EQUIDISTRIBUTION FOR TORSION POINTS OF A DRINFELD MODULE

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Abstract. We prove an equidistribution result for torsion points of Drinfeld modules of generic characteristic. We also show that similar equidistribution statements would provide proofs for the Manin-Mumford and the Bogomolov conjectures for Drinfeld modules.

1. Introduction

Ullmo proved in [14] the Bogomolov Conjecture for curves embedded in their jacobians and Zhang proved in [15] the Bogomolov Conjecture in full generality. In their proofs they obtain an equidistribution result for points of small height on an abelian variety. Bilu proves in [6] a similar equidistribution statement for points of small height on an algebraic torus. If we restrict our attention only to torsion points of an abelian variety or of a power of the multiplicative group, the above mentioned equidistribution results provide a proof for the Manin-Mumford Conjecture for abelian varieties and, respectively, for powers of the multiplicative group.

Using the analogy between abelian varieties and Drinfeld modules, Denis formulated in [8] the Manin-Mumford and the Mordell-Lang conjectures for Drinfeld modules of generic characteristic. Denis-Manin-Mumford problem was answered by Scanlon (see [13] for a proof of the Manin-Mumford Conjecture for Drinfeld modules of generic characteristic using the model theory of difference fields), while Denis-Mordell-Lang conjecture was treated by the author in [10]. Motivated by the same analogy between abelian varieties and Drinfeld modules, Vincent Bosser formulated and proved a Bogomolov statement for Drinfeld modules of generic characteristic in his PhD thesis [4] (see also our Theorem 4.9).

In Section 4 (see our Conjectures 4.4 and 4.7) we ask if the equidistribution results of Bilu, Ullmo and Zhang are valid also in the context of Drinfeld modules. In Section 3 we prove an equidistribution result for torsion points of Drinfeld modules of generic characteristic (see Theorem 2.7). We will also prove in Section 4 that a positive answer to our Conjectures 4.4 and 4.7 would lead to a new proof of the Bogomolov Conjecture and, respectively to a new proof of the Manin-Mumford Conjecture for Drinfeld modules of generic characteristic.

As mentioned before, Vincent Bosser proved the Bogomolov conjecture for Drinfeld modules of generic characteristic in his PhD thesis [4]. Part of his thesis (but not including the Bogomolov problem) were published in [5]. Bosser has a completely different approach than us to the Bogomolov statement for Drinfeld modules. He uses in his argument both diophantine approximation and also Scanlon's result for the Manin-Mumford problem for Drinfeld modules of generic characteristic. Thus a proof of the equidistribution results we propose in Section 4 would lead to proofs using only number theory of both the Manin-Mumford and the Bogomolov problems for Drinfeld modules.

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We also note that the first dimensional case ($g = 1$) of our Conjecture 4.7 follows from Corollary 4.6 in [1]. In that paper, Baker and Hsia prove a first dimensional equidistribution statement in the context of polynomial dynamics. We sketch here the connection between their result and ours. To each additive (i.e. $\mathbb{F}_q$-linear, where q is a power of the prime number p) polynomial

$$P(x) := tx + \sum_{i=1}^{r} a_i x^{q^i},$$

which for simplicity we assume that it has all coefficients in $\mathbb{F}_q(t)$, we associate a Drinfeld module

$$\phi : \mathbb{F}_q[t] \to \mathbb{F}_q(t)\{\tau\}$$

by letting $\phi_t(x) := P(x)$ (see also our definition for Drinfeld modules from Section 2). We also let $\mathbb{C}_\infty$ be the v_∞-adic completion of the algebraic closure of $\mathbb{F}_q((\frac{1}{t}))$ (the valuation v_∞ satisfies $v_\infty(\frac{1}{t}) = 1$; see also our definition of v_∞ from Section 2). Baker and Hsia proved that assuming the v_∞-adic filled Julia set $J_\infty \subset \mathbb{C}_\infty$ associated to $P(x)$ is compact, then the Galois orbits of any sequence of distinct points of small normalized height with respect to P (or equivalently, to ϕ) are equidistributed with respect to the Haar measure on J_∞.

We show next that for a polynomial $P(x)$ as above, the v_∞-adic filled Julia set is compact. Indeed, the derivative $P'(x)$ of $P(x)$ is constant equal to t. Hence, with respect to the absolute v_∞-adic norm on $\mathbb{C}_\infty$,

$$|P'(x)|_\infty = |t|_\infty = p > 1. \tag{1}$$

Therefore, P is uniformly expansive on J_∞ (according to Définition 3 in [3]) and so, by Proposition 16 in [3], J_∞ is compact. Actually, the results in [3] are stated in the case of an algebraically closed complete field of characteristic 0 (such as $\mathbb{C}_p$). However, as pointed out by Robert Benedetto, Bézivin's argument from [3] goes through verbatim for an algebraically closed complete field in any characteristic (see also [2] for a treatment of the rational dynamics for algebraically closed non-archimedean fields of arbitrary characteristic).

Moreover, in the above case, because P is uniformly expansive on J_∞, then J_∞ equals its boundary, which is the (non-filled) Julia set. Even more it is true in this case. As shown in Theorem 3.1 in [11], the Julia set is contained in the topological closure of the periodic points for P (which are torsion points for ϕ). On the other hand, by its definition, the Julia set always contains the topological closure of the repelling periodic points for P. Because of (1), all periodic points for P are repelling. Hence, for an additive polynomial P as above (and so, for the associated Drinfeld module ϕ), the Julia set and the v_∞-adic filled Julia set are both equal to the topological closure of the torsion points of ϕ.

Therefore our Theorem 2.7 (which is a particular case of our Conjecture 4.4) gives a different proof for the fact that the torsion points of ϕ are equidistributed with respect to the Haar measure on the Julia set associated to ϕ as above. Our methods for proving Theorem 2.7 are completely different than the methods used in [1] and [3].

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2. Statement of our equidistribution result

We define first the notion of a Drinfeld module.

Let p be a prime and let q be a power of p. Let $A := \mathbb{F}_q[t]$. Let K be a finitely generated field extension of $\mathbb{F}_q$. We fix a morphism $i : A \to K$. We define the operator τ as the power of the usual Frobenius with the property that for every $x \in K$, $\tau(x) = x^q$. Then we let $K\{\tau\}$ be the ring of polynomials in τ with coefficients from K (the addition is the usual addition, while the multiplication is given by the usual composition of functions).

A Drinfeld module is a morphism $\phi : A \to K\{\tau\}$ for which the coefficient of τ^0 in ϕ_a is $i(a)$ for every $a \in A$, and there exists $a \in A$ such that $\phi_a \neq i(a)\tau^0$. In this case, we also say that ϕ is defined over K. For every field extension $K \subset L$, the Drinfeld module ϕ induces an action on $\mathbb{G}_a(L)$ by $a * x := \phi_a(x)$, for each $a \in A$.

Following the definition from [12], we call ϕ a Drinfeld module of generic characteristic if $\ker(i) = \{0\}$ and we call ϕ a Drinfeld module of finite characteristic if $\ker(i) \neq \{0\}$. If $\ker(i) = \{0\}$, then we extend i to an embedding of $\mathrm{Frac}(A) = \mathbb{F}_q(t)$ into K.

