POINTS OF SMALL HEIGHT ON VARIETIES DEFINED OVER A FUNCTION FIELD

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ABSTRACT. We obtain a Bogomolov type of result for the affine space defined over the algebraic closure of a function field of transcendence degree 1 over a finite field.

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1. Introduction

The Manin-Mumford conjecture, proved by Raynaud [8], asserts that if an irreducible subvariety X of an abelian variety A defined over a number field contains a Zariski dense subset of torsion points of A, then X is a translate of an algebraic subgroup of A by a torsion point. We describe next the Bogomolov conjecture, which is a generalization of the Manin-Mumford conjecture.

Let A be an abelian variety defined over a number field K. We fix an algebraic closure K^{alg} for K and we let $\widehat{\mathrm{h}} : A(K^{\mathrm{alg}}) \to \mathbb{R}_{\geq 0}$ be the Néron height associated to a symmetric, ample line bundle on A. Let X be an irreducible subvariety of A. For each $n \geq 1$, we let

$$X_n = \left\{ x \in X(K^{\mathrm{alg}}) \mid \widehat{\mathrm{h}}(x) < \frac{1}{n} \right\}. \tag{1}$$

The Bogomolov conjecture, which was proved in a special case by Ullmo [10] and in the general case by Zhang [12], asserts that if for every $n \geq 1$, X_n is Zariski dense in X, then X is the translate of an abelian subvariety of A by a torsion point of A. Both Ullmo and Zhang proved the Bogomolov conjecture via an equidistribution statement for points of small height on A. The characteristic 0 function field case of the Bogomolov conjecture was proved by Moriwaki [7], while a generalization of the Bogomolov statement to semi-abelian varieties was obtained by David and Philippon in [5].

The case of Bogomolov conjecture for any power $\mathbb{G}_m^n$ of the multiplicative group was first proved by Zhang in [11]. Other proofs of the Bogomolov conjecture for $\mathbb{G}_m^n$ were given by Bilu [1] and Bombieri and Zannier [2]. This last paper constituted our inspiration for proving here a version of the Bogomolov conjecture for the affine scheme defined over the algebraic closure of a function field of transcendence degree 1 over a finite field (see our Theorem 2.2).

The picture in positive characteristic for the Bogomolov conjecture is much different due to the varieties defined over finite fields. Indeed, if A is a semi-abelian variety defined over a finite field $\mathbb{F}_q$, then every subvariety X of A defined over a finite field contains a Zariski dense subset of torsion points (because $X(\mathbb{F}_q^{\mathrm{alg}}) \subset A(\mathbb{F}_q^{\mathrm{alg}}) = A_{\mathrm{tor}}$ is Zariski dense in X). Because all torsion points have canonical height 0, then each subvariety X defined over $\mathbb{F}_q^{\mathrm{alg}}$ constitutes a counterexample to the obvious translation in positive characteristic of the classical Bogomolov statement. Thus, it is not true in characteristic p that only translates of algebraic tori are accumulating subvarieties of $\mathbb{G}_m^n$ for points of small height. All subvarieties of $\mathbb{G}_m^n$ invariant under a power of the Frobenius are accumulating varieties for points of small height. The group structure of the ambient space $\mathbb{G}_m^n$ disappears from the conclusion of a Bogomolov statement for $\mathbb{G}_m^n$. This motivated our approach to Theorem 2.2 in which the ambient space is simply the affine space, and not an algebraic torus as in [2].

We note that Bosser [3] proved a Bogomolov statement for the additive group scheme in characteristic p under the action of a Drinfeld module of generic characteristic. His result is not yet published, but the main ingredient of his proof was published in [4]. The author formulated in [6] an equidistribution statement for points of small height for Drinfeld modules of generic characteristic (and we also proved in [6] a first instance of our equidistribution statement). Our equidistribution statement is similar with the ones proved by Ullmo [10] and Zhang [12] for abelian varieties. Finally, we note that our Theorem 2.2 can be interpreted as a Bogomolov type statement for Drinfeld modules defined over finite fields.

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2. Statement of our main result

In this section we state our main result Theorem 2.2, which we prove in Section 3.

