THE DYNAMICAL MORDELL-LANG CONJECTURE

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A linear recurrence sequence $\{x_n\}_{n\geq 1}$ of complex numbers has the property that there exist a positive integer N and constants $c_1,\dots,c_N \in \mathbb{C}$ such that, for all $n \geq 1$, $x_{n+N} = c_1x_{n+N-1} + c_2x_{n+N-2} + \cdots + c_Nx_n$. It is natural to ask what is the structure of the set $S := \{n \in \mathbb{N} \colon x_n = 0\}$.

Skolem (in the case each $x_i \in \mathbb{Z}$), later Mahler (in the case each $x_i \in \bar{\mathbb{Q}}$), and finally Lech (in the general case) answered this question by showing that S is a union of at most finitely many (infinite) arithmetic progressions along with a finite set. It is indeed possible that S contains an arithmetic progression, for example, if $N = 3$, $c_1 = c_2 = 0$ and $c_3 = 1$, while $x_2 = 0$, then $\{2 + 3n \colon n \geq 0\} \subseteq S$. We sketch briefly the method of Skolem-Mahler-Lech. There exist complex numbers $r_1,\dots,r_m$ and polynomials $f_1,\dots,f_m \in \mathbb{C}[z]$ such that, for all $n \geq 1$,

$$x_n = f_1(n)r_1^n + \cdots + f_m(n)r_m^n.$$

The numbers r_i are the distinct, nonzero roots of the characteristic equation for the linear recurrence sequence: $x^N - c_1x^{N-1} - \cdots - c_{N-1}x - c_N = 0$, while each polynomial f_i is not constant only if the corresponding r_i is a multiple root of the above equation. We let K be the finitely generated extension of $\mathbb{Q}$ containing each r_i and each coefficient of each f_i. Then one can show that there exists a prime number p and a suitable embedding $K \hookrightarrow \mathbb{Q}_p$ such that each r_i is mapped into a p-adic unit, [1]. Furthermore, there exists $k \in \mathbb{N}$ such that $|r_i^k - 1|_p < 1$ and so, for each $i = 1,\dots,m$, the function $z \mapsto (r_i^k)^z$ is analytic on $\mathbb{Z}_p$. Hence, for each $\ell = 0,\dots,k-1$, we let G_ℓ be the p-adic analytic function given by

$$G_\ell(z) := \sum_{i=1}^{m} r_i^\ell f_i(kz+\ell)\cdot\left(r_i^k\right)^z,$$

for each $z \in \mathbb{Z}_p$, and obtain that, for all $n \in \mathbb{N}$, $x_{nk+\ell} = G_\ell(n)$. Therefore $x_{nk+\ell} = 0$ if and only if $G_\ell(n) = 0$. But, similarly to a nonzero complex analytic function, a nonzero p-adic analytic function does not have infinitely many zeros in a compact set, such as $\mathbb{Z}_p$. So, for each $\ell = 0,\dots,k-1$, either $G_\ell(z) = 0$ has finitely many solutions in $\mathbb{Z}_p$, and therefore in $\mathbb{N}$, or G_ℓ is identically equal to 0, and thus $G_\ell(n) = 0$ for all $n \in \mathbb{N}$. This yields that the set $S = \{n \in \mathbb{N} \colon x_n = 0\}$ is a union of at most finitely many arithmetic progressions (of ratio k) along with a finite set.

The above argument can be formulated also in a geometric setting. Indeed, for any linear map $\Phi : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ (i.e., a N-by-N matrix A with complex entries) and any point $\alpha \in \mathbb{C}^N$, again one can find a prime number p (together with a suitable embedding into $\mathbb{Q}_p$ of α and of all entries of A), a positive integer k, and p-adic analytic maps $G_\ell : \mathbb{Z}_p \longrightarrow {\mathbb{Z}_p}^N, \ell = 0,\dots,k-1$, such that $\Phi^{nk+\ell}(\alpha) = G_\ell(n)$ for all $n \in \mathbb{N}$ (note that $\Phi^n := \Phi \circ \cdots \circ \Phi$, where Φ is composed with itself n times). Hence the same reasoning as above regarding the discreteness of zeros for nontrivial p-adic analytic functions yields that for any (linear) subvariety $V \subset \mathbb{C}^N$, the set $S := \{n \in \mathbb{N} \colon \Phi^n(\alpha) \in V\}$ is a union of at most finitely many arithmetic progressions along with a finite set. An argument identical with this one yields the same conclusion for automorphisms Φ of $\mathbb{P}^N$ defined over $\mathbb{C}$.

