

# ZARISKI DENSE ORBITS FOR ENDOMORPHISMS OF A POWER OF THE ADDITIVE GROUP SCHEME DEFINED OVER FINITE FIELDS

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ABSTRACT. We prove the Zariski dense orbit conjecture in positive characteristic for endomorphisms of  $\mathbb{G}_a^N$  defined over  $\overline{\mathbb{F}_p}$ .

## 1. INTRODUCTION

**1.1. Notation.** We let  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$  denote the set of nonnegative integers.

For any morphism  $\Phi$  on a variety  $X$  and for any integer  $n \geq 0$ , we let  $\Phi^n$  be the  $n$ -th iterate of  $\Phi$  (where  $\Phi^0$  is the identity map  $\text{id} := \text{id}_X$ , by definition). For a point  $x \in X$ , we denote by  $\mathcal{O}_\Phi(x)$  the orbit of  $x$  under  $\Phi$ , i.e., the set of all  $\Phi^n(x)$  for  $n \geq 0$ . When  $\Phi$  is only a rational self-map of  $X$ , the orbit  $\mathcal{O}_\Phi(x)$  of the point  $x \in X$  is well-defined if each  $\Phi^n(x)$  lies outside the indeterminacy locus of  $\Phi$ . For any self-map  $\Phi$  on a variety  $X$ , we say that  $x \in X$  is preperiodic if its orbit  $\mathcal{O}_\Phi(x)$  is finite.

We denote by  $M_{m,n}(R)$  the set of  $m \times n$ -matrices with entries in the ring  $R$ ; we denote by  $\mathbf{I}_m$  the identity  $m \times m$ -matrix.

**1.2. The classical Zariski dense orbit conjecture.** The following conjecture was motivated by a similar question raised by Zhang [Zha06], and it was formulated independently by Medvedev and Scanlon [MS14] and by Amerik and Campana [AC08].

**Conjecture 1.1.** *Let  $X$  be a quasiprojective variety defined over an algebraically closed field  $K$  of characteristic 0 and let  $\Phi : X \dashrightarrow X$  be a dominant rational self-map. Then either there exists  $\alpha \in X(K)$  whose orbit under  $\Phi$  is well-defined and Zariski dense in  $X$ , or there exists a non-constant rational function  $f : X \dashrightarrow \mathbb{P}^1$  such that  $f \circ \Phi = f$ .*

There are several partial results known towards Conjecture 1.1 (for example, see [AC08, BGRS17, BGR17, GH18, GSa, GS19, GS17, GX18, MS14]).

**1.3. The picture in positive characteristic.** If  $K$  has characteristic  $p > 0$ , then Conjecture 1.1 does not hold due to the presence of the Frobenius endomorphism (see [BGR17, Example 6.2] and also, [GS21, Remark 1.2]). Based on the discussion from [GS21], the authors proposed the following conjecture as a variant of Conjecture 1.1 in positive characteristic.

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**Conjecture 1.2.** *Let  $K$  be an algebraically closed field of positive transcendence degree over  $\overline{\mathbb{F}_p}$ , let  $X$  be a quasiprojective variety defined over  $K$ , and let  $\Phi : X \dashrightarrow X$  be a dominant rational self-map defined over  $K$  as well. Then at least one of the following three statements must hold:*

- (A) *There exists  $\alpha \in X(K)$  whose orbit  $\mathcal{O}_\Phi(\alpha)$  is Zariski dense in  $X$ .*
- (B) *There exists a non-constant rational function  $f : X \dashrightarrow \mathbb{P}^1$  such that  $f \circ \Phi = f$ .*
- (C) *There exist positive integers  $m$  and  $r$ , there exists a variety  $Y$  defined over a finite subfield  $\mathbb{F}_q$  of  $K$  such that  $\dim(Y) \geq \text{trdeg}_{\overline{\mathbb{F}_p}} K + 1$  and there exists a dominant rational map  $\tau : X \dashrightarrow Y$  such that*

$$\tau \circ \Phi^m = F^r \circ \tau,$$

where  $F$  is the Frobenius endomorphism of  $Y$  corresponding to the field  $\mathbb{F}_q$ .

Conjecture 1.2 has been proven in the case of algebraic tori in [GS21] and more generally in the case of all split semiabelian varieties defined over  $\overline{\mathbb{F}_p}$  in [GSb]. For an illustration of the trichotomy in the conclusion of Conjecture 1.2, we refer the reader to [GS21, Example 1.6]. Also, we note that one definitely requires the hypothesis that  $\text{trdeg}_{\overline{\mathbb{F}_p}} K \geq 1$  in Conjecture 1.2 since for any self-map  $\Phi$  defined over  $\overline{\mathbb{F}_p}$ , each point of  $X(\overline{\mathbb{F}_p})$  is preperiodic and therefore, condition (A) cannot hold; on the other hand, there are plenty of examples of maps  $\Phi$  defined over  $\overline{\mathbb{F}_p}$  for which neither condition (B) nor condition (C) would hold.

**1.4. Our results.** We prove the following more precise version of Conjecture 1.2 in the case of group endomorphisms of  $\mathbb{G}_a^N$  defined over  $\overline{\mathbb{F}_p}$ .

**Theorem 1.3.** *Let  $N \in \mathbb{N}$  and let  $L$  be an algebraically closed field of characteristic  $p$  such that  $\text{trdeg}_{\overline{\mathbb{F}_p}} L \geq 1$ . Let  $\Phi : \mathbb{G}_a^N \dashrightarrow \mathbb{G}_a^N$  be a dominant group endomorphism defined over  $\overline{\mathbb{F}_p}$ . Then at least one of the following statements must hold.*

- (A) *There exists  $\alpha \in \mathbb{G}_a^N(L)$  whose orbit under  $\Phi$  is Zariski dense in  $\mathbb{G}_a^N$ .*
- (B) *There exists a non-constant rational function  $f : \mathbb{G}_a^N \dashrightarrow \mathbb{P}^1$  such that  $f \circ \Phi = f$ .*
- (C) *There exist positive integers  $m$  and  $r$ , a positive integer  $N_0$  greater than or equal to  $\text{trdeg}_{\overline{\mathbb{F}_p}} L + 1$  and a dominant group homomorphism  $\tau : \mathbb{G}_a^N \dashrightarrow \mathbb{G}_a^{N_0}$  such that*

$$(1.3.1) \quad \tau \circ \Phi^m = F^r \circ \tau,$$

where  $F$  is the usual Frobenius endomorphism of  $\mathbb{G}_a^{N_0}$  induced by the field automorphism  $x \mapsto x^p$ .

**1.5. Discussion of our proof.** The strategy of our proof for Theorem 1.3 is as follows. Suppose we have a group endomorphism  $\Phi : \mathbb{G}_a^N \dashrightarrow \mathbb{G}_a^N$  defined over  $\mathbb{F}_q$  where  $q = p^\ell$  for some  $\ell \in \mathbb{N}$ . As shown in [Pog17, Proposition 3.9], the endomorphism  $\Phi$  is given by an  $N$ -by- $N$  matrix  $A$  (acting linearly on  $\mathbb{G}_a^N$ ) whose entries are one-variable additive polynomials in the variables  $x_1, \dots, x_N$  (corresponding to the  $N$  coordinate axes of  $\mathbb{G}_a^N$ ), i.e.,

$$(1.3.2) \quad A = (f_{i,j}(x_j))_{1 \leq i,j \leq N},$$

where a polynomial  $f(x)$  is additive if it is of the form

$$(1.3.3) \quad \sum_{k=0}^r c_k x^{p^k};$$

furthermore, since  $\Phi$  is defined over  $\mathbb{F}_q$ , then each coefficient of each additive polynomial  $f_i(x_j)$  belongs to  $\mathbb{F}_q$ . We denote by  $F$  the Frobenius endomorphism (of  $L$ ) corresponding to the finite field  $\mathbb{F}_p$  (i.e.,  $x \mapsto x^p$ ). So, the action of  $\Phi$  is given by a matrix of polynomials (with coefficients in  $\mathbb{F}_q$ ) in the Frobenius operator, i.e., the entries of our matrix  $A$  live in  $\mathbb{F}_q[F]$ . We will study technical properties of  $\mathbb{F}_q[F]$  and of the ring of matrices with entries in  $\mathbb{F}_q[F]$  in Section 3.

Now, for any given point  $\alpha \in \mathbb{G}_a^N(L)$ , the orbit  $\mathcal{O}_\Phi(\alpha)$  is contained in a finitely generated  $\mathbb{F}_p[F]$ -submodule  $\Gamma$  of  $\mathbb{G}_a^N$ . If we assume that  $\mathcal{O}_\Phi(\alpha)$  is not Zariski dense, then it must be contained in some proper subvariety  $V$  of  $\mathbb{G}_a^N$ . Then, we can describe the structure of  $V(L) \cap \Gamma$  using [Ghi08, Theorem 2.6]; for more details, see Section 3.3. Then we will use the fact that  $\Phi$  is integral over the commutative ring  $\mathbb{F}_p[F^\ell]$ ; for more details, see Section 3. We also employ several reductions discussed in Section 4 which allow us to split the action of  $\Phi$  into  $(\Phi_1, \Phi_2)$  where  $\Phi_1$  is given by the diagonal action of powers of the Frobenius endomorphism, while the minimal polynomial of  $\Phi_2$  over  $\mathbb{F}_p[F^\ell]$  has roots that are multiplicatively independent with respect to  $F$ . This helps us reduce Theorem 1.3 into two separate extreme cases which are much more convenient to deal with; the general case in Theorem 1.3 then follows.

**1.6. Discussion of possible extensions.** We note that our approach does not generalize to the case the endomorphism  $\Phi$  of  $\mathbb{G}_a^N$  is defined over an arbitrary field  $L$  of characteristic  $p$ . The reason is that in the case of general group endomorphisms of  $\mathbb{G}_a^N$ , the orbit will not necessarily be contained in a finitely generated  $\mathbb{F}_p[F]$ -submodule of  $\mathbb{G}_a^N$ , and thus, we cannot use the  $F$ -structure theorem proven in [Ghi08, Theorem 2.6] anymore. In the special case when for each  $1 \leq i, j \leq N$ , the linear term  $c_0$  of  $f_i(x_j)$  (i.e.,  $c_0 = f'_i(x_j)$  from equations (1.3.2) and (1.3.3)) is transcendental over  $\mathbb{F}_p$ , then one can reformulate Conjecture 1.2 for  $(\mathbb{G}_a^N, \Phi)$  in the context of Drinfeld modules of generic characteristic. Even though there is a very rich arithmetic theory for Drinfeld modules of generic characteristic built in parallel to the classical Diophantine geometry questions in characteristic 0 (see, for example, [Bre05, CG20, GT07, GT08, Sca02]), there are still several technical difficulties to overcome in this case alone. Furthermore, when we deal with the most general case of an endomorphism of  $\mathbb{G}_a^N$  (defined over a field  $L$  of characteristic  $p$ ), in which case some of the derivatives of the polynomials  $f_i(x_j)$  are in a finite field, while others are transcendental over  $\mathbb{F}_p$ , then there are significant more complications since then we would be dealing with a mixed arithmetic structure coming from the action of Drinfeld modules of both generic characteristic and also of special characteristic (see [Ghi05] for a sample of difficulties arising in the context of Drinfeld modules of special characteristic).

**1.7. Plan for our paper.** In Section 4 we use the technical results proven in Sections 2 and 3 to show that instead of proving Theorem 1.3 for the dynamical system  $(\mathbb{G}_a^N, \Phi)$ , we can instead prove it for the dynamical system  $(\mathbb{G}_a^{N_1} \times \mathbb{G}_a^{N_2}, (\Phi_1, \Phi_2))$  where  $N = N_1 + N_2$ ,  $\Phi_1$  is a group endomorphism of  $\mathbb{G}_a^{N_1}$  given by the coordinate-wise action of powers of the Frobenius endomorphism, and the minimal polynomial of  $\Phi_2$  over  $\mathbb{F}_p[F^\ell]$  has roots that are multiplicatively

independent with respect to  $F$ . In other words, we reduce Theorem 1.3 to Proposition 5.2. We conclude our paper by proving Proposition 5.2 in Section 5.

## 2. TECHNICAL BACKGROUND

In this Section we gather some useful results for our proofs which come from two different sources: the theory of skew fields (see Section 2.1 and more generally [Coh95]) and the  $F$ -structure theorem proven by Moosa-Scanlon [MS04] in order to describe intersections of subvarieties of  $\mathbb{G}_m^N$  with finitely generated subgroups of  $\mathbb{G}_m^N(\overline{\mathbb{F}_p(t)})$  (see Lemma 2.4 in Section 2.2).

**2.1. Some results about splitting matrices over skew fields.** In this Section we state useful results about splitting matrices over skew fields which will be used later in our proofs; for more details about skew fields, we refer the reader to [Coh95].

**Fact 2.1.** *Let  $K$  be a skew field with centre  $k$ . Suppose that  $A$  is a matrix in  $M_{n,n}(K)$  which is algebraic over  $k$ . Let  $f(x) = f_0(x)f_1(x)$  be the minimal polynomial of  $A$  over  $k$  where  $f_0, f_1 \in k[x]$  are coprime. Then,  $A$  has a conjugate of the form*

$$A_0 \oplus A_1$$

such that the minimal polynomials of  $A_0$  and  $A_1$  over  $k$  are equal to  $f_0$  and  $f_1$ , respectively.

*Proof.* Using the arguments after [Coh95, Corollary 8.3.4, p. 381–382],  $A$  must have a conjugate of the form

$$B_1 \oplus \cdots \oplus B_r,$$

where each  $B_i$  corresponds to an elementary divisor, say  $q_i$ . Since  $f_0$  and  $f_1$  are coprime, each  $q_i$  must divide exactly one of  $f_0$  and  $f_1$  and be coprime with respect to the other one. So, assume without loss of generality that for some  $i \geq 0$  we have

$$q_j \mid f_0 \text{ and } (q_j, f_1) = 1,$$

for all  $j \leq i$  and

$$q_j \mid f_1 \text{ and } (q_j, f_0) = 1,$$

for  $j > i$ . Letting

$$A_0 = B_1 \oplus \cdots \oplus B_i, \text{ and } A_1 := B_{i+1} \oplus \cdots \oplus B_r,$$

gives us the desired conclusion.  $\square$

**Fact 2.2.** *Let  $K$  be a skew field with centre  $k$  and  $A \in M_{n,n}(K)$  be a matrix with a minimal polynomial equal to  $(x - \alpha)^r$  for some  $\alpha \in k$  and  $r \in \mathbb{N}$ . Then, there exist an invertible matrix  $P \in M_{n,n}(K)$  such that*

$$P^{-1}AP = J_{\alpha,r_1} \bigoplus \cdots \bigoplus J_{\alpha,r_m},$$

where  $J_{\alpha,s}$  is the  $s$ -by- $s$  Jordan canonical matrix having unique eigenvalue  $\alpha$  and its only nonzero entries away from the diagonal being the entries in positions  $(i, i+1)$  (for  $i = 1, \dots, s-1$ ), which are all equal to 1.

*Proof.* This is a consequence of the the discussion after [Coh95, Corollary 8.3.4, p. 381–384]; also, see [Coh95, Theorem 8.3.6].  $\square$

**2.2. A special type of Diophantine equation.** In this Section we prove Lemma 2.4 that gives an asymptotic upper bound on the number of solutions of a special type of equation given by equation (2.4.1). This bound will be instrumental in our proof Theorem 1.3. We start with an easy result which will be used in our proof for Lemma 2.4. The result is actually a consequence of Vandermonde determinants and still holds if we know that the equation (2.3.1) holds for  $r$  consecutive  $n$ 's.

**Lemma 2.3.** *Let  $K$  be a field. Suppose that distinct non-zero elements  $\lambda_1, \dots, \lambda_r \in K$  are given. Moreover, suppose that for some  $N > 0$  and  $c_1, \dots, c_r \in K$  we have*

$$(2.3.1) \quad c_1 \lambda_1^n + \dots + c_r \lambda_r^n = 0,$$

for every  $n \geq N$ . Then,  $c_1 = \dots = c_r = 0$ .

Before proving the main result of this Section, we recall that for a subset  $S \subseteq \mathbb{N}$ , the (upper asymptotic) density (also called *natural density*) of  $S$  is defined as

$$\mu(S) := \limsup_{n \rightarrow \infty} \frac{\#\{m \in S : m \leq n\}}{n}.$$

**Lemma 2.4.** *Let  $K = \overline{\mathbb{F}_p(t)}$ , let  $r \in \mathbb{N}$  and let  $c_0, c_1, \dots, c_r \in K$ . Suppose that  $\lambda \in K \setminus \{0\}$  is multiplicatively independent with respect to  $t$ . Let  $S$  be the set of positive integers  $m$  for which there exist positive integers  $n_1, \dots, n_r$  such that*

$$(2.4.1) \quad \lambda^m = c_0 + \sum_{i=1}^r c_i t^{n_i}.$$

Then the natural density of  $S$  is equal to zero.

*Proof.* Solving equation (2.4.1) is equivalent with analyzing the intersection of the hyperplane  $V \subset \mathbb{G}_m^{1+r}$  given by the equation:

$$y = c_0 + \sum_{i=1}^r c_i x_i$$

with the subgroup  $\Gamma$  of  $\mathbb{G}_m^{1+r}$  spanned by  $(\lambda, 1, \dots, 1), (1, t, 1, \dots, 1), \dots, (1, \dots, 1, t)$ . Then by Moosa-Scanlon's  $F$ -structure theorem (see [MS04, Theorem B]), we know the intersection is a union of finitely many sets  $R_1, \dots, R_u$ ; furthermore, each set  $R$  from the list  $R_1, \dots, R_u$  is of the form

$$R := \gamma_0 \cdot S(\gamma_1, \dots, \gamma_s; \delta_1, \dots, \delta_s) \cdot H,$$

where  $H$  is a subgroup of  $\Gamma$ , while  $\gamma_i \in \mathbb{G}_m^{1+r}(K)$  for each  $i = 0, \dots, s$  and  $\delta_j \in \mathbb{N}$  for each  $j = 1, \dots, s$ , and the set  $S(\gamma_1, \dots, \gamma_s; \delta_1, \dots, \delta_s)$  is defined as follows:

$$S(\gamma_1, \dots, \gamma_s; \delta_1, \dots, \delta_s) := \left\{ \prod_{j=1}^s \gamma_j^{p^{k_j \delta_j}} : k_i \in \mathbb{N} \text{ for } i = 1, \dots, s \right\}.$$

So, the set  $R$  consists of points of the form

$$(2.4.2) \quad \gamma_0 \cdot \prod_{j=1}^s \gamma_j^{p^{k_j \delta_j}} \cdot \epsilon,$$

where  $\epsilon$  is in the subgroup  $H$ ; moreover, there exists a positive integer  $\ell$  such that  $\gamma_i^\ell \in \Gamma$  for each  $i = 0, \dots, m$ . For more details regarding the  $F$ -sets structure for the intersection of a subvariety of a semiabelian variety with a finitely generated subgroup, we refer the reader both to [MS04] and also to [Ghi08], for further refinements of Moosa-Scanlon's original result.

Now, in order to show that the set of all positive integers  $m$  for which equation (2.4.1) is solvable has natural density equal to 0 (as a subset of  $\mathbb{N}$ ), it suffices to prove that the projection of  $H$  on the first coordinate of  $\mathbb{G}_m^{1+r}$  is trivial; this way, the set of those  $m$ 's that satisfy equation (2.4.1) would be a sum of powers of  $p$  (due to equation (2.4.2)) and thus, it would have natural density zero.

Let us assume, for the sake of contradiction, that  $H$  projects non-trivially on the first coordinate of  $\mathbb{G}_m^{1+r}$ . This means that there exists some tuple

$$(m_0, \ell_1, \dots, \ell_r)$$

with  $m_0 \neq 0$  such that a coset of the cyclic subgroup  $H_0$  spanned by

$$(\lambda^{m_0}, t^{\ell_1}, \dots, t^{\ell_r})$$

would be contained in the intersection of  $V$  with the subgroup  $\Gamma$ .

So, letting  $\ell_0 := 0$ , there must exist some constants  $d_i \in \overline{\mathbb{F}_p(t)}$  (depending on the  $c_i$ 's) such that for all positive integers  $n$ , we have

$$(2.4.3) \quad \lambda^{m_0 n} = \sum_{i=0}^r d_i t^{\ell_i n}.$$

Combining the terms with the same exponent, we may assume without loss of generality that the powers  $\ell_1, \dots, \ell_r$  are distinct. Using equation (2.4.3) and Lemma 2.3 we get that  $\lambda^{m_0 n}$  must be equal to  $t^{\ell_i n}$  for some  $i = 0, 1, \dots, r$  (note that not all of the  $d_i$ 's could equal 0 since  $\lambda \neq 0$ ), which means that  $\lambda$  is multiplicative dependent with respect to  $t$ . This contradicts our hypothesis and thus delivers the conclusion of Lemma 2.4.  $\square$

### 3. ARITHMETIC AND ALGEBRAIC PROPERTIES FOR RINGS INVOLVING THE FROBENIUS OPERATOR

From now on in this paper, we let  $p$  be a prime number and let  $F$  be the Frobenius operator corresponding to the field  $\mathbb{F}_p$ .

**3.1. Operators involving the Frobenius operator.** We consider the polynomial ring  $\mathbb{F}_p[F]$  whose elements are operators of the form  $\sum_{i=0}^n a_i F^i$  which act on any field  $L$  of characteristic  $p$  as follows:

$$(3.0.1) \quad \left( \sum_{i=0}^n a_i F^i \right) (x) = \sum_{i=0}^n a_i x^{p^i} \text{ for } x \in L.$$

Since  $F$  leaves invariant each element of  $\mathbb{F}_p$ , we can identify  $\mathbb{F}_p[F]$  with a polynomial ring in one variable over  $\mathbb{F}_p$ ; in particular, we can consider its fraction field, denoted  $\mathbb{F}_p(F)$ . For an

algebraically closed field  $L$  (of characteristic  $p$ ), the action of an element  $u$  of  $\mathbb{F}_p(F)$ , which is of the form

$$(3.0.2) \quad u := \frac{\sum_{i=0}^n a_i F^i}{\sum_{j=0}^m b_j F^j}$$

(for some non-negative integers  $m$  and  $n$  and moreover, the denominator in equation (3.0.2) is nonzero and is coprime with respect to the numerator) can be interpreted as a finite-to-finite map  $\varphi_u : L \rightarrow L$  which has the property that to each element  $x \in L$ , it associates the finitely many elements  $y \in L$  for which

$$(3.0.3) \quad \left( \sum_{j=0}^m b_j F^j \right) (y) = \left( \sum_{i=0}^n a_i F^i \right) (x).$$

**3.2. Linearly independent elements with respect to the maps from the polynomial ring in the Frobenius operator.** In this Section, we let  $L$  be an algebraically closed field of positive transcendence degree over  $\mathbb{F}_p$ . The following notion is used in our proof of Proposition 5.2, which is a key technical step in deriving our main result (Theorem 1.3). First, we note that similar to our construction of the ring of operators  $\mathbb{F}_p[F]$  from Section 3.1, we can construct the non-commutative ring of operators  $\overline{\mathbb{F}_p}[F]$ .

**Definition 3.1.** *Given elements  $\delta_1, \dots, \delta_\ell$  and  $\gamma_1, \dots, \gamma_k$  in  $L$ , we say that  $\gamma_1, \dots, \gamma_k$  are linearly independent from  $\delta_1, \dots, \delta_\ell$  over  $\overline{\mathbb{F}_p}[F]$  if whenever there exist polynomial operators  $P_1(F), \dots, P_k(F) \in \overline{\mathbb{F}_p}[F]$  and  $Q_1(F), \dots, Q_\ell(F) \in \overline{\mathbb{F}_p}[F]$  such that*

$$\sum_{i=1}^k P_i(F)(\gamma_i) = \sum_{j=1}^{\ell} Q_j(F)(\delta_j),$$

then we must have that  $P_1(F) = \dots = P_k(F) = 0$ .

Moreover, in the special case  $\ell = 1$  and  $\delta_1 = \{0\}$ , then the above condition simply translates into asking that the points  $\gamma_1, \dots, \gamma_k$  are linearly independent over  $\overline{\mathbb{F}_p}[F]$ .

The following result will be used in our proof of Proposition 5.2.

**Proposition 3.2.** *For any positive integers  $k$  and  $\ell$ , and any given elements  $\delta_1, \dots, \delta_\ell \in L$ , there exist  $\gamma_1, \dots, \gamma_k \in L$  which are linearly independent from  $\delta_1, \dots, \delta_\ell$  over  $\overline{\mathbb{F}_p}[F]$ .*

*Proof.* We let  $L_0 \subset L$  be a finitely generated extension of  $\overline{\mathbb{F}_p}$  containing  $\delta_1, \dots, \delta_\ell$ . Then we view  $L_0$  as the function field of a projective, smooth variety  $V$  defined over  $\overline{\mathbb{F}_p}$ . We let  $\Omega_V$  be the set of inequivalent absolute values corresponding to the irreducible divisors of  $V$ . Since there are only finitely many places of  $V$  where the  $\delta_j$ 's have poles, we can choose  $k$  elements  $\gamma_i \in L_0$  (for  $i = 1, \dots, k$ ) such that there exist absolute values  $|\cdot|_{v_i} \in \Omega_V$  (for  $i = 1, \dots, k$ ) satisfying the following properties:

- (i)  $|\delta_j|_{v_i} \leq 1$  for each  $1 \leq j \leq \ell$  and each  $1 \leq i \leq k$ ;
- (ii)  $|\gamma_i|_{v_i} > 1$  for each  $i = 1, \dots, k$ ; and
- (iii)  $|\gamma_i|_{v_j} \leq 1$  for each  $j \neq i$ .

Conditions (i)-(iii) can be achieved since there exist infinitely many absolute values in  $\Omega_V$  and so, we can proceed inductively on  $k$ , each time choosing an element  $\gamma_i$  which has a pole at some place of  $V$  where none of the  $\delta_j$ 's and also none of the  $\gamma_1, \dots, \gamma_{i-1}$  have poles.

Now, we claim that the elements  $\gamma_1, \dots, \gamma_k$  are linearly independent from  $\delta_1, \dots, \delta_\ell$  over  $\overline{\mathbb{F}_p}[F]$ . Indeed, if there exist polynomial operators  $P_1(F), \dots, P_k(F) \in \overline{\mathbb{F}_p}[F]$  and  $Q_1(F), \dots, Q_\ell(F) \in \overline{\mathbb{F}_p}[F]$  such that

$$(3.2.1) \quad \sum_{i=1}^k P_i(F)(\gamma_i) = \sum_{j=1}^{\ell} Q_j(F)(\delta_j),$$

then we assume there exists some  $i_0 \in \{1, \dots, k\}$  such that  $P_{i_0}(F) \neq 0$  and we will derive a contradiction. Indeed, using conditions (ii)-(iii) above, we get that

$$(3.2.2) \quad \left| \sum_{i=1}^k P_i(F)(\gamma_i) \right|_{v_{i_0}} = |P_{i_0}(F)(\gamma_{i_0})|_{v_{i_0}} > 1.$$

Note that in order to derive inequality (3.2.2), we use the fact that if  $|\gamma|_v > 1$ , then for any nonzero polynomial operator  $P(F) \in \overline{\mathbb{F}_p}[F]$  of degree  $D \geq 0$  (in the operator  $F$ ), we have that

$$|P(F)(\gamma)|_v = |\gamma|_v^{p^D}.$$

On the other hand, using condition (i) above, we get that

$$(3.2.3) \quad \left| \sum_{j=1}^{\ell} Q_j(F)(\delta_j) \right|_{v_{i_0}} \leq 1.$$

Inequalities (3.2.2) and (3.2.3) yield a contradiction along with equality (3.2.1). This shows that indeed, the elements  $\gamma_1, \dots, \gamma_k$  must be linearly independent from  $\delta_1, \dots, \delta_\ell$  over  $\overline{\mathbb{F}_p}[F]$ , which concludes our proof for Proposition 3.2.  $\square$

**3.3. A Mordell-Lang type theorem for the additive group scheme.** We will employ in our proof of Proposition 5.2 a Mordell-Lang type theorem for the additive group scheme, which was proven in [Ghi08, Theorem 2.6]. Before stating our technical result (see Proposition 3.3), we need to introduce some notation.

Let  $N$  be a positive integer and we extend the action of the ring of operators  $\mathbb{F}_p[F]$  on  $\mathbb{G}_a^N$  acting diagonally. Let  $L$  be an algebraically closed field of characteristic  $p$ . Inspired by the definition of  $F$ -sets introduced by Moosa and Scanlon in [MS04], we define the following subsets of  $\mathbb{G}_a^N(L)$ . So, for points  $\gamma_1, \dots, \gamma_r \in \mathbb{G}_a^N(L)$  and positive integers  $k_1, \dots, k_r$ , we define

$$(3.2.4) \quad S(\gamma_1, \dots, \gamma_r; k_1, \dots, k_r) := \left\{ \sum_{i=1}^r F^{n_i k_i}(\gamma_i) : n_i \in \mathbb{N} \text{ for } i = 1, \dots, r \right\}.$$

The following result is proven in [Ghi08, Theorem 2.6].

**Proposition 3.3.** *Let  $X \subseteq \mathbb{G}_a^N$  be an affine variety defined over an algebraically closed field  $L$  of characteristic  $p$ . Let  $F$  be the usual Frobenius map  $x \mapsto x^p$  and we extend the action*



of  $\mathbb{F}_p[F]$  to  $\mathbb{G}_a^N$  acting diagonally. Let  $\Gamma \subset \mathbb{G}_a^N(L)$  be a finitely generated  $\mathbb{F}_p[F]$ -submodule. Then the intersection  $X(L) \cap \Gamma$  is a finite union of sets of the form

$$(3.3.1) \quad \gamma_0 + S(\gamma_1, \dots, \gamma_r; k_1, \dots, k_r) + H,$$

for some points  $\gamma_0, \gamma_1, \dots, \gamma_r \in \mathbb{G}_a^N(L)$  and positive integers  $k_1, \dots, k_r$ , where  $S(\gamma_1, \dots, \gamma_r; k_1, \dots, k_r)$  is defined as in equation (3.2.4), while  $H$  is an  $\mathbb{F}_p[F]$ -submodule of  $\Gamma$ .

Proposition 3.3 can be viewed as a Mordell-Lang type statement for the additive group scheme, in the same spirit as Moosa-Scanlon's main result from [MS04] (which is a Mordell-Lang theorem for semiabelian varieties defined over finite fields). Actually, the proof of [Ghi08, Theorem 2.6] followed the exact strategy employed by Moosa-Scanlon for proving [MS04, Theorem 7.8]. Both [Ghi08, Theorem 2.6] and [MS04, Theorem B] (and also their common generalization for arbitrary commutative algebraic groups proven in [BGM]) are extensions in positive characteristic of the classical Mordell-Lang Theorem proven by Faltings [Fal94] (for abelian varieties) and by Vojta [Voj96] (for semiabelian varieties).

*Remark 3.4.* We make a couple of important observations regarding Proposition 3.3. First, the statement in [Ghi08, Theorem 2.6] assumed the points in  $\Gamma$  live in a finitely generated, regular extension of some given finite field; this can always be achieved and also it is not essential for the proof, as observed in [MS04, Remark 7.11] (and also noted before the statement of [BGM, Theorem 2.2]).

Second, just as shown in [MS04, Lemma 2.7], one can prove that the points  $\gamma_0, \gamma_1, \dots, \gamma_r$  corresponding to an  $F$ -set as appearing in the intersection of  $X(L) \cap \Gamma$  from equation (3.3.1) live in the  $\mathbb{F}_p[F]$ -division hull of  $\Gamma$ , i.e., there exists some nonzero polynomial  $P(F) \in \mathbb{F}_p[F]$  such that

$$(3.4.1) \quad P(F)(\gamma_i) \in \Gamma \text{ for } i = 0, 1, \dots, r.$$

**3.4. A non-commutative ring of operators.** From now on, we fix  $q = p^\ell$  for some given  $\ell \in \mathbb{N}$ . Then the polynomial ring of operators  $\mathbb{F}_q[F]$  acting as in equation (3.0.1) is no longer a commutative ring since for some  $c \in \mathbb{F}_q \setminus \mathbb{F}_p$ , we have that  $c^p \neq c$ .

From now on, we define  $K := \mathbb{F}_q[F] \otimes_{\mathbb{F}_p[F^\ell]} \mathbb{F}_p(F^\ell)$  and so, any element in  $K$  can be written as

$$(3.4.2) \quad \sum_{i=0}^{\ell-1} a_i F^i,$$

where  $a_i \in \mathbb{F}_q[F^\ell] \otimes_{\mathbb{F}_p[F^\ell]} \mathbb{F}_p(F^\ell) \subseteq \mathbb{F}_q(F^\ell)$ ; note that  $\mathbb{F}_q(F^\ell)$  is a field since  $\mathbb{F}^\ell$  fixes each element of  $\mathbb{F}_q$ .

**3.5. Matrices of operators.** In this Section, we study matrices whose entries are themselves operators from  $K$  (see the notation from Section 3.4).

**Notation 3.5.** Let  $A$  be a matrix in  $M_{n,n}(K)$  for some  $n \in \mathbb{N}$ . Using equation (3.4.2), we can find unique matrices  $A_0, \dots, A_{\ell-1} \in M_{n,n}(\mathbb{F}_q(F^\ell))$  such that

$$A = \sum_{i=0}^{\ell-1} A_i F^i.$$

From now on, we will use the matrices  $A_0, \dots, A_{\ell-1}$  each one of them belonging to  $M_{n,n}(\mathbb{F}_q(F^\ell))$  in order to identify the matrix  $A \in M_{n,n}(K)$ . Furthermore, for convenience, we will often use the  $n \times n\ell$ -matrix  $(A_0, \dots, A_{\ell-1}) \in M_{n,n\ell}(\mathbb{F}_q(F^\ell))$  to identify the matrix  $A \in M_{n,n}(K)$ .

The next result is a simple consequence of multiplying matrices from  $M_{n,n}(K)$  and keeping track of the decomposition of their action as given in Notation 3.5.

**Proposition 3.6.** *Given a matrix  $A \in M_{n,n}(K)$  there exists a unique matrix  $\tilde{A} \in M_{n\ell,n\ell}(\mathbb{F}_q(F^\ell))$  such that for every matrix  $B \in M_{n,n}(K)$  we have*

$$(3.6.1) \quad ((BA)_0, \dots, (BA)_{\ell-1}) = (B_0, \dots, B_{\ell-1}) \cdot \tilde{A}.$$

*Proof.* The uniqueness of  $\tilde{A}$  is obvious due to equation (3.6.1) since we can take each matrix  $B_i$  to be any arbitrary matrix in  $M_{n,n}(\mathbb{F}_q(F^\ell))$ .

Now, we identify as in Notation 3.5 the matrix  $A \in M_{n,n}(K)$  with the vector of matrices  $(A_0, \dots, A_{\ell-1})$ , each one of them in  $M_{n,n}(\mathbb{F}_q(F^\ell))$ . We define the function  $\mathbf{red} : \{0, \dots, \ell - 1\} \times \{0, \dots, \ell - 1\} \rightarrow \{0, \dots, \ell - 1\}$  to be the map given by

$$(i, j) \mapsto (i - j) \pmod{\ell}$$

We define then

$$\tilde{A} = \left( F^j A_{\mathbf{red}(i,j)} F^{\mathbf{red}(i,j)-i} \right)_{0 \leq i, j \leq \ell-1},$$

which we view as an  $\ell \times \ell$ -matrix whose entries are themselves matrices from  $M_{n,n}(\mathbb{F}_q[F])$ . The fact that the entries of  $\tilde{A}$  lie inside  $\mathbb{F}_q(F^\ell)$  is clear from the fact that

$$F^j A_{\mathbf{red}(i,j)} F^{\mathbf{red}(i,j)-i} = (F^j A_{\mathbf{red}(i,j)} F^{-j}) F^{j+\mathbf{red}(i,j)-i}.$$

Indeed, the entries of  $F^j A_{\mathbf{red}(i,j)} F^{-j}$  all lie inside  $\mathbb{F}_q(F^\ell)$  since the entries of  $A_{\mathbf{red}(i,j)}$  lie inside  $\mathbb{F}_q(F^\ell)$ ; furthermore,  $F^{j+\mathbf{red}(i,j)-i}$  is a non-negative power of  $F^\ell$ .  $\square$

The following result is an immediate consequence of the definition of  $\tilde{A} \in M_{n\ell,n\ell}(\mathbb{F}_q(F^\ell))$  for any given matrix  $A \in M_{n,n}(K)$  satisfying the conclusion of Proposition 3.6.

**Proposition 3.7.** *The map  $M_{n,n}(K) \rightarrow M_{n\ell,n\ell}(\mathbb{F}_q(F^\ell))$  given by*

$$A \mapsto \tilde{A}$$

*is an embedding of  $\mathbb{F}_p[F^\ell]$ -algebras.*

**3.6. A skew field.** Finally, in this Section we prove that  $K$  is a skew field with center  $\mathbb{F}_p(F^\ell)$ .

The next result is an easy consequence of Proposition 3.7.

**Corollary 3.8.** *For every  $A \in M_{n,n}(\mathbb{F}_q[F])$  there exists a monic polynomial  $Q(x) \in \mathbb{F}_p[F^\ell][x]$  such that  $Q(A) = 0$ .*

*Proof.* Since the  $n\ell \times n\ell$ -matrix  $\tilde{A}$  has its entries in the commutative ring  $\mathbb{F}_q[F^\ell]$ , then the classical Cayley-Hamilton's theorem yields the existence of a monic polynomial with coefficients in  $\mathbb{F}_q[F^\ell]$  which kills the matrix  $\tilde{A}$ . Because  $\mathbb{F}_q[F^\ell]$  is itself integral over  $\mathbb{F}_p[F^\ell]$ , then we can find a monic polynomial  $Q(x) \in \mathbb{F}_p[F^\ell][x]$  such that  $Q(\tilde{A}) = 0$ . Then, Proposition 3.7 yields that  $Q(A) = 0$  as well.  $\square$

**Proposition 3.9.** *For any  $P(F) \in \mathbb{F}_q[F]$  there exists a nonzero polynomial  $Q(F) \in \mathbb{F}_q[F]$  such that  $Q(F)P(F) \in \mathbb{F}_p[F^\ell]$ .*

*Proof.* We regard  $P(F)$  as a matrix in  $M_{1,1}(\mathbb{F}_q[F])$ . Since  $P$  is non-zero, then Proposition 3.7 yields that  $\tilde{P}$  must be an invertible  $\ell \times \ell$ -matrix. Therefore, there exists a vector  $(Q_0, \dots, Q_{\ell-1})$  with coordinates in  $\mathbb{F}_q[F^\ell]$  such that

$$(3.9.1) \quad (Q_0, \dots, Q_{\ell-1}) \cdot \tilde{P} = (\alpha, 0, \dots, 0).$$

for some non-zero  $\alpha \in \mathbb{F}_q[F^\ell]$ . If we let  $Q$  be the (nonzero) polynomial in  $\mathbb{F}_q[F^\ell]$  corresponding to  $(Q_0, \dots, Q_{\ell-1})$ , then equation (3.9.1) implies that  $Q_1(F^\ell) := Q(F)P(F) \in \mathbb{F}_q[F^\ell]$ . Since  $\mathbb{F}_q$  is a finite extension of  $\mathbb{F}_p$ , there must exist another nonzero polynomial  $Q_2 \in \mathbb{F}_q[F^\ell]$  such that  $Q_2(F^\ell)Q_1(F^\ell) \in \mathbb{F}_p[F^\ell]$ . So,

$$Q_2(F^\ell)Q(F)P(F) \in \mathbb{F}_p[F^\ell]$$

as desired.  $\square$

Finally, the desired conclusion about  $K$  being a skew field with center  $\mathbb{F}_p(F^\ell)$  follows as an immediate consequence of Proposition 3.9.

**Corollary 3.10.**  $\mathbb{F}_q[F] \otimes_{\mathbb{F}_p[F^\ell]} \mathbb{F}_p(F^\ell)$  is a skew field and  $\mathbb{F}_p(F^\ell)$  is its centre.

#### 4. REDUCTIONS FOR OUR MAIN RESULT

**Proposition 4.1.** *In order to prove Theorem 1.3 for the dynamical system  $(\mathbb{G}_a^N, \Phi)$ , it suffices to prove Theorem 1.3 for the dynamical system  $(\mathbb{G}_a^N, \Phi^n)$  for some  $n \in \mathbb{N}$ .*

*Proof.* It is clear that if condition (C) holds for an iterate of  $\Phi$  then it also holds for  $\Phi$ . The fact that if conditions (A) and (B) hold for an iterate of  $\Phi$  then they also hold for  $\Phi^n$  follows from [BGRS17, Lemma 2.1].  $\square$

**Notation 4.2.** *Let  $h$  be an element in  $\overline{\mathbb{F}_p}[F]$  and let  $N$  be a positive integer. We let  $[h]$  denote the group endomorphism of  $\mathbb{G}_a^N$  given by the coordinate-wise action of  $h$ .*

Also, as a matter of notation throughout our paper, we will often use  $\vec{x}$  to denote the point  $x \in \mathbb{G}_a^N$  just so it would be more convenient when using a group endomorphism  $\Phi$  of  $\mathbb{G}_a^N$  corresponding to some matrix  $A \in M_{N,N}(\overline{\mathbb{F}_p}[F])$ , because then we would write  $A\vec{x}$  to denote  $\Psi(\vec{x})$ .

**Definition 4.3.** *We call  $\Psi : \mathbb{G}_a^N \rightarrow \mathbb{G}_a^N$  a finite-to-finite map (defined over  $\overline{\mathbb{F}_p}$ ) if there exists a nonzero element  $h \in \overline{\mathbb{F}_p}[F]$  with the property that  $[h] \circ \Psi$  is a group endomorphism of  $\mathbb{G}_a^N$ . In other words, there exists a matrix  $B \in M_{N,N}(\overline{\mathbb{F}_p}[F])$  such that for each point  $x \in \mathbb{G}_a^N$ , the finite-to-finite map  $\Psi$  associates to the point  $x$  the finitely many points  $y \in \mathbb{G}_a^N$  such that*

$$[h](\vec{y}) = B\vec{x}.$$

**Proposition 4.4.** *Let  $K = \mathbb{F}_q[F] \otimes_{\mathbb{F}_p[F^\ell]} \mathbb{F}_p(F^\ell)$  where  $q = p^\ell$  for some  $\ell \in \mathbb{N}$  (see also Section 3.4). Let  $\Phi : \mathbb{G}_a^N \rightarrow \mathbb{G}_a^N$  be a dominant group endomorphism of  $\mathbb{G}_a^N$  defined over  $\mathbb{F}_q$ . Then, there exists  $n \in \mathbb{N}$  and there exist non-negative integers  $N_0$  and  $N_1$  such that*

$N = N_0 + N_1$ , along with a dominant group endomorphism  $\Phi_0 : \mathbb{G}_a^{N_0} \longrightarrow \mathbb{G}_a^{N_0}$  corresponding to the matrix

$$(4.4.1) \quad A_0 := J_{F^{n_1 \ell}, m_1} \oplus \cdots \oplus J_{F^{n_s \ell}, m_s},$$

and a finite-to-finite map  $\Phi_1 : \mathbb{G}_a^{N_1} \longrightarrow \mathbb{G}_a^{N_1}$  (see also Definition 4.3) corresponding to a matrix  $A_1 \in M_{N_1, N_1}(K)$ , where the minimal polynomial of  $A_1$  over  $\mathbb{F}_p[F^\ell]$  has roots that are multiplicatively independent with respect to  $F^\ell$ , there exists a dominant group endomorphism  $g : \mathbb{G}_a^N \longrightarrow \mathbb{G}_a^{N_0} \times \mathbb{G}_a^{N_1}$  defined over  $\mathbb{F}_q$ , and there exists a nonzero element  $h \in \mathbb{F}_p[F^\ell]$  such that the next diagram commutes

$$(4.4.2) \quad \begin{array}{ccc} \mathbb{G}_a^N & \xrightarrow{[h] \circ \Phi^{mn}} & \mathbb{G}_a^N \\ \downarrow g & & \downarrow g \\ \mathbb{G}_a^{N_0} \times \mathbb{G}_a^{N_1} & \xrightarrow{[h] \circ (\Phi_0^m, \Phi_1^m)} & \mathbb{G}_a^{N_0} \times \mathbb{G}_a^{N_1}, \end{array}$$

for all  $m \in \mathbb{N}$ . In particular,  $[h] \circ (\Phi_0^m, \Phi_1^m)$  is a well-defined group endomorphism for all  $m \in \mathbb{N}$ .

*Proof.* Suppose that  $\Phi$  corresponds to a matrix  $A \in M_{N, N}(\mathbb{F}_q[F])$ . For some suitable power  $\Phi^n$  of  $\Phi$  we have that the roots of the minimal polynomial of  $A^n$  over  $\mathbb{F}_p(F^\ell)$ , say  $r(x) \in \mathbb{F}_p(F^\ell)[x]$ , are either a non-negative integer power of  $F^\ell$  or multiplicatively independent with respect to  $F$ . Indeed, note that the roots of the minimal polynomial of  $A$  are integral over  $\mathbb{F}_p[F^\ell]$  (see also Section 3.6) and so, if a root  $u_0$  is multiplicatively dependent with respect to  $F^\ell$ , then a power  $u_0^n$  (for  $n \in \mathbb{N}$ ) must be of the form  $F^{\ell j_0}$  for some non-negative integer  $j_0$ .

So, with the above assumption regarding  $A^n$  and its minimal polynomial  $r(x)$ , then we can write  $r(x) = r_0(x)r_1(x)$  where  $r_0(x)$  is a polynomial whose roots are (non-negative integer) powers of  $F^\ell$  and  $r_1(x)$  is a polynomial whose roots are multiplicatively independent with respect to  $F$ . Using Facts 2.1 and 2.2 (see Section 2.1) along with Corollary 3.10, there must exist an invertible matrix  $P \in M_{N, N}(K)$  such that

$$(4.4.3) \quad PA^n P^{-1} = A_0 \oplus A_1$$

where  $A_0$  corresponds to a matrix of the form (4.4.1) and the minimal polynomial of  $A_1$  over  $\mathbb{F}_p(F^\ell)$  is  $r_1$ . Using Equation (4.4.3) we have

$$(4.4.4) \quad PA^{mn} P^{-1} = A_0^m \oplus A_1^m,$$

for every positive integer  $m$ . Due to the definition of  $K$  there exists a nonzero  $u \in \mathbb{F}_p[F^\ell]$  such that  $uP \in \mathbb{F}_q[F]$ ; also, because  $A_1$  is integral over  $\mathbb{F}_p[F^\ell]$ , there exists a nonzero  $h \in \mathbb{F}_p[F^\ell]$  such that each  $hA_1^m$  (for  $m \in \mathbb{N}$ ) has entries in  $\mathbb{F}_q[F]$ . Therefore, if we let  $g$  be the group endomorphism corresponding to the matrix  $uP$ , using equation (4.4.4) we will get a commutative diagram of the form (4.4.2). This concludes our proof of Proposition 4.4.  $\square$

The following result is an easy consequence of Proposition 4.4 and of the fact that for any positive integer  $a$ , we have that  $\binom{p^a}{i} = 0$  in  $\mathbb{F}_p$  whenever  $0 < i < p^a$ .

**Corollary 4.5.** *In Proposition 4.4, at the expense of replacing the positive integer  $n$  by a multiple, we may assume without loss of generality that  $\Phi_0$  corresponds to a diagonal matrix*

of the form

$$A_0 = F^{n_1 \ell} \mathbf{I}_{m_1} \bigoplus \cdots \bigoplus F^{n_s \ell} \mathbf{I}_{m_s}$$

where,  $n_1, \dots, n_s$  are distinct non-negative integers and  $m_1, \dots, m_s$  are non-negative integers.

*Proof.* Let  $p^a$  be a power of  $p$  that is greater than all  $m_1, \dots, m_s$  in the statement of Proposition 4.4. Then, replacing  $n$  by  $np^a$  and combining the Jordan blocks corresponding to the same power of  $F$  will deliver the desired conclusion.  $\square$

Let  $\Phi$  be a dominant endomorphism of  $\mathbb{G}_a^N$ , let  $n \in \mathbb{N}$ , let  $h \in \mathbb{F}_p[F^\ell]$  and  $\Phi_1 : \mathbb{G}_a^{N_1} \rightarrow \mathbb{G}_a^{N_1}$  be as in the statement of Proposition 4.4, while  $\Phi_0 : \mathbb{G}_a^{N_0} \rightarrow \mathbb{G}_a^{N_0}$  has the form as in Corollary 4.5. With the above notation, we prove the next three technical lemmas.

**Lemma 4.6.** *Suppose that there exists some  $i \in \{1, \dots, s\}$  such that  $n_i = 0$ , and also  $m_i > 0$ ; in particular, this means that  $N_0 \geq 1$  with the notation as in Corollary 4.5. Then there exists a non-constant rational function  $f : \mathbb{G}_a^N \rightarrow \mathbb{P}^1$  such that  $f \circ \Phi^n = f$ .*

*Proof.* Suppose without loss of generality that  $n_1 = 0$ . Let  $\pi : \mathbb{G}_a^N \rightarrow \mathbb{G}_a$  be the projection onto the first coordinate. Then, using Equation (4.4.2) we must have

$$\pi \circ g \circ [h] \circ \Phi^n = [h] \circ \pi \circ g.$$

However, since  $\pi$  and  $g$  are both defined over  $\mathbb{F}_q$ , the map  $[h]$  commutes with both of them. So, we have

$$\pi \circ g \circ [h] \circ \Phi^n = \pi \circ g \circ [h].$$

Hence,  $\pi \circ g \circ [h]$  defines a non-constant rational function that is left invariant by  $\Phi^n$ .  $\square$

**Lemma 4.7.** *Suppose that the numbers  $n_1, \dots, n_s$  are all positive and  $\max\{m_1, \dots, m_s\} \geq \text{trdeg}_{\mathbb{F}_p} L + 1$ . Then there exist integers  $r \geq 1$  and  $M \geq \text{trdeg}_{\mathbb{F}_p} L + 1$ , and there exists a dominant group homomorphism  $\tau : \mathbb{G}_a^N \rightarrow \mathbb{G}_a^M$  such that  $\tau \circ \Phi^n = F^r \circ \tau$ .*

*Proof.* Suppose without loss of generality that  $m_1 \geq \text{trdeg}_{\mathbb{F}_p} L + 1$ . Let  $\pi$  be the projection map onto the first  $m_1$  coordinates of  $\mathbb{G}_a^N$ . Using the equation (4.4.2) we must have

$$\pi \circ g \circ [h] \circ \Phi^n = [h] \circ F^{n_1 \ell} \circ \pi \circ g.$$

Since  $g$ , and  $\pi$  are defined over  $\mathbb{F}_q$ , they must commute with  $[h]$ ; also, they all commute with  $F^{n_1 \ell}$ . Hence, we have

$$\pi \circ g \circ [h] \circ \Phi^n = F^{n_1 \ell} \circ \pi \circ g \circ [h].$$

So, the map  $\tau := \pi \circ g \circ [h]$  has the desired property.  $\square$

**Lemma 4.8.** *Let  $L$  be an algebraically closed field of characteristic  $p$ . If there exists a point  $\alpha := (\alpha_0, \alpha_1)$  with  $\alpha_0 \in \mathbb{G}_a^{N_0}(L)$  and  $\alpha_1 \in \mathbb{G}_a^{N_1}(L)$  such that*

$$\mathcal{O} := \{([h] \circ (\Phi_0^m, \Phi_1^m))(\alpha_0, \alpha_1) : m \geq 0\}$$

*is Zariski dense in  $\mathbb{G}_a^{N_0+N_1}$ , then there also exists a point  $\beta \in \mathbb{G}_a^N(L)$  such that  $\mathcal{O}_\Phi(\beta)$  is Zariski dense in  $\mathbb{G}_a^N$ .*

*Proof.* Choose  $\beta$  such that  $g(\beta) = \alpha$  (note that  $g$  is a dominant group endomorphism). Then the commutative diagram (4.4.2) along with the fact that  $g$  and  $[h]$  are dominant group endomorphisms yields that the orbit of  $\beta$  under  $\Phi^n$  must be Zariski dense in  $\mathbb{G}_a^N$ . Since  $\mathcal{O}_{\Phi^n}(\beta) \subseteq \mathcal{O}_{\Phi}(\beta)$ , we obtain the desired conclusion in Lemma 4.8.  $\square$

Lemmas 4.6, 4.7, and 4.8 along with Proposition 4.1 will reduce Theorem 1.3 to Proposition 5.2 stated and proved in the next Section.

## 5. PROOF OF THEOREM 1.3

In this Section we conclude the proof of our main result. We work under the hypotheses of Theorem 1.3. We start by stating a useful result, which is a special case of [GS21, Proposition 4.1].

**Proposition 5.1.** *Let  $L$  be an algebraically closed field of transcendence degree  $d > 0$  over  $\mathbb{F}_p$ . Let  $\Phi : \mathbb{G}_a^N \rightarrow \mathbb{G}_a^N$  be a dominant group endomorphism corresponding to the matrix*

$$A = F^{n_1} \mathbf{I}_{m_1} \bigoplus \cdots \bigoplus F^{n_s} \mathbf{I}_{m_s}$$

where  $m_1, \dots, m_s, n_1, \dots, n_s$  are positive integers and  $n_1, \dots, n_s$  are distinct. Then there exists a point  $\alpha \in \mathbb{G}_a^N(L)$  such that every infinite subset of  $\mathcal{O}_{\Phi}(\alpha)$  is Zariski dense in  $\mathbb{G}_a^N$  if and only if

$$(5.1.1) \quad \max\{m_1, \dots, m_s\} \leq d.$$

Finally, we can state the technical reformulation of Theorem 1.3, which will allow us to prove the desired conclusion in our main result.

**Proposition 5.2.** *Let  $N_0$  and  $N_1$  be non-negative integers, let  $q := p^\ell$ , let  $L$  be an algebraically closed field which has transcendence degree over  $\mathbb{F}_p$  equal to  $d > 0$ , and let  $K = \mathbb{F}_q[F] \otimes_{\mathbb{F}_p[F^\ell]} \mathbb{F}_p(F^\ell)$ . Let  $\Phi_0 : \mathbb{G}_a^{N_0} \rightarrow \mathbb{G}_a^{N_0}$  be a dominant group endomorphism corresponding to the matrix*

$$(5.2.1) \quad A := F^{n_1} \mathbf{I}_{m_1} \bigoplus \cdots \bigoplus F^{n_s} \mathbf{I}_{m_s},$$

(for some non-negative integers  $s, n_1, \dots, n_s$ , while  $N_0 = \sum_{i=1}^s m_i$ ) and  $\Phi_1 : \mathbb{G}_a^{N_1} \rightarrow \mathbb{G}_a^{N_1}$  be a finite-to-finite map corresponding to a matrix  $A_1 \in M_{N_1, N_1}(K)$ , where the minimal polynomial of  $A_1$  over  $\mathbb{F}_p[F^\ell]$  has roots that are multiplicatively independent with respect to  $F^\ell$ . Suppose there exists a non-zero element  $h \in \mathbb{F}_p[F^\ell]$  such that  $[h] \circ (\Phi_0^n, \Phi_1^n)$  is a well-defined dominant group endomorphism of  $\mathbb{G}_a^{N_0+N_1}$  for each  $n \in \mathbb{N}$ . Then, one of the following statements must hold:

- (i)  $N_0 \geq 1$  and one of the numbers  $n_1, \dots, n_s$  is equal to zero.
- (ii) The numbers  $n_1, \dots, n_s$  are all positive and  $\max\{m_1, \dots, m_s\} > d$ .
- (iii) There exists a point  $\alpha := (\alpha_0, \alpha_1)$  with  $\alpha_0 \in \mathbb{G}_a^{N_0}(L)$  and  $\alpha_1 \in \mathbb{G}_a^{N_1}(L)$  such that

$$(5.2.2) \quad \mathcal{O} := \{([h] \circ (\Phi_0^n, \Phi_1^n))(\alpha_0, \alpha_1) : n \geq 0\}$$

is Zariski dense in  $\mathbb{G}_a^{N_0+N_1}$ .

As noted at the end of Section 4, lemmas 4.6, 4.7, 4.8 reduce Theorem 1.3 to Proposition 5.2, which we will prove next.

*Proof of Proposition 5.2.* First of all, as noted also in Section 4, for a point  $\gamma \in \mathbb{G}_a^k(L)$  (for some non-negative integer  $k$ ), we will use the notation  $\vec{\gamma}$  in order to emphasize that the point  $\vec{\gamma} \in \mathbb{G}_a^k(L)$  is a vector consisting of  $k$  elements from  $L$ .

We will prove Proposition 5.2 by assuming that if conditions (i) and (ii) do not hold, then condition (iii) must hold. If we assume that conditions (i) and (ii) do not hold, then by Proposition 5.1 there must exist a point  $\vec{\alpha}_0 \in \mathbb{G}_a^{N_0}(L)$  such that any infinite subset of

$$\mathcal{O}_{\Phi_0}(\vec{\alpha}_0) := \{\Phi_0^n(\vec{\alpha}_0) : n \geq 0\},$$

is Zariski dense in  $\mathbb{G}_a^{N_0}$ . Now, choose a point  $\vec{\alpha}_1 \in \mathbb{G}_a^{N_1}(L)$  whose coordinates are linearly independent with respect to the coordinates of  $\vec{\alpha}_0$  over  $\overline{\mathbb{F}_p}[F]$  (see Proposition 3.2). Note that if  $N_0 = 0$ , then our only requirement is that the coordinates of  $\vec{\alpha}_1$  are linearly independent over  $\overline{\mathbb{F}_p}[F]$  (see the second part of Definition 3.1).

We let  $\vec{\alpha} := (\vec{\alpha}_0, \vec{\alpha}_1)$ . Suppose for the sake of contradiction that the Zariski closure of  $\mathcal{O}$  (from equation (5.2.2)) in  $\mathbb{G}_a^{N_0+N_1}$  is a proper subvariety, say  $V$ . Let

$$\Gamma := \{(B_0\vec{\alpha}_0, B_1\vec{\alpha}_1) : B_0 \in M_{N_0, N_0}(\mathbb{F}_q[F]) \text{ and } B_1 \in M_{N_1, N_1}(\mathbb{F}_q[F])\}.$$

Then  $\Gamma$  is a finitely generated  $\mathbb{F}_p[F]$ -module that contains  $\mathcal{O}$ . Therefore, according to Proposition 3.3, the intersection  $V(L) \cap \Gamma$  is contained in the union of finitely many sets of the form

$$(5.2.3) \quad \vec{\beta} + S(\vec{\gamma}_1, \dots, \vec{\gamma}_r; \delta_1, \dots, \delta_r) + H,$$

where  $H$  is an  $\mathbb{F}_p[F]$ -submodule of  $\Gamma$ , and  $S(\vec{\gamma}_1, \dots, \vec{\gamma}_r; \delta_1, \dots, \delta_r)$  is a sum of  $F$ -orbits of the points  $\vec{\gamma}_i \in \mathbb{G}_a^{N_0+N_1}(L)$  (for some given positive integers  $\delta_i$ , as in equation (3.2.4)), i.e.,

$$S(\vec{\gamma}_1, \dots, \vec{\gamma}_r; \delta_1, \dots, \delta_r) = \left\{ \sum_{i=1}^r F^{n_i \delta_i}(\vec{\gamma}_i) : n_i \in \mathbb{N} \text{ for } i = 1, \dots, r \right\}.$$

Furthermore, as noted in Remark 3.4 (see equation (3.4.1)), there exists a nonzero polynomial  $P(F) \in \mathbb{F}_p[F]$  such that

$$(5.2.4) \quad P(F) \left( \vec{\beta} \right) := (B_0\vec{\alpha}_0, B_1\vec{\alpha}_1),$$

and for each  $i = 1, \dots, r$ , we have

$$(5.2.5) \quad P(F) \left( \vec{\alpha}_i \right) := (C_{0,i}\vec{\alpha}_0, C_{1,i}\vec{\alpha}_1),$$

for some  $B_0, C_{0,1}, \dots, C_{0,r} \in M_{N_0, N_0}(\mathbb{F}_q[F])$  and  $B_1, C_{1,1}, \dots, C_{1,r} \in M_{N_1, N_1}(\mathbb{F}_q[F])$ . Furthermore, since  $\mathbb{F}_p[F]$  is a finite integral extension of  $\mathbb{F}_p[F^\ell]$ , then (at the expense of multiplying  $P(F)$  by a suitable nonzero element of  $\mathbb{F}_p[F]$ , which would only replace the matrices  $B_i$  and  $C_{i,j}$  by other matrices with entries in  $\mathbb{F}_q[F]$ ) we may assume that  $P(F) \in \mathbb{F}_p[F^\ell]$ .

We let  $U$  be a set of the form (5.2.3) that contains the subset

$$(5.2.6) \quad \mathcal{O}_S := \{([h] \circ (\Phi_0^n, \Phi_1^n))(\alpha_0, \alpha_1) : n \in S\},$$

of  $\mathcal{O}$  where  $S$  is a subset of  $\mathbb{N}$  that has a positive natural density.



Now, since  $H$  is an  $\mathbb{F}_p[F]$ -submodule of  $\mathbb{G}_a^{N_0+N_1}$ , then its Zariski closure  $\overline{H}$  is an algebraic subgroup of  $\mathbb{G}_a^{N_0+N_1}$  defined over  $\mathbb{F}_p$ . So, let  $\vec{v} = (\vec{v}_0, \vec{v}_1)$  with  $\vec{v}_0 \in \mathbb{F}_p[F]^{N_0}$  and  $\vec{v}_1 \in \mathbb{F}_p[F]^{N_1}$  such that

$$(5.2.7) \quad \vec{v}_0^T \vec{x}_0 + \vec{v}_1^T \vec{x}_1 = 0,$$

for all  $(\vec{x}_0, \vec{x}_1) \in \overline{H}$  (where always  $\vec{v}^T$  denotes the transpose of  $\vec{v}$ ). Note that since  $\overline{H}$  is an algebraic subgroup of  $\mathbb{G}_a^{N_0+N_1}$  defined over  $\mathbb{F}_p$ , then  $\overline{H}$  is the zero locus of finitely many equations of the form (5.2.7). Using both equations (5.2.7) and (5.2.3) along with equations (5.2.4) and (5.2.5) and with the fact that the operator  $P(F)$  leaves invariant the entries in both  $\vec{v}_0$  and  $\vec{v}_1$ , we obtain that

$$(5.2.8) \quad \sum_{i=0}^1 P(F) (\vec{v}_i^T \cdot ([h] \circ \Phi_i^n) (\vec{\alpha}_i)) = \sum_{i=0}^1 \left( \vec{v}_i^T B_i \vec{\alpha}_i + \sum_{j=1}^r \vec{v}_i^T F^{n_j \delta_j} C_{i,j} \vec{\alpha}_j \right).$$

for all  $n \in S$ . Since for each  $j = 0, 1$ , we have that  $\Phi_j$  corresponds to the matrix  $A_j \in M_{N_j, N_j}(K)$ , and then using that the set of the coordinates of  $\vec{\alpha}_1$  are linearly independent from the set of coordinates of  $\vec{\alpha}_0$  over  $\overline{\mathbb{F}_p}[F]$ , then writing  $h_1 := P(F) \cdot h \in \mathbb{F}_p[F^\ell]$ , we get:

$$(5.2.9) \quad \vec{v}_1^T \left( h_1 A_1^n - B_1 - \sum_{j=1}^r F^{n_j \delta_j} C_{1,j} \right) = \vec{0} \in M_{N_1, 1}(K) \text{ for all } n \in S.$$

At the expense of replacing  $S$  with a subset of  $S$  with a positive natural density, we may assume that each  $n_j \delta_j$  (for  $j = 1, \dots, r$ ) has the same remainder modulo  $\ell$  for all  $n \in S$ . This allows us to rewrite equation (5.2.9) as an equation of the form

$$(5.2.10) \quad \vec{v}_1^T \left( h_1 A_1^n - B_1 - \sum_{j=1}^r F^{m_j \ell} C'_{1,j} \right) = \vec{0},$$

for some matrices  $C'_{1,j} \in M_{N_1, N_1}(\mathbb{F}_q[F])$  depending on the matrices  $C_{1,j}$ . Let  $\mathcal{V}$  be the  $N_1 \times N_1$ -matrix whose rows are all equal to  $\vec{v}_1^T$ . So, we have

$$(5.2.11) \quad \mathcal{V} \left( h_1 A_1^n - B_1 - \sum_{j=1}^r F^{m_j \ell} C'_{1,j} \right) = 0 \in M_{N_1, N_1}(K).$$

Applying the operator  $\sim$  (defined as in Notation 3.5 from Section 3.5) to equation (5.2.11) and also using the fact that  $h_1 \in \mathbb{F}_p[F^\ell]$ , we get

$$(5.2.12) \quad \tilde{\mathcal{V}} \left( h_1 \tilde{A}_1^n - \tilde{B}_1 - \sum_{j=1}^r F^{m_j \ell} \tilde{C}'_{1,j} \right) = 0.$$

Now suppose that  $\mathcal{V}$  is non-zero. Then we must have some nonzero row  $\vec{u}^T$  in  $\tilde{\mathcal{V}}$ ; so, we get

$$(5.2.13) \quad \vec{u}^T \left( h_1 \tilde{A}_1^n - \tilde{B}_1 - \sum_{j=1}^r F^{m_j \ell} \tilde{C}'_{1,j} \right) = 0.$$

We note that equation (5.2.13) is similar to [GS21, Lemma 4.5, equation (4.5.1)] (after transposing both sides). So, by proceeding exactly as in the proof of [GS21, Lemma 4.5], and



using the fact that the roots of the minimal polynomial of  $A_1$  (and thus, also of  $\tilde{A}_1$ ) are multiplicatively independent with respect to  $F^\ell$ , equation (5.2.13) leads to an equation of the form

$$(5.2.14) \quad a\lambda^n = c_0 + \sum_{j=1}^r c_j F^{m_j \ell},$$

for some  $a, c_0, c_1, \dots, c_r \in \overline{\mathbb{F}_p(F^\ell)}$  with  $a \neq 0$  and some eigenvalue  $\lambda$  of  $A_1$ , which is thus multiplicatively independent with respect to  $F^\ell$ . Note that  $\overline{\mathbb{F}_p(F^\ell)}$  is isomorphic to  $\overline{\mathbb{F}_p(t)}$  for some transcendental element  $t$  since  $\mathbb{F}_p(F^\ell)$  is naturally isomorphic to  $\mathbb{F}_p(t)$ . However, by Lemma 2.4, the set of  $n$ 's for which equation (5.2.14) is solvable for some positive integers  $m_j$  must have a natural density equal to zero which contradicts our choice of the subset  $S$ .

This means that for any vector  $\vec{v} = (\vec{v}_0, \vec{v}_1)$  with  $\vec{v}_0 \in \mathbb{F}_p[F]^{N_0}$  and  $\vec{v}_1 \in \mathbb{F}_p[F]^{N_1}$  such that  $\vec{v}_0^T \vec{x}_0 + \vec{v}_1^T \vec{x}_1 = 0$ , for all  $(\vec{x}_0, \vec{x}_1) \in \overline{H}$  we must have  $\vec{v}_1 = \vec{0}$ . Hence,  $\overline{H} := G_0 \times \mathbb{G}_a^{N_1}$  for some algebraic subgroup  $G_0 \subseteq \mathbb{G}_a^{N_0}$ .

Thus,  $\overline{U}$  (the Zariski closure of the set  $U$  containing the elements from equation (5.2.6)), which is itself a subset of  $V$ , must be a set of the form  $W \times \mathbb{G}_a^{N_1}$  for some closed subset  $W \subseteq \mathbb{G}_a^{N_0}$  since  $\{0\} \times \mathbb{G}_a^{N_1}$  is contained in the stabilizer of  $\overline{U}$  (because  $\overline{H} = G_0 \times \mathbb{G}_a^{N_1}$ ). On the other hand, note that  $W$  contains

$$\{([h] \circ \Phi_0^n)(\vec{\alpha}_0) : n \in S\},$$

which must be Zariski dense in  $\mathbb{G}_a^{N_0}$  because of our choice of  $\vec{\alpha}_0$  and the fact that  $[h]$  is a dominant endomorphism of  $\mathbb{G}_a^{N_0}$ . So, we conclude that  $\overline{U} = \mathbb{G}_a^{N_0} \times \mathbb{G}_a^{N_1}$  which contradicts the fact that  $V$  is a proper subvariety of  $\mathbb{G}_a^{N_0} \times \mathbb{G}_a^{N_1}$ . This contradiction completes our proof for Proposition 5.2.  $\square$

Since we proved that Theorem 1.3 reduces to Proposition 5.2, this concludes our proof for Theorem 1.3.

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