For each finite extension K of $\mathbb{F}_p(t)$, we construct the usual set of valuations M_K and the associated local heights h_v on K. For the reader's convenience we sketch this classical construction (for more details, see Chapter 2 in [9]). Let $R := \mathbb{F}_p[t]$. For each irreducible polynomial $P \in R$ we let v_P be the valuation on $\mathbb{F}_p(t)$ given by $v_P\left(\frac{Q_1}{Q_2}\right) = \mathrm{ord}_P(Q_1) - \mathrm{ord}_P(Q_2)$ for every nonzero $Q_1, Q_2 \in R$, where $\mathrm{ord}_P(Q_i)$ is the order of the polynomial Q_i at P. Also, we construct the valuation v_∞ on $\mathbb{F}_p(t)$ given by $v_\infty\left(\frac{Q_1}{Q_2}\right) = \deg(Q_2) - \deg(Q_1)$ for every nonzero $Q_1, Q_2 \in R$. We let the *degree* of v_P be $d(v_P) = \deg(P)$ for every irreducible polynomial $P \in R$ and we also let $d(v_\infty) = 1$. Then, for every nonzero $x \in \mathbb{F}_p(t)$, we have the sum formula $\sum_{v \in M_{\mathbb{F}_p(t)}} d(v) \cdot v(x) = 0$.

Let K be a finite extension of $\mathbb{F}_p(t)$. We normalize each valuation w from M_K so that the range of w is the entire $\mathbb{Z}$. For $w \in M_K$, if $v \in M_{\mathbb{F}_p(t)}$ lies below w, then $e(w|v)$ represents the corresponding ramification index, while $f(w|v)$ represents the relative residue degree. Also, we define $d(w) = \frac{f(w|v)d(v)}{[K:\mathbb{F}_p(t)]}$. Let $x \in K$. We define the local height of x at w as $\mathrm{h}_w(x) = -d(w)\min\{w(x), 0\}$. Finally, we define the (global) height of x as $\mathrm{h}(x) = \sum_{w \in M_K} \mathrm{h}_w(x)$.

We extend the above heights to every affine space $\mathbb{A}^n$ defined over $\mathbb{F}_p(t)^{\mathrm{alg}}$. Let K be a finite extension of $\mathbb{F}_p(t)$ and let $P = (x_1, \dots, x_n) \in \mathbb{A}^n_K$. We define the local height of P at w as $\mathrm{h}_w(P) = \mathrm{h}_w(x_1, \dots, x_n) = \max_{i=1}^n \mathrm{h}_w(x_i)$. We define the (global) height of P as $\mathrm{h}(P) = \sum_{w \in M_K} \mathrm{h}_w(P)$.

The following proposition contains standard results on the Weil height h.

Proposition 2.1. *For every $P, Q \in \mathbb{A}^n_{\mathbb{F}_p(t)^{\mathrm{alg}}}$, the following statements are true:*

(i) $\mathrm{h}(P) = 0$ *if and only if* $P \in \mathbb{A}^n_{\mathbb{F}_p^{\mathrm{alg}}}$.

(ii) $\mathrm{h}(P + Q) \le \mathrm{h}(P) + \mathrm{h}(Q)$ *(triangle inequality). Moreover, if $x_1, x_2 \in \mathbb{F}_p(t)^{\mathrm{alg}}$, then* $\mathrm{h}(x_1 + x_2) \le \mathrm{h}(x_1, x_2)$.

Proof. The results of Proposition 2.1 are classical, possibly with the exception of the "moreover" part of *(ii)*. Hence we show next how to obtain that statement. For each place v, $v(x_1 + x_2) \ge \min\{v(x_1), v(x_2)\}$. Thus $\mathrm{h}_v(x_1 + x_2) \le \max\{\mathrm{h}_v(x_1), \mathrm{h}_v(x_2)\} = \mathrm{h}_v(x_1, x_2)$. Therefore $\mathrm{h}(x_1 + x_2) \le \mathrm{h}(x_1, x_2)$. ∎

The following theorem is our main result.

Theorem 2.2. *Let X be an affine subvariety of $\mathbb{A}^n$ defined over $\mathbb{F}_p(t)^{\mathrm{alg}}$. Let Y be the Zariski closure of the set $X(\mathbb{F}_p^{\mathrm{alg}})$, i.e. Y is the largest $\mathbb{F}_p^{\mathrm{alg}}$-subvariety of X.*

There exists a positive constant C, depending only on X, such that if $P \in X(\mathbb{F}_p(t)^{\mathrm{alg}})$ and $\mathrm{h}(P) < C$, then $P \in Y(\mathbb{F}_p(t)^{\mathrm{alg}})$.

