psi}$ on $\mathrm{Gr}_{\dim(W)}\left(T_{\tilde{E},P}\right)$, where $T_{\tilde{E},P}$ is the tangent space of $\tilde{E}$ at P and $\mathrm{Gr}_{\dim(W)}\left(T_{\tilde{E},P}\right)$ is the corresponding Grassmannian. Since each ψ_i is an isogeny of E_i of same degree, we get that $\tilde{\psi}$ is a polarizable endomorphism of $\tilde{E}$ and so, [GTZ11, Theorem 2.1] yields that W is preperiodic under the action of $\tilde{\psi}$. Therefore $V = p(W)$ is preperiodic under the action of

$$(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n)),$$

as desired in the conclusion of Theorem 1.3. □

2.4. **Strategy for our proof.** The remaining sections of our paper are dedicated to proving Theorem 2.2. The setup is as follows:

- $n > 2$ and $H \subset (\mathbb{P}^1)^n$ is a hypersurface projecting dominantly onto each subset of $(n-1)$ coordinate axes;
- $f_1, \dots, f_n$ are rational functions of degrees larger than 1 acting coordinatewise on $(\mathbb{P}^1)^n$; and
- H contains a Zariski dense set either of preperiodic points or of points of small height (see (2) in Theorem 1.2) under the action of $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$.

If at least one of the functions f_i is not exceptional, then we will derive a contradiction. Now, if some f_i is conjugated to a monomial or ±Chebyshev polynomial, then we prove that each of the n rational functions must be conjugated to a monomial or ±Chebyshev polynomial. Similarly, if one of the f_i's is a Lattés map, then we prove that each f_i must be a Lattés map. We obtain this goal (see Theorem 6.1) by showing a similitude between the Julia sets of each one of the rational functions f_i. In turn, the relation between the Julia sets is a consequence of a powerful equidistribution theorem for points of small height.

More precisely, using the equidistribution theorem of [Yua08] for points of small height on a variety (see [CL06] for the case of curves and also [BR06] and [FRL06] for the case of $\mathbb{P}^1$), we prove that under the above hypotheses for H and the f_i's, then the measures $\hat{\mu}_i$ induced on H by the invariant measures corresponding to the dynamical systems

$$\left((\mathbb{P}^1)^{n-1}, f_1 \times \cdots f_{i-1} \times f_{i+1} \times \cdots f_n\right)$$

are equal (for each $i = 1, \dots, n$). Using a careful study of the local analytic maps which preserve (locally) the Julia set of a rational map (which is not exceptional), we obtain the conclusion of Theorem 1.2. Even though our arguments resemble the ones we employed in [GNY] to treat the case of plane curves (i.e., $n = 2$), there are significant new complications in our analysis.

Indeed, using Yuan's arithmetic equidistribution theorem [Yua08] for points with small height on a space of dimension $n \geq 2$, we first get connections for the $(n-1, n-1)$-currents (coming from dynamics) on a hypersurface $H \subset (\mathbb{P}^1)^n$. From these connections, we are able to construct many symmetries for the aforementioned $(n-1, n-1)$-current. A further analysis of the symmetries for such an $(n-1, n-1)$-current yields additional symmetries of the Julia set on the 1-dimensional slices of $(\mathbb{P}^1)^n$. Applying the rigidity of the symmetries of the Julia set on the 1-dimensional slices, we are able to derive the rigidity of the symmetries of the entire $(n-1, n-1)$-current, from which we derive the desired conclusion regarding H and the dynamical system $(f_1, \dots, f_n)$ (see the proof of Theorem 6.1). It is precisely the study of the rigidity of this $(n-1, n-1)$-current (for $n > 2$) which provides the new proof of Medvedev's result [Med07], which otherwise could not have been obtained from the arguments from our previous paper [GNY].

Also, in order to finish the proof of Theorem 2.2 by showing that the hypotheses of Theorem 6.1 are met, we need to know that for a hypersurface $H \subset (\mathbb{P}^1)^n$ as in Theorem 2.2, for each point $(a_1, \dots, a_n) \in H$,

(2.4.1) if $a_1, \dots, a_{n-1}$ are preperiodic, then also a_n is preperiodic.

If $n = 2$, this fact was known for quite some time (see [Mim13] which publishes the findings of Mimar's PhD thesis [Mim97] from 20 years ago). However, if $n > 2$, in order to prove (2.4.1) (see our Proposition 5.2 in the case each f_i and also H are defined over $\bar{\mathbb{Q}}$), we need to use arithmetic versions of the Hodge Index Theorem proved by Faltings [Fal84] and by Hriljac [Hri85] for arithmetic surfaces and proved by Moriwaki [Mor96] for higher dimensional arithmetic varieties, and also, we use crucially the new Arithmetic Hodge Index Theorem proved by Yuan and Zhang [YZ16]. Furthermore, in order to derive (2.4.1) in the general case (over $\mathbb{C}$) we need a specialization argument based on a result of Yuan-Zhang [YZa, YZb] regarding the specialization of a Zariski dense set of preperiodic points for a polarizable endomorphism defined over a base curve.

3. Complex dynamics and height functions

In this section, we introduce the Julia set of a rational function, some of its properties and also the arithmetic height functions associated to an algebraic dynamical system.

3.1. **The Julia set.** Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function defined over $\mathbb{C}$ of degree $d_f \geq 2$. The *Julia set* J_f is the set of points $x \in \mathbb{P}^1_{\mathbb{C}}$ for which the dynamics is chaotic under the iteration of f. The Julia set J_f is closed, nonempty and invariant under f. Let x be a periodic point in a cycle of exact period n; then the *multiplier* λ of this cycle (or of the periodic point x) is the derivative of f^n at x. A cycle is *repelling* if its multiplier has absolute value greater than 1. All but finitely many cycles of f are repelling, and repelling cycles are in the Julia set J_f. Locally, at a repelling fixed point x with multiplier λ, we can conjugate f to the linear map $z \to \lambda \cdot z$ near $z = 0$ (note that $\lambda \neq 0$ since the point is assumed to be repelling). For more details about the dynamics of a rational function, we refer the reader to Milnor's book [Mil00].

There is a probability measure μ_f on $\mathbb{P}^1_{\mathbb{C}}$ associated to f, which is the unique f-invariant measure achieving maximal entropy $\log d_f$; see [Bro65, Lyu83, FLM83, Man83]. Also μ_f is the unique measure satisfying

$$\mu_f(f(A)) = d_f \cdot \mu_f(A) \tag{3.1.1}$$

for any Borel set $A \subset \mathbb{P}^1_{\mathbb{C}}$ with f injective when restricted on A. The support of μ_f is J_f, and $\mu_f(x) = 0$ for any $x \in \mathbb{P}^1_{\mathbb{C}}$. Moreover, μ_f has continuous potential, in the sense that locally there is a continuous subharmonic function $u(x)$ such that the $(1,1)$-current satisfies

$$dd^c u(x) = d\mu_f(x),$$

and then (3.1.1) is equivalent to

$$dd^c u \circ f(x) = d_f \cdot d\mu_f(x).$$

3.2. **Measures on a hypersurface associated to a dynamical system.** Let

$$\hat{f}(x_1, \cdots, x_n) := (f_1(x_1), \cdots, f_n(x_n))$$

be an endormorphism of $(\mathbb{P}^1_{\mathbb{C}})^n$ with f_i being a rational function of degree $d_i \geq 2$ for $1 \leq i \leq n$. For $i = 1, \cdots, n$, denote

$$\tilde{f}_i := (f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n) \tag{3.2.1}$$

as an endormorphism of $(\mathbb{P}^1_{\mathbb{C}})^{n-1}$ with invariant measure

$$\tilde{\mu}_i := \mu_{f_1} \times \cdots \mu_{f_{i-1}} \times \mu_{f_{i+1}} \times \cdots \mu_{f_n}. \tag{3.2.2}$$

Let $H \subset (\mathbb{P}^1_{\mathbb{C}})^n$ be an irreducible hypersurface projecting dominantly onto any subset of $(n-1)$ coordinates, i.e., the canonical projections $\hat{\pi}_i : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$ (where for each $i = 1, \cdots, n$, $\hat{\pi}_i$ is the projection of $(\mathbb{P}^1)^n$ onto the $(n-1)$ coordinates forgetting the i-th coordinate axis) restrict to dominant morphisms $(\hat{\pi}_i)|_H : H \longrightarrow (\mathbb{P}^1)^{n-1}$. By abuse of notation, we denote the restriction $(\hat{\pi}_i)|_H$ also by $\hat{\pi}_i$. We define probability measures $\hat{\mu}_i$ (for $i = 1, \cdots, n$) on H corresponding to the dynamical system $\left((\mathbb{P}^1_{\mathbb{C}})^{n-1}, \tilde{f}_i\right)$. More precisely, for each $i = 1, \cdots, n$, we pullback $\tilde{\mu}_i$ by $\hat{\pi}_i$ to get a measure $\hat{\pi}_i^* \mu_i$ on H so that

$$\hat{\pi}_i^* \tilde{\mu}_i(A) := \tilde{\mu}_i(\hat{\pi}_i(A))$$

for any Borel set $A \subset H$ such that $\hat{\pi}_i$ is injective on A. Another way to interpret this is that for $t = (x_1, \cdots, x_n) \in H$, we have that $d(\hat{\pi}_i^* \tilde{\mu}_i)(t)$ is an $(n-1, n-1)$-current on H given by

$$d\hat{\pi}_i^* \tilde{\mu}_i(t) = dd^c u_1(x_1) \wedge \cdot \cdot dd^c u_{i-1}(x_{i-1}) \wedge dd^c u_{i+1}(x_{i+1}) \cdot \cdot \wedge dd^c u_n(x_n)$$

where u_j is a locally defined continuous subharmonic function with $dd^c u_j = d\mu_{f_j}$ for each $j = 1, \ldots, n$. Hence we get the probability measures on H:

$$\hat{\mu}_i := \hat{\pi}_i^* \tilde{\mu}_i / \deg(\hat{\pi}_i) \text{ for } i = 1, \cdots, n.$$

Similarly, one has that

$$\tilde{f}_i^* \tilde{\mu}_i = d_1 \cdots d_{i-1} \cdot d_{i+1} \cdots d_n \cdot \tilde{\mu}_i \text{ for } i = 1, \cdots, n.$$

3.3. **Symmetries of the Julia set.** Let ζ be a meromorphic function on some disc $B(a, r)$ of radius r centred at a point $a \in J_f$. We say that ζ is a *symmetry* on J_f if it satisfies the following properties:

- $x \in B(a, r) \cap J_f$ if and only if $\zeta(x) \in \zeta(B(a, r)) \cap J_f$; and
- if J_f is either a circle, a line segment, or the entire sphere, there is a constant $\alpha > 0$ such that for any Borel set A where $\zeta|_A$ is injective, one has $\mu_f(\zeta(A)) = \alpha \cdot \mu_f(A)$.

A family $\mathcal{S}$ of symmetries of J_f on $B(a, r)$ is said to be *nontrivial* if $\mathcal{S}$ is normal on $B(a, r)$ and no infinite sequence $\{\zeta_n\} \subset \mathcal{S}$ converges to a constant function. A rational function is *post-critically finite* (sometimes called critically finite), if each of its critical points has finite forward orbit, i.e. all critical points are preperiodic. According to Thurston [Thu85, DH93], there is an orbifold structure on $\mathbb{P}^1$ corresponding to each post-critically finite map. A rational function is post-critically finite with *parabolic* orbifold if and only if it is exceptional; or equivalently its Julia set is smooth (a circle, a line segment or the entire sphere) with smooth maximal entropy measure on it; see [DH93].

3.4. **The height functions.** Let K be a number field and $\overline{K}$ be the algebraic closure of K. The number field K is naturally equipped with a set Ω_K of pairwise inequivalent nontrivial absolute values, together with positive integers N_v for each $v \in \Omega_K$ such that

- for each $\alpha \in K^*$, we have $|\alpha|_v = 1$ for all but finitely many places $v \in \Omega_K$.
- every $\alpha \in K^*$ satisfies the *product formula*

$$\prod_{v \in \Omega_K} |\alpha|_v^{N_v} = 1 \tag{3.4.1}$$

For each $v \in \Omega_K$, let K_v be the completion of K at v, let $\overline{K}_v$ be the algebraic closure of K_v and let $\mathbb{C}_v$ denote the completion of $\overline{K}_v$. We fix an embedding of $\overline{K}$ into $\mathbb{C}_v$ for each $v \in \Omega_K$; hence we have a fixed extension of $|\cdot|_v$ on $\overline{K}$. If v is archimedean, then $\mathbb{C}_v \cong \mathbb{C}$. Let $f \in K(z)$ be a rational function with degree $d \geq 2$. There is a *canonical height* $\widehat{h}_f$ on $\mathbb{P}^1(\overline{K})$ given by

$$\widehat{h}_f(x) := \frac{1}{[K(x) : K]} \lim_{n\to\infty} \sum_{T \in \mathrm{Gal}(\overline{K}/K)\cdot X} \sum_{v \in \Omega_K} N_v \cdot \frac{\log \|F^n(T)\|_v}{d^n} \tag{3.4.2}$$

where $F : K^2 \to K^2$ and X are homogenous lifts of f and respectively $x \in \mathbb{P}^1(\overline{K})$, while $\|(z_1, z_2)\|_v := \max\{|z_1|_v, |z_2|_v\}$. By product formula (3.4.1), the height $\widehat{h}_f$ does not depend on the choice of the homogenous lift F and therefore it is well-defined. As proven in [CS93], $\widehat{h}_f(x) \geq 0$ with equality if and only if x is preperiodic under the iteration of f.