The above extension of the original Skolem-Mahler-Lech method to the geometric setting works since one has explicit formulas for the n-th iterate of a point under an automorphism of $\mathbb{P}^N$. Quite surprisingly, one can extend this p-adic method to any automorphism of an affine variety, [1], and even further to any étale endomorphism Φ of any quasiprojective variety X, [2], even though in these cases there are *no* explicit formulas for the n-th iterate of a point $\alpha \in X(\mathbb{C})$. Nevertheless, one can show that there exist finitely many p-adic analytic functions G_ℓ that parametrize the orbit of α under Φ. Then again we obtain that for any subvariety $V \subseteq X$, the set $S := \{n \in \mathbb{N} \colon \Phi^n(\alpha) \in V(\mathbb{C})\}$ is a union of at most finitely many arithmetic progressions along with a finite set. It is natural to ask whether the above conclusion holds for any endomorphism of any variety.

The Dynamical Mordell-Lang Conjecture. *Let X be a quasiprojective variety defined over $\mathbb{C}$, let $V \subseteq X$ be a subvariety, let $\alpha \in X(\mathbb{C})$ be a point, and let $\Phi : X \longrightarrow X$ be an endomorphism. Then the set $S = \{n \in \mathbb{N} \colon \Phi^n(\alpha) \in V(\mathbb{C})\}$ is a union of at most finitely many arithmetic progressions along with a finite set.*

An alternative formulation of this statement is to say that whenever $V \subseteq X$ contains no positive dimensional periodic subvariety periodic under Φ, then the above set S must be finite. The name of this conjecture comes from its connection with the Mordell-Lang Conjecture of arithmetic geometry (now a theorem due to Faltings), which states that the intersection of a subvariety of a semiabelian variety G with a finitely generated subgroup Γ of $G(\mathbb{C})$ is a union of at most finitely many cosets of subgroups of Γ. If Γ is a cyclic group generated by a point $\gamma \in G(\mathbb{C})$, the Mordell-Lang Conjecture reduces to the one stated above applied to the translation-by-γ map $\Phi : G \longrightarrow G$. Also, there are counterexamples to the Dynamical Mordell-Lang conjecture if X is defined over a field of positive characteristic, [3].

There are only a few partial results known for the above conjecture besides the case of étale endomorphisms, [2]; we list some of the cases below:

(1) if $X = (\mathbb{P}^1)^N$ and $\Phi(x_1, \dots, x_N) = (\varphi_1(x_1), \dots, \varphi_N(x_N))$ for some classes of rational maps $\varphi_i \in \mathbb{C}(z)$ and some classes of subvarieties $V \subseteq X$, [4], [5]. For each such result, if the conditions on V are milder, then the conditions on the φ_i are stricter. For example, the Dynamical Mordell-Lang Conjecture holds for any subvariety $V \subseteq X$ defined over $\bar{\mathbb{Q}}$ if $\varphi_i(z) = z^2 + c_i$ for some $c_i \in \mathbb{Z}$, for each $i = 1, \dots, N$, [4]. In the opposite direction, the Dynamical Mordell-Lang Conjecture holds for complex curves $V \subseteq X$ if $\varphi_1 = \varphi_2 = \cdots = \varphi_N \in \mathbb{C}[z]$ is a polynomial with no periodic ramified points, except for the point at infinity.

(2) if $X = \mathbb{A}^N$, V is a complex line, and $\Phi(x_1, \dots, x_N) = (f_1(x_1), \dots, f_N(x_N))$, where each $f_i \in \mathbb{C}[z]$, [7].

(3) if Φ is a *generic* endomorphism of $X = \mathbb{P}^N$ defined over $\mathbb{C}$, [6].

(4) if Φ is a birational polynomial morphism of the complex plane, [8].

Only for the results in (1) one relies heavily on the use of the Skolem-Mahler-Lech method to find suitable p-adic parametrizations of the orbit of α under Φ. In general, it is *very* difficult to find such parametrizations since one needs to find a prime p such that the orbit of α does not meet the ramification locus of Φ modulo p. Heuristically it is even expected that this method might never work for endomorphisms of $\mathbb{P}^N$ if $N \geq 5$, [4]. Also, for (2)–(4), the methods of proof do not seem to allow generalizations. So, one would need a *new* approach to prove the Dynamical Mordell-Lang Conjecture in its full generality. Finally, in [3] it is shown that in the Dynamical Mordell-Lang Conjecture the set S is *always* a union of at most finitely many arithmetic progressions along with a set T of Banach density 0. However, proving that T is actually finite seems currently beyond the reach.

References

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