Remark 2.3. The result of Theorem 2.2 extends to any closed projective subvariety X of a projective space $\mathbb{P}^n$. Indeed, we cover $\mathbb{P}^n$ by finitely many open affine spaces $\{U_i\}_i$, and then apply Theorem 2.2 to each $X \cap U_i$ (which is a closed subvariety of the affine space U_i).

3. Proof of our main result

Unless otherwise stated, all our subvarieties are closed. We start with a definition.

Definition 3.1. We call *reduced* a non-constant polynomial $f \in \mathbb{F}_p[t][X_1, \ldots, X_n]$, whose coefficients a_i have no non-constant common divisor in $\mathbb{F}_p[t]$. For each finite extension K of $\mathbb{F}_p(t)$, we define the *local height* $\mathrm{h}_w(f)$ of f at a place $w \in M_K$ as $\max_i \mathrm{h}_w(a_i)$. Then we define the (global) height $\mathrm{h}(f)$ of f as $\sum_{w \in M_K} \mathrm{h}_w(f)$. Note that our definition is independent of K, as $\mathrm{h}(f)$ equals the maximum of the degrees of the coefficients $a_i \in \mathbb{F}_p[t]$ of f.

Our proof of Theorem 2.2 goes through a series of lemmas.

Lemma 3.2. *Let $f \in \mathbb{F}_p[t][X_1, \ldots, X_n]$ be a reduced polynomial of total degree d. For every k such that $p^k \geq 2\,\mathrm{h}(f)$, if $(x_1, \ldots, x_n) \in \mathbb{A}^n_{\mathbb{F}_p(t)^{\mathrm{alg}}}$ satisfies $f(x_1, \ldots, x_n) = 0$, then either*

$$\mathrm{h}(x_1, \ldots, x_n) \geq \frac{1}{2d}$$

or

$$f(x_1^{p^k}, \ldots, x_n^{p^k}) = 0.$$

Proof. Let k satisfy the inequality from the statement of Lemma 3.2. Let $(x_1, \ldots, x_n) \in \mathbb{A}^n_{\mathbb{F}_p(t)^{\mathrm{alg}}}$ be a zero of f. We let $f = \sum_i a_i M_i$, where the a_i's are the nonzero coefficients of f and the M_i's are the corresponding monomials of f. For each i, we let $m_i := M_i(x_1, \ldots, x_n)$.

Assume $f(x_1^{p^k}, \ldots, x_n^{p^k}) \neq 0$.

We let $K = \mathbb{F}_p(t)(x_1, \ldots, x_n)$. If $\zeta = f(x_1^{p^k}, \ldots, x_n^{p^k})$, then (because $\zeta \neq 0$)

$$\sum_{w \in M_K} d(w) w(\zeta) = 0. \tag{2}$$

Because $f(x_1, \ldots, x_n) = 0$, we get $\zeta = \zeta - f(x_1, \ldots, x_n)^{p^k}$ and so,

$$\zeta = \sum_i (a_i - a_i^{p^k}) m_i^{p^k}. \tag{3}$$

Claim 3.3. For every $g \in \mathbb{F}_p[t]$, $\left(t^{p^k} - t\right) \mid \left(g^{p^k} - g\right)$.

Proof of Claim 3.3. Let $g := \sum_{j=0}^m b_j t^j$. Then $g^{p^k} = \sum_{j=0}^m b_j t^{jp^k}$. The proof of Claim 3.3 is immediate because for every $j \in \mathbb{N}$, $\left(t^{p^k} - t\right) \mid \left(t^{jp^k} - t^j\right)$. □

Using the result of Claim 3.3 and equation (3), we get

$$\zeta = (t^{p^k} - t) \sum_i b_i m_i^{p^k}, \tag{4}$$

where $b_i = \frac{a_i - a_i^{p^k}}{t^{p^k} - t} \in \mathbb{F}_p[t]$. Let S be the set of valuations $w \in M_K$ such that w lies above an irreducible factor (in $\mathbb{F}_p[t]$) of $t^{p^k} - t$. For each $w \in S$,

$$d(w) \cdot w(\zeta) \geq d(w) \cdot w(t^{p^k} - t) - dp^k\,\mathrm{h}_w(x_1, \ldots, x_n), \tag{5}$$

because for each i, $w(b_i) \geq 0$ (as $b_i \in \mathbb{F}_p[t]$ and w does not lie over v_∞) and also,

$$d(w) \cdot w(m_i^{p^k}) \geq -dp^k \operatorname{h}_w(x_1, \ldots, x_n),$$

as the total degree of M_i is at most d.