4. Adelic metrized line bundles and the equidistribution of points of small height

In this section, we setup the height functions and state the equidistribution theorem for points of small height, which would be used later in proving the main theorems of this article. The main tool we use here is the arithmetic equidistribution theorem for points with small height on algebraic varieties (see [Yua08]).

4.1. Adelic metrized line bundle.

Let $\mathcal{L}$ be an ample line bundle of an irreducible projective variety V over a number field K. As in Subsection 3.4, K is naturally equipped with absolute values $|\cdot|_v$ for $v \in \Omega_K$. A *metric* $\|\cdot\|_v$ on $\mathcal{L}$ is a collection of norms, one for each $t \in V(K_v)$, on the fibres $\mathcal{L}(t)$ of the line bundle, with

$$\|\alpha s(t)\|_v = |\alpha|_v \|s(t)\|_v$$

for any section s of $\mathcal{L}$. An *adelic (semipositive) metrized line bundle* $\overline{\mathcal{L}} = \{\mathcal{L}, \{\|\cdot\|_v\}_{v\in\Omega_K}\}$ over $\mathcal{L}$ is a collection of metrics on $\mathcal{L}$, one for each place $v \in \Omega_K$, satisfying certain continuity and coherence conditions; see [Zha95a, Zha95b, YZ16].

There are various adelic metrized line bundles; the simplest adelic (semipositive) metrized line bundle is the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ equipped with metrics $\|\cdot\|_v$ (for each $v \in \Omega_K$), which evaluated at a section $s := u_0Z_0 + u_1Z_1$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ (where u_0, u_1 are scalars and Z_0, Z_1 are the canonical sections of $\mathcal{O}_{\mathbb{P}^1}(1)$) is given by

$$\|s([z_0 : z_1])\|_v := \frac{|u_0z_0 + u_1z_1|_v}{\max\{|z_0|_v, |z_1|_v\}}.$$

Furthermore, we can define other metrics on $\mathcal{O}_{\mathbb{P}^1}(1)$ corresponding to a rational function f of degree $d \geq 2$ defined over K. We fix a homogenous lift $F : K^2 \to K^2$ of f with homogenous degree d. For $j \geq 1$, write $F^j = (F_{0,j}, F_{1,j})$. For each place $v \in \Omega_K$, we can define a metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ as

$$\|s([z_0 : z_1])\|_{v,F,j} := \frac{|u_0z_0 + u_1z_1|_v}{\max\{|F_{0,j}(z_0, z_1)|_v, |F_{1,j}(z_0, z_1)|_v\}^{1/d^j}}, \tag{4.1.1}$$

where $s = u_0Z_0 + u_1Z_1$ with u_0, u_1 scalars and Z_0, Z_1 canonical sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Hence $\{\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_{v,F,j}\}_{v\in\Omega_K}\}$ is an adelic metrized line bundle over $\mathcal{O}_{\mathbb{P}^1}(1)$.

A sequence $\{\mathcal{L}, \{\|\cdot\|_{v,j}\}_{v\in\Omega_K}\}_{j\geq1}$ of adelic metrized line bundles over an ample line bundle $\mathcal{L}$ on a variety V is convergent to $\{\mathcal{L}, \{\|\cdot\|_v\}_{v\in\Omega_K}\}$, if for all j and all but finitely many $v \in \Omega_K$, we have that $\|\cdot\|_{v,j} = \|\cdot\|_v$, and moreover, $\left\{\log \frac{\|\cdot\|_{v,j}}{\|\cdot\|_v}\right\}_{j\geq1}$ converges to 0 uniformly on $V(\mathbb{C}_v)$ for all $v \in \Omega_K$. The limit $\{\mathcal{L}, \{\|\cdot\|_v\}_{v\in\Omega_K}\}$ is an adelic metrized line bundle. Also, the tensor product of two (adelic) metrized line bundles is again a (adelic) metrized line bundle.

A typical example of a convergent sequence of adelic metrized line bundles is $\{\{\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_{v,F,j}\}_{v\in\Omega_K}\}\}_{j\geq 1}$ which converges to the metrized line bundle denoted by

$$\overline{\mathcal{L}}_F := \{\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_{v,F}\}_{v\in\Omega_K}\} \tag{4.1.2}$$

(see [BR06] and also see [Zha95b, Theorem 2.2] for the more general case of a polarizable endomorphism f of a projective variety).

As usual, we let $\tilde{f} = (f_1, \cdots, f_n)$ with f_i being a rational function of degree $d_i \geq 2$ defined over the number field K for $1 \leq i \leq n$. Fix a homogenous lift F_i for each f_i and denote

$$\tilde{F} := (F_1, \cdots, F_n).$$

We let π_i be the i-th coordination projection map from $(\mathbb{P}^1)^n$ to $\mathbb{P}^1$. We construct an adelic metrized line bundle on $(\mathbb{P}^1)^n$ as follows

$$\overline{\mathcal{L}}_{\tilde{F}} := \{\mathcal{L}_{\tilde{F}}, \|\cdot\|_{v,\tilde{F}}\} := (\pi_1^*\overline{\mathcal{L}}_{F_1}) \otimes (\pi_2^*\overline{\mathcal{L}}_{F_2}) \cdots \otimes (\pi_n^*\overline{\mathcal{L}}_{F_n}). \tag{4.1.3}$$

where the metric $\|\cdot\|_{v,\tilde{F}}$ on $\mathcal{L}_{\tilde{F}}$ is the one inherited from the metrics $\|\cdot\|_{v,F_i}$ on $\mathcal{O}_{\mathbb{P}^1}(1)$ for $1 \leq i \leq n$.

4.2. **Equidistribution of small points.** For a semipositive metrized line bundle $\overline{\mathcal{L}}$ on a (irreducible and projective) variety V defined over a number field K, the height for $t \in V(\overline{K})$ is given by

$$\widehat{h}_{\overline{\mathcal{L}}}(t) = \frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot t|} \sum_{y\in \operatorname{Gal}(\overline{K}/K)\cdot t} \sum_{v\in\Omega_K} -N_v \cdot \log\|s(y)\|_v \tag{4.2.1}$$

where $|\operatorname{Gal}(\overline{K}/K)\cdot t|$ is the number of points in the Galois orbits of t, and s is any meromorphic section of $\mathcal{L}$ with support disjoint from $\operatorname{Gal}(\overline{K}/K)\cdot t$. A sequence of points $t_j \in V(\overline{K})$ is *small* if $\lim_{j\to\infty} \widehat{h}_{\overline{\mathcal{L}}}(t_j) = \widehat{h}_{\overline{\mathcal{L}}}(V)$, and is *generic* if no subsequence of t_j is contained in a proper Zariski closed subset of V; see [Zha95b] for more details on constructing the height for any irreducible subvariety Y of V (which is denoted by $\widehat{h}_{\overline{\mathcal{L}}}(Y)$). We use the following equidistribution result due to Yuan [Yua08] in the case of an arbitrary projective variety.

Theorem 4.1. [Yua08, Theorem 3.1] *Let V be a projective irreducible variety of dimension n defined over a number field K, and let $\overline{\mathcal{L}}$ be a metrized line bundle over V such that $\mathcal{L}$ is ample and the metric is semipositive. Let $\{t_n\}$ be a generic sequence of points in $V(\overline{K})$ which is small. Then for any $v \in \Omega_K$, the Galois orbits of the sequence $\{t_j\}$ are equidistributed in the analytic space $V^{an}_{\mathbb{C}_v}$ with respect to the probability measure $d\mu_v = c_1(\overline{\mathcal{L}})^n_v/\deg_{\mathcal{L}}(V)$.*

When v is archimedean, $V^{an}_{\mathbb{C}_v}$ corresponds to $V(\mathbb{C})$ and the curvature $c_1(\overline{\mathcal{L}})_v$ of the metric $\|\cdot\|_v$ is given by $c_1(\overline{\mathcal{L}})_v = \frac{\partial\overline{\partial}}{\pi i}\log\|\cdot\|_v$. If v is a non-archimedean place, then $V^{an}_{\mathbb{C}_v}$ is the Berkovich space associated to $V(\mathbb{C}_v)$, and Chambert-Loir [CL06] constructed an analog for the curvature on $V^{an}_{\mathbb{C}_v}$.

The precise meaning of the equidistribution statement in Theorem 4.1 is that

$$\lim_{j\to\infty}\frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot t_j|}\sum_{y\in\operatorname{Gal}(\overline{K}/K)\cdot t_j}\delta_y=\mu_v, \tag{4.2.2}$$

where δ_y is the point mass probability measure supported on $y\in V^{an}_{\mathbb{C}_v}$, while the limit from (4.2.2) is the weak limit for the corresponding probability measures on the compact space $V^{an}_{\mathbb{C}_v}$.

4.3. **Some examples.** For the dynamical system $(\mathbb{P}^1, f)$ corresponding to a rational function f defined over a number field K and of degree $d_f\ge 2$, at an archimedean place v, it is well known that the curvature of the limit of the metrized line bundles

$$\{\mathcal{O}_{\mathbb{P}^1}(1),\{\|\cdot\|_{v,F,j}\}_{v\in\Omega_K}\}_{j\ge 1}$$

is a $(1,1)$-current given by $d\mu_f$, which is independent on the choice of F. Combining the definition (3.4.2) of the canonical height $\widehat{h}_f$ of f, with the height (4.2.1) of points for an adelic metrized line bundle and the definition (4.1.1, 4.1.2) of $\overline{\mathcal{L}}_F$, we get

$$\widehat{h}_{\overline{\mathcal{L}}_F}(x)=\widehat{h}_f(x)$$

which is independent of the choice for the lift F of f.

We conclude this section by noting that in the case of the metrized line bundle $\overline{\mathcal{L}}_{\tilde{F}}$ on $(\mathbb{P}^1)^n$ associated to an endomorphism $\tilde{f}$ of $(\mathbb{P}^1)^n$ (see subsection 4.1), at an archimedean place v, the (n,n)-current satisfies the formula:

$$c_1(\overline{\mathcal{L}}_{\tilde{F}})^n_v=n!\cdot d\tilde{\mu}, \tag{4.3.1}$$

where $\tilde{\mu}=\mu_{f_1}\times\cdots\times\mu_{f_n}$ is the invariant measure on $(\mathbb{P}^1_{\mathbb{C}_v})^n$ associated to the endomorphism $\tilde{f}=(f_1,\cdots,f_n)$. To see this, we first notice that since v is archimedean, then $\mathbb{C}_v=\mathbb{C}$ and so, by taking $\frac{\partial\overline{\partial}}{\pi i}\log\|\cdot\|_{v,\tilde{F}}$ we get

$$c_1(\overline{\mathcal{L}}_{\tilde{F}})_v=dd^c(u_1(x_1)+\cdots+u_n(x_n)), \tag{4.3.2}$$

where $u_i(x_i)$ is a locally defined continuous subharmonic function on $\mathbb{P}^1_{\mathbb{C}_v}$ with $dd^cu_i=d\mu_{f_i}$ for $1\le i\le n$. Hence

$$c_1(\overline{\mathcal{L}}_{\tilde{F}})^n_v=n!\cdot dd^cu_1(x_1)\wedge\cdots\wedge dd^cu_n(x_n)=n!\cdot d\tilde{\mu},$$

and so, the equality from (4.3.1) follows. Moreover, for a point $t=(a_1,\cdots,a_n)\in(\mathbb{P}^1)^n(\overline{K})$, from (4.1.3) we see that

$$\widehat{h}_{\overline{\mathcal{L}}_{\tilde{F}}}(t)=\widehat{h}_{f_1}(a_1)+\cdots+\widehat{h}_{f_n}(a_n). \tag{4.3.3}$$

5. Measures and heights on a hypersurface

In this section we study the measures and the corresponding heights on a hypersurface in $(\mathbb{P}^1)^n$; this allows us to obtain two important technical ingredients (Theorem 5.1 and Proposition 5.2) which will later be used in proving Theorem 2.2. So, let $\hat{f} = (f_1, \cdots, f_n)$ be an endomorphism of $(\mathbb{P}^1)^n$ defined over a number field K, with degrees $d_i \geq 2$ for each rational function f_i (for $1 \leq i \leq n$). Also, let $H \subset (\mathbb{P}^1)^n$ be an irreducible hypersurface defined over K, which projects dominantly onto each subset of $(n-1)$ coordinate axes.

5.1. **Adelic metrized line bundles on the hypersurface.** For each $i = 1, \ldots, n$, as in (3.2.1), we let $\tilde{f}_i$ be the endomorphism of $(\mathbb{P}^1)^{n-1}$ given by forgetting the i-th coordinate axis (along with the action of f_i) in the dynamical system $\left((\mathbb{P}^1)^n, \hat{f}\right)$. Let $\tilde{F}_i$ be a homogenous lift of $\tilde{f}_i$ as in Subsection 3.2 and then similar to (4.1.3), we construct an adelic metrized line bundle $\overline{\mathcal{L}}_{\tilde{F}_i}$ on $(\mathbb{P}^1)^{n-1}$ such that when v is archimedean, we have

$$c_1(\overline{\mathcal{L}}_{\tilde{F}_i})_v^{n-1} = (n-1)! \cdot d\tilde{\mu}_i$$

(for each $1 \leq i \leq n$), where the probability measure $\tilde{\mu}_i$ on $(\mathbb{P}^1_{\mathbb{C}_v})^{n-1}$ is the one appearing in (3.2.2).

