ZARISKI DENSE ORBITS FOR REGULAR SELF-MAPS OF TORI IN POSITIVE CHARACTERISTIC

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Abstract. We formulate a variant in characteristic p of the Zariski dense orbit conjecture previously posed by Zhang, Medvedev-Scanlon and Amerik-Campana for rational self-maps of varieties defined over fields of characteristic 0. So, in our setting, let K be an algebraically closed field, which has transcendence degree $d \geq 1$ over $\mathbb{F}_p$. Let X be a variety defined over K, endowed with a dominant rational self-map Φ. We expect that either there exists a variety Y defined over a finite subfield $\mathbb{F}_q$ of $\overline{\mathbb{F}_p}$ of dimension at least $d+1$ and a dominant rational map $\tau : X \dashrightarrow Y$ such that $\tau \circ \Phi^m = F^r \circ \tau$ for some positive integers m and r, where F is the Frobenius endomorphism of Y corresponding to the field $\mathbb{F}_q$, or either there exists $\alpha \in X(K)$ whose orbit under Φ is well-defined and Zariski dense in X, or there exists a non-constant $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$. We explain why the new condition in our conjecture is necessary due to the presence of the Frobenius endomorphism in case X is isotrivial. Then we prove our conjecture for all regular self-maps on $\mathbb{G}_m^N$.

1. Introduction

1.1. **Notation.** We let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the set of nonnegative integers. For any self-map Φ on a variety X and for any integer $n \geq 0$, we let Φ^n be the n-th iterate of Φ (where Φ^0 is the identity map $\mathrm{id} := \mathrm{id}_X$, by definition). For a point $x \in X$ with the property that each point $\Phi^n(x)$ avoids the indeterminacy locus of Φ, we denote by $\mathcal{O}_\Phi(x)$ the orbit of x under Φ, i.e., the set of all $\Phi^n(x)$ for $n \geq 0$.

1.2. **The classical Zariski dense orbit conjecture.** The following conjecture was motivated by a similar question raised by Zhang [Zha06] and was formulated by Medvedev and Scanlon [MS14] and by Amerik and Campana [AC08].

Conjecture 1.1. *Let X be a quasiprojective variety defined over an algebraically closed field K of characteristic 0 and let $\Phi : X \dashrightarrow X$ be a dominant rational self-map. Then either there exists $\alpha \in X(K)$ whose orbit under Φ is well-defined and Zariski dense in X, or there exists a non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.*

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One sees immediately that if there exists a non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$, then no orbit $\mathcal{O}_\Phi(\alpha)$ can be Zariski dense in X. So, the entire difficulty in Conjecture 1.1 is proving that when there is no non-constant rational function f invariant under Φ, then one may indeed find a point α with a Zariski dense orbit. Conjecture 1.1 was proven (see [AC08, BGR17]) in the case K is uncountable. However, if K is countable, then Conjecture 1.1 is very difficult and only a few special cases are known (see [BGRS17, GH18, GS17, GS21, GS19, GX18, MS14, Xie19]). The main difficulty comes from the fact (as proven in [AC08, BGR17]) that, from a strictly geometric point of view, there exist countably many proper subvarieties of X one needs to avoid in order to find a point with a Zariski dense orbit; thus, when K is countable, one needs to exploit the arithmetic dynamics of the setting from Conjecture 1.1 in order to find a point whose orbit is Zariski dense.

1.3. **A variant of the conjecture in positive characteristic.** The picture in characteristic p is very much different due to the presence of the Frobenius endomorphism for any variety X defined over a finite field (see [BGR17, Example 6.2] and also the next Remark).

Remark 1.2. If X is any variety defined over $\mathbb{F}_p$, then there exists no non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ invariant under the Frobenius endomorphism $F : X \longrightarrow X$ (corresponding to the field automorphism $x \mapsto x^p$); however, unless $\mathrm{trdeg}_{\mathbb{F}_p} K \geq \dim(X)$, there is no point in $X(K)$ with a Zariski dense orbit in X (each orbit of a point $\alpha \in X(K)$ lives in a subvariety $Y \subseteq X$ defined over $\mathbb{F}_p$ of dimension $\dim(Y) = \mathrm{trdeg}_{\mathbb{F}_p} L$, where L is the minimal field extension of $\mathbb{F}_p$ for which $\alpha \in X(L)$). Note that the Frobenius endomorphism is very special in the sense that for most maps one can expect a dense orbit for a point defined over a field extension of $\overline{\mathbb{F}_p}$ with a transcendence degree smaller than $\dim(X)$.

The discussion from Remark 1.2 motivates the following conjecture.

Conjecture 1.3. *Let K be an algebraically closed field of positive transcendence degree over $\overline{\mathbb{F}_p}$, let X be a quasiprojective variety defined over K, and let $\Phi : X \dashrightarrow X$ be a dominant rational self-map defined over K as well. Then at least one of the following three statements must hold:*

(A) *There exists $\alpha \in X(K)$ whose orbit $\mathcal{O}_\Phi(\alpha)$ is Zariski dense in X.*
(B) *There exists a non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.*
(C) *There exist positive integers m and r, there exists a variety Y defined over a finite subfield $\mathbb{F}_q$ of $\overline{\mathbb{F}_p}$ such that $\dim(Y) \geq \mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ and there exists a dominant rational map $\tau : X \dashrightarrow Y$ such that*
$$\tau \circ \Phi^m = F^r \circ \tau,$$
where F is the Frobenius endomorphism of Y corresponding to the field $\mathbb{F}_q$.

Remark 1.4. Note that if X is any variety defined over $\overline{\mathbb{F}_p}$, endowed with some endomorphism Φ, then each point $\alpha \in X(\overline{\mathbb{F}_p})$ would be preperiodic under the action of Φ and so, the trichotomy from Theorem 1.3 cannot hold. Hence, it is necessary to assume that K is transcendental over $\overline{\mathbb{F}_p}$ in Conjecture 1.3.

We note that in [BGR17, Theorem 1.2], it was proven that if there is no non-constant rational function f invariant under Φ, then any point $\alpha \in X(K)$ outside a countable union of proper subvarieties of X would have a Zariski dense orbit. So, in particular, [BGR17, Theorem 1.2] proves Conjecture 1.3 whenever K is uncountable, which leaves once again the case when K is countable as the outstanding open case in Conjecture 1.3.

1.4. **Our results.** We prove our Conjecture 1.3 in the case of regular self-maps Φ of $\mathbb{G}_m^N$.

Theorem 1.5. *Let $N \in \mathbb{N}$ and let K be an algebraically closed field of characteristic p such that $\operatorname{trdeg}_{\overline{\mathbb{F}_p}} K \geq 1$. Let $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ be a dominant regular self-map defined over K. Then at least one of the following statements must hold.*

(A) *There exists $\alpha \in \mathbb{G}_m^N(K)$ whose orbit under Φ is Zariski dense in $\mathbb{G}_m^N$.*
(B) *There exists a non-constant rational function $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.*
(C) *There exist positive integers m and r, a connected algebraic subgroup Y of $\mathbb{G}_m^N$ (defined over a finite field $\mathbb{F}_q$) of dimension at least equal to $\operatorname{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ and a dominant regular map $\tau : \mathbb{G}_m^N \longrightarrow Y$ such that*

$$\tau \circ \Phi^m = F^r \circ \tau, \tag{1.5.1}$$

where F is the usual Frobenius endomorphism of Y induced by the field automorphism $x \mapsto x^q$.

Remark 1.2 shows that indeed condition (C) is necessary due to the presence of the Frobenius endomorphism of $\mathbb{G}_m^N$; we also illustrate the trichotomy from the conclusion of our Theorem 1.5 in the next series of examples.

Example 1.6. Let p be an odd prime number, let K be the algebraic closure of $\mathbb{F}_p(t)$, let $(\beta_1, \beta_2, \beta_3) \in \mathbb{G}_m^3(K)$ and let $\Phi : \mathbb{G}_m^3 \longrightarrow \mathbb{G}_m^3$ be a regular map defined over K.

(1) If Φ is the translation map given by $(x_1, x_2, x_3) \mapsto (\beta_1 x_1, \beta_2 x_2, \beta_3 x_3)$, then condition (B) in Theorem 1.5 holds if and only if $\beta_1, \beta_2, \beta_3$ are multiplicatively dependent, i.e., there exist integers c_1, c_2, c_3, not all equal to 0 such that $\prod_{i=1}^3 \beta_i^{c_i} = 1$. If the β_j's are multiplicatively independent, then conclusion (A) from Theorem 1.5 holds; clearly, conclusion (C) does not hold in this example.

(2) If Φ is given by $(x_1, x_2, x_3) \mapsto (\beta_1 x_1^p, \beta_2 x_2^p, \beta_3 x_3^2)$, then condition (B) from Theorem 1.5 does not hold; this can be seen either directly, or by invoking our Theorem 2.2 since conjugating Φ by a suitable translation yields the group endomorphism $(x_1, x_2, x_3) \mapsto (x_1^p, x_2^p, x_3^2)$. Therefore, no non-constant fibration can be invariant under Φ. However, also conclusion (A) from Theorem 1.5 does not hold for this example. Indeed, for any point

$$\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{G}_m^3(K),$$

the orbit of α under Φ is contained in some proper subvariety of $\mathbb{G}_m^3$ of the form $V \times \mathbb{G}_m$, where $V \subset \mathbb{G}_m^2$ is a curve. More precisely, V is the translation by the point $\left(\beta_1^{-1/(p-1)}, \beta_2^{-1/(p-1)}\right)$ of a curve defined over $\mathbb{F}_p$ containing the point $\left(\alpha_1 \beta_1^{1/(p-1)}, \alpha_2 \beta_2^{1/(p-1)}\right)$.

(3) If Φ is given by $(x_1, x_2, x_3) \mapsto \left(\beta_1 x_1^p, \beta_2 x_2^{p^2}, \beta_3 x_3^{p^3}\right)$, then conclusion (A) from Theorem 1.5 holds (neither conclusions (B) nor (C) hold for this example) and so, there exists a point in $\mathbb{G}_m^3(K)$ with a Zariski dense orbit.

The strategy of our proof for Theorem 1.5 is as follows. Each regular self-map Φ of $\mathbb{G}_m^N$ is a composition of a group endomorphism $\varphi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ with a translation τ_y (by a point $y \in \mathbb{G}_m^N(K)$) (see [Iit76, Theorem 2]). Then for each point $\alpha \in \mathbb{G}_m^N(K)$, the entire orbit $\mathcal{O}_\Phi(\alpha)$ lies in a finitely generated subgroup Γ of $\mathbb{G}_m^N(K)$. Assuming $\mathcal{O}_\Phi(\alpha)$ is not Zariski dense in $\mathbb{G}_m^N$, then it means its Zariski closure $Z \subset \mathbb{G}_m^N$ is a proper subvariety. Since $Z(K) \cap \Gamma$ is Zariski dense in Z, then the result of [Hru96, Theorem 1.1] yields that Z is a finite union of translates of subvarieties defined over $\overline{\mathbb{F}_p}$. This property yields some useful information regarding the endomorphism φ in connection with the translation τ_y. However, in order to obtain even more precise information (which in turn delivers the desired conclusion in Theorem 1.5) we employ the F-structure result of Moosa and Scanlon [MS04, Theorem B] regarding the intersection of a finitely generated subgroup with a subvariety of $\mathbb{G}_m^N$ (see Theorem 2.1 and also [CGSZ21, Section 2.2] for a concise description of the main result from [MS04]).

The same strategy employed in our proof of Theorem 1.5 should extend with appropriate modification to the general case when we replace $\mathbb{G}_m^N$ by a split semiabelian variety G defined over a finite field (for example, one would need to employ the results of [Ghi08] to describe the intersection of a subvariety of G with a finitely generated subgroup, plus there exist additional complications due to the larger, possibly non-commutative ring of endomorphisms for an abelian variety defined over a finite field). However, the variant of Theorem 1.5 in the context of isotrivial abelian varieties defined over a field K of transcendence degree 1 over $\overline{\mathbb{F}_p}$ is already quite difficult since the proof of one of the main technical ingredients in our proof of Theorem 1.5 (see Proposition 4.1) does not extend to the abelian case;

instead, the Diophantine question that arises in the case of abelian varieties defined over a function field with a positive transcendence degree is challenging. So, we expect the extension of Theorem 1.5 to the case of isotrivial abelian varieties defined over a function field of a positive transcendence degree to be quite difficult. Furthermore, the case of a non-isotrivial abelian variety defined over a function field of positive characteristic will have additional complications since even the structure of the intersection between a subvariety of such an abelian variety with a finitely generated subgroup is significantly more delicate. Finally, the general case in Conjecture 1.3 when X is an arbitrary variety is expected to be at least as difficult as the general case in Conjecture 1.1. As kindly pointed out by the referee, when X is an algebraic group, one often finds that if there is no Zariski dense orbit under the action of Φ, then there exists some suitable algebraic group Y endowed with a dominant map $g : X \longrightarrow Y$ with the property that $g \circ \Phi = g$; so, when X is not an algebraic group, then Conjecture 1.3 is significantly harder. The increased difficulty for the characteristic p variant of our conjecture is not surprising since quite a few arithmetic conjectures turned out to be very difficult in characteristic p, even more so than in characteristic 0; for example, we mention the variant of the Dynamical Mordell-Lang Conjecture, which was shown to be *very* difficult in characteristic p even for the case of regular self-maps of tori (see [CGSZ21] and the more general discussion from [BGT16, Chapter 13]).