For each $w \in M_K \setminus S$, because $\zeta = \sum_i a_i m_i^{p^k}$,

$$d(w) \cdot w(\zeta) \geq -\operatorname{h}_w(f) - dp^k \operatorname{h}_w(x_1, \ldots, x_n). \tag{6}$$

Adding all inequalities from (5) and (6) we obtain

$$0 = \sum_{w \in M_K} d(w) \cdot w(\zeta) \geq -\operatorname{h}(f) - dp^k \operatorname{h}(x_1, \ldots, x_n) + \sum_{\substack{w \in M_K \\ w(t^{p^k} - t) > 0}} d(w) \cdot w(t^{p^k} - t). \tag{7}$$

By the coherence of the valuations on $\mathbb{F}_p(t)^{\mathrm{alg}}$,

$$\sum_{\substack{w \in M_K \\ w(t^{p^k} - t) > 0}} d(w) \cdot w(t^{p^k} - t) = \sum_{\substack{v \in M_{\mathbb{F}_p(t)} \\ v(t^{p^k} - t) > 0}} d(v) \cdot v(t^{p^k} - t) = -v_\infty(t^{p^k} - t) = p^k.$$

Thus, inequality (7) yields

$$0 \geq -\operatorname{h}(f) - dp^k \operatorname{h}(x_1, \ldots, x_n) + p^k$$

and so,

$$\operatorname{h}(x_1, \ldots, x_n) \geq \frac{1}{d} - \frac{\operatorname{h}(f)}{dp^k}. \tag{8}$$

Because k was chosen such that $p^k \geq 2\operatorname{h}(f)$, we conclude

$$\operatorname{h}(x_1, \ldots, x_n) \geq \frac{1}{2d}. \tag{9}$$

□

Lemma 3.4. *Let k be a positive integer. Let K be a finite field extension of $\mathbb{F}_p(t)$ and let $f \in K[X_1, \ldots, X_n]$ be an irreducible polynomial. If $f(X_1, \ldots, X_n) \mid f(X_1^{p^k}, \ldots, X_n^{p^k})$, then there exists $a \in K \setminus \{0\}$ such that $af \in \mathbb{F}_{p^k}[X_1, \ldots, X_n]$.*

Proof. Let Z be the zero set for f. Let F be the Frobenius on $\mathbb{F}_p$. The hypothesis on f shows that for every $P \in Z(K^{\mathrm{alg}})$, $F^k P \in Z(K^{\mathrm{alg}})$. Hence $F^k Z \subset Z$. Because Z is irreducible (as f is irreducible) and $\dim(F^k Z) = \dim(Z)$, we conclude $F^k Z = Z$. Therefore Z is defined over the fixed field $\mathbb{F}_{p^k}$ of F^k. Moreover, Z is defined over $\mathbb{F}_{p^k} \cap K$. Thus there exists a polynomial $g \in \mathbb{F}_{p^k}[X_1, \ldots, X_n]$ such that $g = a \cdot f$, for some nonzero $a \in K$. □

Lemma 3.5. *Let $X \subset \mathbb{A}^n$ be an affine variety of dimension less than n defined over $\mathbb{F}_p(t)^{\mathrm{alg}}$. There exists a positive constant C, depending only on X, and there exists an affine $\mathbb{F}_p^{\mathrm{alg}}$-variety $Z \subset \mathbb{A}^n$ of dimension less than n, which also depends only on X, such that for every $P \in X(\mathbb{F}_p(t)^{\mathrm{alg}})$, either $P \in Z(\mathbb{F}_p(t)^{\mathrm{alg}})$ or $\operatorname{h}(P) \geq C$.*

Remark 3.6. The only difference between Lemma 3.5 and Theorem 2.2 is that we do not require Z be contained in X.