We note that usually, in the definition of a Drinfeld module, A is the ring of functions defined on a projective non-singular curve C, regular away from a closed point $\infty \in C$. For our definition of a Drinfeld module, $C = \mathbb{P}^1_{\mathbb{F}_q}$ and ∞ is the usual point at infinity on $\mathbb{P}^1$. On the other hand, every ring of functions A as above contains $\mathbb{F}_q[t]$ as a subring, where t is a non-constant function in A.

Before stating our result, we need to introduce several technical ingredients.

Definition 2.1. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. We call the modular transcendence degree of ϕ the smallest integer $d \geq 0$ such that a Drinfeld module isomorphic to ϕ is defined over a field of transcendence degree d over $\mathbb{F}_q$.

For the remaining of this paper, unless otherwise stated, $\phi : A \to K\{\tau\}$ is a Drinfeld module of generic characteristic.

Let v_∞ be the valuation on $F := \mathbb{F}_q(t)$ given by the negative of the degree of any nonzero rational function, i.e.

$$v_\infty\left(\frac{f}{g}\right) = \deg(g) - \deg(f) \text{ for every nonzero } f, g \in \mathbb{F}_q[t].$$

We fix an extension of v_∞ on K (we recall that $F \subset K$, as ϕ is a Drinfeld module of generic characteristic) and we denote it also by v_∞. We let K_∞ be the completion of K at v_∞. We denote by $F_\infty := \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ the completion of F inside K_∞. We fix an algebraic closure K_∞^{alg} of K_∞ and extend v_∞ to a valuation on K_∞^{alg}. Finally, we let $\mathbb{C}_\infty$ be the completion of K_∞^{alg} at v_∞. As shown in [12], $\mathbb{C}_\infty$ is an algebraically closed field. We let K^{alg} and K^{sep} be the algebraic and respectively, the separable closure of K inside K_∞^{alg}.

We define the set $\mathrm{End}_{K^{\mathrm{sep}}}(\phi)$ of endomorphisms of ϕ as the set of all $f \in K^{\mathrm{sep}}\{\tau\}$ such that $f\phi_a = \phi_a f$, for every $a \in A$. As shown in Chapter 4 of [12], there exists a finite separable extension L of K such that each endomorphism $f \in K^{\mathrm{alg}}\{\tau\}$ of ϕ has coefficients in L. Moreover, $\mathrm{End}_{K^{\mathrm{sep}}}(\phi)$ is a finite extension of A (if we identify $a \in A$ with $\phi_a \in K\{\tau\}$).

We define the torsion submodule of ϕ as

$$\phi_{\mathrm{tor}} = \{x \in K^{\mathrm{alg}} \mid \text{ there exists } a \in A \setminus \{0\} \text{ such that } \phi_a(x) = 0\}.$$

For each nonzero $a \in A$, we let $\phi[a] = \{x \in K^{\mathrm{alg}} \mid \phi_a(x) = 0\}$. Because $A = \mathbb{F}_q[t]$ is a PID, for each $x \in \phi_{\mathrm{tor}}$ there exists a unique monic polynomial $a \in A$ such that $\phi_a(x) = 0$, and if $a' \in A$ satisfies $\phi_{a'}(x) = 0$, then $a|a'$. We call a the *order* of x. Note that by construction, the order of a torsion point is always a monic polynomial in t. Also, we will always identify the greatest common divisor in A of a number of polynomials in $\mathbb{F}_q[t]$ by the monic generator of the principal ideal of A generated by them. Finally, we note that because ϕ is a Drinfeld module of generic characteristic, $\phi_{\mathrm{tor}} \subset K^{\mathrm{sep}}$.

As shown by Theorem 4.6.9 of [12], there exists an A-lattice $\Lambda \subset \mathbb{C}_\infty$ associated to the generic characteristic Drinfeld module ϕ. Let e_ϕ be the exponential function defined in 4.2.3 of [12] which gives a continuous (in the v_∞-adic topology) isomorphism

$$\mathrm{e}_\phi : \mathbb{C}_\infty / \Lambda \to \mathbb{C}_\infty .$$

The torsion submodule of ϕ in $\mathbb{C}_\infty$ is isomorphic naturally through ${\mathrm{e}_\phi}^{-1}$ to $(F \otimes_A \Lambda)/\Lambda$.

Notation 2.2. We let T be the closure of $\phi_{\mathrm{tor}} \subset \mathbb{C}_\infty$ in the v_∞-adic topology of $\mathbb{C}_\infty$. As explained in Section 1, T equals the Julia set associated to ϕ (or equivalently, to ϕ_t).

Then the restriction of e_ϕ on $(F_\infty \otimes_A \Lambda)/\Lambda$ gives an isomorphism between $(F_\infty \otimes_A \Lambda)/\Lambda$ and T.

Let r be the rank of Λ. Then $(F \otimes_A \Lambda)/\Lambda \simeq (F/A)^r$. Let $z_1, \ldots, z_r$ be a fixed A-basis of Λ. Using Proposition 4.6.3 of [12], $(F_\infty \otimes_A \Lambda)/\Lambda$ is isomorphic to $\left(\mathbb{F}_q((\frac{1}{t}))/\mathbb{F}_q[t]\right)^r$. Then we have the isomorphism

$$\mathrm{E}_\phi : \left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t]\right)^r \to T \text{ given by}$$

$$\mathrm{E}_\phi(\gamma_1, \ldots, \gamma_r) := \mathrm{e}_\phi(\gamma_1 z_1 + \cdots + \gamma_r z_r), \text{ for each } \gamma_1, \ldots, \gamma_r \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t].$$

We construct the following group isomorphism

$$\sigma : \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t] \to \frac{1}{t} \cdot \mathbb{F}_q[[\frac{1}{t}]], \text{ given by}$$

$$\sigma\left(\sum_{i \geq -n} \alpha_i \left(\frac{1}{t}\right)^i\right) = \sum_{i \geq 1} \alpha_i \left(\frac{1}{t}\right)^i, \tag{2}$$

for every natural number n and for every $\sum_{i \geq -n} \alpha_i \left(\frac{1}{t}\right)^i \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ (obviously, σ vanishes on $\mathbb{F}_q[t]$). The group $\frac{1}{t} \cdot \mathbb{F}_q[[\frac{1}{t}]]$ is a topological group with respect to the restriction of v_∞ on $\frac{1}{t} \cdot \mathbb{F}_q[[\frac{1}{t}]]$. Hence, the isomorphism σ^{-1} induces a topological group structure on $\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t]$. Therefore, σ becomes a continuous isomorphism of topological groups. We endow $\left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t]\right)^r$ with the corresponding product topology. The isomorphism σ extends diagonally to another continuous isomorphism, which we also call

$$\sigma : \left(\mathbb{F}_q((\frac{1}{t}))/\mathbb{F}_q[t]\right)^r \to \left(\frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]]\right)^r =: G.$$

Moreover, using that e_ϕ is a continuous morphism, we conclude

$$\text{(3)} \qquad \mathrm{E}_\phi\, \sigma^{-1} : G \to T \text{ is a continuous isomorphism.}$$

Notation 2.3. Let μ be the Haar measure on G, normalized so that its total mass is 1. Let $\nu := (\mathrm{E}_\phi\, \sigma^{-1})_* \,\mu$ be the induced measure on T (i.e. $\nu(V) := \mu\left(\sigma\, {\mathrm{E}_\phi}^{-1}(V)\right)$ for every measurable $V \subset T$).