For each $i = 1, \ldots, n$, we recall from Subsection 3.2 that the projection

$$\hat{\pi}_i : H \longrightarrow (\mathbb{P}^1)^{n-1}$$

is the one given by forgetting the i-th coordinates; $\hat{\pi}_i$ is a finite, dominant morphism (due to our assumption on H). We let

$$\overline{\mathcal{L}}_{\hat{F}_i} := \hat{\pi}_i^* \overline{\mathcal{L}}_{\tilde{F}_i} \tag{5.1.1}$$

be an adelic metrized line bundle on H, which is the pullback of the adelic metrized line bundle $\overline{\mathcal{L}}_{\tilde{F}_i}$ (on $(\mathbb{P}^1)^{n-1}$) by the morphism $\hat{\pi}_i$.

5.2. **Height functions on the hypersurface.** For each $i = 1, \cdots, n$ and each $t = (a_1, \cdots, a_n) \in H(\overline{K})$, using (4.3.3) we conclude that

$$\widehat{h}_{\overline{\mathcal{L}}_{\hat{F}_i}}(t) = \widehat{h}_{f_1}(a_1) + \cdots + \widehat{h}_{f_{i-1}}(a_{i-1}) + \widehat{h}_{f_{i+1}}(a_{i+1}) + \cdots \widehat{h}_{f_n}(a_n). \tag{5.2.1}$$

Hence $\widehat{h}_{\overline{\mathcal{L}}_{\hat{F}_i}}(t) \geq 0$ with equality if and only if a_j is preperiodic under f_j for each $j \neq i$ with $1 \leq j \leq n$. So, if the set of all $t \in H(\overline{K})$ for which $\widehat{h}_{\overline{\mathcal{L}}_{\hat{F}_i}}(t) = 0$ is Zariski-dense on H, then each essential minima $e_j\left(\overline{\mathcal{L}}_{\hat{F}_i}\right)$ (for $j = 1, \ldots, n$, defined as in [Zha95b]) are equal to 0. Therefore, using the inequality from [Zha95b, Theorem 1.10], we conclude that

$$\widehat{h}_{\overline{\mathcal{L}}_{\hat{F}_i}}(H) = 0. \tag{5.2.2}$$

5.3. **Equal measures on the hypersurface.** Now we are ready to prove the following result.

Theorem 5.1. *Suppose that there is a generic sequence of points* $t_j = (x_{1,j}, \cdots, x_{n,j}) \in H(\overline{K})$ *such that*

$$\lim_{j\to\infty} \widehat{h}_{f_1}(x_{1,j}) + \cdots + \widehat{h}_{f_n}(x_{n,j}) = 0.$$

Then $\hat{\mu}_1 = \hat{\mu}_2 = \cdots = \hat{\mu}_n$.

Proof. This is an immediate consequence of Theorem 4.1 applied to the sequence of points $t_j = (x_{1,j}, \cdots, x_{n,j}) \in H(\overline{K})$ with respect to the adelic metrized line bundles $\overline{\mathcal{L}}_{\hat{F}_i}$ for $1 \le i \le n$. Indeed, when v is archimedean, using (5.2.2) and the assumption on the points $t_j \in H$ we get that the Galois orbits of t_j in H equidistribute with respect to the probability measures $\hat{\mu}_i$ on $H(\mathbb{C})$ for each $i \in \{1, \dots, n\}$. Hence $\hat{\mu}_1 = \hat{\mu}_2 = \cdots = \hat{\mu}_n$. □

5.4. **Preperiodic points on hypersurfaces.** In this section we prove the following important result; we thank Xinyi Yuan and Shouwu Zhang for several very helpful conversations regarding its proof.

Proposition 5.2. *Let* $n \ge 2$ *be an integer, let* $f_1, \dots, f_n \in \mathbb{C}(x)$ *be rational functions of degrees* $d_i \ge 2$ *and let* $H \subset (\mathbb{P}^1)^n$ *be an irreducible hypersurface which projects dominantly onto any subset of* $(n-1)$ *coordinate axes. Assume:*

(1) *either that* H *contains a Zariski dense set of preperiodic points under the action of* $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$ *and that* $d_1 = \cdots = d_n$;
(2) *or that* $f_1, \dots, f_n \in \bar{\mathbb{Q}}(x)$, *that* H *is defined over* $\bar{\mathbb{Q}}$, *and that there exists a Zariski dense sequence of points* $(x_{1,j}, \dots, x_{n,j}) \in H(\bar{\mathbb{Q}})$ *such that* $\lim_{j\to\infty} \sum_{i=1}^n \widehat{h}_{f_i}(x_{i,j}) = 0$, *where* $\widehat{h}_{f_i}$ *is the canonical height with respect to the rational function* f_i.

Then there exists $i \in \{1, \dots, n\}$ *such that for any* $(a_1, \dots, a_n) \in H(\mathbb{C})$ *with* a_j *being preperiodic under the action of* f_j *for each* $j \in \{1, \dots, n\} \setminus \{i\}$, *we must have that also* a_i *is preperiodic under the action of* f_i.

We prove first that hypothesis (2) in Proposition 5.2 yields the desired conclusion, and then we prove that part (1) may be reduced to part (2) in Proposition 5.2 through a specialization result of Yuan-Zhang [YZa, YZb].

Proof of Proposition 5.2 assuming hypothesis (2) holds. Since the case $n = 2$ was proven in [Mim97] (see also [Mim13]), from now on, we assume $n > 2$. We assume each $f_i \in \bar{\mathbb{Q}}(x)$ and also that H is defined over $\bar{\mathbb{Q}}$.

We use the notation as in Subsection 5.1; so, we consider the adelic metrics $\overline{\mathcal{L}}_{\hat{F}_i}$ (for $i = 1, \dots, n$) on H, defined as in (5.1.1). For the sake of simplifying our notation, we will denote from now on the tensor product of two line bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ as $\mathcal{M}_1 + \mathcal{M}_2$. We denote by $\widehat{\mathrm{Pic}}(H)$ the group of (adelic) metrized line bundles on H.

Lemma 5.3. *There exist real numbers c_i (for $i = 1, \ldots, n$) not all equal to 0 such that the metrized line bundle*

$$\overline{\mathcal{L}}_0 := c_1 \cdot \overline{\mathcal{L}}_{\hat{F}_1} + \cdots + c_n \overline{\mathcal{L}}_{\hat{F}_n} \in \overline{\mathrm{Pic}}(H) \otimes \mathbb{R} \tag{5.4.1}$$

has the property that $\overline{\mathcal{L}}_0 \cdot x = \widehat{h}_{\overline{\mathcal{L}}_0}(x) = 0$ for each $x \in H(\bar{\mathbb{Q}})$.

Proof of Lemma 5.3. We thank Shouwu Zhang for suggesting us the proof of this Lemma, which follows the idea used in the proof of [YZ16, Theorem 4.13].

We let $\hat{\mathcal{L}}_i \in \mathrm{Pic}(H)$ be the line bundle supporting $\overline{\mathcal{L}}_{\hat{F}_i}$, i.e.,

$$\hat{\mathcal{L}}_i := \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \cdots \otimes \pi_{i-1}^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_{i+1}^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \cdots \otimes \pi_n^* \mathcal{O}_{\mathbb{P}^1}(1),$$

where π_j is the induced projection map of H onto the j-th coordinate axis of $(\mathbb{P}^1)^n$ (for each $j = 1, \ldots, n$).

Claim 5.4. There exist real constants $c_1, \ldots, c_n$ (not all equal to 0) such that the line bundle $\mathcal{L}_0 := \sum_{i=1}^n c_i \hat{\mathcal{L}}_i \in \mathrm{Pic}(H) \otimes \mathbb{R}$ is numerically trivial.

Proof of Claim 5.4. The main ingredient in our proof is a result on arithmetic intersections, which generalizes the classical Hodge Index Theorem (see [YZ16, Theorem 5.20]). We let

$$\mathcal{L}_1 := \sum_{i=1}^n \hat{\mathcal{L}}_i \in \mathrm{Pic}(H); \tag{5.4.2}$$

then $\mathcal{L}_1$ is ample (since it is the pullback of an ample line bundle on $(\mathbb{P}^1)^n$ under the natural inclusion map $H \hookrightarrow (\mathbb{P}^1)^n$). We find the real numbers c_i so that $\mathcal{L}_0 := \sum_{i=1}^n c_i \hat{\mathcal{L}}_i$ satisfies the following two conditions:

(A) $\mathcal{L}_0 \cdot \mathcal{L}_1^{n-2} = 0$ and
(B) $\mathcal{L}_0^2 \cdot \mathcal{L}_1^{n-3} = 0$.

Condition (A) above yields a linear relation between the unknowns c_i. On the other hand, condition (B) yields a quadratic form in the variables c_i. This quadratic form is not positive-definite since (from the generalization of the Hodge Index Theorem proven in [Fal84, Hri85, Mor96]) we know that generically, for any line bundle $\mathcal{M}$ satisfying $\mathcal{M} \cdot \mathcal{L}_1^{n-2} = 0$, we have that $\mathcal{M}^2 \cdot \mathcal{L}_1^{n-3} \leq 0$. Also, this quadratic form is not negative definite since $\mathcal{L}_1^2 \cdot \mathcal{L}_1^{n-3} = \mathcal{L}_1^{n-1} > 0$ (because $\mathcal{L}_1$ is ample). Therefore, there exist real numbers c_i, not all equal to 0 such that $\mathcal{L}_0$ satisfies both conditions (A) and (B) above. Then [YZ16, Theorem 5.20] yields that $\mathcal{L}_0$ is numerically trivial, as claimed. ☐

Let now $c_1, \ldots, c_n \in \mathbb{R}$ satisfy the conclusion of Claim 5.4 and define

$$\overline{\mathcal{L}}_0 := \sum_{i=1}^n c_i \overline{\mathcal{L}}_{\hat{F}_i} \in \overline{\mathrm{Pic}}(H) \otimes \mathbb{R}.$$

We consider next the adelic metrized line bundle $\overline{\mathcal{L}}_1 := \sum_{i=1}^n \overline{\mathcal{L}}_{\hat{F}_i}$; note that the generic fiber of $\overline{\mathcal{L}}_1$ is the ample line bundle $\mathcal{L}_1$ from (5.4.2). Using our hypothesis (2) from Proposition 5.2, i.e., the existence of a Zariski dense set of points on H of height tending to 0, we obtain that each of the successive minima $e_j(\overline{\mathcal{L}}_1) = 0$ for $j = 0, \dots, n-1$. Note that for each $j = 0, \dots, n$, we have

$$e_j(\overline{\mathcal{L}}_1) := \sup_{\substack{Y \subset H \\ \mathrm{codim}_H(Y) = j}} \inf_{x \in (H \setminus Y)(\bar{\mathbb{Q}})} \widehat{h}_{\overline{\mathcal{L}}_1}(x)$$

and so, indeed hypothesis (2) of Proposition 5.2 yields that $e_j(\overline{\mathcal{L}}_1) = 0$. In particular, $e_n(\overline{\mathcal{L}}_1) = 0$ and thus $\overline{\mathcal{L}}_1^n = 0$. The exact same argument applied for each $i_1, i_2 = 1, \dots, n$ and for each $m_1, m_2 \in \mathbb{N}$ to the metrized line bundle $\overline{\mathcal{L}}_{i_1,i_2,m_1,m_2} := \overline{\mathcal{L}}_1 + m_1 \overline{\mathcal{L}}_{\hat{F}_{i_1}} + m_2 \overline{\mathcal{L}}_{\hat{F}_{i_2}}$ yields again

$$\left(\overline{\mathcal{L}}_1 + m_1 \overline{\mathcal{L}}_{\hat{F}_{i_1}} + m_2 \overline{\mathcal{L}}_{\hat{F}_{i_2}} \right)^n = 0. \tag{5.4.3}$$

Keeping i_1 and i_2 fixed and letting m_1 and m_2 vary in $\mathbb{N}$, we see that equation (5.4.3) yields that $\overline{\mathcal{L}}_1^{j_0} \cdot \overline{\mathcal{L}}_{\hat{F}_{i_1}}^{j_1} \cdot \overline{\mathcal{L}}_{\hat{F}_{i_2}}^{j_2} = 0$ for each non-negative integers j_0, j_1, j_2 such that $j_0 + j_1 + j_2 = n$. Hence

$$\overline{\mathcal{L}}_0^2 \cdot \overline{\mathcal{L}}_1^{n-2} = 0; \tag{5.4.4}$$

moreover, because the numbers c_i satisfy the construction from Claim 5.4 (see condition (A) in the proof of the aforementioned Claim), we also have that

$$\mathcal{L}_0 \cdot \mathcal{L}_1^{n-2} = 0. \tag{5.4.5}$$

Furthermore, since each $\overline{\mathcal{L}}_{\hat{F}_i}$ is semipositive, we obtain that (with the terminology from [YZ16]) $\overline{\mathcal{L}}_0$ is $\overline{\mathcal{L}}_1$-bounded, i.e., there exists $m \in \mathbb{N}$ (any integer larger than $\max_i |c_i|$ would work) such that both $m \cdot \overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_0$ and $m \cdot \overline{\mathcal{L}}_1 + \overline{\mathcal{L}}_0$ are semipositive.