Proof of Lemma 3.5. Let K be the smallest field extension of $\mathbb{F}_p(t)$ such that X is defined over K. Let p^m be the inseparable degree of the extension $K/\mathbb{F}_p(t)$ ($m \geq 0$). Let

$$X_1 = \bigcup_{\sigma} X^{\sigma},$$

where σ denotes any field morphism $K \to \mathbb{F}_p(t)^{\mathrm{alg}}$ over $\mathbb{F}_p(t)$. The variety X_1 is an $\mathbb{F}_p(t)^{1/p^m}$-variety. Also, X_1 depends only on X. Thus, if we prove Lemma 3.5 for X_1, then our result will hold also for $X \subset X_1$. Hence we may and do assume that X is defined over $\mathbb{F}_p(t)^{1/p^m}$.

We let F be the Frobenius on $\mathbb{F}_p$. The variety $X' = F^m X$ is an $\mathbb{F}_p(t)$-variety, which depends only on X. Assume we proved Lemma 3.5 for X' and let C' and Z' be the positive constant and the $\mathbb{F}_p^{\mathrm{alg}}$-variety, respectively, associated to X', as in the conclusion of Lemma 3.5. Let $P \in X(\mathbb{F}_p(t)^{\mathrm{alg}})$. Then $P' := F^m(P) \in X'(\mathbb{F}_p(t)^{\mathrm{alg}})$. Thus, either

$$\mathrm{h}(P') \geq C' \text{ or}$$

$$P' \in Z'(\mathbb{F}_p(t)^{\mathrm{alg}}).$$

In the former case, because $\mathrm{h}(P) = \frac{1}{p^m}\,\mathrm{h}(P')$, we obtain a lower bound for the height of P, depending only on X (note that m depends only on X). In the latter case, if we let Z be the $\mathbb{F}_p^{\mathrm{alg}}$-subvariety of $\mathbb{A}^n$, obtained by extracting the p^m-roots of the coefficients of a set of polynomials (defined over $\mathbb{F}_p^{\mathrm{alg}}$) which generate the vanishing ideal for Z', we get $P \in Z(\mathbb{F}_p(t)^{\mathrm{alg}})$. By its construction, Z depends only on X and so, we obtain the conclusion of Lemma 3.5.

Thus, from now on in this proof, we assume X is an $\mathbb{F}_p(t)$-variety. We proceed by induction on n.

The case $n = 1$ is obvious, because any subvariety of $\mathbb{A}^1$, different from $\mathbb{A}^1$, is a finite union of points. Thus we may take $Z = X(\mathbb{F}_p^{\mathrm{alg}})$, (which is also a finite union of points) and $C := \min_{P \in (X\setminus Z)(\mathbb{F}_p(t)^{\mathrm{alg}})} \mathrm{h}(P)$. By construction, $C > 0$ (there are finitely many points in $(X \setminus Z)\,(\mathbb{F}_p(t)^{\mathrm{alg}})$ and they all have positive height by Proposition 2.1 (i)). If there are no points in $X(\mathbb{F}_p(t)^{\mathrm{alg}}) \setminus X(\mathbb{F}_p^{\mathrm{alg}})$, then we may take $C = 1$, say.

Remark 3.7. The above argument proves the case $n = 1$ for Theorem 2.2, because the variety Z that we chose is a subvariety of X.

We assume Lemma 3.5 holds for $n - 1$ and we prove it for n ($n \geq 2$). We fix a set of defining polynomials for X which contains polynomials $P_i \in \mathbb{F}_p[t][X_1, \ldots, X_n]$ for which

$$\max_i \deg(P_i)$$

is minimum among all possible sets of defining polynomials for X (where $\deg P_i$ is the total degree of P_i). We may assume all of the polynomials we chose are reduced. If all of them have coefficients from a finite field, i.e. $\mathbb{F}_p$, then Lemma 3.5 holds with $Z = X$ and C any positive constant.

Assume there exists a reduced polynomial $f \notin \mathbb{F}_p[X_1, \ldots, X_n]$ in the fixed set of defining equations for X. Let $\{f_i\}_i$ be the set of all the $\mathbb{F}_p(t)$-irreducible factors of f. For each i let H_i be the zero set of f_i. Then X is contained in the finite union $\cup_i H_i$. The polynomials f_i depend only on f. Thus it suffices to prove Lemma 3.5 for each H_i. Hence we may and do assume X is the zero set of a reduced $\mathbb{F}_p(t)$-irreducible polynomial $f \notin \mathbb{F}_p[X_1, \ldots, X_n]$.

Let $P = (x_1, \dots, x_n) \in X(\mathbb{F}_p(t)^{\text{alg}})$. We apply Lemma 3.2 to f and P and conclude that either

$$\text{h}(P) \geq \frac{1}{2\deg(f)} \tag{10}$$

or there exists k depending only on $\text{h}(f)$ such that

$$f(x_1^{p^k}, \dots, x_n^{p^k}) = 0. \tag{11}$$

If (10) holds, then we obtained a good lower bound for the height of P (depending only on the degree of f).

Assume (11) holds. Because f is an irreducible and reduced polynomial, whose coefficients are not all in $\mathbb{F}_p$, Lemma 3.4 yields that $f(X_1, \dots, X_n)$ cannot divide $f(X_1^{p^k}, \dots, X_n^{p^k})$. We know f has more than one monomial because it is reduced and not all of its coefficients are in $\mathbb{F}_p$. Without loss of generality, we may assume f has positive degree in X_n. Because f is irreducible, the resultant R of the polynomials $f(X_1, \dots, X_n)$ and $f(X_1^{p^k}, \dots, X_n^{p^k})$ with respect to the variable X_n is nonzero. Moreover, R depends only on f (we recall that k depends only on $\text{h}(f)$).

The nonzero polynomial $R \in \mathbb{F}_p(t)[X_1, \dots, X_{n-1}]$ vanishes on $(x_1, \dots, x_{n-1})$. Applying the induction hypothesis to the hypersurface $R = 0$ in $\mathbb{A}^{n-1}$, we conclude there exists an $\mathbb{F}_p^{\text{alg}}$-variety Z, strictly contained in $\mathbb{A}^{n-1}$, depending only on R (and so, only on X) and there exists a positive constant C, depending only on R (and so, only on X) such that either

$$\text{h}(x_1, \dots, x_{n-1}) \geq C \text{ or} \tag{12}$$

$$(x_1, \dots, x_{n-1}) \in Z(\mathbb{F}_p(t)^{\text{alg}}). \tag{13}$$

If (12) holds, then $\text{h}(x_1, \dots, x_{n-1}, x_n) \geq \text{h}(x_1, \dots, x_{n-1}) \geq C$ and we have a height inequality as in the conclusion of Lemma 3.5. If (13) holds, then $(x_1, \dots, x_n) \in (Z \times \mathbb{A}^1)(\mathbb{F}_p(t)^{\text{alg}})$ and $Z \times \mathbb{A}^1$ is an $\mathbb{F}_p^{\text{alg}}$-variety, strictly contained in $\mathbb{A}^n$, as desired in Lemma 3.5. This proves the inductive step and concludes the proof of Lemma 3.5. □

The following result is an immediate corollary of Lemma 3.5.

Corollary 3.8. *Let X be a proper subvariety of $\mathbb{A}^n$ defined over $\mathbb{F}_p(t)^{\text{alg}}$. There exists a positive constant C and a proper subvariety $Z \subset \mathbb{A}^n$ defined over $\mathbb{F}_p^{\text{alg}}$, such that the pair (C, Z) satisfies the conclusion of Lemma 3.5, and moreover Z is minimal with this property (with respect to the inclusion of subvarieties of $\mathbb{A}^n$).*

Proof. Let (C_1, Z_1) and (C_2, Z_2) be two pairs of a positive constant and a proper subvariety of $\mathbb{A}^n$ defined over $\mathbb{F}_p^{\text{alg}}$, such that both pairs satisfy the conclusion of Lemma 3.5. Clearly, $(\min\{C_1, C_2\}, Z_1 \cap Z_2)$ also satisfies the conclusion of Lemma 3.5. Using the fact that there exists no infinite descending chain (with respect to the inclusion) of subvarieties of $\mathbb{A}^n$, we obtain the conclusion of Corollary 3.8. □

We are ready now to prove Theorem 2.2.

Proof of Theorem 2.2. If $X = \mathbb{A}^n$, the conclusion is immediate. Therefore, assume from now on in this proof that X is strictly contained in $\mathbb{A}^n$.

We prove Theorem 2.2 by induction on n. The case $n = 1$ was already proved during the proof of Lemma 3.5 (see Remark 3.7).

We assume Theorem 2.2 holds for $n-1$ and we will prove that it also holds for n ($n \geq 2$). Let C and Z be as in the conclusion of Corollary 3.8 for X. Also, we recall that Y, as defined in the statement of Theorem 2.2, is the largest $\mathbb{F}_p^{\mathrm{alg}}$-subvariety of X. Our goal is to show that $Z \subset X$, because this would mean that $Z \subset Y$, as Y is the largest subvariety of X defined over $\mathbb{F}_p^{\mathrm{alg}}$.