Because μ is a probability measure then ν is also a probability measure. Because μ is a Haar measure on G and $\mathrm{E}_\phi\, \sigma^{-1}$ is a group ismorphism, then ν is a Haar measure on T.

Definition 2.4. For each $x \in K^{\mathrm{sep}}$, we denote by $O(x)$ the (finite) orbit of x under $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$.

Definition 2.5. Given $x \in K^{\mathrm{sep}}$, we define a probability measure $\overline{\delta}_x$ on $\mathbb{C}_\infty$ by

$$\overline{\delta}_x = \frac{1}{\#O(x)} \sum_{y \in O(x)} \delta_y,$$

where $\#O(x)$ represents, as always, the cardinality of the set $O(x)$ and δ_y is the Dirac measure on $\mathbb{C}_\infty$ supported on $\{y\}$.

Before we can state our equidistribution result (Theorem 2.7), we need to define the concept of weak convergence for a sequence of probability measures on a metric space.

Definition 2.6. A sequence $\{\lambda_k\}$ of probability measures on a metric space S weakly converges to λ if for any bounded continuous function $f : S \to \mathbb{R}$, $\int_S f d\lambda_k \to \int_S f d\lambda$ as $k \to \infty$. In this case we use the notation $\lambda_k \xrightarrow{w} \lambda$.

Theorem 2.7. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic and of modular transcendence degree at least* 2*. Assume* $\mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi) = A$*. Let $\{x_k\}$ be a sequence of distinct torsion points in ϕ. Then $\overline{\delta}_{x_k} \xrightarrow{\mathrm{w}} \nu$.*

Remark 2.8. If $x \in \phi_{\mathrm{tor}}$, then $O(x) \subset T$ and so, the measure $\overline{\delta}_x$ is supported on T. Therefore, the conclusion of Theorem 2.7 should be interpreted as follows: for each x_k as in Theorem 2.7, $\overline{\delta}_{x_k}$ is a measure supported on T and as $k \to \infty$, the probability measures $\overline{\delta}_{x_k}$ converge weakly to the normalized Haar measure ν on T.

Remark 2.9. We will explain during the proof of Theorem 2.7 why the hypothesis on the modular transcendence degree is needed in our proof. However, we note that the modular transcendence degree of a Drinfeld module ϕ of generic characteristic is at least 1, because no Drinfeld module isomorphic to ϕ is defined over a finite field (in that case, the Drinfeld module would be of finite characteristic).

3. Proof of the main theorem

We continue in this section with the notation from Section 2.

Proof of Theorem 2.7. Let b_k be the order of x_k for each k. Because the elements of the sequence $\{x_k\}$ are distinct, $\deg(b_k) \to \infty$.

For each k we let $O_k := \sigma\, \mathrm{E}_\phi{}^{-1}(O(x_k)) \subset G$. Then we define by $\overline{\delta}_k$ the associated probability measure on G, equally supported on O_k:

$$\overline{\delta}_k := \frac{1}{\# O_k} \sum_{y \in O_k} \delta_y.$$

Because $\mathrm{E}_\phi\, \sigma^{-1} : G \to T$ is a continuous isomorphism, we conclude that it suffices to show

$$\overline{\delta}_k \xrightarrow{w} \mu. \tag{4}$$

Let f be any continuous, real valued function on G. Because G is a totally disconnected, compact space, f is a finite $\mathbb{R}$-linear combination of characteristic functions on open subsets of G. Hence, it suffices to prove (4) for characteristic functions of open subsets of G. Thus, for each such open subset U, we need to show

$$\frac{\#(O_k \cap U)}{\# O_k} \to \mu(U) \text{ as } k \to \infty. \tag{5}$$

Let U be an open subset of G. Then U is a finite union of sets of the form

$$\left(a_1\left(\frac{1}{t}\right), \ldots, a_r\left(\frac{1}{t}\right)\right) + \left(\frac{1}{t^{n_1+1}}\mathbb{F}_q\left[\left[\frac{1}{t}\right]\right], \ldots, \frac{1}{t^{n_r+1}}\mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right), \tag{6}$$

where $a_i \in \frac{1}{t}\mathbb{F}_q[\frac{1}{t}]$ is a polynomial of degree at most n_i for each i, and $n_1, \ldots, n_r$ are positive integers. Hence we may and do assume that U equals a set as in (6). Then our goal is to show

$$\frac{\#(O_k \cap U)}{\# O_k} \to \mu(U) = q^{-\sum_{i=1}^r n_i}, \text{ as } k \to \infty. \tag{7}$$

Let $\widehat{A} := \prod_P A_{(P)}$ denote the profinite completion of A (where P runs over all the monic irreducible polynomials of $A = \mathbb{F}_q[t]$ and $A_{(P)}$ represents the completion of A at the prime ideal (P)). We let

$$\pi : \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K}) \to \mathrm{GL_r}\left(\widehat{\mathrm{A}}\right)$$

be the natural representation on the adèlic Tate module of ϕ. Let Γ be its image. Theorem 3 of [7] proves that under the hypothesis that $\mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(A) = A$ and that ϕ has modular transcendence degree at least equal to 2, the group Γ is open in $\mathrm{GL_r}\left(\widehat{\mathrm{A}}\right)$. Thus our assumptions on ϕ guarantee that

$$\left[\mathrm{GL_r}\left(\widehat{\mathrm{A}}\right) : \Gamma\right] < \aleph_0. \tag{8}$$

Using (8) we conclude there exist finitely many irreducible monic polynomials $P_1, \ldots, P_l \in A$ and there exists a natural number m such that

$$\prod_{i=1}^{l}\left(I_r + \left(\prod_{i=1}^{l} P_i\right)^m \mathrm{GL}_r\left(A_{(P_i)}\right)\right) \cdot \prod_{P \neq P_i} \mathrm{GL}_r\left(A_{(P)}\right) \subset \Gamma, \tag{9}$$

where $I_r \in \mathrm{GL}_r$ is the identity matrix. The subgroup in (9) has finite index in Γ. Let $H \subset \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$ be the preimage through π^{-1} of the subgroup of Γ from (9). Then $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$ is a finite union of cosets of H.

Notation 3.1. We let $P := \left(\prod_{i=1}^{l} P_i\right)^m$. Also, let $L := \deg(P)$.

Let O be one of the sets O_k. Let $b \in \mathbb{F}_q[t]$ be the monic polynomial which is the order of x_k. Then all the torsion points $y \in O(x_k)$ have the same order b. Therefore, the elements of $O \subset G = \left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t]\right)^r$ are of the form

$$\left(\frac{b_1}{b}, \ldots, \frac{b_r}{b}\right), \tag{10}$$

where $b_i \in \mathbb{F}_q[t]$ and for each i,

$$\deg(b_i) < \deg(b) \tag{11}$$

and moreover, the greatest common divisor

$$(b_1, \ldots, b_r, b) = 1. \tag{12}$$

Using (9) we will determine the proportion of all elements in O of the form (10) which also satisfy (11) and (12). Also, (9) will allow us to compute $\#(O \cap U)$, which in turn will lead to the proof of our Theorem 2.7.