Since $\overline{\mathcal{L}}_1$ may not necessarily be arithmetically positive, we alter $\overline{\mathcal{L}}_1$ by adding to it an arbitrarily positive metrized line bundle $\iota^*(\mathcal{C})$ where $\mathcal{C}$ is a positive metrized line bundle on $\mathrm{Spec}(\bar{\mathbb{Q}})$ and $\iota : H \longrightarrow \mathrm{Spec}(\bar{\mathbb{Q}})$ is the structure morphism (for a similar application, see the proof of [YZ16, Theorem 4.13]). Then $\overline{\mathcal{L}}_0$ would still be $\overline{\mathcal{L}}_1'$-bounded with respect to this new metrized line bundle $\overline{\mathcal{L}}_1' := \overline{\mathcal{L}}_1 + \iota^*(\mathcal{C})$. Because the generic fiber of $\overline{\mathcal{L}}_0$ is numerically trivial (according to our choice of the numbers c_i satisfying the conclusion of Claim 5.4), then (5.4.4) and (5.4.5) yield

$$\mathcal{L}_0 \cdot \left(\mathcal{L}_1'\right)^{n-1} = 0 \text{ and } \overline{\mathcal{L}}_0^2 \cdot \left(\overline{\mathcal{L}}_1'\right)^{n-2} = 0. \tag{5.4.6}$$

Thus the hypotheses of [YZ16, Theorem 3.2] are verified and so, we obtain that the metrized line bundle $\overline{\mathcal{L}}_0$ is itself numerically trivial, i.e., $\widehat{h}_{\overline{\mathcal{L}}_0}(x) = 0$ for each $x \in H(\bar{\mathbb{Q}})$. This concludes the proof of Lemma 5.3. $\square$

So, by Lemma 5.3, there exist suitable constants $c_i \in \mathbb{R}$ (for $i = 1, \dots, n$), not all equal to 0 such that the metrized line bundle $\overline{\mathcal{L}}_0 := c_1 \cdot \overline{\mathcal{L}}_{\hat{F}_1} + \cdots + c_n\overline{\mathcal{L}}_{\hat{F}_n} \in \overline{\mathrm{Pic}}(H) \otimes \mathbb{R}$ is numerically trivial on H and therefore, for each $\alpha \in H(\bar{\mathbb{Q}})$, we have that $\overline{\mathcal{L}}_0 \cdot \alpha = 0$, i.e.,

$$\sum_{i=1}^{n} c_i \cdot \widehat{h}_{\overline{\mathcal{L}}_{\hat{F}_i}}(\alpha) = 0. \tag{5.4.7}$$

Since not all c_i are equal to 0, then there exists some $i_0 \in \{1, \dots, n\}$ with the property that

$$c_1 + \cdots + c_{i_0-1} + c_{i_0+1} + \cdots + c_n \neq 0. \tag{5.4.8}$$

Now, for any $\alpha := (a_1, \dots, a_n) \in H(\bar{\mathbb{Q}})$ and for any $i = 1, \dots, n$, we have that

$$\widehat{h}_{\overline{\mathcal{L}}_{\hat{F}_i}}(\alpha) = \widehat{h}_{f_1}(a_1) + \cdots + \widehat{h}_{f_{i-1}}(a_{i-1}) + \widehat{h}_{f_{i+1}}(a_{i+1}) + \cdots + \widehat{h}_{f_n}(a_n), \tag{5.4.9}$$

as shown in (5.2.1). Now, if

$$\widehat{h}_{f_1}(a_1) = \cdots = \widehat{h}_{f_{i_0-1}}(a_{i_0-1}) = \widehat{h}_{f_{i_0+1}}(a_{i_0+1}) = \cdots = \widehat{h}_{f_n}(a_n) = 0,$$

then (5.4.7), (5.4.8) and (5.4.9) yield that also $\widehat{h}_{f_{i_0}}(a_{i_0}) = 0$, as claimed in the conclusion of Proposition 5.2. This concludes our proof of Proposition 5.2 assuming each rational function f_i along with the hypersurface H are defined over $\bar{\mathbb{Q}}$. ☐

Proof of Proposition 5.2 assuming hypothesis (1) holds. We let $K \subset \mathbb{C}$ be a finitely generated extension of $\bar{\mathbb{Q}}$ such that each $f_i \in K(x)$ and also H is defined over K. We argue by induction on $r := \mathrm{trdeg}_{\bar{\mathbb{Q}}} K$; the case $r = 0$ is already proved using Proposition 5.2 with hypothesis (2). Hence, we assume the conclusion of Proposition 5.2 holds whenever $r < s$ (for some $s \in \mathbb{N}$) and we prove that it also holds when $r = s$. We know there exists an infinite sequence S of points $\alpha_j \in H(\mathbb{C})$ such that α_j has its i-th coordinate preperiodic under the action of f_i (for each $i = 1, \dots, n$). Also, we let

$$d := \deg(f_1) = \deg(f_2) = \cdots = \deg(f_n).$$

Then we let K_0 be a subfield $\bar{\mathbb{Q}} \subset K_0 \subset K$ such that $\mathrm{trdeg}_{K_0} K = 1$ and we let $\mathcal{C}$ be a curve defined over K_0 whose function field is K (at the expense of replacing both K_0 and K by finite extensions, we may assume $\mathcal{C}$ is a projective, smooth, geometrically irreducible curve). We fix some algebraic closures $\overline{K_0} \subset \overline{K}$ of our fields.

There exists a Zariski dense, open subset $C \subseteq \mathcal{C}$ such that we may view each f_i as a base change of an endomorphism $f_{i,C}$ of $\mathbb{P}^1_C$; similarly, H is the base change of a hypersurface $H_C \subset (\mathbb{P}^1_C)^n$, while S is the base change of a subset $S_C \subset H_C$. For each geometric point $t \in C\left(\overline{K_0}\right)$, the objects H_C, $f_{i,C}$ and S_C have reductions H_t, $f_{i,t}$ and respectively S_t, such that $S_t \subset H_C$ consists of points with their i-th coordinate preperiodic under the action of $f_{i,C}$, for each $i = 1, \dots, n$.

Claim 5.5. There exists a Zariski dense, open subset $C_0 \subseteq C \subseteq \mathcal{C}$ such that for each $t \in C_0\left(\overline{K_0}\right)$, the set S_t is Zariski dense in H_t.

Proof of Claim 5.5. We let $\Psi := (f_1, \dots, f_{n-1})$ be the coordinatewise action of these rational functions on the first $n-1$ coordinates of $(\mathbb{P}^1)^n$; since $d_1 = \cdots = d_{n-1} = d > 1$, we know that Ψ is a polarizable endomorphism of $(\mathbb{P}^1)^{n-1}$. We let $\tilde{S}$ be the projection of the set S on the first $n-1$ coordinate axes of $(\mathbb{P}^1)^n$; because $S \subset H$ is dense and H projects dominantly onto the first $n-1$ coordinate axes, we conclude that $\tilde{S} \subset (\mathbb{P}^1)^{n-1}$ is also dense. Note that each point of $\tilde{S}$ is a preperiodic point for Ψ. As before, we let $\tilde{S}_t$ be the specialization of the set $\tilde{S}_C$ at some point $t \in C_0\left(\overline{K_0}\right)$.

As proven in [YZb, Theorem 4.7] (see also [YZa, Lemma 3.2.3]), the set $\tilde{S}_t \subset (\mathbb{P}^1)^{n-1}$ is still Zariski dense for all the $\overline{K_0}$-points t of a dense open subset $C_0 \subseteq C$. Here it is the only point in our argument where we use that $d_1 = \cdots = d_n$ because Yuan-Zhang [YZb, YZa] show that specializing a Zariski dense set of preperiodic points for a *polarizable* endomorphism yields also a Zariski dense set of preperiodic points for all specializations in a dense, open subset of the base; in their proof, they employ a result of Faber [Fab09] and of Gubler [Gub08] regarding the equidistribution of subvarieties of a given polarizable dynamical system (X, Φ) with respect to the invariant measure of Φ. (As an aside, we note that the results of [YZa] were recently published in [YZ16], while [YZb] has been updated to [YZc] using slightly different arguments.) Finally, since $\tilde{S}_t \subset (\mathbb{P}^1)^{n-1}$ is Zariski dense, then the Zariski closure of S_t must have dimension $n-1$ because S_t projects to $\tilde{S}_t$ on the first $n-1$ coordinate axes of $(\mathbb{P}^1)^n$. Hence $S_t \subset H_t$ is Zariski dense, which concludes the proof of Claim 5.5. □

Let C_0 be the Zariski dense, open subset of C satisfying the conclusion of Claim 5.5. At the expense of perhaps shrinking C_0 to a smaller, dense, open subset, we may assume that

$$\deg(f_{i,t}) = d > 1 \text{ for all } i = 1, \dots, n \text{ and each } t \in C_0\left(\overline{K_0}\right). \tag{5.4.10}$$

For each $t \in C_0(\bar{\mathbb{Q}})$, our inductive hypothesis (which can be applied since each f_i and also H are defined over $\overline{K_0}$ and $\operatorname{trdeg}_{\bar{\mathbb{Q}}}\overline{K_0} < s$) yields the existence of some index $i_t \in \{1, \dots, n\}$ which has the property that for each $\alpha \in H_t(\bar{\mathbb{Q}})$, if we know that the j-th coordinate of α is preperiodic under the action of $f_{j,t}$ for each $j \in \{1, \dots, n\} \setminus \{i_t\}$, then also the i_t-th coordinate of α is preperiodic under the action of $f_{i_t,t}$.

Let $h_{\mathcal{C}}(\cdot)$ be a height function for the points on $\mathcal{C}\left(\overline{K_0}\right)$ corresponding to a divisor of degree 1 on $\mathcal{C}$, constructed with respect to the Weil height on $\overline{K_0}$. Note that if $\operatorname{trdeg}_{\mathbb{Q}} K_0 \geq 1$, then we construct the Weil height on the function field $K_0/\bar{\mathbb{Q}}$ as in [BG06]. At the expense of replacing C_0 by an infinite subset U_0 for which

$$\sup_{t \in U_0} h_{\mathcal{C}}(t) = +\infty, \tag{5.4.11}$$

we may even assume that for each $t \in U_0$, there is the same index $i_0 := i_t \in \{1, \dots, n\}$ satisfying the above property. We show next that this index i_0 would satisfy the conclusion of Proposition 5.2 for H.

Indeed, let $\alpha = (a_1, \dots, a_n) \in H(\overline{K})$ with the property that for each $j \in \{1, \dots, n\} \setminus \{i_0\}$, we have that a_j is preperiodic under the action of f_j. Then for each $t \in U_0$ we have that each $a_{j,t}$ (for $j \in \{1, \dots, n\} \setminus \{i_0\}$) is preperiodic for $f_{j,t}$ and so, also $a_{i_0,t}$ is preperiodic under the action of $f_{i_0,t}$. Therefore, the canonical height

$$\widehat{h}_{f_{i_0,t}}(a_{i_0,t}) = 0, \tag{5.4.12}$$

where $\widehat{h}_{f_{i_0,t}}(\cdot)$ is the canonical height corresponding to the rational function $f_{i_0,t}$ (which has degree larger than 1; see (5.4.10)), constructed using the Weil height on $\overline{K_0}$. Using [CS93, Theorem 4.1], we have that

$$\lim_{h_{\mathcal{C}}(t)\to\infty} \frac{\widehat{h}_{f_{i_0,t}}(a_{i_0,t})}{h_{\mathcal{C}}(t)} = \widehat{h}_{f_{i_0}}(a_{i_0}), \tag{5.4.13}$$

where $\widehat{h}_{f_{i_0}}(\cdot)$ is the canonical height of f_{i_0} constructed with respect to the function field K/K_0. Equations (5.4.11), (5.4.12) and (5.4.13) yield that

$$\widehat{h}_{f_{i_0}}(a_{i_0}) = 0. \tag{5.4.14}$$

If $f_{i_0} \in K(x)$ is not isotrivial over K_0, then [Bak09] (see also [Ben05] for the case of polynomials) yields that (5.4.14) is equivalent with saying that a_{i_0} is preperiodic under the action of f_{i_0}, as desired. Now, if f_{i_0} is isotrivial over K_0, then there exists a linear transformation

$$\nu : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \text{ (defined over } \overline{K})$$

such that $\nu^{-1} \circ f_{i_0} \circ \nu \in \overline{K_0}(x)$. If $\nu^{-1}(a_{i_0}) \in \overline{K_0}$, then since we know there exists even a single specialization t such that $a_{i_0,t}$ is preperiodic for f_t, we get that also a_{i_0} is preperiodic for f_{i_0}. On the other hand, if $\nu^{-1}(a_{i_0}) \notin \overline{K_0}$, then $\nu^{-1}(a_{i_0})$ cannot be preperiodic for $\nu^{-1} \circ f_{i_0} \circ \nu \in \overline{K_0}(x)$ and so, a_{i_0} is not preperiodic for f_{i_0}, contradiction. This concludes the proof of Proposition 5.2 under hypothesis (1). $\square$

6. Hypersurfaces having a Zariski dense set of preperiodic points

In this section, we prove Theorem 6.1, which (essentially) says that there is no hypersurface H containing a Zariski dense set of preperiodic points under the coordinatewise action of some rational functions f_i, along with some additional technical conditions. To make things simple, we work on a hypersurface $H \subset (\mathbb{P}^1)^{n+1}$ of dimension n and use the following notation

$$\tilde{x} = (x_1, \cdots, x_n),\ \underline{x} = (x_1, \cdots, x_{n-1})$$

and hence $\tilde{a} = (a_1, \cdots, a_n)$, $\underline{a} = (a_1, \cdots, a_{n-1})$, etc. We denote by $D(a, r) \subset \mathbb{C}$ the usual disk of radius r centered at a; also, we use the following notation for polydiscs:

$$D_{n-1}(\underline{a},r)=D(a_1,r)\times\cdots D(a_{n-1},r) \text{ and } D_n(\tilde{a},r)=D_{n-1}(\underline{a},r)\times D(a_n,r).$$

For the benefit of our readers, we split our proof of Theorem 6.1 in several subsections, each one presenting a different step in our argument.