We sketch briefly the plan for our paper. In Section 2 we state a precise version of our Theorem 1.5 (see Theorems 2.5 and 2.6 which refine Theorem 1.5). In Section 3, we prove Theorem 3.7 which solves Theorem 1.5 in the special case Φ is a composition of a translation with a unipotent group endomorphism. in Section 4, using Theorem 3.7 (along with a general reduction provided by our Proposition 3.12 from Section 3.3), we complete the proof of Theorem 1.5 (along with Theorems 2.5 and 2.6).

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2. Additional results and some technical reductions

2.1. **Generalities.** As a matter of notation, we use $\mathrm{id}|_X$ to denote the identity map on the variety (or more general, the set) X. For N-by-N matrices we use $\mathrm{id} := \mathrm{id}_N$ to denote the corresponding identity matrix.

For any field K and any finitely generated subgroup $\Gamma \subset \mathbb{G}_m(K)$, we say that $x \in \mathbb{G}_m(K)$ is *multiplicatively independent from* Γ if there is no nonzero integer m such that $x^m \in \Gamma$. Similarly, given $\gamma_1, \dots, \gamma_r \in \mathbb{G}_m(K)$, we say that x is *multiplicatively independent from* $\gamma_1, \dots, \gamma_r$ if x is multiplicatively independent with respect to the subgroup of $\mathbb{G}_m(K)$ spanned by the γ_i's.

More generally, we say that $x_1, \dots, x_s \in \mathbb{G}_m(K)$ are *multiplicatively independent from* $\gamma_1, \dots, \gamma_r$ if the subgroup of $\mathbb{G}_m(K)$ generated by the x_i's has trivial intersection with the subgroup generated by the γ_j's.

For any point $y \in \mathbb{G}_m^N$, we let $\tau_y : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ be the translation-by-y automorphism, i.e., $\tau_y(x) := y \cdot x$ for each $x \in \mathbb{G}_m^N$. Furthermore, in our paper we find useful to use vector notation for the points of $\mathbb{G}_m^N$, i.e., from now on, the point $x \in \mathbb{G}_m^N$ will be denoted as $\vec{x} := (x_1, \dots, x_N)$.

Since $\mathrm{End}(\mathbb{G}_m^N) \xrightarrow{\sim} M_{N,N}(\mathbb{Z})$, we have that each dominant endomorphism φ of $\mathbb{G}_m^N$ is identified by an invertible N-by-N matrix A with integer entries such that

$$\varphi(\vec{x}) = \vec{x}^A, \tag{2.0.1}$$

i.e., $\varphi(x_1, \dots, x_N) = \left(\prod_{i=1}^N x_i^{a_{1,i}}, \dots, \prod_{i=1}^N x_i^{a_{N,i}}\right)$. We will often identify the group endomorphism φ of $\mathbb{G}_m^N$ with its corresponding N-by-N matrix A as in (2.0.1). Furthermore, we recall that any regular self-map of $\mathbb{G}_m^N$ is a composition of a translation $\tau_{\vec{y}}$ with a group endomorphism φ.

For a point $\vec{\alpha} := (\alpha_1, \dots, \alpha_N) \in \mathbb{G}_m^N(K)$ and some vector $\vec{v} := (v_1, \dots, v_N) \in \mathbb{Z}^N$, we let

$$\vec{\alpha}^{\vec{v}} := \prod_{i=1}^N \alpha_i^{v_i}. \tag{2.0.2}$$

In particular, given a group endomorphism φ corresponding to a matrix A as in (2.0.1), given a point $\vec{\alpha} \in \mathbb{G}_m^N(K)$ and also given a vector $\vec{v}$ with integer entries, we have

$$\varphi\left(\vec{\alpha}\right)^{\vec{v}} = \left(\vec{\alpha}\right)^{A^t \vec{v}}, \tag{2.0.3}$$

where A^t represents the transpose of the matrix A. Also, for any $\vec{\alpha} \in \mathbb{G}_m^N(K)$ and any $k \in \mathbb{Z}$, we let $\vec{\alpha}^k$ be the k-th power of the point $\vec{\alpha}$ in $\mathbb{G}_m^N(K)$.

For a regular self-map $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ given by $\vec{x} \mapsto \vec{\beta} \cdot \vec{x}^A$, a simple computation yields the formula for the n-th iterate:

$$\Phi^n(\vec{x}) = \left(\vec{\beta}\right)^{\sum_{j=0}^{n-1} A^j} \cdot \left(\vec{x}\right)^{A^n}. \tag{2.0.4}$$

Finally, we state a special case of the Moosa-Scanlon structure theorem [MS04, Theorem B] which will be used repeatedly in our proofs.

Theorem 2.1. *Let K be an algebraically closed field of positive characteristic p, let N be a positive integer, let $V \subset \mathbb{G}_m^N$ be a subvariety defined over K and let $\Gamma \subset \mathbb{G}_m^N(K)$ be a finitely generated subgroup. Then, $V(K) \cap \Gamma$ is a finite union of sets of the form*

$$U := \vec{\gamma} \cdot S(\vec{\eta}_1, \dots, \vec{\eta}_r; \delta_1, \dots, \delta_r) \cdot H, \tag{2.1.1}$$

where there exists some positive integer m such that

$$\vec{\gamma}^m, \vec{\eta}_1^m, \dots, \vec{\eta}_r^m \in \Gamma,$$

the δ_j's are positive integers, H is a subgroup of Γ and

$$S(\vec{\eta}_1,\dots,\vec{\eta}_r;\delta_1,\dots,\delta_r) := \left\{ \prod_{j=1}^{r} (\vec{\eta}_j)^{p^{\delta_j n_j}} : n_j \in \mathbb{N}_0 \text{ for } j = 1,\dots,r \right\}.$$

2.2. **A more precise statement for Theorem 1.5.** Our strategy for proving Theorem 1.5 is as follows. We will first prove Theorem 1.5 for group endomorphisms and then use our result to infer the general case in Theorem 1.5.

The next result is used in deriving a more precise statement for Theorem 1.5 in the case where Φ is a group endomorphism.

Theorem 2.2. *Let $N \in \mathbb{N}$, let K be an algebraically closed field of characteristic p and let Φ be a dominant group endomorphism of $\mathbb{G}_m^N$ defined over K. Then the following statements are equivalent:*

(i) *For some $\ell \in \mathbb{N}$, the kernel of $\Phi^\ell - \mathrm{id}$ is positive dimensional.*
(ii) *There exists a non-constant rational function $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.*

If the equivalent conditions (i)-(ii) do not hold, and also assuming that $\mathrm{trdeg}_{\overline{\mathbb{F}_p}} K \ge N$, then for any point $\vec{\alpha} \in \mathbb{G}_m^N(K)$ with the property that its coordinates $\alpha_1,\dots,\alpha_N$ are algebraically independent over $\overline{\mathbb{F}_p}$, we have that any infinite subset of $\mathcal{O}_\Phi(\vec{\alpha})$ is Zariski dense in $\mathbb{G}_m^N$.

Remark 2.3. Condition (i) from Theorem 2.2 tells us that for a dominant group endomorphism Φ of $\mathbb{G}_m^N$ (defined over an arbitrary algebraically closed field K), we have that Φ preserves a non-constant fibration (as in condition (ii) from Theorem 2.2) if and only if the matrix $A \in M_{N,N}(\mathbb{Z})$ corresponding to Φ (as in (2.0.1)) has an eigenvalue which is a root of unity.

Remark 2.4. Given a group endomorphism $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ corresponding to some (invertible) matrix $A \in M_{N,N}(\mathbb{Z})$ (see (2.0.1)), we see that condition (C) from the conclusion of Theorem 1.5 is equivalent with asking that there are $k := \mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ Jordan blocks in the Jordan canonical form for A corresponding to eigenvalues $\lambda_1,\dots,\lambda_k$ with the property that for some positive integers m and r, we have that

$$\lambda_1^m = \cdots = \lambda_k^m = p^r. \tag{2.4.1}$$

Note that the eigenvalues $\lambda_1,\dots,\lambda_k$ may be equal; we are only asking that they correspond to distinct Jordan blocks for A. This observation will be used throughout our proof of Theorem 1.5.

Using Remarks 2.3 and 2.4, we see that in the case of group endomorphisms, Theorem 1.5 is equivalent with the following result which is a stronger version of Theorem 2.2.

Theorem 2.5. *Let K be an algebraically closed field of positive transcendence degree over $\overline{\mathbb{F}_p}$, let $N \in \mathbb{N}$ and let Φ be a dominant group endomorphism of $\mathbb{G}_m^N$ corresponding to some matrix $A \in M_{N,N}(\mathbb{Z})$. Assume the following two conditions are met:*

(1) *there is no eigenvalue λ of A which is a root of unity.*
(2) *there does not exist $k := \mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ Jordan blocks in the Jordan canonical form of A corresponding to eigenvalues $\lambda_1, \dots, \lambda_k$ satisfying the equation*

$$\lambda_1^m = \cdots = \lambda_k^m = p^r, \tag{2.5.1}$$

for some positive integers m and r.

Then there exists $\vec{\alpha} \in \mathbb{G}_m^N(K)$ whose orbit under Φ is Zariski dense in $\mathbb{G}_m^N$. Furthermore, given any finitely generated subgroup $\Gamma \subset \mathbb{G}_m(K)$, one can choose $\vec{\alpha} \in \mathbb{G}_m^N(K)$ such that

(i) *the subgroup spanned by $\alpha_1, \dots, \alpha_N$ (the coordinates of $\vec{\alpha}$) has trivial intersection with Γ; and*
(ii) *any infinite subset of $\mathcal{O}_\Phi(\vec{\alpha})$ is Zariski dense in $\mathbb{G}_m^N$.*

The next result provides a more precise form in the conclusion of Theorem 1.5 for a dominant regular self-map of $\mathbb{G}_m^N$.

Theorem 2.6. *Let $N \in \mathbb{N}$, let K be an algebraically closed field of positive transcendence degree over $\overline{\mathbb{F}_p}$, let $\vec{\beta} \in \mathbb{G}_m^N(K)$, let $\varphi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ be a dominant group endomorphism corresponding to some matrix $A \in M_{N,N}(\mathbb{Z})$, and let $\Phi := \tau_{\vec{\beta}} \circ \varphi$. Assume there does not exist $k := \mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ Jordan blocks in the Jordan canonical form of A corresponding to eigenvalues $\lambda_1, \dots, \lambda_k$ satisfying the equation $\lambda_1^m = \cdots = \lambda_k^m = p^r$ for some positive integers m and r.*

Then the following statements are equivalent:

(i) *There is a non-constant rational function $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.*
(ii) *There is no $\vec{\alpha} \in \mathbb{G}_m^N(K)$ whose orbit $\mathcal{O}_\Phi(\vec{\alpha})$ is Zariski dense in $\mathbb{G}_m^N$.*
(iii) *There exists a positive integer ℓ and there exists a nonzero vector $\vec{v} \in \mathbb{Z}^N$ such that*

$$(A^\ell)^t \cdot \vec{v} = \vec{v} \text{ and } \vec{\beta}^{\left(\sum_{j=0}^{\ell-1} A^j\right)^t \cdot \vec{v}} = 1. \tag{2.6.1}$$

Remark 2.7. We explain here the relevance of condition (iii) from Theorem 2.6. The existence of a nonzero vector $\vec{v} \in \mathbb{Z}^N$ satisfying (2.6.1) means that for each $n \in \mathbb{N}$ and for each $\vec{\alpha} \in \mathbb{G}_m^N(K)$, we have that (see (2.0.4))

$$\begin{aligned}\Phi^{n\ell}(\vec{\alpha})^{\vec{v}} &= \vec{\beta}^{\left(\sum_{j=0}^{n\ell-1} A^j\right)^t \cdot \vec{v}} \cdot \vec{\alpha}^{\left(A^{n\ell}\right)^t \cdot \vec{v}} \\ &= \vec{\beta}^{\left(\sum_{j=0}^{\ell-1} A^j\right)^t \cdot \left(\sum_{i=0}^{n-1} A^{i\ell}\right)^t \vec{v}} \cdot \vec{\alpha}^{\vec{v}} \text{ (by condition (iii))} \\ &= \vec{\beta}^{\left(\sum_{j=0}^{\ell-1} A^j\right)^t \cdot (n\vec{v})} \cdot \vec{\alpha}^{\vec{v}} \text{ (by condition (iii))} \\ &= \vec{\alpha}^{\vec{v}} \text{ (by condition (iii))}.\end{aligned}$$

Therefore, $\mathcal{O}_{\Phi^\ell}(\vec{\alpha})$ is contained in a coset of the proper algebraic subgroup $H \subset \mathbb{G}_m^N$ given by the (nontrivial) equation $\vec{x}^{\vec{v}} = 1$ (actually, H is invariant under Φ^ℓ according to the above computation). Thus $\mathcal{O}_\Phi(\alpha)$ must be contained in a proper subvariety of $\mathbb{G}_m^N$ (which is a finite union of cosets of H) and so, it can never be Zariski dense in $\mathbb{G}_m^N$. Furthermore, one can find the non-constant rational function $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ which is invariant under Φ arguing identically as in the proof of [GS21, Theorem 1.2] where a similar condition (iii) was given in the general case of split semiabelian varieties defined over a field of characteristic 0.

So, the implications (i)⇒(ii) and (iii)⇒(i) from Theorem 2.6 hold with identical proof for regular self-maps of tori regardless of the characteristic of the field. The interesting features of Theorem 2.6 is that one can prove the implication (ii)⇒(iii) in positive characteristic as well. In characteristic 0, the proof of (ii)⇒(iii) from [GS21, Theorem 1.2] employed the classical Mordell-Lang theorems for semiabelian varieties (as established by [Lau84, Fal94, Voj96]) and it was incomparably much easier than the proof of our Theorem 2.6. Indeed, in characteristic p, since the classical Mordell-Lang theorems do not hold (see [Hru96]), one needs to employ a significantly more complicated approach in order to establish the same equivalence as the one stated in our Theorem 2.6.

3. Useful reductions for the general case and the proof of a special case

3.1. **General strategy for our proofs.** We first describe the general approach to proving our results. So, we write $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ as $\tau_{\vec{\beta}} \circ \varphi$ for some point $\vec{\beta} \in \mathbb{G}_m^N$ and some group endomorphism φ of $\mathbb{G}_m^N$, which corresponds to some (invertible) N-by-N matrix A with integer entries.

Using [BGRS17, Lemma 2.1], in order to prove our results, we can always replace Φ by a suitable iterate, i.e., for any given $\ell \in \mathbb{N}$,

- there exists a Zariski dense orbit under the action of Φ if and only if there exists a Zariski dense orbit under the action of Φ^ℓ; and
- Φ leaves invariant a non-constant rational function if and only if Φ^ℓ leaves invariant a non-constant rational function.

When we replace Φ by Φ^ℓ, the group endomorphism φ is replaced by φ^ℓ (and thus, the matrix A is replaced by A^ℓ), while the point $\vec{\beta}$ is replaced by

$\vec{\beta}^{\sum_{j=0}^{\ell-1} A^j}$ (see (2.0.4)). The advantage in our approach is that now we can assume the following: for each eigenvalue λ of A, we have that either $\lambda = 1$, or λ is not a root of unity.

We prove our results separately in the two cases outlined above, i.e., we deal separately with the case when A is a unipotent matrix, and with the case when A has no eigenvalue root of unity. In the latter case, the advantage is that no matter what is the translation $\tau_{\vec{\beta}}$ appearing in Φ, then we can conjugate Φ by another suitable translation $\tau_{\vec{\gamma}}$ so that

$$\Psi := \tau_{\vec{\gamma}}^{-1} \circ \Phi \circ \tau_{\vec{\gamma}} \tag{3.0.1}$$

is actually a group endomorphism of $\mathbb{G}_m^N$. Indeed, we choose $\vec{\gamma}$ such that

$$\vec{\gamma}^{\mathrm{id}-A} = \vec{\beta}$$

(note that $\mathrm{id} - A$ is an ivertible N-by-N matrix since we assume in this case that A does not have eigenvalues which are roots of unity) and then we see that Ψ defined as in (3.0.1) is indeed a group endomorphism. Since our results are invariant if we replace the self-map Φ by a conjugate of itself with an automorphism of $\mathbb{G}_m^N$ (see [GS19, Lemma 3.1]), the case when A has no eigenvalue root of unity reduces to proving our result for group endomorphisms (i.e., we are left to proving Theorem 2.5).

The case when $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ is given by a composition of a translation $\tau_{\vec{\beta}}$ with a group endomorphism φ corresponding to a unipotent N-by-N matrix A is treated in the next section.

3.2. **The case of unipotent maps.** We start by defining the main property we are investigating in this paper.

Definition 3.1. *Let $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ be a dominant regular self-map defined over an algebraically closed field K. We say that Φ has property $\mathcal{P}_K$ if either there exists a non-constant rational function $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$, or there exists a point $\vec{\alpha} \in \mathbb{G}_m^N(K)$ with a Zariski dense orbit under Φ, or there exist positive integers m and r, a connected algebraic subgroup Y of $\mathbb{G}_m^N$ of dimension at least equal to $\mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ defined over a finite subfield $\mathbb{F}_q \subset K$ and a dominant regular map $\tau : G \longrightarrow Y$ such that*

$$\tau \circ \Phi^m = F^r \circ \tau, \tag{3.1.1}$$

where F is the usual Frobenius endomorphism of $\mathbb{G}_m^N$ induced by the field automorphism $x \mapsto x^q$.

Next we establish a useful reduction in all of our proofs.

Proposition 3.2. *Let $N \in \mathbb{N}$ and let $A, B \in M_{N,N}(\mathbb{Z})$ be invertible matrices with the property that there exists an invertible matrix $Q \in M_{N,N}(\mathbb{Q})$ such that $B = Q^{-1}AQ$. Let φ and ψ be group endomorphisms of $\mathbb{G}_m^N$ corresponding to the matrices A and B, respectively.*

Let K be an algebraically closed field of characteristic p. For each $\vec{\gamma} \in \mathbb{G}_m^N(K)$, we let $\Phi_{\vec{\gamma}} := \tau_{\vec{\gamma}} \circ \varphi$ and $\Psi_{\vec{\gamma}} := \tau_{\vec{\gamma}} \circ \psi$ be the corresponding dominant regular self maps on $\mathbb{G}_m^N$ defined over K.

Let $k \in \mathbb{N}$ such that both matrices kQ and kQ^{-1} have integer entries. We let $g : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ be the group endomorphism corresponding to the matrix kQ. Then for each $\vec{\beta} \in \mathbb{G}_m^N(K)$, we have that if $\Phi_{g(\vec{\beta})}$ has property $\mathcal{P}_K$, then $\Psi_{\vec{\beta}}$ has property $\mathcal{P}_K$.

Remark 3.3. We note that if the matrices Q and Q^{-1} have integer entries, then the result of Proposition 3.2 follows immediately from [GS19, Lemma 3.1] since we can consider the group automorphism $g : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ corresponding to the matrix Q and then $\Psi_{\vec{\beta}} = g^{-1} \circ \Phi_{g(\vec{\beta})} \circ g$, which means that $\Psi_{\vec{\beta}}$ has property $\mathcal{P}_K$ if and only if $\Phi_{g(\vec{\beta})}$ has property $\mathcal{P}_K$.

Proof of Proposition 3.2. Let $\vec{\beta} \in \mathbb{G}_m^N(K)$. Lemmas 3.4 , 3.5, and 3.6 deliver the desired conclusion in Proposition 3.2. The next commutative diagram will be used in our proofs for Lemmas 3.4 , 3.5, and 3.6.

(3.3.1)
$$\begin{tikzcd}
\mathbb{G}_m^N \arrow[r, "\psi"] \arrow[d, "g"] \arrow[rr, bend left, "\Psi_{\vec{\beta}}"] & \mathbb{G}_m^N \arrow[r, "\tau_{\vec{\beta}}"] \arrow[d, "g"] & \mathbb{G}_m^N \arrow[d, "g"] \\
\mathbb{G}_m^N \arrow[r, "\varphi"] \arrow[rr, bend right, "\Phi_{g(\vec{\beta})}"'] & \mathbb{G}_m^N \arrow[r, "\tau_{g(\vec{\beta})}"] & \mathbb{G}_m^N
\end{tikzcd}$$

Lemma 3.4. *If there exists a non-constant rational function which is invariant under $\Phi_{g(\vec{\beta})}$, then there exists a non-constant rational function which is invariant under $\Psi_{\vec{\beta}}$.*

Proof of Lemma 3.4. Let $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ be a non-constant rational function such that

$$f \circ \Phi_{g(\vec{\beta})} = f. \tag{3.4.1}$$

Let $f_1 := f \circ g$ (which is still a non-constant rational function since g is a dominant group endomorphism). By the commutative diagram (3.3.1) and the equation 3.4.1, we get

$$f_1 \circ \Psi_{\vec{\beta}} = f \circ g \circ \Psi_{\vec{\beta}} = f \circ \Phi_{g(\vec{\beta})} \circ g = f \circ g = f_1,$$

thus proving the lemma. □

Lemma 3.5. *If there exists a K-point with a Zariski dense orbit under $\Phi_{g(\vec{\beta})}$, then there exists a K-point with a Zariski dense orbit under $\Psi_{\vec{\beta}}$.*

Proof of Lemma 3.5. Let $\vec{\alpha} \in \mathbb{G}_m^N(K)$ whose orbit under $\Phi_{g(\vec{\beta})}$ is Zariski dense in $\mathbb{G}_m^N$. Since g is a dominant group endomorphism, there exists some

$\vec{\gamma} \in \mathbb{G}_m^N(K)$ such that $g(\vec{\gamma}) = \vec{\alpha}$. Then for each $n \in \mathbb{N}$, using the commutative diagram (3.3.1) we have that

$$\Phi^n_{g(\vec{\beta})}(\vec{\alpha}) = \Phi^n_{g(\vec{\beta})} \circ (g(\vec{\gamma})) = g(\Psi^n_{\vec{\beta}}(\vec{\gamma})).$$

Since g is finite morphism, the orbit of $\vec{\gamma}$ under $\Psi_{\vec{\beta}}$ must be Zariski dense in $\mathbb{G}_m^N$, as claimed in Lemma 3.5. $\square$

Lemma 3.6. *If there exists a connected algebraic subgroup Y of $\mathbb{G}_m^N$ with a dimension larger than $\operatorname{trdeg}_{\overline{\mathbb{F}_p}} K$ and a dominant regular map $\tau : \mathbb{G}_m^N \longrightarrow Y$ such that $\Phi_{g(\beta)}$ satisfies equation (3.1.1) for some positive integers m and r, then for the dominant regular map $\tau \circ g : \mathbb{G}_m^N \longrightarrow Y$ we must have*

$$\tau \circ g \circ \Psi^m_\beta = F^r \circ \tau \circ g.$$

Proof. Using the diagram (3.3.1) we have

$$\begin{aligned} \tau \circ g \circ \Psi^m_\beta &= \tau \circ \Phi^m_{g(\beta)} \circ g \\ &= F^r \circ \tau \circ g \text{ by (3.1.1)}, \end{aligned}$$

which concludes our proof of Lemma 3.6. $\square$

Combining Lemmas 3.4, 3.5 and 3.6 yields the desired conclusion for Proposition 3.2. $\square$

Theorem 3.7. *Let $N \in \mathbb{N}$, let K be an algebraically closed field which is a transcendental extension of $\overline{\mathbb{F}_p}$, let φ be a unipotent group endomorphism of $\mathbb{G}_m^N$, let $\vec{\beta} \in \mathbb{G}_m^N(K)$ and let $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ be the dominant regular self-map given by $\Phi := \tau_{\vec{\beta}} \circ \varphi$. Then Φ has property $\mathcal{P}_K$.*

Before proving Theorem 3.7, we first recall the definition of upper asymptotic density of a subset of non-negative integers.

Definition 3.8. *Given a subset U of the set of non-negative integers, the upper asymptotic density of U is given by*

$$\limsup_{m\to\infty} \frac{\#\{0 \le n \le m : n \in U\}}{m}.$$

Remark 3.9. Upper asymptotic densities will appear frequently in the rest of the paper. So, from now on, for the sake of simplifying our notation, we will refer to the upper asymptotic density of some subset $U \subseteq \mathbb{N}_0$ simply as *density* of U and also, denote it by $d(U)$.

Proof of Theorem 3.7. Using Proposition 3.2 (along with the fact that any unipotent metrix with integer entries can be conjugate through a matrix with rational entries to its Jordan canonical form), we may assume from now on, that the matrix A corresponding to the group endomorphism φ is in Jordan canonical form.

The following result provides a precise criterion for the trichotomy in property $\mathcal{P}_K$ satisfied by a self-map $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ of the form $\Phi = \tau_{\vec{\beta}} \circ \varphi$, where φ is a group endomorphism corresponding to a unipotent matrix A in

Jordan canonical form. Before stating our result, we recall the notation $J_{\lambda,m}$ which denotes a Jordan canonical block of dimension $m \geq 1$ corresponding to the eigenvalue λ.

Proposition 3.10. *Let K be an algebraically closed field, which is transcendental over $\overline{\mathbb{F}_p}$ and let $\Phi : \mathbb{G}_m^N(K) \to \mathbb{G}_m^N(K)$ be given by*

$$(x_1, \dots, x_n) \longmapsto \left(\beta_1 \prod_{i=1}^{N} x_i^{a_{1,i}}, \dots, \beta_{i_\ell} \prod_{i=1}^{N} x_i^{a_{N,i}}\right), \tag{3.10.1}$$

where $a_{i,j}$ are the entries of a matrix $A := J_{1,i_1} \bigoplus J_{1,i_2-i_1} \bigoplus \cdots \bigoplus J_{1,i_\ell - i_{\ell-1}}$ (where $1 \leq i_1 < i_2 < \cdots < i_\ell = N$) and $(\beta_1, \dots, \beta_N) \in \mathbb{G}_m^N(K)$. Then, the following statements are equivalent:

(i) *There is a non-constant rational function $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.*
(ii) *There is no $\vec{\alpha} \in \mathbb{G}_m^N(K)$ whose orbit is Zariski dense in $\mathbb{G}_m^N(K)$.*
(iii) *$\beta_{i_1}, \dots, \beta_{i_\ell}$ are multiplicatively dependent.*

Proof. As noted already in [AC08, MS14, BGR17], we have that (i)⇒(ii). Now, in order to prove that (ii)⇒(iii), it suffices to show that if $\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_\ell}$ are multiplicatively independent then we can find a point in $\mathbb{G}_m^N(K)$ with a Zariski dense orbit. Note there exists a vector $\vec{\gamma}$ such that

$$\left(\vec{\gamma}\right)^{A - \mathrm{id}_N} = (\beta_1, \dots, \beta_{i_1 - 1}, 1, \beta_{i_1+1}, \dots, \beta_{i_\ell - 1}, 1)\,.$$

It is easy to check that the map $\tau_{\vec{\gamma}} \circ \Phi \circ \tau_{\vec{\gamma}}^{-1}$ is given by

$$\vec{x} \mapsto \vec{\beta}' \vec{x}^A, \text{ where } \vec{\beta}' := (1, \dots, 1, \beta_{i_1}, 1, \dots, 1, \beta_{i_\ell}) \in \mathbb{G}_m^N(K). \tag{3.10.2}$$

Therefore, after conjugating Φ with $\tau_{\vec{\gamma}}$ (see also [GS19, Lemma 3.1]), we may assume without loss of generality that

$$(\beta_1, \dots, \beta_N) = (1, \dots, 1, \beta_{i_1}, 1, \dots, 1, \beta_{i_\ell}), \tag{3.10.3}$$

i.e., $\beta_k = 1$ unless $k = i_j$ for some $j = 1, \dots, \ell$. We choose a point

$$\vec{\alpha} := (\alpha_1, \dots, \alpha_{i_1-1}, 1, \alpha_{i_1+1}, \dots, \alpha_{i_2-1}, 1, \dots, \alpha_{i_\ell-1}, 1) \in \mathbb{G}_m^N(K), \tag{3.10.4}$$

such that $\alpha_{i_1}, \dots, \alpha_{i_1-1}, \beta_{i_1}, \alpha_{i_1+1}, \dots, \alpha_{i_\ell-1}, \beta_{i_\ell}$ are multiplicatively independent (note that since $\mathrm{trdeg}_{\overline{\mathbb{F}_p}} K > 0$, we can find arbitrarily many multiplicatively independent elements of K). We let

$$\vec{\eta} = (\alpha_1, \dots, \alpha_{i_1-1}, \beta_{i_1}, \alpha_{i_1+1}, \dots, \alpha_{i_\ell-1}, \beta_{i_\ell}).$$

Then the orbit of α under Φ consists of points of the following form:

$$\mathcal{O}_\Phi(\alpha) = \left\{\vec{\beta}^{A^{n-1} + \cdots + A + \mathrm{id}} \vec{\alpha}^{A^n} : n \in \mathbb{N}_0\right\}.$$

We claim that the orbit of $\vec{\alpha}$ under Φ is Zariski dense. We argue by contradiction, and therefore assume that its Zariski closure V is a proper subvariety of $\mathbb{G}_m^N$.

We let $\Gamma \subset \mathbb{G}_m^N$ be the finitely generated group consisting of all elements of the form $\vec{\eta}^E$ where E is any N-by-N matrix with integer entries; clearly,

$\mathcal{O}_\Phi(\vec{\alpha}) \subseteq \Gamma$. By Theorem 2.1, we know that $V \cap \Gamma$ is a union of finitely many sets of the form

$$U := \vec{\gamma} \cdot S(\vec{\eta}_1, \dots, \vec{\eta}_r; \delta_1, \dots, \delta_r) \cdot H, \tag{3.10.5}$$

where there exists some positive integer m such that

$$\vec{\gamma}^m, \vec{\eta}_1^m, \dots, \vec{\eta}_r^m \in \Gamma, \tag{3.10.6}$$

the δ_j's are positive integers, and H is a subgroup of Γ.

Because $\mathcal{O}_\Phi(\vec{\alpha})$ is contained in finitely many sets of the form (3.10.5), then there must exist a given set U of the form (3.10.5) for which the following subset of $\mathbb{N}_0$:

$$S = \{n \in \mathbb{N}_0 \colon \Phi^n(\vec{\alpha}) \in U\}$$

has positive density $d(S)$ (see Remark 3.9 regarding our notation for upper asymptotic density of subsets of $\mathbb{N}_0$).

The algebraic closure of H must be an algebraic group G contained in the stabilizer of the variety W, which is the Zariski closure of U. Since V is a proper subvariety and $W \subseteq V$, then G must also be a proper algebraic subgroup of $\mathbb{G}_m^N$. So, there must exist a nonzero vector $\vec{v} \in \mathbb{Z}^N$ such that

$$(\vec{\epsilon})^{\vec{v}} = 1 \text{ for each } \vec{\epsilon} \in H. \tag{3.10.7}$$

Let $n \in S$; so, $\Phi^n(\alpha) \in U$ (see (3.10.5)). Equation (3.10.6) yields that

$$\vec{\gamma}^m = \vec{\eta}^C \text{ and } \vec{\eta}_i^m = \vec{\eta}^{B_i} \text{ for each } i = 1, \dots, r,$$

where $C, B_1, \dots, B_r \in M_{N,N}(\mathbb{Z})$ and so,

$$\Phi^n(\vec{\alpha})^m = (\vec{\eta})^{C + \sum_{j=1}^r p^{\delta_j n_j} B_j} \cdot \vec{\epsilon}_n \tag{3.10.8}$$

for some nonnegative integers n_j and some $\vec{\epsilon}_n \in H$. So, combining (3.10.8) with (3.10.7) yields

$$\Phi^n(\vec{\alpha})^{m\vec{v}} = \vec{\eta}^{\left(C + \sum_{j=1}^r p^{\delta_j n_j} B_j\right)^t \cdot \vec{v}}. \tag{3.10.9}$$

On the other hand, we know that $\Phi^n(\vec{\alpha}) = \vec{\beta}^{\sum_{j=0}^{n-1} A^j} \cdot \vec{\alpha}^{A^n}$ (see (2.0.4)) and we also compute:

(3.10.10)
$$A^n = \begin{pmatrix} 1 & \binom{n}{1} & \cdots & \binom{n}{i_1 - 1} \\ 0 & 1 & \cdots & \binom{n}{i_1 - 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \bigoplus \cdots \bigoplus \begin{pmatrix} 1 & \binom{n}{1} & \cdots & \binom{n}{i_\ell - i_{\ell-1} - 1} \\ 0 & 1 & \cdots & \binom{n}{i_\ell - i_{\ell-1} - 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and so,

(3.10.11)

$$A^{n-1}+\cdots+\mathrm{id}=\begin{pmatrix} n & \binom{n}{2} & \cdots & \binom{n}{i_1} \\ 0 & n & \cdots & \binom{n}{i_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix}\bigoplus\cdots\bigoplus\begin{pmatrix} n & \binom{n}{2} & \cdots & \binom{n}{i_\ell-i_{\ell-1}} \\ 0 & n & \cdots & \binom{n}{i_\ell-i_{\ell-1}-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix}.$$

Therefore, using (3.10.9) along with formulas (3.10.10) and (3.10.11), we obtain that for each $n\in S$, we have

$$\vec{\beta}^{m\left(\sum_{i=0}^{n-1}A^i\right)^t\cdot\vec{v}}\cdot\vec{\alpha}^{(mA^n)^t\cdot\vec{v}}=(\vec{\eta})^{C^t\vec{v}+\sum_{j=1}^r p^{n_j\delta_j}B_j^t\vec{v}}. \tag{3.10.12}$$

Now, both sides in (3.10.12) consist of products of powers of

$$\alpha_1,\dots,\alpha_{i_1-1},\beta_{i_1},\alpha_{i_1+1},\dots,\alpha_{i_2-1},\beta_{i_2},\alpha_{i_2+1},\dots,\alpha_{i_\ell-1},\beta_{i_\ell} \tag{3.10.13}$$

and since the N elements of $\mathbb{G}_m(K)$ from (3.10.13) are multiplicatively independent, then it means that the exponents of each α_i and each β_{i_j} appearing in the left-hand side of (3.10.12) must match the corresponding exponent of the α_i, respectively of β_{i_j} appearing in the right-hand side of (3.10.12).

Now, since $\vec{v}:=(v_1,\dots,v_N)$ is nonzero, then there is some $1\le k\le\ell$ such that the tuple $(v_{i_{k-1}+1},\dots,v_{i_k})$ is nonzero (where we denoted $i_0:=0$ for convenience). We use equations (3.10.10) and (3.10.11) to compute the exponent of β_{i_k} appearing in the left-hand side of (3.10.12) and then comparing it with the exponent of β_{i_k} from the right-hand side of (3.10.12), we get

$$m\left(v_{i_{k-1}+1}\cdot\binom{n}{i_k-i_{k-1}}+v_{i_{k-1}+2}\cdot\binom{n}{i_k-i_{k-1}-1}+\cdots+v_{i_k}\cdot\binom{n}{1}\right)$$

$$=b_0+\sum_{j=1}^r b_jp^{\delta_jn_j}, \tag{3.10.14}$$

for some integers $b_0,\dots,b_r$ which are independent of n (and only depend on the entries of the matrices $C,B_1,\dots,B_r$ and the entries of the vector $\vec{v}$). Since the tuple $\left(v_{i_{k-1}+1},\dots,v_{i_k}\right)$ is nonzero, then the polynomial

$$P(n):=m\cdot\sum_{j=1}^{i_k-i_{k-1}}v_{i_{k-1}+j}\cdot\binom{n}{i_k-i_{k-1}+1-j} \tag{3.10.15}$$

must be non-constant. So, equations (3.10.15) and (3.10.14) yield that each element $n\in S$ must satisfy an equation of the form:

$$P(n)=b_0+\sum_{j=1}^r b_jp^{\delta_jn_j}, \tag{3.10.16}$$

for some $n_j\in\mathbb{N}_0$. Because P is non-constant (while the δ_j's are positive integers and the b_j's are given), [GOSS21b, Theorem 1.1] yields that $d(S)=$

0, therefore contradicting our assumption that S has positive density. Hence, indeed $\mathcal{O}_\Phi(\vec{\alpha})$ must be Zariski dense in $\mathbb{G}_m^N$, as desired for showing the implication (ii)$\Rightarrow$(iii).

Finally, in order to prove that (iii)$\Rightarrow$(i), we know that there exists a nonzero $\vec{v} \in \mathbb{Z}^\ell$ such that $\prod_{j=1}^{\ell} \beta_{i_j}^{v_j} = 1$ since the β_{i_j}'s are multiplicatively dependent. Therefore, the non-constant rational function

$$f(x_1, \dots, x_N) := \prod_{j=1}^{\ell} x_{i_j}^{v_j}$$

is invariant under Φ (note that $i_\ell = N$ with our notation from Proposition 3.10). This concludes our proof for Proposition 3.10. $\square$

Proposition 3.10 finishes the proof of Theorem 3.7. $\square$

Remark 3.11. Our proof of Proposition 3.10 shows that for a regular self-map Φ as given in (3.10.1), if in addition the vector β has the form (3.10.3) with $\beta_{i_1}, \dots, \beta_{i_\ell}$ multiplicatively independent, then for any point $\vec{\alpha} \in \mathbb{G}_m^N(K)$ as in (3.10.4) such that

$$\alpha_1, \dots, \alpha_{i_1-1}, \beta_{i_1}, \alpha_{i_1+1}, \dots, \alpha_{i_2-1}, \beta_{i_2}, \alpha_{i_2+1}, \dots, \alpha_{i_\ell-1}, \beta_{i_\ell}$$

are multiplicatively independent, $\mathcal{O}_\Phi(\vec{\alpha})$ is Zariski dense. Furthermore, our proof of Proposition 3.10 yields the stronger statement that for a point $\vec{\alpha} \in \mathbb{G}_m^N(K)$ as in (3.10.4), for *any subset* $S \subseteq \mathbb{N}_0$ of positive density, the set

$$\{\Phi^n(\vec{\alpha}) \colon n \in S\}$$

is actually Zariski dense in $\mathbb{G}_m^N$. The strength of this refined result coming from Theorem 3.7 allows us to prove an important reduction step in Theorem 2.6 (see Proposition 3.12).

3.3. **The split case.** The following result is instrumental in proving our Theorem 2.6 by reducing it to our Theorem 3.7 combined with Theorem 2.5.

Proposition 3.12. *Let K be an algebraically closed field of characteristic $p > 0$, let $N_1, N_2 \in \mathbb{N}$, let $N := N_1 + N_2$, let D be an invertible N_2-by-N_2 matrix with integer entries, whose eigenvalues are not roots of unity, let B be a unipotent N_1-by-N_1 matrix in Jordan canonical form, i.e.,*

$$B := J_{1,i_1} \oplus J_{1,i_2-i_1} \oplus \cdots \oplus J_{1,i_s-i_{s-1}},$$

where $i_s = N_1$, and let $\vec{\beta} := (1, \dots, 1, \beta_{i_1}, 1, \dots, 1, \beta_{i_s}) \in \mathbb{G}_m^{N_1}(K)$. Let

$$\vec{\gamma} := (\gamma_1, \dots, \gamma_{i_1-1}, 1, \gamma_{i_1+1}, \dots, \gamma_{i_2-1}, 1, \gamma_{i_2+1}, \dots, \gamma_{i_s-1}, 1) \in \mathbb{G}_m^{N_1}(K)$$

and let $\vec{\alpha} := (\alpha_1, \dots, \alpha_{N_2}) \in \mathbb{G}_m^{N_2}(K)$. Assume the following elements of $\mathbb{G}_m(K)$ are multiplicatively independent:

$$\gamma_1, \dots, \gamma_{i_1-1}, \beta_1, \gamma_{i_1+1}, \dots, \gamma_{i_s-1}, \beta_{i_s}, \tag{3.12.1}$$

and define

$$\vec{\eta} = (\gamma_1, \dots, \gamma_{i_1-1}, \beta_1, \gamma_{i_1+1}, \dots, \gamma_{i_s-1}, \beta_{i_s}).$$

Also, assume that the α_i's are multiplicatively independent from the elements from (3.12.1), i.e., letting Γ be the subgroup of $\mathbb{G}_m(K)$ spanned by the elements from (3.12.1) and letting Λ be the subgroup of $\mathbb{G}_m(K)$ spanned by the α_i's, then $\Gamma \cap \Lambda = \{1\}$.

Let $\Phi_1 : \mathbb{G}_m^{N_1} \longrightarrow \mathbb{G}_m^{N_1}$ be the regular map defined by

$$\vec{x} \mapsto \vec{\beta} \cdot (\vec{x})^B \text{ for each } \vec{x} \in \mathbb{G}_m^{N_1}.$$

Let Φ_2 be the group endomorphism of $\mathbb{G}_m^{N_2}$ given by $\vec{x} \mapsto (\vec{x})^D$ for each $\vec{x} \in \mathbb{G}_m^{N_2}$, and let Φ be the regular self-map of $\mathbb{G}_m^N := \mathbb{G}_m^{N_1} \oplus \mathbb{G}_m^{N_2}$ given by $\Phi_1 \oplus \Phi_2$.

Assume that for any positive density subset $S \subseteq \mathbb{N}_0$, the set

$$\{\Phi_2^n(\vec{\alpha}) \colon n \in S\}$$

is Zariski dense in $\mathbb{G}_m^{N_2}$. Then $\mathcal{O}_\Phi(\vec{\gamma} \oplus \vec{\alpha})$ is Zariski dense in $\mathbb{G}_m^N$.

Proof. Assume $\mathcal{O}_\Phi(\vec{\gamma} \oplus \vec{\alpha})$ is not Zariski dense in $\mathbb{G}_m^N$ and thus, let $V \subset \mathbb{G}_m^N$ be its Zariski closure.

Let $\Delta := \Gamma^{N_1} \times \Lambda^{N_2} \subset \mathbb{G}_m^N(K)$; then $\mathcal{O}_\Phi(\vec{\gamma} \oplus \vec{\alpha}) \subseteq \Delta$. Then $V \cap \Delta$ is a finite union of sets of the form (3.10.5), i.e., sets of the form

$$U := \vec{\eta}_0 \cdot S(\vec{\eta}_1, \dots, \vec{\eta}_r; \delta_1, \dots, \delta_r) \cdot H, \tag{3.12.2}$$

where there exists some positive integer m such that

$$\vec{\eta}_0^m, \vec{\eta}_1^m, \dots, \vec{\eta}_r^m \in \Delta, \tag{3.12.3}$$

while the δ_j's are positive integers and H is a subgroup of Δ. Because the entire orbit of $\vec{\gamma} \oplus \vec{\alpha}$ under Φ is contained in the union of finitely many sets as the one from (3.12.2), there must exist some set U as in (3.12.2) containing $\Phi^n(\vec{\gamma} \oplus \vec{\alpha})$ for all integers n in some subset $S \subseteq \mathbb{N}_0$ of positive density.

Now, assume there exists some nonzero vector $\vec{v}_1 \in \mathbb{Z}^{N_1}$ and some vector $\vec{v}_2 \in \mathbb{Z}^{N_2}$ such that for the vector $\vec{v} := \vec{v}_1 \oplus \vec{v}_2 \in \mathbb{Z}^N$, we have that $(\vec{\epsilon})^{\vec{v}} = 1$ for each $\vec{\epsilon} \in H$. We argue as in the proof of Proposition 3.10 and get that for each $n \in S$, we have

$$\Phi^n(\vec{\gamma} \oplus \vec{\alpha})^{m\vec{v}} = (\vec{\eta} \oplus \vec{\alpha})^{\left(C + \sum_{j=1}^r p^{\delta_j n_j} B_j\right)^t \cdot \vec{v}}, \tag{3.12.4}$$

for some suitable N-by-N matrices $C, B_1, \dots, B_r$ with integer entries. Now, using that $\vec{v}_1$ is a nonzero vector, along with our hypothesis that the β_{i_j}'s and the γ_j's are multiplicatively independent, while the α_i's are multiplicatively independent from the β_{i_j}'s and the γ_j's, then arguing exactly as in the proof of Proposition 3.10 (see equations (3.10.14), (3.10.15) and (3.10.16)) we get that there exists some non-constant polynomial P and some integers b_j such that for each $n \in S$, there are non-negative integers n_j such that

$$P(n) = b_0 + \sum_{j=1}^r b_j p^{\delta_j n_j}. \tag{3.12.5}$$

Since S has positive density, this yields a contradiction to the conclusion of [GOSS21b, Theorem 1.1]. Therefore there is no nonzero vector $\vec{v}_1 \in \mathbb{Z}^{N_1}$

such that for some $\vec{v}_2 \in \mathbb{Z}^{N_2}$, we have that $\vec{v} = \vec{v}_1 \oplus \vec{v}_2$ kills each element of H. We let $G \subseteq \mathbb{G}_m^N$ be the Zariski closure of H; then G is an algebraic subgroup. Now, the fact that any vector $\vec{v} \in \mathbb{Z}^N$ which kills each element of G must have its first N_1 entries equal to 0 yields that $G = \mathbb{G}_m^{N_1} \times G_2$ for some algebraic subgroup $G_2 \subset \mathbb{G}_m^{N_2}$.

So, letting W be the Zariski closure of U in $\mathbb{G}_m^N$, then its stabilizer must contain G and therefore, it contains $\mathbb{G}_m^{N_1}$ (seen as a subgroup of $\mathbb{G}_m^N$ under the natural embedding $\vec{x} \mapsto \vec{x} \oplus \vec{1}_{\mathbb{G}_m^{N_2}}$); i.e., for each $\vec{\epsilon}_1 \in \mathbb{G}_m^{N_1}$ and each $\vec{\mu} \in W$, we have that $\vec{\epsilon} \cdot \vec{\mu} \in W$. Hence $W = \mathbb{G}_m^{N_1} \times Z$, for some subvariety $Z \subseteq \mathbb{G}_m^{N_2}$. However, Z must contain each $\Phi_2^n(\vec{\alpha})$ for $n \in S$ and $S \subseteq \mathbb{N}_0$ is a set of positive density; then our hypothesis yields that $Z = \mathbb{G}_m^{N_2}$. Therefore, $W = \mathbb{G}_m^N$ and so, indeed $\mathcal{O}_\Phi(\vec{\gamma} \oplus \vec{\alpha})$ must be Zariski dense in $\mathbb{G}_m^N$. $\square$

4. Proof of Theorem 1.5

We start this Section by proving a preliminary result used in the proof of Theorem 2.5 and then we will proceed to proving Theorems 2.5 and 2.6.

Proposition 4.1. *Let K be an algebraically closed field of transcendence degree $d \geq 1$ over $\overline{\mathbb{F}_p}$. Let $\Phi : \mathbb{G}_m^N(K) \to \mathbb{G}_m^N(K)$ be given by $\vec{x} \mapsto (\vec{x})^A$, where A is an invertible N-by-N matrix that has a conjugate of the form*

$$\bigoplus_{i=1}^{s} \left(\bigoplus_{j=1}^{\ell_i} J_{p^{n_i}, m_i^{(j)} - m_i^{(j-1)}} \right), \tag{4.1.1}$$

where n_i's are distinct positive integers and $m_i^{(j)}$'s are non-negative integers such that for every $1 \leq i \leq s$ we have

$$0 = m_i^{(0)} < m_i^{(1)} < \cdots < m_i^{(\ell_i)},$$

while

$$\sum_{j=1}^{s} m_j^{(\ell_j)} = N.$$

Then one of the following statements must hold:

(1) *There exists $1 \leq i \leq s$ such that $\ell_i > d$.*
(2) *for any finitely generated subgroup $\Lambda \subset \mathbb{G}_m(K)$ there exists $\vec{\alpha} \in \mathbb{G}_m^N(K)$ such that*
   (i) *the subgroup of $\mathbb{G}_m(K)$ spanned by the α_i's (the coordinates of $\vec{\alpha}$) has trivial intersection with Λ; and*
   (ii) *any infinite subset of $\mathcal{O}_\Phi(\vec{\alpha})$ is Zariski dense in $\mathbb{G}_m^N$.*

Remark 4.2. Note that condition (1) in Proposition 4.1 says precisely that condition (C) from Theorem 1.5 holds for the given map Φ.

Proof of Proposition 4.1. Suppose that condition (1) does not hold. We will prove the next lemma which reduces the problem to the case where A is equal to a matrix of the form (4.1.1).

Lemma 4.3. *It suffices to prove that condition (2) holds in the case where A is equal to a matrix of the form (4.1.1).*

Proof of Lemma 4.3. Since A has a conjugate of the form (4.1.1), there must exist a group endomorphism Ψ corresponding to a matrix of the form (4.1.1) and a dominant group endomorphism $g : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ such that the next diagram commutes

$$\begin{array}{ccc} \mathbb{G}_m^N & \xrightarrow{\Phi} & \mathbb{G}_m^N \\ \downarrow g & & \downarrow g \\ \mathbb{G}_m^N & \xrightarrow{\Psi} & \mathbb{G}_m^N. \end{array} \tag{4.3.1}$$

Suppose that $\vec{\alpha}$ satisfies conditions (i) and (ii) with respect to the group endomorphism Ψ. We choose $\vec{\beta} \in \mathbb{G}_m^N$ such that $g(\vec{\beta}) = \vec{\alpha}$. Using Lemma 3.5 the orbit of $\vec{\beta}$ under Φ must be Zariski dense in $\mathbb{G}_m^N$. Now suppose for the sake of contradiction that there exists some non-zero vector $\vec{v} \in \mathbb{Z}^N$ such that $\vec{\beta}^{\vec{v}} \in \Lambda \setminus \{0\}$. Let g correspond to a matrix $B \in M_{N,N}(\mathbb{Z})$ which is invertible as g is dominant. So, there must exist a non-zero integer m and a non-zero vector $\vec{v}' \in \mathbb{Z}^N$ such that $B^t\vec{v}' = m\vec{v}$. This implies that

$$\vec{\alpha}^{\vec{v}'} = \vec{\beta}^{B^t\vec{v}'} = \vec{\beta}^{m\vec{v}} \in \Lambda \setminus \{0\},$$

which contradicts the assumption that $\vec{\alpha}$ satisfies condition (ii). This concludes our proof of Lemma 4.3. $\square$

Therefore, from now on we may assume without loss of generality that A is equal to a matrix of the form (4.1.1). Choose $t_1, \dots, t_d \in \mathbb{G}_m(K)$ that are algebraically independent over $\overline{\mathbb{F}_p}$ and moreover, the subgroup of $\mathbb{G}_m(K)$ generated by $t_1, \dots, t_d$ has trivial intersection with Λ. We claim that any infinite subset of the orbit of

$$\vec{\alpha} := \vec{\alpha}_1 \oplus \cdots \oplus \vec{\alpha}_s \in \mathbb{G}_m^N(K),$$

where

$$\vec{\alpha}_i := (\underbrace{t_1, \dots, t_1}_{m_i^{(1)} \text{ times}}, \underbrace{t_2, \dots, t_2}_{m_i^{(2)} - m_i^{(1)} \text{ times}}, \dots, \underbrace{t_{\ell_i}, \dots, t_{\ell_i}}_{m_i^{(\ell_i)} - m_i^{(\ell_i - 1)} \text{ times}})$$

under Φ is Zariski dense. Note that for every $1 \le i \le s$, $\vec{\alpha}_i$ is well-defined since $\ell_i < d$. We also note that due to our choice for $t_1, \dots, t_d$, the entries of $\vec{\alpha}$ satisfy conclusion (i) from Proposition 4.1.

Now, suppose that there exists an infinite subset $S \subseteq \mathbb{N}_0$ with the property that the Zariski closure of the set $\{\Phi^n(\vec{\alpha}) \colon n \in S\}$ is a proper subvariety $V \subset \mathbb{G}_m^N$; we will derive a contradiction, which will thus show that $\vec{\alpha}$ also satisfies conclusion (ii) from Proposition 4.1.

Let Γ_0 be the finitely generated subgroup of $\mathbb{G}_m(K)$ generated by $t_1, \dots, t_d$ and let $\Gamma := \Gamma_0^N \subset \mathbb{G}_m^N(K)$. Then $\mathcal{O}_\Phi(\vec{\alpha}) \subseteq \Gamma$ and furthermore, by Theorem

2.1, $V \cap \Gamma$ is a finite union of sets of the form (3.10.5), i.e., sets of the form:

$$U := \vec{\gamma}^{-1} \cdot S(\vec{\eta}_1, \dots, \vec{\eta}_r; \delta_1, \dots, \delta_r) \cdot H, \tag{4.3.2}$$

where there exists some positive integer m such that

$$\vec{\gamma}^m, \vec{\eta}_1^m, \dots, \vec{\eta}_r^m \in \Gamma, \tag{4.3.3}$$

the δ_j's are positive integers, and H is a subgroup of Γ. We also recall that $S(\vec{\eta}_1, \dots, \vec{\eta}_r; \delta_1, \dots, \delta_r)$ consists of all points of the form $\prod_{j=1}^r (\vec{\eta}_j)^{p^{\delta_j n_j}}$ for any nonnegative integers $n_1, \dots, n_r$.

Since S is an infinite subset of $\mathbb{N}_0$ and each $\Phi^n(\vec{\alpha})$ belongs to a set as in (4.3.2), then the pigeonhole principle guarantees that at the expense of replacing S by an infinite subset of it, we may assume that each $\Phi^n(\vec{\alpha})$ are contained in the same set U as in (4.3.2).

Lemma 4.4. *The Zariski closure of the set U from (4.3.2) is of the form $\gamma^{-1} \cdot W$, where $W \subset \mathbb{G}_m^N$ is a proper subvariety defined over $\overline{\mathbb{F}_p}$.*

Proof of Lemma 4.4. The Zariski closure of the subgroup H from (4.3.2) is an algebraic subgroup of $\mathbb{G}_m^N$ and therefore, it is defined over $\overline{\mathbb{F}_p}$. Also, the Zariski closure of the set $S(\vec{\eta}_1, \dots, \vec{\eta}_r; \delta_1, \dots, \delta_r)$ is invariant under a suitable power of the Frobenius endomorphism (more precisely, it is invariant under F^δ, where δ is the least common multiple of all the positive integers δ_j). Therefore, the Zariski closure of $S(\vec{\eta}_1, \dots, \vec{\eta}_r; \delta_1, \dots, \delta_r) \cdot H$ must be a subvariety W defined over $\overline{\mathbb{F}_p}$. Furthermore, W is a proper subvariety of $\mathbb{G}_m^N$ since, according to our assumption, also $V \subset \mathbb{G}_m^N$ is a proper subvariety (and $\gamma^{-1} \cdot W \subseteq V$). This concludes our proof of Lemma 4.4. $\square$

Lemma 4.4 yields the existence of a polynomial $g(x) \in \overline{\mathbb{F}_p}[x_1, \dots, x_N]$ such that $g(\gamma \cdot \vec{x})$ vanishes at each point $\Phi^n(\vec{\alpha})$ for $n \in S$. Let $g(\vec{x}) := \sum_{i=1}^M a_i (\vec{x})^{\vec{v}_i}$, where the vectors $\vec{v}_i \in \mathbb{Z}^N$ are distinct and each $a_i \in \overline{\mathbb{F}_p}$ is nonzero.

Let $\vec{\eta} := (t_1, \dots, t_d) \in \mathbb{G}_m^d(K)$ and choose a point $\vec{\eta}_0 := (t_1', \dots, t_d') \in \mathbb{G}_m^d(K)$ where $(t_i')^m = t_i$ for every $1 \le i \le d$; in particular, $(\vec{\eta}_0)^m = \vec{\eta}$. Also, note that $t_1', \dots, t_d'$ are algebraically independent over $\overline{\mathbb{F}_p}$. Similarly, define

$$\vec{\alpha}_0 := \vec{\alpha}_1' \oplus \cdots \oplus \vec{\alpha}_s' \in \mathbb{G}_m^N(K),$$

where

$$\vec{\alpha}_i' := (\underbrace{t_1', \dots, t_1'}_{m_i^{(1)} \text{ times}}, \underbrace{t_2', \dots, t_2'}_{m_i^{(2)} - m_i^{(1)} \text{ times}}, \dots, \underbrace{t_{\ell_i}', \dots, t_{\ell_i}'}_{m_i^{(\ell_i)} - m_i^{(\ell_i - 1)} \text{ times}}).$$

Since $\vec{\gamma}^m \in \Gamma$, there must exist an N-by-d matrix B with integer entries such that $\vec{\gamma}^m = \vec{\eta}^B$. This implies that $\vec{\gamma} = \vec{\zeta} \cdot (\vec{\eta}_0)^B$ where $\zeta \in \mathbb{G}_m^N(K)$ is a point of order dividing m; in particular, $\vec{\zeta} \in \mathbb{G}_m^N(\overline{\mathbb{F}_p})$. Also, for every $1 \le k \le d$ define $\vec{u}_k$ to be a vector in $\mathbb{Z}^N$ whose i-th coordinate is equal to 1 whenever the i-th coordinate of α is equal to t_k and it is 0 otherwise. Then, for every $n \in S$ we must have

$$\begin{aligned}
0 &= \sum_{i=1}^{M} a_i \vec{\alpha}^{(A^n)^t \vec{v}_i} \vec{\gamma}^{\vec{v}_i} \\
&= \sum_{i=1}^{M} \left(a_i (\vec{\zeta})^{\vec{v}_i} \right) (\vec{\alpha}_0)^{m (A^n)^t \vec{v}_i} \, (\vec{\eta}_0)^{B^t \vec{v}_i} \\
&= \sum_{i=1}^{M} c_i \cdot \prod_{k=1}^{d} (t'_k)^{m\left((A^n)^t \vec{v}_i\right) \cdot \vec{u}_k + (B^t \vec{v}_i)_k}, \qquad (4.4.1)
\end{aligned}$$

where $c_i := a_i \cdot (\vec{\zeta})^{\vec{v}_i} \in \overline{\mathbb{F}_p}$ and $(B^t \vec{v}_i)_k$ denotes the k-th coordinate of $B^t \vec{v}_i$ for every $1 \le i \le M$. Since $t'_1, \dots, t'_d$ are algebraically independent over $\overline{\mathbb{F}_p}$ there must exist $i < j$ such that

$$m\big((A^n)^t \vec{v}_i\big) \cdot \vec{u}_k + (B^t \vec{v}_i)_k = m\big((A^n)^t \vec{v}_j\big) \cdot \vec{u}_k + (B^t \vec{v}_j)_k$$

for every $1 \le k \le d$, which implies that

$$m\big((A^n)^t (\vec{v}_i - \vec{v}_j)\big) \cdot \vec{u}_k + (B^t (\vec{v}_i - \vec{v}_j))_k = 0 \qquad (4.4.2)$$

for every $1 \le k \le d$. But because there are only finitely many pairs (i, j) of indices in $\{1, \dots, M\}$, by the pigeonhole principle, there is a pair (i, j) and an infinite subset $S_0 \subset S$ such that for every $n \in S_0$, (4.4.2) holds. Let $\vec{w} := \vec{v}_i - \vec{v}_j \in \mathbb{Z}^N$ and $(B^t \vec{w})_k = c_k \in \mathbb{Z}$ for every $1 \le k \le d$. So, for each $n \in S_0$ and every $1 \le k \le d$ we have

$$m((A^n)^t \vec{w}) \cdot \vec{u}_k + c_k = 0. \qquad (4.4.3)$$

For each $n \in \mathbb{N}$, we have that A^n equals

$$\bigoplus_{i=1}^{s} \left(\bigoplus_{j=1}^{\ell_i} \begin{pmatrix} p^{n \cdot n_i} & \binom{n}{1} p^{(n-1) \cdot n_i} & \cdots & \binom{n}{m_i^{(j)} - m_i^{(j-1)} - 1} p^{(n - (m_i^{(j)} - m_i^{(j-1)}) + 1) \cdot n_i} \\ 0 & p^{n \cdot n_i} & \cdots & \binom{n}{m_i^{(j)} - m_i^{(j-1)} - 2} p^{(n - (m_i^{(j)} - m_i^{(j-1)}) + 2) \cdot n_i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^{n \cdot n_i} \end{pmatrix} \right).$$

Let $\vec{w} := (w_1, \dots, w_N)$. Since $\vec{w}$ is nonzero, we let w_r be the first nonzero entry of $\vec{w}$ from the left. Due to the definition of each $\vec{u}_k$, we have that there exists a unique $1 \le k \le d$ such that the r-th coordinate $(\vec{u}_k)_r$ of $\vec{u}_k$ is non-zero. Also, there exist unique integers $1 \le i' \le s$ and $1 \le j' \le \ell_{i'}$ such that

$$\sum_{q=1}^{i'-1} m_q^{(\ell_q)} + m_{i'}^{(j'-1)} < r \le \sum_{q=1}^{i'-1} m_q^{(\ell_q)} + m_{i'}^{(j')}.$$

Then, the coefficient of $\binom{n}{r-m_{i'}^{(j'-1)}-1}p^{\left(n-\left(r-m_{i'}^{(j'-1)}\right)+1\right)\cdot n_{i'}}$ in $(A^n)^t\vec{w}\cdot\vec{u}_k$ (note that we have a dot product of vectors) is equal to

$$w_r(\vec{u}_k)_r, \tag{4.4.4}$$

which is non-zero as both w_r and $(\vec{u}_k)_r$ are non-zero. Furthermore, there is no other (nonzero) term in $(A^n)^t\vec{w}\cdot\vec{u}_k$ containing $p^{n\cdot n_{i'}}$ multiplied by a polynomial in n of degree greater than or equal to $r-m_{i'}^{(j'-1)}-1$ (note that this is a consequence of our choice for the coordinates of $\vec{\alpha}$). Therefore we get

$$m((A^n)^t\vec{w}\cdot\vec{u}_k)=\sum_{i=1}^{s}Q_i(n)p^{n\cdot n_i},$$

where each Q_i is a polynomial with coefficients in $\overline{\mathbb{Q}}$ and furthermore, $Q_{i'}(n)$ is nonzero of degree $r-m_{i'}^{(j'-1)}-1$. Thus, equation (4.4.3) becomes

$$\sum_{i=1}^{s}Q_i(n)p^{n\cdot n_i}+c=0 \tag{4.4.5}$$

for every $n\in S_0$. But, the left-hand side of (4.4.5) is the general term of a non-degenerate linear recurrence which can have only finitely many solutions (see [Sch03] for a thorough treatment of the famous Skolem-Mahler-Lech problem represented by equation (4.4.5)) since not all of the Q_i's are identically equal to zero and furthermore, because the n_i's are distinct positive integers, the quotient of any two p^{n_i} appearing in the equation (4.4.5) is not equal to a root of unity and also no p^{n_i} is a root of unity (note that the characteristic roots of the linear recurrence sequence from (4.4.5) belong to the set $\{1,p^{n_1},\dots,p^{n_s}\}$). This contradicts the fact that S_0 is an infinite set. So, any infinite subset of the orbit of $\vec{\alpha}$ under Φ must be Zariski dense in $\mathbb{G}_m^N$, which concludes our proof of Proposition 4.1. ∎

The next lemma will be used in the proof of Theorem 2.5.

Lemma 4.5. *Let p be a prime number, let $N\in\mathbb{N}$, let $\vec{v}\in\mathbb{Z}^N$, let $\delta_1,\dots,\delta_r\in\mathbb{N}$, and let $A,B_1,\dots,B_r,C$ be N-by-N matrices with integers entries such that A is invertible and moreover, none of the eigenvalues of A are multiplicatively dependent with respect to p. If there exists an infinite subset $S\subseteq\mathbb{N}$ with the property that for each $n\in S$, there exist $n_1,\dots,n_r\in\mathbb{N}_0$ such that*

$$A^n\vec{v}=C\vec{v}+\sum_{i=1}^{r}p^{n_i\delta_i}B_i\vec{v}, \tag{4.5.1}$$

then $\vec{v}$ must be the zero vector.

Proof. Note that there exists a matrix P such that $A=P^{-1}DP$ where $D=J_{\lambda_1,i_1}\bigoplus J_{\lambda_2,i_2-i_1}\bigoplus\cdots\bigoplus J_{\lambda_\ell,i_\ell-i_{\ell-1}}$ ($i_\ell=N$). Thus, equation (4.5.1) becomes

$$(P^{-1}D^nP)\vec{v} = C\vec{v} + \sum_{i=1}^{r} p^{n_i\delta_i} B_i\vec{v},$$

which is equivalent to

$$D^nP\vec{v} = (PC)\vec{v} + \sum_{i=1}^{r} p^{n_i\delta_i}(PB_i)\vec{v}. \tag{4.5.2}$$

Now suppose for the sake of contradiction that $\vec{v}$ is nonzero. This implies that $P\vec{v}$ is nonzero (since P is invertible). Let j be the first nonzero coordinate of $P\vec{v}$ from the right. Let $i_0 = 0$ and suppose that $i_{s-1} < j \le i_s$ for some $1 \le s \le \ell$. Comparing the j-th coordinate of both sides of equation (4.5.2) we get that there exist $a, c_1, \dots, c_r \in \overline{\mathbb{Q}}$ with $a \neq 0$ such that

$$a\lambda_s^n = c_1p^{n_1\delta_1} + \cdots + c_rp^{n_r\delta_r}.$$

for each $n \in S$. This fact contradicts [CGSZ21, Theorem 5.1 (A)]; therefore, $\vec{v}$ must indeed be the zero vector, as claimed in Lemma 4.5. □

Proof of Theorem 2.5. Let $\Gamma \subset \mathbb{G}_m(K)$ be a finitely generated subgroup. We first prove a useful reduction.

Lemma 4.6. *It suffices to prove Theorem 2.5 after replacing Φ by an iterate Φ^ℓ (for some $\ell \in \mathbb{N}$).*

Proof of Lemma 4.6. So, assume conditions (i)-(ii) are satisfied for the starting point $\vec{\alpha}$ (with respect to Γ) and for the endomorphism Φ^ℓ (for some given $\ell \in \mathbb{N}$). We claim that $\vec{\alpha}$ will also satisfy conditions (i)-(ii) in Theorem 2.5 for the endomorphism Φ. Clearly, condition (i) is unaffected since it refers strictly about the coordinates of the given starting point $\vec{\alpha}$. Now, in order to check condition (ii), we let $S \subseteq \mathbb{N}_0$ be an infinite subset and we want to prove that

$$U_S := \{\Phi^n(\vec{\alpha})\colon n \in S\} \tag{4.6.1}$$

is Zariski dense in $\mathbb{G}_m^N$. In particular, there exists $i_0 \in \{0, \dots, \ell-1\}$ such that the set

$$S_{i_0} := \{n \in S\colon n \equiv i_0 \pmod{\ell}\}$$

is an infinite subset. Since condition (ii) is verified by $\left(\Phi^\ell, \vec{\alpha}\right)$, then the set

$$U_{S,i_0} := \left\{\Phi^{n-i_0}(\vec{\alpha})\colon n \in S_{i_0}\right\} \tag{4.6.2}$$

must be Zariski dense in $\mathbb{G}_m^N$. Because Φ^{i_0} is a dominant group endomorphism, then also $\Phi^{i_0}(U_{S,i_0}) \subseteq U_S$ (see (4.6.1) and (4.6.2)) is Zariski dense in $\mathbb{G}_m^N$, as desired in the conclusion of Lemma 4.6. □

Using Lemma 4.6 (and therefore after replacing Φ by a suitable iterate), we may assume that the matrix A corresponding to the endomorphism Φ

has the property that for each of its eigenvalues λ, if λ is multiplicatively dependent with respect to p, then actually,

$$\lambda = p^m \text{ for some } m \in \mathbb{N}. \tag{4.6.3}$$

Note that the exponent m from (4.6.3) can be chosen indeed to be a *positive* integer since $m = 0$ would lead to A having eigenvalues root of unity (which is not allowed by hypothesis (1) in Theorem 2.5), while a *negative* integer would mean that λ from (4.6.3) would not be an algebraic integer (which contradicts the fact that λ is an eigenvalue of a matrix with integer entries).

We let $g \in \mathbb{Z}[x]$ be the minimal polynomial for the endomorphism Φ. We let $h_1(x)$ be the polynomial with integer coefficients, which is a factor of $g(x)$ having all the roots (with corresponding multiplicities) of $g(x)$ which are of the form (4.6.3). Then we can write $g(x) := h_1(x) \cdot h_2(x)$, where also the polynomial $h_2(x)$ has integer coefficients. Furthermore, $h_1(x)$ and $h_2(x)$ are coprime polynomials. We let $G_1 = h_1(\Phi)\big(\mathbb{G}_m^N\big)$ and $G_2 = h_2(\Phi)\big(\mathbb{G}_m^N\big)$. Then G_1 and G_2 are both connected algebraic subgroups of $\mathbb{G}_m^N$. Since h_1 and h_2 are coprime, then there exist polynomials with integer coefficients Q_1 and Q_2 along with some positive integer ℓ_0 such that

$$Q_1(x) \cdot h_1(x) + Q_2(x) \cdot h_2(x) = \ell_0,$$

which means that G_1 and G_2 are complementary subtori of $\mathbb{G}_m^N$, in the sense that $G_1 \simeq \mathbb{G}_m^k$ and $G_2 \simeq \mathbb{G}_m^{N-k}$, for some integer $k \in \{0, \dots, N\}$ and moreover, $\mathbb{G}_m^N = G_1 \cdot G_2$, while $G_1 \cap G_2$ is finite (consisting only of points of order dividing ℓ_0). Furthermore, Φ induces endomorphisms of both G_1 and G_2; call them Φ_1, respectively Φ_2. In addition, the minimal polynomial of Φ_1 is $h_2(x)$, while the minimal polynomial of Φ_2 is $h_1(x)$. Also, if we let $\iota : G_1 \times G_2 \longrightarrow \mathbb{G}_m^N$ be the map given by $(x_1, x_2) \mapsto x_1 \cdot x_2$ (note that G_1 and G_2 are subgroups of $\mathbb{G}_m^N$), then the following diagram commutes

$$\begin{array}{ccc} G_1 \times G_2 & \xrightarrow{(\Phi_1,\Phi_2)} & G_1 \times G_2 \\ \big\downarrow \iota & & \big\downarrow \iota \\ \mathbb{G}_m^N & \xrightarrow{\quad\Phi\quad} & \mathbb{G}_m^N. \end{array} \tag{4.6.4}$$

Note that ι is a finite morphism of degree ℓ_0. We now prove the following lemma.

Lemma 4.7. *It suffices to prove the conclusion of Theorem 2.5 for the action of $\Psi := (\Phi_1, \Phi_2)$ on $G_1 \times G_2$.*

Proof of Lemma 4.7. For a given finitely generated subgroup $\Gamma \subset \mathbb{G}_m(K)$, we let $\tilde{\Gamma} := \iota^{-1}(\Gamma^N)$ and then let $\Gamma_1 \subset \mathbb{G}_m(K)$ be the finitely generated subgroup spanned by the projections of $\tilde{\Gamma}$ onto each coordinate of $G_1 \times G_2 \xrightarrow{\sim} \mathbb{G}_m^N$.

Assume there exists a point $(\vec{x}_1, \vec{x}_2) \in (G_1 \times G_2)(K)$ satisfying the conclusions (i)-(ii) of Theorem 2.5 with respect to (Φ_1, Φ_2) and the subgroup

Γ_1. We claim that $\vec{x} := \iota(\vec{x}_1, \vec{x}_2) \in \mathbb{G}_m^N(K)$ satisfies the conclusions (i)-(ii) of Theorem 2.5 with respect to the endomorphism Φ and the subgroup Γ.

Indeed, first of all, condition (i) is satisfied by $\vec{x}$ with respect to the subgroup Γ since the same condition is satisfied by $(\vec{x}_1, \vec{x}_2)$ and subgroup Γ_1. As for condition (ii) in Theorem 2.5, we let $S \subseteq \mathbb{N}_0$ be an infinite subset. Since by our hypothesis, the set

$$\{(\Phi_1, \Phi_2)^n (\vec{x}_1, \vec{x}_2) : n \in S\}$$

is Zariski dense in $G_1 \times G_2$, then its image under ι will be Zariski dense in $\mathbb{G}_m^N$, thus proving the desired condition (ii) for Φ, as claimed in Lemma 4.7. $\square$

Now, $G_1 \times G_2$ is itself isomorphic to $\mathbb{G}_m^k \times \mathbb{G}_m^{N-k}$; our argument thus far has been similar to the proof of our Proposition 3.2 in order to justify that we can work with a dominant group endomorphism $\Psi = (\Phi_1, \Phi_2)$ where $\Phi_1 : \mathbb{G}_m^k \to \mathbb{G}_m^k$ and $\Phi_2 : \mathbb{G}_m^{N-k} \to \mathbb{G}_m^{N-k}$ given by $\vec{x}_1 \mapsto \vec{x}_1^{A_1}$ and respectively, $\vec{x}_2 \mapsto \vec{x}_2^{A_2}$. Moreover, the minimal polynomials of A_1 and A_2 are $h_2(x)$ and $h_1(x)$, respectively.

We pick a starting point $(\vec{x}_1, \vec{x}_2)$ for the action of Ψ on $\mathbb{G}_m^k(K) \times \mathbb{G}_m^{N-k}(K)$ of the following form:

- $\vec{x}_2 \in \mathbb{G}_m^{N-k}(K)$ satisfies both conditions (i)-(ii) from the conclusion of Proposition 4.1 with respect to the finitely generated subgroup $\Gamma \subset \mathbb{G}_m(K)$ (note that because of Condition (2) in the hypothesis of Theorem 2.5, Condition (2) in Proposition 4.1 must hold); and
- $\vec{x}_1$ has its k coordinates multiplicatively independent among themselves and also, the subgroup of $\mathbb{G}_m(K)$ generated by the coordinates of $\vec{x}_1$ has trivial intersection with the subgroup spanned by Γ and the coordinates of $\vec{x}_2$.

Let

$$\Lambda := \left\{ \left(\vec{x}_1^{E_1}, \vec{x}_2^{E_2} \right) : E_1 \in M_{N_1,N_1}(\mathbb{Z}) \text{ and } E_2 \in M_{N_2,N_2}(\mathbb{Z}) \right\}; \tag{4.7.1}$$

then Λ is finitely generated and all the points in $\mathcal{O}_\Psi(\vec{x}_1, \vec{x}_2)$ lie in Λ.

We let $S \subseteq \mathbb{N}_0$ be an arbitrary infinite subset; we will prove that the set

$$U := \{\Psi^n(\vec{x}_1, \vec{x}_2) : n \in S\}$$

must be Zariski dense in $\mathbb{G}_m^N$. If U is not Zariski dense, then we let $V \subset \mathbb{G}_m^N$ be its Zariski closure. Using Theorem 2.1, there must exist a set of the form (3.10.5) containing infinitely many elements of U. So, at the expense of replacing S by a still infinite subset (and thus replacing the set U with its corresponding infinite subset), we may assume without loss of generality that there exists a set

$$\mathcal{F} := \vec{\lambda} \cdot S(\vec{\eta}_1, \dots, \vec{\eta}_r; \delta_1, \dots, \delta_r) \cdot H, \tag{4.7.2}$$

containing U. Now, regarding the set $\mathcal{F}$, just as before, there exists a positive integer m such that

$$\vec{\lambda}^m, \vec{\eta}_1^m, \dots, \vec{\eta}_r^m \in \Lambda, \tag{4.7.3}$$

while the δ_j's are positive integers and H is a subgroup of Λ.

Since we assumed that U is not Zariski dense in $\mathbb{G}_m^N$, then V is a proper subvariety of $\mathbb{G}_m^N$ and in particular, the Zariski closure of H must be a proper algebraic subgroup of $\mathbb{G}_m^N$; so, there exists a nonzero $\vec{v} \in \mathbb{Z}^n$ with the property that

$$\text{for each } \vec{h} \in H \text{ we have } \left(\vec{h}\right)^{\vec{v}} = 1. \tag{4.7.4}$$

We write $\vec{v} = (\vec{v}_1, \vec{v}_2) \in \mathbb{Z}^{N_1} \times \mathbb{Z}^{N_2}$. Using (4.7.3), there exist matrices $B_i \in M_{N_1,N_1}(\mathbb{Z})$ and $C_i \in M_{N_2,N_2}(\mathbb{Z})$ (for $i = 1, \dots, r$) along with matrices $D_j \in M_{N_j,N_j}(\mathbb{Z})$ for $j = 1, 2$ such that

$$\vec{\eta}_i^m = \left(\vec{x}_1^{B_i}, \vec{x}_2^{C_i}\right) \text{ for each } i = 1, \dots, r \text{ and } \vec{\lambda}^m = \left(\vec{x}_1^{D_1}, \vec{x}_2^{D_2}\right).$$

So, for each $n \in S$, using that $\Psi^n(\vec{x}_1, \vec{x}_2) \in \mathcal{F}$, we must have some some nonnnegative integers n_i (for $i = 1, \dots, r$) such that

$$\begin{aligned} \Psi^n(\vec{x}_1, \vec{x}_2)^{m\vec{v}} &= (\Phi_1^n(\vec{x}_1), \Phi_2^n(\vec{x}_2))^{m\vec{v}} \\ &= \Phi_1^n(x_1)^{m\vec{v}_1} \cdot \Phi_2^n(x_2)^{m\vec{v}_2} \\ &= \vec{x}_1^{m(A_1^n)^t\vec{v}_1} \cdot \vec{x}_2^{m(A_2^n)^t\vec{v}_2} \\ &= \vec{x}_1^{D_1^t\vec{v}_1 + \sum_{i=1}^r p^{n_i\delta_i} B_i^t\vec{v}_1} \cdot \vec{x}_2^{D_2^t\vec{v}_2 + \sum_{i=1}^r p^{n_i\delta_i} C_i^t\vec{v}_2}, \end{aligned} \tag{4.7.5}$$

where in (4.7.5) we also used (4.7.4). Since the coordinates of $\vec{x}_1$ are multiplicatively independent among themselves, and also multiplicatively independent with respect to the coordinates of $\vec{x}_2$ we must have that

$$m(A_1^n)^t\vec{v}_1 = D_1^t\vec{v}_1 + \sum_{i=1}^r p^{n_i\delta_i} B_i^t\vec{v}_1, \tag{4.7.6}$$

for every $n \in S$. Hence, since none of the eigenvalues of A_1 are multiplicatively dependent with respect to p, Lemma 4.5 yields that we must have $\vec{v}_1 = 0$. So, this means that for any vector $\vec{v} = \vec{v}_1 \oplus \vec{v}_2 \in \mathbb{Z}^N$ with the property that $\left(\vec{h}\right)^{\vec{v}} = 1$ for each point $\vec{h}$ in the Zariski closure $\overline{H}$ of H inside $G_1 \oplus G_2$, we must have that $\vec{v}_1$ is the zero vector in $\mathbb{Z}^{N_1}$. Therefore, $\overline{H}$ is an algebraic group of the form $G_1 \oplus \overline{H}_2$ for some algebraic subgroup $\overline{H}_2 \subseteq G_2$.

So, the Zariski closure W of the set $\mathcal{F}$ (which is itself contained in the Zariski closure of the set U) must be of the form $G_1 \oplus W_2$ for some subvariety $W_2 \subseteq G_2$ because $G_1 \oplus \vec{1}_{G_2}$ is contained in the stabilizer of W. However, W_2 contains all the points $\Phi_2^n(x_2)$ for $n \in S$. Then using the fact that S is an infinite subset of $\mathbb{N}_0$ along with Proposition 4.1, we conclude that W_2 must be the entire G_2. So, actually W must be the entire $G_1 \oplus G_2 = \mathbb{G}_m^N$, which means that any infinite subset of the orbit of $(\vec{x}_1, \vec{x}_2)$ under (Φ_1, Φ_2) must be Zariski dense in $G_1 \times G_2$.

This concludes our proof of Theorem 2.5. $\square$

Proof of Theorem 2.6. As noted before (see [AC08, MS14, BGR17]) we have that (i)⇒(ii).

Our strategy for proving that (ii)⇒(iii) is to assume that condition (iii) does not hold and then prove the existence of a point with a Zariski dense orbit.

First note that there exists a suitable power A^{n_0} of A (for some $n_0 \geq 1$) such that each eigenvalue of A^{n_0} is either equal to 1 or it is not a root of unity and each eigenvalue of A^{n_0} which is multiplicatively dependent with respect to p is actually of the form p^m for some $m \in \mathbb{N}_0$. Next we prove that condition (iii) from Theorem 2.6 is not changed when replacing Φ by Φ^{n_0}.

Lemma 4.8. *Let $n_0 \in \mathbb{N}$. If condition (iii) from Theorem 2.6 is not met for the regular self-map $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$, then condition (iii) is also not met for $\Phi^{n_0} : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$.*

Proof of Lemma 4.8. When we replace Φ by Φ^{n_0}, then we replace A by A^{n_0} and also, replace $\vec{\beta}$ by

$$\vec{\beta}_1 := \vec{\beta}^{\sum_{j=0}^{n_0-1} A^j}. \tag{4.8.1}$$

Now, we assume there exists a nonzero vector $\vec{v} \in \mathbb{Z}^N$ such that condition (iii) is met for Φ^{n_0}, i.e., for some $\ell \in \mathbb{N}$ we have:

$$\left(A^{n_0\ell}\right)^t \vec{v} = \vec{v} \text{ and } \left(\vec{\beta}_1\right)^{\left(\sum_{j=0}^{\ell-1} A^{n_0 j}\right)^t \vec{v}} = 1. \tag{4.8.2}$$

But then using (4.8.1), we see that

$$1 = \vec{\beta}_1^{\left(\sum_{j=0}^{\ell-1} A^{n_0 j}\right)^t \cdot \vec{v}} = \vec{\beta}^{\left(\sum_{j=0}^{n_0\ell-1} A^j\right)^t \cdot \vec{v}},$$

thus proving (in connection with (4.8.2)) that condition (iii) would be met for Φ, contradiction. This concludes our proof of Lemma 4.8. ☐

Lemma 4.8 allows us to replace Φ by Φ^{n_0} and therefore, it suffices to find a point $\vec{\alpha} \in \mathbb{G}_m^N(K)$ with a Zariski denese orbit under Φ^{n_0}; note that then also $\mathcal{O}_\Phi(\vec{\alpha})$ would be Zariski dense in $\mathbb{G}_m^N$. So, from now on, we work under the hypothesis that

- each eigenvalue of the matrix A corresponding to the group endomorphism φ (where $\Phi = \tau_{\vec{\beta}} \circ \varphi$) is either equal to 1 or it is not a root of unity; and
- each eigenvalue of A which is multiplicatively dependent with respect to p is actually of the form p^m for some $m \in \mathbb{N}_0$.

This hypothesis yields that there exists an invertible matrix P with rational entries such that

$$P^{-1}AP = J_{1,i_1} \bigoplus J_{1,i_2-i_1} \bigoplus \cdots \bigoplus J_{1,i_s-i_{s-1}} \bigoplus D, \tag{4.8.3}$$

where D is an invertible matrix satisfying the following properties:

- no eigenvalue of D is a root of unity;

- there does not exist $k := \mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ Jordan blocks in the Jordan canonical form of A corresponding to eigenvalues $\lambda_1, \dots, \lambda_k$ satisfying the equation $\lambda_1^m = \cdots = \lambda_k^m = p^r$ for some positive integers m and r.

We write $N_1 := i_s$ and $N_2 := N - N_1$; so, N_1 is the dimension of the unipotent matrix B appearing on the right-hand side of equation (4.8.3), while N_2 is the dimension of the matrix D. According to Proposition 3.2 (especially, see Lemma 3.5), it suffices to prove the existence of a point with a Zariski dense orbit under the assumption that the matrix A actually has the form from the right-hand side of (4.8.3). Furthermore, condition (iii) is unchanged when we replace A by another matrix with integer entries of the form $P^{-1}AP$ where P and P^{-1} have rational entries. Indeed, one can choose some positive integer m such that both mP and mP^{-1} have integer entries and then condition (iii) holds for the triple $\left(A, \vec{\beta}, \vec{v}\right)$ (and $\ell \in \mathbb{N}$) if and only if condition (iii) holds for $\left(P^{-1}AP, \vec{\beta}^{m(P^{-1})^t}, mP^t\vec{v}\right)$ (and the same integer ℓ).

So, from now on, we may assume that the matrix A corresponding to the endomorphism φ is itself equal to $B \bigoplus D$.

For any vector $\vec{x} = (x_1, \dots, x_N) \in \mathbb{G}_m^N(K)$, we let $\vec{x}_B$ and $\vec{x}_D$ denote $(x_1, \dots, x_{N_1})$ and $(x_{N_1+1}, \dots, x_N)$ respectively. Since D has no eigenvalues that are equal to a root of unity we can choose a vector $\vec{\gamma}_D \in \mathbb{G}_m^{N_2}(K)$ such that $\left(\vec{\gamma}_D\right)^{D-\mathrm{id}_{N_2}} = \vec{\beta}_D$. Also, there is a vector $\vec{\gamma}_B$ such that

$$\left(\vec{\gamma}_B\right)^{B-\mathrm{id}_{N_1}} = \left(\beta_1, \dots, \beta_{i_1-1}, 1, \beta_{i_1+1}, \dots, \beta_{i_s-1}, 1\right).$$

Let $\vec{\gamma} := \vec{\gamma}_B \oplus \vec{\gamma}_D \in \mathbb{G}_m^N$. It is easy to check that the map $\tau_{\vec{\gamma}} \circ \Phi \circ \tau_{\vec{\gamma}}^{-1}$ is given by

$$\vec{x} \mapsto \vec{\beta}'_B \vec{x}^B \oplus \vec{x}^D, \text{ where } \vec{\beta}'_B := (1, \dots, 1, \beta_{i_1}, 1, \dots, 1, \beta_{i_s}) \in \mathbb{G}_m^{N_1}(K). \tag{4.8.4}$$

According to [GS19, Lemma 3.1], it suffices to prove that there exists a point with a Zariski dense orbit for the regular self-map on $\mathbb{G}_m^N$ given by (4.8.4). We also note that condition (iii) is unchanged when replacing A and $\vec{\beta}$ by A and $\vec{\beta}'$, where $\vec{\beta}' := \vec{\beta}'_B \oplus \vec{0}$. Indeed, for any positive integer ℓ, we have that

$$(A^\ell)^t \vec{v} = \vec{v} \text{ if and only if } (B^\ell)^t \vec{v}_B = \vec{v}_B \text{ and } \vec{v}_D = 0$$

since 1 is not an eigenvalue of D^ℓ. Moreover, every eigenvector of $(B^\ell)^t$ corresponding to 1 must be of the form

$$(0, \dots, 0, v_{i_1}, 0, \dots, 0, v_{i_s}). \tag{4.8.5}$$

So, for a vector as in (4.8.5), we have that

$$\left(\vec{\beta}\right)^{\left(\sum_{i=0}^{\ell-1} A^i\right)^t \vec{v}} = 1 \text{ if and only if } \left(\vec{\beta}'\right)^{\left(\sum_{i=0}^{\ell-1} A^i\right)^t \vec{v}} = 1.$$

Therefore, in the proof of the implication (ii)⇒(iii) we may assume from now on that $\vec{\beta}_B = \vec{\beta}'_B$ and $\vec{\beta}_D = (0, \dots, 0)$ (see (4.8.4)).

Since condition (iii) does not hold, in particular we have that

$$\beta_{i_1}, \dots, \beta_{i_s} \text{ are multiplicatively independent.} \tag{4.8.6}$$

Indeed, otherwise we would have some nonzero vector $\vec{w} \in \mathbb{Z}^s$ such that $\prod_{j=1}^{s} \beta_{i_j}^{w_j} = 1$ and so, letting

$$\vec{v} := (0, \dots, 0, w_1, 0, \dots, 0, w_2, 0, \dots, 0, w_s, 0, \dots, 0) \in \mathbb{Z}^N$$

be the vector whose only possibly nonzero entries are its i_j-th entries (for $j = 1, \dots, s$), we immediately see that

$$A^t \vec{v} = \vec{v} \text{ and } \vec{\beta}^{\vec{v}} = 1. \tag{4.8.7}$$

Thus showing that condition (iii) holds in this case. So, indeed, since we assumed that condition (iii) does not hold, then we must have that the β_{i_j}'s are multiplicatively independent (as claimed in (4.8.6)). We let

$$\vec{\alpha}_B := (\gamma_1, \dots, \gamma_{i_1-1}, 1, \gamma_{i_1+1}, \dots, \gamma_{i_2-1}, 1, \gamma_{i_2+1}, \dots, \gamma_{i_s-1}, 1) \in \mathbb{G}_m^{N_1}(K)$$

where the γ_j's are multiplicatively independent and also multiplicatively independent with respect to the β_{i_j}'s.

Since the eigenvalues of D satisfy the hypotheses of Theorem 2.5, then we can find $\vec{\alpha}_D \in \mathbb{G}_m^{N_2}(K)$ which satisfies conditions (i)-(ii) from the conclusion of Theorem 2.5 with respect to the subgroup Γ of $\mathbb{G}_m^K$ spanned by all the β_{i_j}'s and all the γ_j's. In particular, this means that writing $\vec{\alpha}_D := (\alpha_1, \dots, \alpha_{N_2})$, we have that the α_j's (along with the β_{i_j}'s and the γ_j's) satisfy the hypotheses of Proposition 3.12. Hence, the orbit of $\vec{\alpha} := \vec{\alpha}_B \oplus \vec{\alpha}_D \in \mathbb{G}_m^N(K)$ under Φ must be Zariski dense in $\mathbb{G}_m^N$, as claimed. This concludes our proof of Theorem 2.6. □

As noted in Section 2, Theorem 1.5 is a consequence of Theorem 2.6.

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