For each $i \in \{1, \ldots, l\}$, let β_i be the exponent of P_i in the prime decomposition of b. Let $b' := \prod_{i=1}^{l} P_i^{\beta_i'}$ be the monic polynomial which is the greatest common divisor between b and P. Obviously, $\beta_i' \leq \beta_i$. For each $i \in \{1, \ldots, l\}$, let

$$\left(\frac{\alpha_1}{P_i^{\beta_i}}, \ldots, \frac{\alpha_r}{P_i^{\beta_i}}\right) \tag{13}$$

be an element of the P_i-power part of O. Then (9) shows that the image through H of the point from (13) consists of all elements of the form

$$\left(\frac{\alpha_1 + P_i^{\beta_i'} q_1}{P_i^{\beta_i}}, \ldots, \frac{\alpha_r + P_i^{\beta_i'} q_r}{P_i^{\beta_i}}\right) \tag{14}$$

for polynomials $q_1, \ldots, q_r$ satisfying $\deg(\alpha_i + P_i^{\beta_i'} q_i) < \deg(P_i^{\beta_i})$.

For a monic irreducible polynomial $Q \in A$, different than $P_1, \ldots, P_l$, let Q^β be the maximal power of Q dividing b. Then, by (9), the Q-power part of O consists of all torsion points of order Q^β.

We let C be the collection of all r-uples of polynomials of the form

$$(\alpha_1, \ldots, \alpha_r) \text{ with } \deg(\alpha_i) < \deg(b') \text{ for all } i \tag{15}$$

such that for each such r-uple there exists $y \in O$ of the form

$$\left(\frac{\alpha_1 + b' q_1}{b}, \ldots, \frac{\alpha_r + b' q_r}{b}\right), \tag{16}$$

where for each i, $q_i \in \mathbb{F}_q[t]$ and

$$\deg(\alpha_i + b' q_i) < \deg(b) \tag{17}$$

and

$$(\alpha_1 + b' q_1, \ldots, \alpha_r + b' q_r, b) = 1. \tag{18}$$

If $b' = 1$, then the only r-uple as in (15) is $(\alpha_1, \ldots, \alpha_r) = (0, \ldots, 0)$.

Because $b'|b$, condition (18) shows that

$$(\alpha_1, \ldots, \alpha_r, b') = 1. \tag{19}$$

Clearly, C is a finite set because there are finitely many r-uples of the form (15) without even asking any extra conditions. Moreover, the cardinality of C, even though it depends on O, it is bounded above independently of O by (15) and by $\deg(b') \leq \deg(P) = L$. Using (9) and our analysis for the action of $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$ on the different Q-power parts of O, we conclude that if O contains an element of the form (16) for some r-uple $(\alpha_1, \ldots, \alpha_r) \in C$, then O contains *all* elements of the form (16) corresponding to $(\alpha_1, \ldots, \alpha_r)$ which also satisfy conditions (17) and (18).

Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ be a fixed r-uple in C. Let $O^{\overline{\alpha}}$ be the set of all elements in G of the form (16), corresponding to this fixed r-uple $\overline{\alpha}$, and satisfying (17) and (18). As explained in the above paragraph,

$$O = \bigcup_{\overline{\alpha} \in C} O^{\overline{\alpha}}. \tag{20}$$

Because $|C|$ is bounded above in terms of L, in order to show (7) it suffices to show for every $\overline{\alpha} \in C$ that

$$\frac{\#(O^{\overline{\alpha}} \cap U)}{\# O^{\overline{\alpha}}} \to q^{-\sum_{i=1}^{r} n_i} \text{ as } \deg(b) \to \infty. \tag{21}$$

We define the Möbius function μ on the set of all monic polynomials in $\mathbb{F}_q[t]$ by

$$\mu(1) = 1,$$

$$\mu(Q_1 Q_2 \ldots Q_n) = (-1)^n \text{ if } Q_1, \ldots, Q_n \text{ are distinct irreducible, non-constant polynomials,}$$

$$\mu(f) = 0 \text{ if } f \text{ is not squarefree.}$$

In this proof, the letter μ also appears as denoting the measure on G. This should not be confused with the above defined Möbius function, as the measure μ is always evaluated on subsets of G, while the Möbius function μ is always evaluated on monic polynomials.

It is immediate to see that for every nonzero polynomial $f \in \mathbb{F}_q[t]$ (monic or not)

$$\sum_{g|f} \mu(g) = 1 \text{ if } \deg(f) = 0 \text{ and it is 0 otherwise.} \tag{22}$$

Of course, g in (22) is a monic polynomial and in general, when we will sum over divisors of a polynomial f, we will always include only the monic polynomials dividing f.

Uaing (17) and (18), in order to count the number of elements of $O^{\overline{\alpha}}$, we compute

$$\sum_{\substack{q_1, \ldots, q_r \\ \deg(\alpha_1 + b'q_1) < \deg(b), \ldots, \deg(\alpha_r + b'q_r) < \deg(b)}} \left(\sum_{d | (\alpha_1 + b'q_1, \ldots, \alpha_r + b'q_r, b)} \mu(d) \right). \tag{23}$$

Indeed, using (22), we obtain that the inner sum in (23) equals 1 if and only if the greatest common divisor $(\alpha_1 + b'q_1, \ldots, \alpha_r + b'q_r, b) = 1$, otherwise the inner sum equals 0. Changing the order of summation in (23) we obtain

$$\sum_{d|b} \mu(d) \cdot \left(\sum_{\substack{q_1, \ldots, q_r \\ \deg(\alpha_1 + b'q_1) < \deg(b), \ldots, \deg(\alpha_r + b'q_r) < \deg(b) \\ d | (\alpha_1 + b'q_1), \ldots, d | (\alpha_r + b'q_r)}} 1 \right). \tag{24}$$

We evaluate the inner sum taken into account that for its computation, $\alpha_1, \ldots, \alpha_r$ and b, b' and d are all fixed. We also take into account that if d and b' are not coprime, then the inner sum is 0 as shown by (19) (because otherwise $(\alpha_1, \ldots, \alpha_r, b') \neq 1$). On the other hand, if d and b' are coprime, then each congruence

$$\alpha_i + b'q_i \equiv 0 \pmod{d} \text{ has one incongruent solution } q_i \text{ modulo } d. \tag{25}$$

For each i, let s_i be the unique solution to the congruence (25) with $\deg(s_i) < \deg(d)$. Then all the solutions q_i to the congruence (25), which we count in the inner sum from (24), are of the form

$$q_i = s_i + dq_i', \tag{26}$$

for some polynomials q_i' such that

$$\deg(\alpha_i + b'q_i) < \deg(b). \tag{27}$$

Using (26) we get

$$\alpha_i + b'q_i = \alpha_i + b's_i + b'dq_i'. \tag{28}$$

Because the inner sum of (24) is nonzero only if $(d, b') = 1$ and because $d|b$, we need to estimate the inner sum of (24) only when $d|\frac{b}{b'}$ (and even then, the inner sum still might be 0). So, for us, $\deg(d) \leq \deg(b) - \deg(b')$. This shows

$$\deg(\alpha_i + b's_i) \leq \max\{\deg(\alpha_i), \deg(b') + \deg(s_i)\} < \max\{\deg(b'), \deg(b') + \deg(d)\} \leq \deg(b).$$

Therefore, for each i and for each polynomial q_i' of degree less than $\deg(b) - \deg(b'd)$, the degree of (28) is less than the degree of b. Hence, we have $q^{(\deg(b)-\deg(b'd))r}$ choices for r-uples $(q_1', \ldots, q_r')$. So, we compute the inner sum in (24) and obtain

$$q^{r(\deg(b)-\deg(b'd))} = q^{r(\deg(b)-\deg(b'))-r\deg(d)}, \tag{29}$$

if $(d, b') = 1$, while if $(d, b') \neq 1$, the inner sum in (24) is 0. We use (29) in (24) and obtain

$$\sum_{\substack{d|b \\ (d,b')=1}} \mu(d) q^{r(\deg(b)-\deg(b'))-r\deg(d)} = q^{r(\deg(b)-\deg(b'))} \sum_{\substack{d|b \\ (d,b')=1}} \mu(d) q^{-r\deg(d)}. \tag{30}$$

We observe that the result we obtained in (30) is independent of the particular choice of the r-uple $\overline{\alpha}$. Now we compute, for the same fixed r-uple $\overline{\alpha}$, the number of elements of $O^{\overline{\alpha}}$ which are also in U. As explained before, using (9), we need to count the number of all elements of U of the form (16), corresponding to this fixed r-uple $\overline{\alpha}$, and satisfying (17) and (18).

For each $i \in \{1, \ldots, r\}$, we let $a_i'(t) := t^{n_i} a_i(\frac{1}{t})$. Then $a_i'(t)$ is a polynomial in t of degree less than n_i (we are using the fact that $a_i(\frac{1}{t}) \in \frac{1}{t}\mathbb{F}_q[\frac{1}{t}]$ is a polynomial of degree at most n_i). The requirement for an element of $O^{\overline{\alpha}}$ to lie in U is given by

$$v_\infty\left(\frac{\alpha_i + b'q_i}{b} - \frac{a_i'}{t^{n_i}}\right) \geq n_i + 1 \text{ for each } i. \tag{31}$$

Inequality (31) is equivalent with

$$\deg\left(t^{n_i}(\alpha_i + b'q_i) - ba_i'\right) \leq \deg(b) - 1. \tag{32}$$

Thus when we count the number of elements of $O^{\overline{\alpha}} \cap U$ we obtain the same sum as in (23), only that now we have the extra assumption (32) on top of the other restrictions on q_i. Note that (32) already yields (17), because otherwise

$$\deg(t^{n_i}(\alpha_i + b'q_i)) \geq \deg(b) + n_i > \max\{\deg(ba_i'), \deg(b) - 1\},$$

contradicting thus (32).

Again we can change the order of summation in (23) and obtain (24), only that the inner summation is over all q_i which also satisfy (32). We obtain once again that the corrsponding inner sum to a divisor $d|b$ is nonzero only if d and b' are coprime. Hence we obtain again (26) and we use it in (32) to get

$$\deg\left(t^{n_i}(\alpha_i + b's_i + b'dq_i') - ba_i'\right) \leq \deg(b) - 1. \tag{33}$$

We are interested in estimating $\#(O^{\overline{\alpha}} \cap U)$ when $\deg(b)$ is much larger than the degree of b' (and implicitly, much larger than the degrees of the α_i, as $\deg(\alpha_i) < \deg(b')$) and also much larger than the numbers n_i. This is the case because our goal is to prove (21), and as $\deg(b) \to \infty$, U stays fixed. We recall that the degree of b' is bounded by L, while the numbers n_i depend only on U.

We go back now to (32). We know that $\deg(q_i) < \deg(b) - \deg(b')$ (see (17)). Thus, using (26), we deduce as before (when we evaluated $\#O^{\overline{\alpha}}$) that $\deg(q_i') < \deg(b) - \deg(b') - \deg(d)$. Therefore, let

$$q_i' = \sum_{j=0}^{\deg(b)-\deg(b')-\deg(d)-1} \gamma_j^{(i)} t^j \in \mathbb{F}_q[t], \tag{34}$$

where some of the $\gamma_j^{(i)}$ could be 0 (including some of the top coefficients).

Let $n_0 := \max\{n_1, \dots, n_r\} + 1$. Assume for the moment that

$$\deg(d) \leq \deg(b) - L - n_0 < \deg(b) - \deg(b') - n_i, \tag{35}$$

where in the second inequality from (35) we also used the fact that $b'|P$ (and so, $\deg(b') \leq L$). Hence

$$\deg(t^{n_i} b'dq_i') \leq \deg(b) + n_i - 1, \tag{36}$$

with equality if $\gamma_i^{(\deg(b)-\deg(b')-\deg(d)-1)} \neq 0$ (see (34)). Hence, under our assumption (35), we obtain that the top n_i coefficients $\gamma_j^{(i)}$ of q_i' are determined uniquely by the coefficients of d, b', b, s_i, α_i and a_i' and by condition (33), while the remaining $(\deg(b) - \deg(b') - \deg(d) - n_i)$ coefficients $\gamma_j^{(i)}$ can be arbitrary elements of $\mathbb{F}_q$. Therefore, under the assumption (35), we obtain that the inner sum in (24) associated to elements in $O^{\overline{\alpha}} \cap U$ equals

$$q^{\sum_{i=1}^r (\deg(b)-\deg(b')-\deg(d)-n_i)}, \tag{37}$$

if $(b', d) = 1$, while if $(b', d) > 1$, the inner sum in (24) associated to elements in $O^{\overline{\alpha}} \cap U$ is 0.

Next we analyze the case $\deg(d) > \deg(b) - L - n_0$. In this case, (33) shows that $\deg(q_i') < L + n_0$, because otherwise

$$\deg(t^{n_i} b'dq_i') > n_i + \deg(b) > \max\{\deg(t^{n_i}\alpha_i), \deg(t^{n_i}b's_i), \deg(ba_i'), \deg(b) - 1)\},$$

which would contradict (33) (we also used in the above inequality that $\deg(b) \geq \deg(b'd) > \deg(b's_i)$, because d divides b, but d is coprime with $b'|b$). Thus, the inner sum in (24) corresponding to elements in $O^{\overline{\alpha}} \cap U$ can contribute at most $q^{r(L+n_0)}$ and this computation is without even taking into consideration the actual restrictions on the coefficients imposed by (33).

Combining our findings in both cases with respect to assumption (35), we conclude that the number of elements in $O^{\overline{\alpha}} \cap U$ for a fixed $\overline{\alpha} \in C$ is

$$\sum_{\substack{d|b \\ (d,b')=1 \\ \deg(d)\leq\deg(b)-L-n_0}} \mu(d) q^{r(deg(b)-\deg(b')-\deg(d))-\sum_{i=1}^{r} n_i} + \sum_{\substack{d|b \\ (d,b')=1 \\ \deg(d)>\deg(b)-L-n_0}} \mathsf{O}(q^{r(L+n_0)}), \tag{38}$$

where the O-notation in the above second sum is the classical one and it refers in our context to the fact that the summand in the second sum from (38) is bounded in absolute value by a constant times $q^{r(L+n_0)}$ regardless of b. We note that the O-notation has nothing to do with our notation for orbits or for the set O, as the O-notation will always have attached to it, in parenthesis, a certain real number.

Because of the degree condition for d in the second sum in (38), we know there are at most q^{L+n_0} possibilities for d (for each fixed b), because $\deg\left(\frac{b}{d}\right) < L + n_0$. Hence, the second sum from (38) is of the order of $q^{(r+1)(L+n_0)}$. Introducing this estimate in (38) and adding and subtracting from (38) the quantity

$$\sum_{\substack{d|b \\ (d,b')=1 \\ \deg(d)>\deg(b)-L-n_0}} \mu(d) q^{r(deg(b)-\deg(b')-\deg(d))-\sum_{i=1}^{r} n_i},$$

we obtain the following estimate for (38):

$$\sum_{\substack{d|b \\ (d,b')=1}} \mu(d) q^{r(deg(b)-\deg(b'd))-\sum_{i=1}^{r} n_i} - \sum_{\substack{d|b \\ (d,b')=1 \\ \deg(d)>\deg(b)-L-n_0}} \mu(d) q^{r(deg(b)-\deg(b'd))-\sum_{i=1}^{r} n_i} + \mathsf{O}\left(1\right).$$

Note that in the O-estimate from above, we replaced $\mathsf{O}\left(q^{(r+1)(L+n_0)}\right)$ by $\mathsf{O}(1)$, because q, r and L are always fixed, while n_0 is fixed the moment we fix U. We estimate the second sum from above and we easily conclude it is also $\mathsf{O}\left(q^{(r+1)(L+n_0)}\right) = \mathsf{O}(1)$. Hence the above sum is

$$q^{r(\deg(b)-\deg(b'))-\sum_{i=1}^{r} n_i} \sum_{\substack{d|b \\ (d,b')=1}} \mu(d) q^{-r\deg(d)} + \mathsf{O}\left(1\right). \tag{39}$$

The sum in (39) is the number of elements in $O^{\overline{\alpha}} \cap U$ for a fixed $\overline{\alpha}$. The estimate (39) is independent of the r-uple $\overline{\alpha} \in C$. We show next that the quotient of (39) by (30) tends to $q^{-\sum_{i=1}^{r} n_i}$, as $\deg(b) \to \infty$ (see (21)). The numbers r, L, n_0 are constant as $\deg(b) \to \infty$. So, in order to prove (21), we use (30) and (39) and we are done if we show

$$q^{r(\deg(b)-\deg(b'))} \sum_{\substack{d|b \\ (d,b')=1}} \mu(d) q^{-r\deg(d)} \to \infty \tag{40}$$

as $\deg(b) \to \infty$. Let b_0 be the product of all the powers of irreducible polynomials dividing b other than powers of the polynomials $P_1, \ldots, P_l$. Thus b_0 is the largest divisor of b coprime with P (and so, coprime with b'). Then the sum in (40) can be rewritten as

$$\sum_{d|b_0} \mu(d) q^{-r\deg(d)}. \tag{41}$$

The sum in (41) equals

$$\prod_{\substack{c \text{ irreducible and monic} \\ c|b_0 \\ \deg(c)\geq 1}} \left(1 - \frac{1}{q^{r\deg(c)}}\right). \tag{42}$$

We observe that all the factors in the product (42) are less than 1 and so, if we extend the product (42) to include also the possible prime divisors of b from the set $\{P_1, \dots, P_l\}$, we can only decrease our product. So, to prove (40), it suffices to show

$$q^{r(\deg(b)-\deg(b'))} \prod_{\substack{c \text{ irreducible and monic} \\ c|b \\ \deg(c)\geq 1}} \left(1 - \frac{1}{q^{r\deg(c)}}\right) \to \infty \tag{43}$$

as $\deg(b) \to \infty$. We note that $\deg(b') \leq L$ (as it was remarked previously in our proof, because $b'|P$ and $\deg(P) = L$). So, we only need to show

$$q^{r\deg(b)} \prod_{\substack{c \text{ irreducible and monic} \\ c|b \\ \deg(c)\geq 1}} \left(1 - \frac{1}{q^{r\deg(c)}}\right) \to \infty \text{ as } \deg(b) \to \infty. \tag{44}$$

Consider the powers of the irreducible monic polynomials c_i dividing b:

$$\prod_{i=1}^{s} c_i^{e_i}.$$

Then the left hand side of (44) reads

$$\prod_{i=1}^{s} q^{r(e_i-1)\deg(c_i)} \left(q^{r\deg(c_i)} - 1\right). \tag{45}$$

Because ϕ has modular transcendence degree at least 2, the rank r of ϕ is at least 2 (otherwise, ϕ_t is a polynomial in τ of degree 1, which means that ϕ is isomorphic over K^{alg} with a Drinfeld module ψ for which $\psi_t = t\tau^0 + \tau$; hence ϕ would be isomorphic with a Drinfeld module defined over F). Because $r \geq 2$, we obtain $q^{r\deg(c_i)} - 1 \geq q^{\frac{r\deg(c_i)}{2}}$ for each i.

As $\deg(b) \to \infty$, then $\sum_{i=1}^{s} e_i \deg(c_i) \to \infty$, which proves

$$\prod_{i=1}^{s} q^{r(e_i-1)\deg(c_i)} \left(q^{r\deg(c_i)} - 1\right) \geq \prod_{i=1}^{s} q^{r(e_i-1)\deg(c_i)} \cdot q^{\frac{r\deg(c_i)}{2}} \geq q^{\frac{r\sum_{i=1}^{s} e_i\deg(c_i)}{2}} \to \infty. \tag{46}$$

This concludes the proof of (21), which proves Theorem 2.7. □

Remarks 3.2. Because we use Theorem 3 of [7] in our proof we needed to impose the two extra hypothesis on ϕ: that it has modular transcendence degree at least 2 and that its endomorphism ring equals A. As remarked by Breuer and Pink in a *Note* after the proof of their Theorem 3 in [7], their result is conjectured to be true without the extra assumption on the modular transcendence degree. In that case, our proof of Theorem 2.7 would show the equidistribution result for every Drinfeld module ϕ of generic characteristic for which $\mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi) = A$.

Note that even if we used also in the last part of our argument the fact that $r \geq 2$, the only place where we used crucially the fact that ϕ has modular transcendence degree at least 2 was in Breuer's and Pink's result. The limit in (44) is infinite even if $r = 1$ (in which case ϕ would have modular transcendence degree 1).

We note that the hypothesis on ϕ not to have complex multiplication, i.e. $\mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi) = A$ is a natural hypothesis, which is verified by most Drinfeld modules. We need this hypothesis because otherwise the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ in $\mathrm{GL_r}\left(\widehat{\mathrm{A}}\right)$ does not have finite index (see Theorem 8 in [7]). We need this finite index hypothesis because we use in an essential way the fact that the Galois orbit of each torsion point x of order $b \in A$ contains "almost" all torsion points of order b.

4. The Bogomolov and the Manin-Mumford Theorems for Drinfeld modules

In this section, all subvarieties are *closed* subvarieties.

We continue with the notation as in Section 3. Hence, let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic. For each positive integer g, we let ϕ act on $\mathbb{G}_a^g$ diagonally. Therefore, we may define just as before the torsion points of the action of ϕ on $\mathbb{G}_a^g$ as all the g-uples $(x_1, \dots, x_g)$ for which there exists a nonzero $a \in A$ such that $\phi_a(x_i) = 0$ for all i. We believe a similar equidistribution result as our Theorem 2.7 holds for the torsion points of $\mathbb{G}_a^g$. Before stating our conjecture, we require the following definitions.

Definition 4.1. An algebraic ϕ-submodule of $\mathbb{G}_a^g$ is a K^{sep}-algebraic subvariety of $\mathbb{G}_a^g$ which is invariant under the action of ϕ.

Definition 4.2. A torsion subvariety of $\mathbb{G}_a^g$ is a translate of an irreducible algebraic ϕ-submodule of $\mathbb{G}_a^g$ by a torsion point.

Definition 4.3. A sequence of points $\{x_k\} \subset \mathbb{G}_a^g(K^{\mathrm{sep}})$ is strict if any proper torsion subvariety of $\mathbb{G}_a^g$ contains x_k for only finitely many values of k.

For a point $x \in \mathbb{G}_a^g(K^{\mathrm{sep}})$, we let as before $O(x)$ denote the (finite) orbit of x under the diagonal action of $\mathrm{Gal(K^{sep}/K)}$ on $\mathbb{G}_a^g(K^{\mathrm{sep}})$. We also define the associated probability measure $\overline{\delta}_x$ on ${\mathbb{C}_\infty}^g$ for such an orbit $O(x)$:

$$\overline{\delta}_x := \frac{1}{\#O(x)} \sum_{y \in O(x)} \delta_y.$$

Finally, we denote by $\nu^{(g)}$ the product measure on T^g corresponding to ν taken g times. We note that $\nu^{(g)}$ is the Haar measure of total mass 1 on T^g.

Conjecture 4.4. Let ϕ be a Drinfeld module of generic characteristic. Assume $\mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi) = A$. Let $\{x_k\}$ be a strict sequence of torsion points in $\mathbb{G}_a^g(K^{\mathrm{sep}})$, for some $g \geq 1$. Then $\overline{\delta}_{x_k} \xrightarrow{w} \nu^{(g)}$.

Remarks 4.5. We use in Conjecture 4.4 the same convention as explained in Remark 2.8, regarding the measures $\overline{\delta}_{x_k}$. Because their support is contained in T^g, we interpret them as probability measures on T^g, rather than as probability measures on the larger space ${\mathbb{C}_\infty}^g$.

We did not include in our Conjecture 4.4 the hypothesis that ϕ has modular transcendence degree at least 2 (as we did in our Theorem 2.7), because, as mentioned before, it is believed that Breuer's and Pink's result holds without this extra hypothesis.

We require in our Conjecture 4.4 the hypothesis on the sequence $\{x_k\}$ being strict because otherwise the support of the limit measure would lie on a proper subvariety of T^g. In the case $g = 1$, our hypothesis in Theorem 2.7 that the sequence $\{x_k\} \subset K^{\mathrm{alg}}$ contains distinct torsion points suffices for the condition that the sequence is strict (because the torsion subvarieties of $\mathbb{G}_a$ are the torsion points of ϕ). Actually, our proof of Theorem 2.7 follows precisely the same way under the slightly weaker hypothesis that the sequence $\{x_k\} \subset \phi_{\mathrm{tor}}$ contains each torsion point of ϕ at most finitely many times (this condition being equivalent with the condition that $\{x_k\}$ is strict).

A positive answer to our Conjecture 4.4 would provide a proof to the following result (the Manin-Mumford Theorem for Drinfeld modules of generic characteristic).

Theorem 4.6. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic. Assume $\mathrm{End}_{\mathrm{K^{sep}}}(\phi) = A$. Let $g \geq 1$ and let X be an irreducible K^{sep}-subvariety of $\mathbb{G}_a^g$ (i.e., an irreducible closed subvariety of the g-dimensional affine space). If $X(K^{\mathrm{sep}}) \cap \phi_{\mathrm{tor}}^g$ is Zariski dense in X, then X is a torsion subvariety of $\mathbb{G}_a^g$.*

As mentioned in Section 1, Theorem 4.6 was proved by Scanlon in [13] using the methods of model theory of difference fields. His result is valid even without the extra assumption that the endomorphism ring of ϕ equals A. However, a positive answer to Conjecture 4.4 would provide a completely different proof of Theorem 4.6, given purely in the language of number theory.

Moreover, we believe that an equidistribution result, similar to the results proved by Bilu, Ullmo and Zhang, holds also for points of small height associated to the action of a Drinfeld module (see [9] for the definition of the height $\widehat{\mathrm{h}}$ associated to a Drinfeld module). The only explicit property of heights which we will use later is that the height of a point in an affine space is the sum of the heights of its coordinates.

Conjecture 4.7. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic and let $g \geq 1$. Assume $\mathrm{End}_{\mathrm{K^{sep}}}(\phi) = A$. Let $\{x_k\} \subset \mathbb{G}_a^g(K^{\mathrm{sep}})$ be a strict sequence. If $\lim_{k\to\infty} \widehat{\mathrm{h}}(x_k) = 0$, then $\overline{\delta}_{x_k} \xrightarrow{w} \nu^{(g)}$.

Remarks 4.8. The measures $\overline{\delta}_{x_k}$ are probability measures on ${\mathbb{C}_\infty}^g$, while $\nu^{(g)}$ is the normalized Haar measure on T^g. Therefore, we interpret the conclusion of Conjecture 4.7 as follows: the measures $\overline{\delta}_{x_k}$ converge weakly to the probability measure $\nu^{(g)}$ on ${\mathbb{C}_\infty}^g$, which is supported on T^g (and the restriction of $\nu^{(g)}$ on T^g is a Haar measure).

Conjecture 4.4 is a particular case of Conjecture 4.7 because all the torsion points of a Drinfeld module have height 0. Hence, as explained in Section 1, Theorem 2.7 would follow also from the results of [1] with the extra argument provided by us in Section 1. Moreover, the results of [1] are unconditional and do not require the hypothesis on $\mathrm{End}_{\mathrm{K^{sep}}}(\phi)$ or on the modular transcendence degree of ϕ.

An equidistribution result as Conjecture 4.7 would lead to the following form of the Bogomolov Conjecture in the context of Drinfeld modules of generic characteristic.

Theorem 4.9. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic. Assume $\mathrm{End}_{K^{\mathrm{sep}}}(\phi) = A$. Let $g \geq 1$ and let X be an irreducible K^{sep}-subvariety of $\mathbb{G}_a^g$. For each $n \geq 1$, we let*

$$X_n := \{x \in X(K^{\mathrm{sep}}) \mid \widehat{\mathrm{h}}(x) < \frac{1}{n}\}.$$

If for each $n \geq 1$, X_n is Zariski dense in X, then X is a torsion subvariety of $\mathbb{G}_a^g$.

As noted in Section 1, Theorem 4.9 was proved by Bosser in [4]. Actually, he proved the above result without requiring any hypothesis on the endomorphism ring of ϕ. We will show how Theorem 4.9 follows from Conjecture 4.7. Our methods are completely different than the methods employed in [4]. We require the hypothesis on $\mathrm{End}_{K^{\mathrm{sep}}}(\phi)$ in Theorem 4.9 because we also formulated Conjecture 4.7 under the same hypothesis.

We can also formulate the Manin-Mumford and the Bogomolov questions for Drinfeld modules of finite characteristic. However, we cannot always expect the conclusion be that the variety X is a torsion subvariety. This phenomen appears also in the case of the Mordell-Lang problem for Drinfeld modules (see [10]).

We will show how to deduce Theorem 4.6 and Theorem 4.9 from Conjecture 4.4 and, respectively Conjecture 4.7. More precisely, we will show that knowing the validity of Conjectures 4.4 and 4.7 for all $g \leq N$ yields the conclusions of Theorem 4.6 and, respectively Theorem 4.9 for $g = N$. Our proofs are inspired by the arguments from the proof of Theorem 5.1 of [6].

We first need a relative notion of the condition that a sequence is strict.

Definition 4.10. Let Y be an algebraic ϕ-submodule of $\mathbb{G}_a^N$ and let $\{x_k\}_{k \geq 1} \subset Y(K^{\mathrm{sep}})$ be a sequence of points in Y. We call the sequence $\{x_k\}$ strict relative to Y if any torsion subvariety of Y of dimension smaller than $\dim(Y)$ contains x_k for only finitely many values of k.

We will also use the following result.

Lemma 4.11. *Let Y be a subvariety of $\mathbb{G}_a^g$ such that $\phi_t(Y) \subset Y$. Then each irreducible component of Y is a torsion subvariety. Moreover, each irreducible component of Y is a translate by a torsion point of an irreducible component of Y which passes through* 0.

Proof. Let $Y_1, \dots, Y_m$ be the irreducible components of Y. Then for each $i \in \{1, \dots, m\}$ and for each $n \geq 1$, there exists $j(i,n) \in \{1, \dots, m\}$ such that $\phi_{t^n}(Y_i) = Y_{j(i,n)}$ (because $\phi_{t^n}(Y_i)$ is also an irreducible component of Y). Hence, for each $i \in \{1, \dots, m\}$, there exist positive integers $n_1 < n_2$ (depending on i) such that $\phi_{t^{n_1}}(Y_i) = \phi_{t^{n_2}}(Y_i)$. Thus, $\phi_{t^{n_1}}(Y_i)$ is invariant under $\phi_{t^{n_2 - n_1}}$. Using Lemme 4 of [8] we conclude $\phi_{t^{n_1}}(Y_i)$ is a torsion subvariety. Therefore, using that Y_i is irreducible, we obtain that also Y_i is a torsion subvariety.

Let $Y_i = \alpha + Z$ where α is a torsion point and Z is an algebraic ϕ-submodule. Let $a \in A \setminus \{0\}$ sarisfy $\phi_a(\alpha) = 0$. Then $\phi_a(Y_i) = \phi_a(Z) = Z$ is another irreducible component of Y. Hence, each irreducible component of Y is a translate by a torsion point of an irreducible component of Y which passes through 0. ☐

The following lemma is a classical result, whose proof we include for completeness.

Lemma 4.12. *Let S be an infinite set in K^{alg} and let $n \geq 1$. Then S^n (the cartesian product of S with itself n times) is Zariski dense in $\mathbb{G}_a^n$.*

Proof. We prove the statement of our lemma by induction on n. For $n = 1$, the statement is clear, as every infinite set is Zariski dense in the 1-dimensional affine space. Next we assume the lemma holds for n and we will prove it for $n + 1$.

Let $f \in K^{\text{alg}}[X_1, \dots, X_{n+1}]$ be a polynomial vanishing on S^{n+1}. We will prove $f = 0$, which will show that indeed, S^{n+1} is Zariski dense in $\mathbb{G}_a^{n+1}$. For each $\alpha \in S$, $f(X_1, \dots, X_n, \alpha)$ vanishes on $S^n \subset \mathbb{G}_a^n$. Using the induction hypothesis, we conclude

$$f(X_1, \dots, X_n, \alpha) = 0. \tag{47}$$

We consider $f \in K^{\text{alg}}(X_1, \dots, X_n)[X_{n+1}]$ as a polynomial of only the variable X_{n+1}. Because (47) holds for the (infinitely many) elements $\alpha \in S$, we conclude $f = 0$, as desired. $\square$

Because $\phi_{\text{tor}} \subset K^{\text{alg}}$ is infinite, we obtain the following consequence of Lemma 4.12.

Corollary 4.13. *For each $n \geq 1$, the torsion submodule ϕ_{tor}^n is Zariski dense in $\mathbb{G}_a^n$.*

Moreover, the following is also true.

Corollary 4.14. *Let $Y \subset \mathbb{G}_a^N$ be a torsion subvariety. Then $\phi_{\text{tor}}^N \cap Y$ is Zariski dense in Y.*

Proof. Let Y be the translate of the irreducible algebraic ϕ-module Y_0 by a torsion point in $\mathbb{G}_a^N$. It suffices to show that $Y_0 \cap \phi_{\text{tor}}^N$ is Zariski dense in Y_0. Let $M := \dim(Y_0)$. Then there exists a suitable finite-to-one, dominant projection π of Y_0 on M coordinates of $\mathbb{G}_a^N$. Then $\pi(Y_0) = \mathbb{G}_a^M$. Moreover, π has finite fibers.

Claim 4.15. The preimage of a torsion point of $\mathbb{G}_a^M$ through π^{-1} is a finite set of torsion points in Y_0.

Proof of Claim 4.15. Let x be a torsion point of $\mathbb{G}_a^M$. Let S_0 be the finite subset of torsion points of $\mathbb{G}_a^M$ containing the orbit of x under the action of ϕ. Because π has finite fibers, $S := \pi^{-1}(S_0)$ is a finite subset of Y_0. Moreover, because S_0 and π are invariant under ϕ, then also S is invariant under ϕ. Hence, S consists of only torsion points (because it is a finite set invariant under ϕ). Therefore, the preimage of x is indeed a finite set of torsion points in Y_0. $\square$

By Corollary 4.13, ϕ_{tor}^M is dense in $\mathbb{G}_a^M$. We conclude that the Zariski closure of $\pi^{-1}(\phi_{\text{tor}}^M) \subset Y_0$ has dimension M. Hence, it equals Y_0 (because Y_0 is irreducible). Thus $\pi^{-1}(\phi_{\text{tor}}^M)$ is a Zariski dense set of torsion points in Y_0 (see Claim 4.15). This concludes the proof of Corollary 4.14. $\square$

We first show that the validity of Conjecture 4.4 for all $g \leq N$ yields the following key result.

Theorem 4.16. *Let Y be an algebraic ϕ-submodule of $\mathbb{G}_a^N$ and let X be a K^{sep}-subvariety of Y. Then X has at most finitely many maximal torsion subvarieties.*

Proof of Theorem 4.16. We prove our theorem by induction on $\dim(Y)$. The case $\dim(Y) = 0$ is trivial, as then Y consists of only finitely many torsion points.

Assume Theorem 4.16 holds for $\dim(Y) < M \leq N$ and we will prove it for varieties of dimension M.

First we note that without loss of generality we may assume $\dim(X) < M$. