ZARISKI DENSE ORBITS FOR REGULAR SELF-MAPS ON SPLIT SEMIABELIAN VARIETIES

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ABSTRACT. We provide a direct proof of the Medvedev-Scanlon's conjecture from [MS14] regarding Zariski dense orbits under the action of regular self-maps on split semiabelian varieties defined over a field of characteristic 0. Besides obtaining significantly easier proofs than the ones previously obtained in [GS17] (for the case of abelian varieties) and [GS19] (for the case of semiabelian varieties), our method allows us to exhibit numerous starting points with Zariski dense orbits, which the methods from [GS17, GS19] could not provide.

1. INTRODUCTION

1.1. **Notation.** Throughout this paper, we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the set of nonnegative integers. As always in arithmetic dynamics, we denote by Φ^n the n-th iterate of the self-map Φ acting on some ambient variety X. For each point x of X, we denote its orbit under Φ by

$$\mathcal{O}_\Phi(x) := \{\Phi^n(x) \colon n \in \mathbb{N}_0\}.$$

1.2. **A conjecture about Zariski dense orbits.** In the early 1990's, Zhang formulated a far-reaching set of conjectures in arithmetic dynamics in parallel to famous questions in arithmetic geometry; hence the genesis of the Dynamical Manin-Mumford and the Dynamical Bogomolov conjectures which generated a lot of research in the past 20 years (for example, see [GT] and the references therein). At that time, Zhang also formulated a very interesting conjecture regarding the existence of Zariski dense orbits under the action of a polarizable endomorphism of a projective variety defined over a number field (which appeared in print as [Zha06, Conjecture 4.1.6]). Later, both Amerik-Campana [AC08] and Medvedev-Scanlon [MS14] formulated a refinement of Zhang's original question regarding Zariski dense orbits as follows.

Conjecture 1.1. *Given a variety X defined over an algebraically closed field K of characteristic 0, endowed with a dominant endomorphism Φ, then we have the following dichotomy:*

(A) *either there exists a point $x \in X(K)$ whose orbit $\mathcal{O}_\Phi(x)$ is Zariski dense in X; or*

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(B) *there exists a non-constant rational map* $f : X \dashrightarrow \mathbb{P}^1$ *such that* $f \circ \Phi = f$.

It is immediate to see that if condition (B) above holds, then no orbit can be Zariski dense; so, the entire difficulty of the conjecture advanced by Zhang, Medvedev-Scanlon and Amerik-Campana is to prove that in the absence of condition (B), there must exist a Zariski dense orbit.

Amerik and Campana [AC08] (see also [BGZ17]) proved Conjecture 1.1 under the assumption that K is uncountable; essentially, in the absence of condition (B) above, the orbit of a very general point (which lies outside countably many special proper subvarieties of X) would have a Zariski dense orbit. However, the case of a countable algebraically closed field K remains open and quite difficult since, a priori, the method of both [AC08] and [BGZ17] does not guarantee the existence of a K-point outside the union of those countably many special proper subvarieties of X. In the past 10 years several partial results were obtained (for example, see [Xie] and the references therein).

1.3. **Our results.** We prove Conjecture 1.1 for regular self-maps of split semiabelian varieties. We recall the definition of a *split semiabelian variety* G (defined over some algebraically closed field K), which is a connected group variety isogenous to a direct product $\mathbb{G}_m^N \times A$ for some $N \in \mathbb{N}_0$ and some abelian variety A. Also, we recall (see [NW14, Theorem 5.1.37]) that any regular self-map on a semiabelian variety G is a composition of a translation with a group endomorphism.

Before stating our result, we define a notion useful for our Theorem 1.2: given two points α and β of some algebraic group G, we say that α is *linearly independent over* $\mathrm{End}(G)$ *from* β if for any two (group) endomorphisms ϕ_1 and ϕ_2 of G, we have that $\phi_1(\alpha) = \phi_2(\beta)$, then ϕ_1 must be the trivial map. Also, for any point β of the algebraic group G, we let $\tau_\beta : G \longrightarrow G$ be the translation-by-β map on G. Finally, we denote by Id the identity map on G.

Theorem 1.2. *Let* G *be a split semiabelian variety defined over an algebraically closed field* K *of characteristic* 0*. Let* $\Phi : G \longrightarrow G$ *be a dominant, regular self-map; we let* $\Phi = \tau_\beta \circ \varphi$ *where* $\beta \in G(K)$ *and* φ *is a group endomorphism of* G*. Then the following statements are equivalent:*

(i) *there exists a non-constant rational function* $f : G \dashrightarrow \mathbb{P}^1$ *such that* $f \circ \Phi = f$*;*
(ii) *there exists no* $\alpha \in G(K)$ *such that* $\mathcal{O}_\Phi(\alpha)$ *is Zariski dense in* G*;*
(iii) *there exists a non-constant group endomorphism* $\Psi : G \longrightarrow G$ *and there exist nonnegative integers* $m < n$ *such that* $\Psi \circ (\varphi^{n-m} - \mathrm{Id}) = 0$ *in* $\mathrm{End}(G)$ *and also,* $\Psi\left(\sum_{j=m}^{n-1} \varphi^j(\beta)\right) = 0$*.*

Furthermore, if none of the above conditions hold, then for each point $\alpha \in G(K)$ *which is linearly independent over* $\mathrm{End}(G)$ *from* β*, we have that* $\mathcal{O}_\Phi(\alpha)$ *is Zariski dense in* G*.*

We note that according to [Vil08, Theorem 5], we can always find algebraic points α in any semiabelian variety G which are linearly independent over $\mathrm{End}(G)$ from any point given point β of G since the group $G(K)$ has infinite rank, while $\mathrm{End}(G)$ is a finite $\mathbb{Z}$-module. We also observe (see Remark 2.1) that Theorem 1.2 holds with almost the same proof verbatim for abelian varieties G defined over an algebraically closed field K of characteristic p, assuming $\mathrm{Tr}_{K/\overline{\mathbb{F}_p}}(G)$ is trivial (the only difference in our proof would be changing the reference of the Mordell-Lang theorems of Faltings [Fal94] and Vojta [Voj96] to the function field version of the Mordell-Lang theorem in characteristic p, as proven by Hrushovski [Hru96]).

Conjecture 1.1 was previously proven in [GS19] for regular self-maps of arbitrary semiabelian varieties defined over an algebraicallly closed field of characteristic 0 (see also [GS17] for the proof in the case of abelian varieties). However, our current proof is much more direct (and simpler); furthermore, our Theorem 1.2 provides explicit points whose orbit is Zariski dense, which is in stark contrast with the results of [GS17, GS19] in which there was no explicit information about the points with Zariski dense orbits. For example, the "furthermore" statement in our Theorem 1.2 yields that for any finitely generated subfield $L \subset K$ for which the group $G(L)$ has sufficiently high rank (note that the group $G(K)$ has infinite rank for an algebraically closed field K), then we can find a point $\alpha \in G(L)$ with a Zariski dense orbit under Φ (assuming Φ does not leave invariant a non-constant rational function). In particular, if $G = \mathbb{G}_m^N$, then our Theorem 1.2 yields that if the equivalent conditions (i)-(iii) do not hold for a regular self-map Φ on $\mathbb{G}_m^N$, then there exist infinitely many multiplicatively independent points $\alpha \in \mathbb{G}_m^N(\mathbb{Q})$ with a Zariski dense orbit under Φ. Conversely, our proof of Theorem 1.2 allows us also to construct a very explicit rational function which is left invariant by Φ when conditions (i)-(iii) hold (see the proof of the implication (iii)$\Rightarrow$(i) in Theorem 1.2, especially the equation (2.0.10)).

Generally, the partial results towards Conjecture 1.1 employed various complicated techniques: from invariant theory (as in [GX18]), to Diophantine arguments in the spirit of the famous theorem of Laurent [Lau84] (as in [GH18]), to deep results regarding the algebraic dynamics on surfaces coupled with the so-called "p-adic arc lemma" (first introduced in the context of the Dynamical Mordell-Lang Conjecture; see [BGT16]), as recently employed by Xie [Xie] in his proof of Conjecture 1.1 for endomorphisms of surfaces. Also, the proofs of Conjecture 1.1 for regular self-maps on abelian varieties or more generally, on semiabelian varieties (see [GS17, GS19]) were quite involved, employing nontrivial arithmetic and geometric results (besides the use of the famous theorems of Faltings [Fal94] and Vojta [Voj96] which solved the classical Mordell-Lang conjectures for abelian, respectively semiabelian varieties). Our proof of Theorem 1.2 also uses the theorems of Faltings [Fal94] and [Voj96] (which is essentially unavoidable for any approach to Theorem 1.2), but then our proof only exploits the Poincaré

Reducibility Theorem for abelian varieties (see [GS17, Fact 3.2]), avoiding all of the much more difficult arithmetic arguments present in the proofs from [GS17, GS19]. Since the Poincaré Reducibility Theorem also holds in the context of algebraic tori and therefore for split semiabelian varieties, we are able to prove the result from our Theorem 1.2. However, the failure of the Poincaré Reducibility Theorem for general semiabelian varieties means that one would still need to use more geometric and arithmetic arguments as in [GS19] in order to prove Conjecture 1.1 in this general case (for example, see [GS19, Section 3], especially the use of minimal dominating semiabelian subvarieties and the construction of topological generators).

2. Proof of our main result

Proof of Theorem 1.2. We note that for each nonnegative integer n and for each $\alpha \in G(K)$, we have that

$$\Phi^n(\alpha) = \left(\sum_{j=0}^{n-1} \varphi^j(\beta)\right) + \varphi^n(\alpha). \tag{2.0.1}$$

In particular, this means that the entire orbit $\mathcal{O}_\Phi(\alpha)$ is contained in a finitely generated subgroup Γ of $G(K)$. Indeed, φ must be integral over the subring $\mathbb{Z}$ of $\mathrm{End}(G)$ and so, there exists some $g \in \mathbb{N}$ and $a_0, \dots, a_{g-1} \in \mathbb{Z}$ such that

$$\varphi^g = \sum_{i=0}^{g-1} a_i \varphi^i. \tag{2.0.2}$$

So, letting Γ be the subgroup of G spanned by $\varphi^i(\alpha)$ and $\varphi^i(\beta)$ for $i = 0, \dots, g-1$, we obtain that $\mathcal{O}_\Phi(\alpha) \subseteq \Gamma$. This observation is crucial for our argument.

We prove the equivalence for the conditions (i)-(iii) by showing that (i)⇒(ii)⇒(iii)⇒(i); the "furthermore" statement from the conclusion of Theorem 1.2 follows as a consequence of our proof of the implication (ii)⇒(iii).

First, we note that (i) always implies (ii) as previously observed in [AC08, MS14, BGZ17]; this statement holds very generally for any dominant, regular self-map on any variety.

Second, we assume (ii) holds and we prove (iii) must also hold. We establish this by proving that if condition (iii) does not hold, then the orbit of any point $\alpha \in G(K)$ which is linearly independent over $\mathrm{End}(G)$ from β must be Zariski dense; in particular, this proves our "furthermore" statement from the conclusion of Theorem 1.2.

So, assume (iii) does not hold and let $\alpha \in G(K)$ be linearly independent over $\mathrm{End}(G)$ from β (such points always exist, see [Vil08], for example). If $\mathcal{O}_\Phi(\alpha)$ is not Zariski dense, then its Zariski closure is a proper subvariety Z of G. On the other hand, since $\mathcal{O}_\Phi(\alpha)$ is contained in a finitely generated subgroup Γ of $G(K)$, we must have that Z is a finite union of cosets of proper algebraic subgroups of G, according to the famous theorems of [Lau84, Fal94, Voj96]. By the pigeonhole principle, there exist integers $0 \le m < n$ and some proper algebraic subgroup H of G such that

$$\Phi^n(\alpha) - \Phi^m(\alpha) \in H. \tag{2.0.3}$$

The Poincaré Reducibility Theorem for abelian varieties (see [GS17, Fact 3.2]), which also holds for algebraic tori and therefore, for split semiabelian varieties yields that there exists a complement C of H in G, i.e., some algebraic subgroup $C \subset G$ such that $C + H = G$ and $C \cap H$ is finite. Thus, considering the projection $G \longrightarrow G/H$ composed with the isogeny $G/H \longrightarrow C$ and finally the embedding $C \hookrightarrow G$, we conclude that there exists a *nontrivial* endomorphism $\Psi \in \mathrm{End}(G)$ such that $H \subseteq \ker(\Psi)$ (note that $\ker(\Psi)$ must be a proper subgroup of G because H is a proper subgroup of G and the index of H in $\ker(\Psi)$ is finite). In particular, we have

$$\Psi\left(\Phi^n(\alpha) - \Phi^m(\alpha)\right) = 0,$$

which coupled with (2.0.1) yields that

$$\left(\Psi \circ (\varphi^n - \varphi^m)\right)(\alpha) = \left(\Psi \circ \left(-\sum_{j=m}^{n-1} \varphi^j\right)\right)(\beta). \tag{2.0.4}$$

Equation (2.0.4) coupled with the fact that α is linearly independent over $\mathrm{End}(G)$ from β yields that $\Psi \circ (\varphi^n - \varphi^m) = 0$. Since Φ is dominant then also φ must be dominant and so, we conclude that actually,

$$\Psi \circ (\varphi^{n-m} - \mathrm{Id}) = 0. \tag{2.0.5}$$

Moreover, using (2.0.5) and (2.0.4), we obtain that

$$\Psi\left(\sum_{j=m}^{n-1} \varphi^j(\beta)\right) = 0. \tag{2.0.6}$$

Equations (2.0.5) and (2.0.6) are exactly the desired conditions from (iii) in the conclusion of Theorem 1.2. This concludes our proof for the implication (ii)⇒(iii); in addition, we see that if condition (iii) does not hold, then the orbit of any point $\alpha \in G(K)$ which is linearly independent over $\mathrm{End}(G)$ from β must be Zariski dense in G.

Finally, we prove the implication (iii)⇒(i). So, we assume there exists a non-constant endomorphism Ψ of G such that for some integers $0 \le m < n$, we have that

$$\Psi \circ \left(\varphi^{n-m} - \mathrm{Id}\right) = 0 \tag{2.0.7}$$

and

$$\Psi\left(\sum_{j=m}^{n-1} \varphi^j(\beta)\right) = 0. \tag{2.0.8}$$

Let $\overline{G} := G/\ker(\Psi)$; then $\overline{G}$ is a positive dimensional semiabelian variety since Ψ is a nontrivial endomorphism of G. Let $h : \overline{G} \dashrightarrow \mathbb{P}^1$ be a non-constant rational function. Let $\pi : G \longrightarrow \overline{G}$ be the natural projection map and let $g := h \circ \pi$; then $g : G \dashrightarrow \mathbb{P}^1$ is a non-constant rational function.

For each of the $k := n - m$ fundamental symmetric functions τ_i (for $i = 1, \dots, k$) on k variables, we let $g_i : G \dashrightarrow \mathbb{P}^1$ be defined by

$$g_i(x) := \tau_i \left(g\left(\Phi^m(x)\right), g\left(\Phi^{m+1}(x)\right), \cdots, g\left(\Phi^{n-1}(x)\right) \right).$$

We claim that for each $x \in G$, we have that

$$g\left(\Phi^m(x)\right) = g\left(\Phi^n(x)\right). \tag{2.0.9}$$

Indeed, using (2.0.1) and also equations (2.0.7) and (2.0.8), we get that $\Phi^n(x) - \Phi^m(x) \in \ker(\Psi)$ and therefore, by the definition of $g = h \circ \pi$, we obtain equality (2.0.9). Thus, for each of the k symmetric functions τ_i, we have that

$$g_i(\Phi(x)) = \tau_i \left(\Phi^{m+1}(x), \cdots, \Phi^n(x) \right) = \tau_i \left(\Phi^m(x), \cdots, \Phi^{n-1}(x) \right) = g_i(x). \tag{2.0.10}$$

Now, if each of the rational functions g_i are constant (for $i = 1, \dots, k$), then since the τ_i's are all the fundamental symmetric functions based on k variables, we conclude that each rational function $G \dashrightarrow \mathbb{P}^1$ given by $x \mapsto g\left(\Phi^{m-1+j}(x)\right)$ for $j = 1, \dots, k$ must be constant. However, since Φ is a dominant endomorphism of G, this would mean that $g : G \dashrightarrow \mathbb{P}^1$ is a constant map, contradiction. Therefore, there exists some non-constant rational function $f := g_i$ (for some $i = 1, \dots, k$); using (2.0.10), we conclude that $f : G \dashrightarrow \mathbb{P}^1$ is a non-constant rational function invariant under Φ.

This concludes our proof for the last implication (iii)⇒(i) and also concludes our proof for Theorem 1.2. □

Remark 2.1. Since all we used about the fact that the algebraically closed field K has characteristic 0 was the fact that the classical Mordell-Lang conjecture has a positive answer for semiabelian varieties G defined over K, we note that the exact same proof works also when K has prime characteristic p, under the additional assumption that G is an abelian variety with trivial $K/\overline{\mathbb{F}_p}$-trace (since in this case, the Mordell-Lang conjecture has affirmative answer for G, as proven by [Hru96]).

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