6.1. Statement of our theorem.

Theorem 6.1. *Let $n\geq 2$, let f_i be rational functions defined over $\mathbb{C}$ of degrees $d_i>1$ (for $1\leq i\leq n+1$), and let $H\subset(\mathbb{P}^1)^{n+1}$ be an irreducible hypersurface defined over $\mathbb{C}$ which projects dominantly onto each subset of n coordinate axes. For each $i=1,\dots,n+1$, let $\tilde{f}_i$ be the coordinatewise action on $(\mathbb{P}^1)^n$ given by*

$$(x_1,\cdot\cdot,x_{i-1},x_{i+1},\cdot\cdot,x_{n+1})\mapsto(f_1(x_1),\cdot\cdot,f_{i-1}(x_{i-1}),f_{i+1}(x_{i+1}),\cdot\cdot,f_{n+1}(x_{n+1}))\,.$$

Let $\hat{\mu}_i$ be the measures on H induced from the dynamical systems $\left((\mathbb{P}^1)^n,\tilde{f}_i\right)$ and assume that $\hat{\mu}_i=\hat{\mu}_{n+1}$ for $1\leq i\leq n$. Also assume that there is a point $(\tilde{a},b_0)\in H\cap\mathbb{C}^{n+1}$ with $\tilde{a}=(a_1,\cdots,a_n)$, such that:

- *a_i is a repelling fixed point of f_i for $1\leq i\leq n$; and*
- *$b_1:=f_{n+1}(b_0)$ is a fixed point of f_{n+1}; and*
- *there is a holomorphic germ $h(\tilde{x})$ at $\tilde{a}$ with $h(\tilde{a})=b_0$, and $(\tilde{x},h(\tilde{x}))\in H(\mathbb{C})$ for all $\tilde{x}\in\mathbb{C}^n$ in a small (complex analytic) neighbourhood of $\tilde{a}$. Moreover, for each $i=1,\cdots,n$ we have that*

$$\beta_i:=\frac{\partial h}{\partial x_i}(\tilde{a})\neq 0.$$

Then the f_i's must be exceptional, and moreover, they are

- *either all of them conjugate to monomials and $\pm$Chebyshev polynomials,*
- *or all of them Lattès maps.*

Proof. As we previously stated, we will prove Theorem 6.1 over the next several subsections of Section 6. To summarize, in Section 6.2 we construct a local symmetry g of the Julia set $J_{f_{n+1}}$ as in (6.2.1) fixing b_0 and with multiplier λ of absolute value greater than one. This, coupled with a refined analysis of the dynamics of the f_i's, allows us in Sections 6.3 and 6.4 to construct two normal families of local symmetries Ψ_ℓ (as in (6.4.1)) and Φ_ℓ (as in (6.4.2)) preserving the (n,n)-current corresponding to $\mu_{f_1}\times\cdots\mu_{f_n}$ (up to a scaling). Using Proposition 6.4, we reduce the symmetries Φ_ℓ of the (n,n)-current to the symmetries $\Phi_\ell(\underline{\alpha},x_n)$ of J_{f_n} for fixed $\underline{\alpha}\in J_{f_1}\times\cdots\times J_{f_{n-1}}$. Finally, combining this with the rigidity (proven by Levin [Lev90]) of the symmetries of the Julia set, we finish the proof of Theorem 6.1 in Section 6.7.

6.2. **Julia sets and invariant measures.** From the assumptions of Theorem 6.1, the multiplier

$$\lambda_i := f_i'(a_i)$$

has absolute value $|\lambda_i| > 1$ for $1 \le i \le n$. So, each a_i is in the support of the Julia set J_{f_i} of f_i, for $i = 1, \ldots, n$. Thus $(\tilde{a}, b_0)$ is in the support of $\hat{\mu}_{n+1}$ and because $\hat{\mu}_n = \hat{\mu}_{n+1}$, we get that $(\tilde{a}, b_0)$ must be in the support of $\hat{\mu}_n$. Therefore, b_0 must be in the support $J_{f_{n+1}}$ of $\mu_{f_{n+1}}$. Hence $b_1 = f_{n+1}(b_0) \in J_{f_{n+1}}$ and so, it has multiplier

$$\rho := f_{n+1}'(b_1)$$

of absolute value $|\rho| \ge 1$. Let j_0 be the local degree of the map $f_{n+1}(x)$ at $x = b_0$, and let $g(x)$ be a holomorphic germ on $\mathbb{P}^1$ at b_0 which is one of the following branches

$$g(x) := f_{n+1}^{-1} \circ f_{n+1} \circ f_{n+1}(x) \tag{6.2.1}$$

satisfying $g(b_0) = b_0$. Although there are j_0 different choices for $g(x)$, in the rest of this section we fix our choice $g(x)$ for such a branch. An easy computation shows that

$$\lambda := g'(b_0) = \sqrt[j_0]{\rho} \tag{6.2.2}$$

is a j_0-th root of the multiplier ρ of f_{n+1} at b_1. Since $\mu_{f_{n+1}}$ admits no atoms on $\mathbb{P}^1$ and $\mu_{f_{n+1}}(f_{n+1}(A)) = d_{n+1} \cdot \mu_{f_{n+1}}(A)$ for any Borel set A with f_{n+1} being injective on A, the definition of $g(x)$ yields that

$$\mu_{f_{n+1}}(g(A)) = d_{n+1} \cdot \mu_{f_{n+1}}(A)$$

for any Borel set A in a small neighborhood of b_0.

Lemma 6.2. *The multiplier λ of $g(x)$ at b_0 has absolute value $|\lambda| > 1$.*

Proof of Lemma 6.2. We first assume that $|\lambda| \le 1$ and then prove the lemma by deriving a contradiction. Using (6.2.2) and the fact that $|\rho| \ge 1$, we get that $|\lambda| = 1$.

Pick a positive integer m with $d_n < d_{n+1}^m$. Let

$$\Phi_{00}(\tilde{x}) := (\underline{x}, h(\underline{x}, f_n(x_n))) \text{ and } \Phi_{11}(\tilde{x}) := (\underline{x}, g^m \circ h(\tilde{x})) \tag{6.2.3}$$

be functions locally defined in a neighborhood of $\tilde{a} \in \mathbb{C}^n$, mapping that small neighborhood of $\tilde{a}$ into a neighborhood of $(\underline{a}, b_0) \in \mathbb{C}^n$. Since $\hat{\mu}_n = \hat{\mu}_{n+1}$, there exists some $c > 0$ with

$$\Phi_{00}^*(\tilde{\mu}_n) = c \cdot d_n \cdot \tilde{\mu}_{n+1} \text{ and } \Phi_{11}^*(\tilde{\mu}_n) = c \cdot d_{n+1}^m \cdot \tilde{\mu}_{n+1}. \tag{6.2.4}$$

The measures $\tilde{\mu}_n$ and $\tilde{\mu}_{n+1}$ (defined in (3.2.2)) appearing in (6.2.4) are restricted on some small neighborhood of $\tilde{a}$ (respectively of $(\underline{a}, b_0)$). Let A be the polydisc given by $A := D_{n-1}(\underline{a}, r_1) \times D(a_n, r_2)$ for very small r_2 and much smaller r_1. We claim that $\Phi_{11}(A) \subset \Phi_{00}(A)$. To see this, let r_2 be very small and we see that $f_n(D(a_n, r_2)) \sim D(a_n, |\lambda_n| r_2)$. As

$|\lambda| = |g'(b_0)| = 1 < |\lambda_n|$ and $\beta_n = \frac{\partial h}{\partial x_n}(\tilde{a}) \neq 0$, using (6.2.3) we can pick some very small r_2 and a much smaller r_1 such that

$$\Phi_{11}(A) \subset D_{n-1}(\underline{a}, r_1) \times D(b_0, r_2 \cdot |\beta_n| \cdot |\lambda_n|^{1/2}) \subset \Phi_{00}(A). \tag{6.2.5}$$

However, combining (6.2.4) with $d_n < d_{n+1}^m$ gives

$$\tilde{\mu}_n(\Phi_{11}(A)) > \tilde{\mu}_n(\Phi_{00}(A)),$$

which is a contradiction. This concludes the proof of Lemma 6.2. □

6.3. **A special sequence of tuples of positive integers.** Now since $|\lambda| > 1$ and $|\lambda_i| > 1$ for $1 \le i \le n$, we can pick a sequence of tuples of positive integers $(j_\ell, j_{1,\ell}, \cdots, j_{n,\ell})$ such that $j_\ell \to \infty$ as $\ell \to \infty$ and moreover,

$$\lim_{\ell\to\infty} \inf \frac{|\lambda_1^{j_{1,\ell}}|}{|\lambda^{j_\ell}|}, \cdots, \lim_{\ell\to\infty} \inf \frac{|\lambda_{n-1}^{j_{1,\ell}}|}{|\lambda^{j_\ell}|} \ge \lim_{\ell\to\infty} \frac{\lambda_n^{j_{n,\ell}}}{\lambda^{j_\ell}} = 1. \tag{6.3.1}$$

It will be useful later in our argument (see Lemma 6.8) that our sequence of tuples $(j_\ell, j_{1,\ell}, \cdots, j_{n,\ell})$ satisfies the following arithmetic property in addition to (6.3.1). We want that for every $N \in \mathbb{N}$, there exist $\ell_2 > \ell_1 > N$ such that

$$j_{\ell_2} = j_{\ell_1} \text{ and } j_{i,\ell_2} = j_{i,\ell_1} \text{ for } 2 \le i \le n, \text{ while } j_{1,\ell_2} = j_{1,\ell_1} + 1. \tag{6.3.2}$$

In order to achieve (6.3.2), we may replace the original sequence of tuples $\{(j_\ell, j_{1,\ell}, \dots, j_{n,\ell})\}_{\ell=1}^\infty$ by the larger sequence $\left\{(j'_\ell, j'_{1,\ell}, \dots, j'_{n,\ell})\right\}_{\ell=1}^\infty$ for which

$$j'_{2\ell-1} = j'_{2\ell} = j_\ell \text{ and } j'_{i,2\ell-1} = j'_{i,2\ell} = j_{i,\ell}$$

for $i = 2, \dots, n$, while

$$j'_{1,2\ell-1} = j_{1,\ell} \text{ and } j'_{1,2\ell} = j_{1,\ell} + 1$$

and still the new sequence $\left\{(j'_\ell, j'_{1,\ell}, \dots, j'_{n,\ell})\right\}_{\ell=1}^\infty$ satisfies (6.3.1) and the fact that $j_\ell \to \infty$ as $\ell \to \infty$. For the sake of simplifying our notation, we will denote our new sequence of tuples also as $\{(j_\ell, j_{1,\ell}, \dots, j_{n,\ell})\}_{\ell=1}^\infty$, but we note that this sequence of tuples satisfies (6.3.2).

6.4. **Local symmetries for the Julia sets.** From [Mil00], we know we can conjugate f_i (for $1 \le i \le n$) and g to linear maps in small neighborhoods of the repelling fixed points. More precisely, there exist holomorphic germs ϕ_i at $x = 0$ satisfying

$$\phi_i(0) = a_i,\ \phi_{n+1}(0) = b_0,\ \phi'_i(0) = \phi'_{n+1}(0) = 1 \text{ for } 1 \le i \le n \text{ and}$$
$$\phi_i^{-1} \circ f_i \circ \phi_i(x) = \lambda_i \cdot x \text{ for } 1 \le i \le n,$$
$$\text{while } \phi_{n+1}^{-1} \circ g \circ \phi_{n+1}(x) = \lambda \cdot x.$$

We notice that for $(x_1, \cdots, x_n)$ in a neighborhood of $\tilde{a} \in \mathbb{C}^n$, we have an equality of germs:

$$g^{j_\ell} \circ h \circ \left(f_1^{-j_{1,\ell}}(x_1), \cdots, f_n^{-j_{n,\ell}}(x_n)\right)$$

$$= \phi_{n+1} \circ \left(\lambda^{j_\ell} \cdot h_\phi \left(\frac{\phi_1^{-1}(x_1)}{\lambda_1^{j_{1,\ell}}}, \cdots, \frac{\phi_n^{-1}(x_n)}{\lambda_n^{j_{n,\ell}}} \right) \right),$$

where $h_\phi := \phi_{n+1}^{-1} \circ h \circ (\phi_1, \cdots, \phi_n)$ and f_i^{-1} is the germ of a branch of the inverse of f_i at $x_i = a_i$ with $f_i^{-1}(a_i) = a_i$. So, using also (6.3.1), then for very small $r_0 > 0$ and all $\tilde{x}$ in the ball $B(\tilde{a}, r_0) \subset \mathbb{C}^n$ of radius r_0, the map

$$\tilde{x} \mapsto g^{j_\ell} \circ h \circ (f_1^{-j_{1,\ell}}, \cdots, f_n^{-j_{n,\ell}})(\tilde{x})$$

is well defined and uniformly bounded on $B(\tilde{a}, r_0)$ for all $\ell \geq 1$. Next, we construct the function

$$\Psi(\tilde{x}) := (x_1, \cdots, x_{n-1}, h(\tilde{x}))$$

for $\tilde{x} = (x_1, \cdots, x_n)$, which is locally one-to-one at $\tilde{x} = \tilde{a}$ since $\beta_n = \frac{\partial h}{\partial x_n}(\tilde{a}) \neq 0$. Shrinking r_0 if necessary, we let

$$\begin{aligned} \Psi_\ell(\tilde{x}) &:= \Psi^{-1} \circ \left(x_1, \cdots, x_{n-1}, g^{j_\ell} \circ h \circ (f_1^{-j_{1,\ell}}, \cdots, f_n^{-j_{n,\ell}})(\tilde{x}) \right) \\ &=: (x_1, \cdots, x_{n-1}, h_\ell(\tilde{x})) \end{aligned} \tag{6.4.1}$$

for all $\tilde{x} \in B(\tilde{a}, r_0)$ and all $\ell \geq 1$, where h_ℓ is some local analytic function on $B(\tilde{a}, r_0)$ satisfying (6.4.1).

Lemma 6.3. *The family of functions $\{h_\ell(\tilde{x})\}_{\ell \geq 1}$ restricted on $B(\tilde{a}, r_0)$ is a normal family.*

Proof of Lemma 6.3. Since $\tilde{x} \mapsto g^{j_\ell} \circ h \circ (f_1^{-j_{1,\ell}}, \cdots, f_n^{-j_{n,\ell}})(\tilde{x})$ is uniformly bounded on $B(\tilde{a}, r_0)$ for all $\ell \geq 1$, then that h_ℓ (defined as in (6.4.1)) is uniformly bounded on $B(\tilde{a}, r_0)$, i.e., there exist $R > 0$ such that

$$h_\ell(B(\tilde{a}, r_0)) \subset B(b_0, R) \subset \mathbb{C}$$

for all $\ell \geq 1$. Hence h_ℓ is a distance non-increasing map from $B(\tilde{a}, r)$ (with respect to the Bergman metric) to $B(b_0, R)$ (with respect to the hyperbolic metric). Thus $\{h_\ell(\tilde{x})\}_{\ell \geq 1}$ is equicontinuous on $B(\tilde{a}, r_0)$, or equivalently, $\{h_\ell(\tilde{x})\}_{\ell \geq 1}$ is a normal family. □

From Lemma 6.3, we can pick a subsequence of $\{\Psi_\ell\}_{\ell \geq 1}$ which converges uniformly on $B(\tilde{a}, r_0)$. By passing to a subsequence, without loss of generality, we can assume that the sequence $\{\Psi_\ell\}_{\ell \geq 1}$ itself converges uniformly to

$$\Psi_0(\tilde{x}) := \lim_{\ell \to \infty} \Psi_\ell(\tilde{x})$$

and satisfies (6.3.2) with $\Psi_0(\tilde{a}) = \tilde{a}$ and $\Psi_0(\tilde{x}) =: (x_1, \cdots, x_{n-1}, h_0(\tilde{x}))$. Since

$$\frac{\partial h_0}{\partial x_n}(\tilde{a}) = \lim_{\ell \to \infty} \frac{\partial h_\ell}{\partial x_n}(\tilde{a}) = \lim_{\ell \to \infty} \frac{\lambda^{j_\ell}}{\lambda_n^{j_{n,\ell}}} = 1 \neq 0,$$

the map Ψ_0 is locally one-to-one at $\tilde{x} = \tilde{a}$. Shrinking r_0 if necessary, we can further assume that the sequence of maps

$$\Psi_0^{-1} \circ \Psi_\ell(\tilde{x}) =: (x_1, \cdots, x_{n-1}, \hbar_\ell(\tilde{x})) =: \Phi_\ell(\tilde{x}) \tag{6.4.2}$$

converges uniformly to the identity map on $B(\tilde{a}, r_0)$ as $\ell \to \infty$. The next goal is to show that Φ_ℓ is the identity map for all large ℓ; see Lemma 6.7.

6.5. **Equal currents.**

Proposition 6.4. *Let r_1 and r_2 be positive real numbers and let $u_1, \cdots, u_n$ and u be continuous subharmonic functions on $D(0, r_1)$, respectively on $D(0, r_2)$. Let θ be a holomorphic map from $D_n(\tilde{0}, r_1)$ to $D(0, r_2)$ and moreover, assume the following two (n,n)-currents satisfy the relation:*

$$dd^c u_1(x_1) \wedge \cdots \wedge dd^c u_n(x_n) = c_0 \cdot dd^c u_1(x_1) \wedge \cdots \wedge dd^c u_{n-1}(x_{n-1}) \wedge dd^c u \circ \theta(\tilde{x})$$

on $D_n(\tilde{0}, r_1)$ for some constant $c_0 > 0$. Then for any given point $\underline{\alpha}$ in the support of $dd^c u_1(x_1) \wedge \cdots \wedge dd^c u_{n-1}(x_{n-1})$, we have the following equality of $(1,1)$-currents on $D(0, r_1)$:

$$dd^c u_n(x_n) = c_0 \cdot dd^c u \circ \theta(\underline{\alpha}, x_n).$$

Proof of Proposition 6.4. Let $\underline{\alpha}$ be a point in the support of $dd^c u_1(x_1) \wedge \cdots \wedge dd^c u_{n-1}(x_{n-1})$. It suffices to show that for any C^∞ real function φ with compact support on $D(0, r_1)$, one has

$$\int_{D(0,r_1)} \varphi(x_n) dd^c u_n(x_n) = c_0 \int_{D(0,r_1)} \varphi(x_n) dd^c u \circ \theta(\underline{\alpha}, x_n).$$

To see this, we let $\underline{\mu}$ be the measure on $D_{n-1}(\underline{0}, r_1)$ with

$$d\underline{\mu}(\underline{x}) := c_0 \cdot dd^c u_1(x_1) \wedge \cdots \wedge dd^c u_{n-1}(x_{n-1})$$

and let $\tilde{\mu}$ be the measure on $D_n(\tilde{0}, r_1)$ with

$$d\tilde{\mu}(\tilde{x}) := d\underline{\mu}(\underline{x}) \wedge \frac{dx_n \wedge d\bar{x}_n}{-4\pi i}$$

For each small positive real number r, we let $\eta_r(\underline{x})$ be a C^∞-function on $D_{n-1}(\underline{0}, r_1)$ satisfying the properties:

- $0 \leq \eta_r \leq 1$;
- η_r is supported on $D_{n-1}(\underline{\alpha}, r)$; and
- $\eta_r = 1$ on $D_{n-1}(\underline{\alpha}, r/2)$.

From the proportionality assumption of the two (n,n)-currents, we get

$$\begin{aligned}
\frac{1}{c_0}\left(\int \eta_r d\underline{\mu}\right) \int \varphi dd^c u_n &= \frac{1}{c_0} \int \eta_r(\underline{x}) \varphi(x_n) d\underline{\mu}(\underline{x}) \wedge dd^c u_n(x_n) \\
&= \int \eta_r(\underline{x}) \varphi(x_n) d\underline{\mu}(\underline{x}) \wedge dd^c u \circ \theta(\tilde{x}) \\
&= \int u \circ \theta(\tilde{x}) d\underline{\mu} \wedge dd^c(\eta_r \varphi) \\
&= \int \eta_r(\underline{x}) u \circ \theta(\tilde{x}) \triangle \varphi(x_n) d\tilde{\mu}(\tilde{x})
\end{aligned} \tag{6.5.1}$$

where $\triangle$ is the Laplacian and the right hand side is integrated over the domain $D_n(\tilde{0}, r_1)$. Similarly we derive that

$$\left(\int \eta_r d\underline{\mu}\right) \int \varphi dd^c u \circ \theta(\underline{\alpha}, x_n) = \int \eta_r(\underline{x}) u \circ \theta(\underline{\alpha}, x_n) \triangle \varphi(x_n) d\tilde{\mu}(\tilde{x}).$$

Now let

$$\Theta_r(\tilde{x}) := \eta_r(\underline{x}) \cdot (u \circ \theta(\underline{\alpha}, x_n) - u \circ \theta(\tilde{x})) \cdot \triangle \varphi(x_n)$$

which is supported on $D_{n-1}(\underline{\alpha}, r) \times D(0, r_1)$. Hence as $u \circ \theta$ is continuous and φ has compact support on $D(0, r_1)$, there exist constants $\epsilon_r \to 0$ as $r \to 0$ such that for any $\tilde{x} \in D_n(\tilde{0}, r_1)$, we have

$$|\Theta_r(\tilde{x})| \leq \eta_r(\underline{x}) \cdot \epsilon_r.$$

Consequently

$$\left| \frac{1}{c_0} \int \varphi dd^c u_n - \int \varphi dd^c u \circ \theta(\underline{\alpha}, x_n) \right| \leq \frac{\int_{D_n(\tilde{0}, r_1)} \eta_r(\underline{x}) \cdot \epsilon_r \; d\tilde{\mu}(\tilde{x})}{\int_{D_{n-1}(\underline{0}, r_1)} \eta_r(\underline{x}) d\underline{\mu}(\underline{x})} = \epsilon_r \cdot c_1$$

with $c_1 = \int_{D(0,r_1)} 1 \cdot \frac{dx_n \wedge d\bar{x}_n}{-4\pi i}$. Now letting $r \to 0$, the conclusion in Proposition 6.4 follows. □

6.6. **The rational functions must be exceptional.** The next result yields half of the conclusion in Theorem 6.1 by showing that if f_{n+1} is an exceptional rational function, then each f_i is exceptional, and moreover, each f_i is either LattLattèes or not, depending on whether f_{n+1} is a Lattès map, or not.

Corollary 6.5. *The following statements hold:*

- *if f_{n+1} is conjugate to a monomial or a $\pm$Chebyshev polynomial, then each f_i (for $i = 1, \dots, n$) is conjugate to a monomial or a $\pm$Chebyshev polynomial.*
- *if f_{n+1} is a Lattès map, then each f_i is a Lattès map.*

Proof of Corollary 6.5. So, we assume that f_{n+1} is exceptional. Without loss of generality, we show that f_n is exceptional as well and moreover, it is Lattès if and only if f_{n+1} is a Lattès map. Since f_i (and f_{n+1}) has continuous potential near a_i (respectively near b_0) and moreover, $a_i \in J_{f_i}$ which is the support of μ_{f_i}, then Proposition 6.4 along with the hypotheses of Theorem 6.1 yield that the map $h(\underline{a}, \cdot)$ which sends a neighborhood of $a_n \in J_{f_n}$ to a neighborhood of $b_0 \in J_{f_{n+1}}$ preserves the measures up to a scaling, i.e., for some $c > 0$

$$h^*(\underline{a}, \cdot)\mu_{f_{n+1}} = c \cdot \mu_{f_n}. \tag{6.6.1}$$

In [Lev90, Theorem 1], it was shown that there exists an infinite nontrivial family of symmetries on J_f if and only if f is post-critically finite with parabolic orbifold; hence (6.6.1) (see also Subsection 3.3) yields that f_n must be exceptional.

By a theorem of Zdunik [Zdu90], a rational function f is Lattès if and only if J_f is $\mathbb{P}^1$ and μ_f is absolutely continuous with respect to Lebesgue measure on $\mathbb{P}^1$; therefore, (6.6.1) yields that f_n is Lattès if f_{n+1} is Lattès.

Assume that f_{n+1} is conjugate either to a monomial or $\pm$Chebyshev polynomial. Then (6.6.1) yields that J_{f_n} is a one-dimensional topological space of Hausdorff dimension 1. According to Hamilton [Ham95], a Julia set which is a one-dimensional topological manifold must be either a circle, closed line segment (up to an automorphism of $\mathbb{P}^1$) or of Hausdorff dimension greater than one; thus J_{f_n} is itself a circle or a closed line segment (up to an automorphism of $\mathbb{P}^1$). This yields that f_n must be conjugated to a monomial or a $\pm$Chebyshev polynomial, which concludes the proof of Corollary 6.5. □

6.7. **Conclusion of our arguments.** Corollary 6.5 yields that all we have left to prove in Theorem 6.1 is that f_{n+1} must be exceptional. So, from now on, we assume that f_{n+1} is non-exceptional and we will derive a contradiction.

Lemma 6.6. *Let $\mathcal{S}$ be the family of symmetries of J_{f_n} on $D(a_n, r)$ for some $r > 0$. Then there exists $\epsilon > 0$ such that for any $\zeta \in \mathcal{S}$ with*

$$\sup_{x \in B(a_n, r)} |\zeta(x) - x| < \epsilon,$$

we must have $\zeta(x) \equiv x$ for $x \in D(a_n, r)$.

Proof of Lemma 6.6. Suppose this lemma is not true, then there exists a sequence of integers $\epsilon_\ell > 0$ with $\epsilon_\ell \to 0$ as ℓ tends to infinity, and a sequence of functions $\zeta_\ell \in \mathcal{S}$, which are not the identity map, such that

$$\sup_{x \in D(a_n, r)} |\zeta_\ell(x) - x| = \epsilon_\ell. \tag{6.7.1}$$

Consequently, $\{\zeta_\ell(x)\}_{\ell \geq 1}$ is a normal family with no subsequence having a constant limit (because ζ_ℓ tends to the identity map as $\ell \to \infty$). By Levin's result [Lev90], $\{\zeta_\ell\}_{\ell \geq 1}$ must consist of finitely many elements, which is a contradiction because there are infinitely many distinct real numbers ϵ_ℓ as in (6.7.1). □

Lemma 6.7. *There exists $N \in \mathbb{N}$, such that Φ_ℓ is the identity map on $B(\tilde{a}, r_0)$ for all $\ell \geq N$.*

Proof of Lemma 6.7. By abuse of notation, let $\tilde{\mu}_{n+1}$ and $\tilde{\mu}_n$ be the measures $\tilde{\mu}_{n+1}$ and $\tilde{\mu}_n$ in (3.2.2) restricted on $D_n(\tilde{a}, r_1)$ and respectively, on $D_n(\tilde{b}, r_2)$ for $\tilde{b} = (a_1, \ldots, a_{n-1}, b_0)$ and small radii r_1, r_2. Since $\hat{\mu}_n = \hat{\mu}_{n+1}$, there exist constants $c_\ell > 0$, such that

$$\Phi_\ell^*(\tilde{\mu}_{n+1}) = c_\ell \cdot \tilde{\mu}_{n+1}.$$

By Proposition 6.4, we see that for any $\underline{\alpha}$ in $D_{n-1}(\underline{a}, r_1) \cap J_{f_1} \times \cdots \times J_{f_{n-1}}$, the map $\hbar_\ell(\underline{\alpha}, \cdot)$ is a symmetry of J_{f_n} on $D(a_n, r_2)$. Moreover, the functions $\hbar_\ell(\tilde{x})$ converge uniformly to $\hbar(\tilde{x}) := x_n$ on $D_n(\tilde{a}, r_1)$ as ℓ tends to infinity.

Applying Lemma 6.6, there exists $N \in \mathbb{N}$, such that for any $\ell \geq N$ and any $\underline{\alpha}$ in $D_{n-1}(\underline{a}, r_1) \cap \big(J_{f_1} \times \cdots \times J_{f_{n-1}}\big)$, we have

$$\hbar_\ell(\underline{\alpha}, x_n) = x_n$$

for each $x_n \in D(a_n, r_1)$. Since a_i is an accumulating point in J_{f_i} for each i (see [Mil00]), when $\ell \geq N$, the zero locus of the equation $\hbar_\ell(\tilde{x}) - x_n = 0$ on $D_n(\tilde{a}, r_1)$ cannot have dimension $\leq n-1$, i.e., $\hbar_\ell(\tilde{x})$ is identically equal to x_n and so, Φ_ℓ is the identity map. This concludes the proof of Lemma 6.7. □

Let N be the positive integer appearing in Lemma 6.7. Pick $\ell_2 > \ell_1 > N$ with $j_{\ell_2} \geq j_{\ell_1}$ and $j_{i,\ell_2} \geq j_{i,\ell_1}$ for $1 \leq i \leq n$. Let

$$m_i := j_{i,\ell_2} - j_{i,\ell_1} \text{ for } 1 \leq i \leq n \text{ and } m_{n+1} := j_{\ell_2} - j_{\ell_1}.$$

Lemma 6.8. *With the above notation for the m_i's, let*

$$H' := (f_1^{m_1}, \cdots, f_{n+1}^{m_{n+1}})(H) \subset (\mathbb{P}^1)^{n+1}_{\mathbb{C}}.$$

Then $(f_1^{m_1}, \cdots, f_{n+1}^{m_{n+1}})(H') = H'$.

Proof of Lemma 6.8. From Lemma 6.7 (see also (6.4.1)), we have that

$$g^{j_{\ell_1}} \circ h \circ (f_1^{-j_{1,\ell_1}}, \cdots, f_n^{-j_{n,\ell_1}})(\tilde{x}) = g^{j_{\ell_2}} \circ h \circ (f_1^{-j_{1,\ell_2}}, \cdots, f_n^{-j_{n,\ell_2}})(\tilde{x})$$

on $D_n(\tilde{a}, r_0)$, or equivalently

$$h(\tilde{x}) = g^{m_{n+1}} \circ h \circ (f_1^{-m_1}, \cdots, f_n^{-m_n})(\tilde{x}). \tag{6.7.2}$$

Let

$$h'(\tilde{x}) := f_{n+1}^{m_{n+1}} \circ h \circ (f_1^{-m_1}, \cdots, f_n^{-m_n})(\tilde{x})$$

on a neighborhood of $\tilde{a}$. Now consider the analytic equation

$$h'(\tilde{x}) - x_{n+1} = 0$$

on a neighbourhood of $(\tilde{a}, b_1) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. The zero set of this equation is an analytic set of dimension n passing through the point $(\tilde{a}, b_1)$. For $\tilde{x}$ close to $\tilde{a}$, the points of the form $(\tilde{x}, h'(\tilde{x}))$ lie on the hypersurface H'. Combining (6.2.1) and (6.7.2), we get

$$h' \circ (f_1^{m_1}, \cdots, f_n^{m_n})(\tilde{x}) = f^{m_{n+1}} \circ h'(\tilde{x}).$$

Hence for points $\tilde{x}$ close to $\tilde{a}$, the points $(f_1^{m_1}, \cdots, f_{n+1}^{m_{n+1}})(\tilde{x}, h'(\tilde{x}))$, which are points on $(f_1^{m_1}, \cdots, f_{n+1}^{m_{n+1}})(H')$ satisfy also the equation $h'(\tilde{x}) - x_{n+1} = 0$. Finally, as both H' and $(f_1^{m_1}, \cdots, f_{n+1}^{m_{n+1}})(H')$ share an analytic set of dimension n in a neighbourhood of $(\tilde{a}, b_1)$, they must be identical. So H' is fixed by the endomorphism $(f_1^{m_1}, \cdots, f_{n+1}^{m_{n+1}})$ of $(\mathbb{P}^1)^{n+1}$, as desired. □

We recall that our sequence of tuples $(j_\ell, j_{1,\ell}, \ldots, j_{n,\ell})$ satisfies condition (6.3.2). Therefore, we can choose some integers $\ell_2 > \ell_1 > N$ such that $j_{\ell_2} = j_{\ell_1}$ and also, $j_{i,\ell_2} = j_{i,\ell_1}$ for $i = 2, \ldots, n$, while $j_{1,\ell_2} = j_{1,\ell_1} + 1$ and then apply Lemma 6.8 to the tuple of integers

$$m_i := j_{i,\ell_2} - j_{i,\ell_1} \text{ for } 1 \leq i \leq n \text{ and } m_{n+1} := j_{\ell_2} - j_{\ell_1}.$$

We have that $m_i = 0$ for each $i = 2, \dots, n+1$, while $m_1 = 1$. Therefore, Lemma 6.8 yields that

$$\left(f_1^2, \mathrm{id}, \cdots, \mathrm{id}\right)(H) = \left(f_1, \mathrm{id}, \cdots, \mathrm{id}\right)(H), \tag{6.7.3}$$

where the action in (6.7.3) on coordinates x_i for $2 \le i \le n+1$ is given by the corresponding identity maps. Equation (6.7.3) yields that H is a hypersurface of the form $\mathbb{P}^1 \times H_0$ (for some hypersurface $H_0 \subset (\mathbb{P}^1)^n$), contradicting thus our hypothesis that H projects dominantly onto any subset of n coordinate axes. Hence f_{n+1} (and thus each of the f_i's, as shown in Corollary 6.5) must be exceptional; this concludes our proof of Theorem 6.1. $\square$

7. Conclusion of our proof

In this Section we finish our proof of Theorem 2.2 and then we prove Theorem 1.4. Since we showed in Proposition 2.1 that it suffices to assume in Theorems 1.1, 1.2 and 1.3 that the subvariety $V \subset (\mathbb{P}^1)^n$ is a hypersurface projecting dominantly onto each subset of $(n-1)$ coordinate axes, then this will conclude our proof for each one of those theorems; note that we proved the above three theorems in Section 2.3 as a consequence of Theorem 2.2.

Proof of Theorem 2.2. So, we have a hypersurface $H \subset (\mathbb{P}^1)^n$ (for some integer $n > 2$) containing a Zariski dense set of points satisfying either hypothesis (1) or hypothesis (2) in Theorem 2.2. Furthermore, H projects dominantly onto any subset of $(n-1)$ coordinate axes of $(\mathbb{P}^1)^n$. We let $\hat{\mu}_i$ (for $i = 1, \dots, n$) be the measures introduced in Subsection 3.2.

Lemma 7.1. *We have $\hat{\mu}_1 = \hat{\mu}_2 = \cdots = \hat{\mu}_n$.*

Proof of Lemma 7.1. If each f_i and also H are defined over $\bar{\mathbb{Q}}$ (i.e, hypothesis (2) in Theorem 2.2 is met), then the conclusion of Lemma 7.1 follows immediately from Theorem 5.1. So, assume now that each f_i and also H are defined over $\mathbb{C}$, and moreover hypothesis (1) in Theorem 2.2 is met; in particular, $\deg(f_1) = \deg(f_2) = \cdots = \deg(f_n)$. We prove the result in this general case using a specialization technique similar to the one employed in the proof of Claim 5.5.

So, we let K be a finitely generated subfield of $\mathbb{C}$ such that each f_i and also H are defined over K, and let $\overline{K}$ be a fixed algebraic closure of K in $\mathbb{C}$. We know there exists an infinite sequence $S := \{(x_{1,j}, \dots, x_{n,j})\} \subset H(\mathbb{C})$ such that each $x_{i,j}$ is a preperiodic point for f_i for $i = 1, \dots, n$ and for each $j \ge 1$. Then the f_i's are base changes of endomorphisms $f_{i,K}$ of $\mathbb{P}^1_K$ (for $i = 1, \dots, n$); similarly, S is the base change of a subset $S_K \subset H(\overline{K})$. We can further extend $f_{i,K}$ to endomorphisms

$$f_{i,U} : \mathbb{P}^1_U \longrightarrow \mathbb{P}^1_U$$

over a variety U over $\mathbb{Q}$ of finite type and with function field K. For each geometric point $t \in U(\bar{\mathbb{Q}})$, the objects $f_{i,U}$ and S_U have reductions $f_{i,t}$ and S_t such that S_t consists of points with coordinates preperiodic under the action of the $f_{i,U}$'s. We also let $\hat{\mu}_{i,t}$ (for $i = 1, \dots, n$) be the probability measures on H_t obtained as pullback through the usual projection map onto $(n-1)$ coordinates (with the exception of the i-th coordinate axis) of the invariant measures on $(\mathbb{P}^1_{\mathbb{C}})^{n-1}$ corresponding to each $f_{j,t}$ for $j \neq i$. As proven in Claim 5.5 (using [YZb, Theorem 4.7] and also [YZa, Lemma 3.2.3]), we obtain that the subset $S_t \subset H_t$ is still Zariski dense for all the $\bar{\mathbb{Q}}$-points t in a dense open subset $U_0 \subseteq U$. Thus, as proven in Theorem 5.1, we conclude that

$$\hat{\mu}_{1,t} = \hat{\mu}_{2,t} = \cdots = \hat{\mu}_{n,t}$$

for each $t \in U_0(\bar{\mathbb{Q}})$. Since $U_0(\bar{\mathbb{Q}})$ is dense in $U(\mathbb{C})$ with respect to the usual archimedean topology, while the measures $\hat{\mu}_{i,t}$ vary continuously with the parameter t (since from the construction, the potential functions of these measures vary continuously with the coefficients of $f_{i,t}$), we conclude that

$$\hat{\mu}_{1,t} = \hat{\mu}_{2,t} = \cdots = \hat{\mu}_{n,t}$$

for all points in $U(\mathbb{C})$ including the point corresponding to the original embedding $K \subset \mathbb{C}$. Thus $\hat{\mu}_1 = \hat{\mu}_2 = \cdots = \hat{\mu}_n$, which concludes the proof of Lemma 7.1. □

Lemma 7.1 yields that the hypotheses of Proposition 5.2 are met and so, we know that there exists an index i, which we assume (without loss of generality) to be n so that for each $\alpha := (a_1, \dots, a_n) \in H(\mathbb{C})$, if a_i is preperiodic under the action of f_i for $i = 1, \dots, n-1$, then also a_n is preperiodic under the action of f_n.