Assume Z is not a subvariety of X. Thus there exists an $\mathbb{F}_p^{\mathrm{alg}}$-irreducible subvariety W of Z, such that $W \cap X$ is a finite union of proper $\mathbb{F}_p(t)^{\mathrm{alg}}$-irreducible subvarieties $\{W_j\}_{j=1}^{l}$ of W. Let $j \in \{1, \dots, l\}$. Note that both W and W_j depend only on X (because Z and $W \cap X$ have finitely many geometrically irreducible components).

Assume $P := (x_1, \dots, x_n) \in W_j(\mathbb{F}_p(t)^{\mathrm{alg}})$. According to Lemma 3.5, $\dim Z < n$ and so, $\dim W =: d < n$. Moreover, $\dim W_j < \dim W$, because both W and W_j are irreducible and W_j is a proper subvariety of W. Without loss of generality, we may assume the projection $\pi : \mathbb{A}^n \to \mathbb{A}^d$, when restricted to W is generically finite-to-one (after relabelling the n coordinates of $\mathbb{A}^n$ we can achieve this anyway).

Let U_j be the Zariski closure of $\pi(W_j)$. Because W_j is a closed subvariety of W of smaller dimension, $\dim U_j < d$. Because W_j depends only on X, U_j depends only on X. Because $d < n$ and U_j is a subvariety strictly contained in $\mathbb{A}^d$, we may apply the inductive hypothesis to U_j. Let $U_{j,0}$ be the largest $\mathbb{F}_p^{\mathrm{alg}}$-subvariety of U_j. We conclude there exists a positive constant C_j depending only on the variety U_j (and so, depending only on the variety X) such that either

$$\mathrm{h}(x_1, \dots, x_d) \geq C_j \tag{14}$$

or

$$(x_1, \dots, x_d) \in U_{j,0}(\mathbb{F}_p(t)^{\mathrm{alg}}). \tag{15}$$

If (14) holds, then $\mathrm{h}(x_1, \dots, x_n) \geq \mathrm{h}(x_1, \dots, x_d) \geq C_j$. If (15) holds, then $(x_1, \dots, x_n) \in \left(U_{j,0} \times \mathbb{A}^{n-d}\right)(\mathbb{F}_p(t)^{\mathrm{alg}})$. The $\mathbb{F}_p^{\mathrm{alg}}$-variety $U_{j,0} \times \mathbb{A}^{n-d}$ intersects W in a subvariety of smaller dimension because

$$\dim(\pi(U_{j,0} \times \mathbb{A}^{n-d})) = \dim(U_{j,0}) < d = \dim(\pi(W)).$$

Let $V_j := \left(U_{j,0} \times \mathbb{A}^{n-d}\right) \cap W$. Then P lies on V_j, and V_j is an $\mathbb{F}_p^{\mathrm{alg}}$-variety (both $U_{j,0}$ and W are $\mathbb{F}_p^{\mathrm{alg}}$-varieties) which is properly contained in W. Moreover, V_j depends only on X, because both W and $U_{j,0} \times \mathbb{A}^{n-d}$ depend only on X.

Hence, for each $P \in W \cap X$, there exists $j \in \{1, \dots, l\}$ such that $P \in W_j(\mathbb{F}_p(t)^{\mathrm{alg}})$. Then

$$\text{either } h(P) \geq C_j, \tag{16}$$

$$\text{or } P \in V_j(\mathbb{F}_p(t)^{\mathrm{alg}}). \tag{17}$$

Let $C' := \min\{C, C_1, \dots, C_l\}$. Then C' is a positive constant which depends only on X. Let Z' be the proper subvariety of Z obtained by replacing the irreducible component W of Z by $\bigcup_{i=1}^{l} V_i$. Then Z' is also a closed subvariety of $\mathbb{A}^n$ defined over $\mathbb{F}_p^{\mathrm{alg}}$. Moreover, because the pair (C, Z) satisfies Lemma 3.5, using also (16) and (17), we conclude that the pair (C', Z') also satisfies the conclusion of Lemma 3.5. This contradicts the minimality of Z which satisfies the conclusion of Corollary 3.8. This contradiction shows that $Z \subset X$ (and so, $Z \subset Y$), which concludes the proof of Theorem 2.2. ∎

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