Since all but finitely many periodic points of a rational map are repelling, and also, there is a Zariski dense open subset of points $\alpha \in H$ such that the restriction of the natural projection map $\pi|_H : H \longrightarrow (\mathbb{P}^1)^{n-1}$ on the first $(n-1)$ coordinate axes is unramified, then we can find a point $(x_{1,0}, \dots, x_{n,0}) \in H(\mathbb{C})$ satisfying the following properties:

(a) $x_{i,0}$ is a periodic repelling point for f_i for each $i = 1, \dots, n-1$; and
(b) there is a non-constant holomorphic germ h_0 defined in a neighborhood of $\tilde{x}_0 := (x_{1,0}, \dots, x_{n-1,0})$, with $h_0(\tilde{x}_0) = x_{n,0}$ and $(\tilde{x}, h_0(\tilde{x})) \in H(\mathbb{C})$ for all $\tilde{x}$ in a small neighborhood of $\tilde{x}_0$. Moreover, we also have that

$$\frac{\partial h_0}{\partial x_i}(\tilde{x}_0) \neq 0 \text{ for each } i = 1, \dots, n-1. \tag{7.0.1}$$

Note that hypothesis (7.0.1) can be achieved since the points satisfying $\frac{\partial h}{\partial x_i} = 0$ live in a proper Zariski closed subset of H (i.e., inequality (7.0.2) is an *open condition* which can be seen from computing the partial derivatives using implicit functions). It is essential in this case to know that H projects dominantly onto each subset of $(n-1)$ coordinates, i.e., H is *not* of the form $\mathbb{P}^1 \times H_0$ for some hypersurface $H_0 \subset (\mathbb{P}^1)^{n-1}$ since otherwise condition 7.0.1 would not necessarily hold.

Proposition 5.2 and condition (a) above yield that $x_{n,0}$ is preperiodic for f_n. At the expense of replacing each f_i by f_i^ℓ (for a suitable positive integer ℓ), we may assume that

- $x_{i,0}$ is a repelling fixed point of f_i for $1 \le i \le n-1$;
- $x_{n,1} := f_n(x_{n,0})$ is a fixed point of f_n; and
- there is a holomorphic germ $h(\tilde{x})$ near $\tilde{x}_0 = (x_{1,0}, \dots, x_{n-1,0})$ with $h(\tilde{x}_0) = x_{n,0}$, and $(\tilde{x}, h(\tilde{x})) \in H(\mathbb{C})$ for all $\tilde{x} \in (\mathbb{P}^1)^{n-1}(\mathbb{C})$ in a small (complex analytic) neighbourhood of $\tilde{x}_0$. Moreover, for each $i = 1, \cdots, n-1$ we have that

$$\beta_i := \frac{\partial h}{\partial x_i}(\tilde{x}_0) \neq 0. \tag{7.0.2}$$

Then all hypotheses in Theorem 6.1 are met; this yields that each f_i must be either all conjugate to monomials and ±Chebyshev polynomials, or they are all Lattès maps, which concludes our proof of Theorem 2.2. ☐

We finish our paper by proving Theorem 1.4.

Proof of Theorem 1.4. First we observe (similar to the proof of Proposition 2.1) that it suffices to prove that each irreducible, preperiodic hypersurface $H \subset (\mathbb{P}^1)^n$ is of the form $\pi_{i,j}^{-1}(C_{i,j})$ (for a pair of indices $i, j \in \{1, \dots, n\}$), where $C_{i,j} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve, which is preperiodic under the action of $(x_i, x_j) \mapsto (f_i(x_i), f_j(x_j))$ (and $\pi_{i,j}$ is the projection of $(\mathbb{P}^1)^n$ onto the (i, j)-th coordinate axes). Indeed, just as in the proof of Proposition 2.1, we obtain that any preperiodic subvariety $V \subset (\mathbb{P}^1)^n$ is a component of an intersection of preperiodic hypersurfaces, thus reducing our proof to the case V is a hypersurface.

Since the case $n = 2$ was proved in [GNY, Theorem 1.1], from now on, we assume $V \subset (\mathbb{P}^1)^n$ is a hypersurface and $n > 2$. Then, at the expense of replacing $\Phi = (f_1, \dots, f_n)$ by an iterate of it and also replacing the hypersurface V by a suitable $\Phi^k(V)$ (for $k \in \mathbb{N}$), we may (and do) assume that V is invariant under the action of Φ. Also, we may assume V projects dominantly onto each subset of $(n-1)$ coordinate axes of $(\mathbb{P}^1)^n$ since otherwise $V = \mathbb{P}^1 \times V_0$ and then we can argue inductively on n (because $V_0 \subset (\mathbb{P}^1)^{n-1}$ would be invariant under the induced action of Φ on those $(n-1)$ coordinate axes). Next we will prove there are *no* such hypersurfaces, thus providing the desired conclusion in Theorem 1.4.

We let $\pi|_V : V \longrightarrow (\mathbb{P}^1)^{n-1}$ be the projection on the first $n-1$ coordinate axes; we know there exists a Zariski open subset $U \subset (\mathbb{P}^1)^{n-1}$ such that $\pi|_V^{-1}(\beta)$ is finite for each $\beta \in U$.

Now, let $\beta := (a_1, \dots, a_{n-1}) \in U(\mathbb{C})$ such that each a_i is periodic under the action of f_i. We claim that each point $\alpha \in V(\mathbb{C})$ satisfying $\pi|_V(\alpha) = \beta$ is preperiodic under the action of Φ, i.e., its last coordinate is preperiodic for f_n. Indeed, since β is periodic, then for some positive integer m, we have that $\Phi^m(\alpha) \in \pi|_V^{-1}(\beta)$ and because $\pi|_V^{-1}(\beta)$ is a finite set, we conclude that the last coordinate of α (and therefore, α itself) must be preperiodic, as claimed.

At the expense of shrinking U to a smaller, but still Zariski dense, open subset, we may even assume $\pi|_V$ is unramified above each point of U. Then we can argue as in the proof of Theorem 2.2 and find a point $(x_{1,0}, \dots, x_{n,0})$ satisfying the conditions:

(a) $x_{i,0}$ is a periodic repelling point for f_i for each $i = 1, \dots, n-1$; and
(b) there is a non-constant holomorphic germ h_0 defined in a neighborhood of $\tilde{x}_0 := (x_{1,0}, \dots, x_{n-1,0})$, with $h_0(\tilde{x}_0) = x_{n,0}$ and $(\tilde{x}, h_0(\tilde{x})) \in V(\mathbb{C})$ for all $\tilde{x}$ in a small neighborhood of $\tilde{x}_0$. Moreover, we also have that $\frac{\partial h_0}{\partial x_i}(\tilde{x}_0) \neq 0$ for each i.

Furthermore, after replacing Φ by yet another iterate, we get that each $x_{i,0}$ is fixed by f_i. Then we meet the hypotheses of Theorem 6.1 and since we assumed that each f_i is non-exceptional, we derive a contradiction. This concludes our proof of Theorem 1.4. □

References

[Bak09] M. Baker, *A finiteness theorem for canonical heights attached to rational maps over function fields*, J. Reine Angew. Math. **626** (2009), 205-233.

[BH05] M. Baker and L.-C. Hsia, *Canonical heights, transfinite diameters, and polynomial dynamics*, J. reine angew. Math. **585** (2005), 61–92.

[BR06] M. Baker and R. Rumely, *Equidistribution of small points, rational dynamics, and potential theory*, Ann. Inst. Fourier (Grenoble) **56** (2006), 625–688.

[Ben05] R. L. Benedetto, *Heights and preperiodic points of polynomials over function fields*, Int. Math. Res. Not. **62** (2005), 3855-3866.

[BG06] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs **4**, Cambridge University Press, Cambridge, 2006. xvi+652 pp.

[Bro65] H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6**(1965), 103–144.

[CS93] G. Call and J. Silverman, *Canonical heights on varieties with morphisms*, Compositio Math. **89**(1993), 163–205.

[CL06] A. Chambert-Loir, *Mesures et équidistribution sur les espaces de Berkovich*, J. Reine Angew. Math. **595** (2006), 215–235.

[DF] R. Dujardin and C. Favre, *The dynamical Manin-Mumford problem for plane polynomial automorphisms*, **19** (2017), no. 11, 3421–3465.

[DH93] A. Douady and J. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta. Math. **171** (1993), 263–297.

[Fab09] X. W. C. Faber, *Equidistribution of dynamically small subvarieties over the function field of a curve*, Acta Arith. **137** (2009), 345–389.

[Fal84] G. Faltings, *Calculus on arithmetic surfaces*, Ann. Math. **119** (1984), 387-424.

[FRL06] C. Favre and J. Rivera-Letelier, *Équidistribution quantitative des points de petite hauteur sur la droite projective*, Math. Ann. **355** (2006), 311–361.

[FLM83] A. Freire, A. Lopes and R. Mañé, *An invariant measure for rational maps*, Bol. Soc. Brasil Mat. **14** (1983) 45-62.

[GN16] D. Ghioca and K. D. Nguyen, *Dynamical Anomalous Subvarieties: Structure and Bounded Height Theorems*, Adv. Math. **288** (2016), 1433–1462.

[GN] D. Ghioca and K. D. Nguyen, *A dynamical variant of the Pink-Zilber conjecture*, submitted for publication (2017), 24 pp.

[GNY] D. Ghioca, K. D. Nguyen and H. Ye, *The Dynamical Manin-Mumford Conjecture and the Dynamical Bogomolov Conjecture for split rational maps*, J. Eur. Math. Soc., 2017, 25 pages, to appear.

[GT10] D. Ghioca and T. J. Tucker, *Proof of a dynamical Bogomolov conjecture for lines under polynomial actions*, Proc. Amer. Math. Soc. **138** (2010), 937–942.

[GTZ11] D. Ghioca, T. J. Tucker and S. Zhang, *Towards a dynamical Manin-Mumford Conjecture*, Int. Math. Res. Not. IMRN 2011, no. 22, 5109–5122.
[Gub08] W. Gubler, *Equidistribution over function fields*, Manuscripta Math. **127** (2008), 485-510.
[Ham95] D. Hamilton, *Length of Julia curves*, Pacific J. Math. **169** (1995), no. 1, 75–93.
[Hri85] P. Hriljac, *Heights and Arakelovs intersection theory*, Amer. J. Math. **107** (1985), 23-38.
[Ino11] H. Inou, *Extending local analytic conjugacies*, Trans. Amer. Math. Soc. **363** (2011), no. 1, 331–343.
[Lau84] M. Laurent, *Equations diophantiennes exponentielles*, Invent. Math. **78** (1984), 299–327.
[Lev90] G. M. Levin, *Symmetries on Julia sets*, Mat. Zametki **48** (1990), 72–79, 159.
[Lyu83] M. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Erg. Th. and Dyn. Sys. **3** (1983), 351–385.
[Man83] R. Mañé, *On the uniqueness of the maximizing measure for rational maps*, Bol. Soc. Bras. Math. **14.2** (1983), 27–43.
[McQ95] M. McQuillan, *Division points on semi-abelian varieties*, Invent. Math. **120** (1995), no. 1, 143–159.
[Med07] A. Medvedev, *Minimal sets in ACFA*, Thesis (Ph.D.)–University of California, Berkeley, 2007, 96 pp.
[MS14] A. Medvedev and T. Scanlon, *Invariant varieties for polynomial dynamical systems*, Ann. of Math. (2) **179** (2014), no. 1, 81–177.
[Mil00] J. Milnor, *Dynamics in one complex variable*, Wiesbaden, Germany: Vieweg. ISBN 3-528-13130-6. (2000).
[Mil04] J. Milnor, *On Lattès maps*, preprint available online at http://arxiv.org/abs/math/0402147.
[Mim97] A. Mimar, *On the preperiodic points of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which lie on a curve*, Thesis (Ph.D.), Columbia University, 1997.
[Mim13] A. Mimar, *On the preperiodic points of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which lie on a curve*, Trans. Amer. Math. Soc. **365** (2013), no. 1, 161–193.
[Mor96] A. Moriwaki, *Hodge index theorem for arithmetic cycles of codimension one*, Math. Res. Lett. **3** (1996), no. 2, 173-183.
[Ray83] M. Raynaud, *Sous-variétés d'une variété abélienne et points de torsion*, Arithmetic and geometry, vol. I, Progr. Math., vol. 35, Birkhäuser, Boston, MA, 1983, pp. 327–352.
[Thu85] W. Thurston, *On the combinatorics of iterated rational maps*, preprint, 1985.
[Ull98] E. Ullmo, *Positivité et discrétion des points algébriques des courbes*, Ann. of Math. (2) **147** (1998), 167–179.
[Yua08] X. Yuan, *Big line bundles over arithmetic varieties*, Invent. Math. **173** (2008), 603–649.
[YZa] X. Yuan and S. Zhang, *Calabi theorem and algebraic dynamics*, preprint, 24 pages.
[YZb] X. Yuan and S. Zhang, *Small points and Berkovich metrics*, 28 pages, preprint available on the webpage: `http://www.math.columbia.edu/~szhang/papers/Preprints.html`
[YZc] X. Yuan and S. Zhang, *The arithmetic Hodge index theorem for adelic line bundles II*, arXiv:1304.3539.
[YZ16] X. Yuan and S. Zhang, *The arithmetic Hodge index theorem for adelic line bundles*, Math. Ann. **367** (2017), no. 3, 1123–1171.
[Zan12] U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Mathematics Studies, vol. 181, Princeton University Press, Princeton, NJ, 2012, With appendixes by David Masser.

[Zdu90] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Inv. Math. **99** (1990) 627–649.

[Zha95a] S. Zhang, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. **8** (1995), 187–221.

[Zha95b] S. Zhang, *Small points and adelic metrics*, J. Alg. Geom. **4** (1995), 281–300.

[Zha98] S. Zhang, *Equidistribution of small points on abelian varieties*, Ann. Math. (2), **147**(1998), 159–165.

[Zha06] S. Zhang, *Distributions in algebraic dynamics*, Survey in Differential Geometry, vol. 10, International Press, 2006, pp. 381–430.

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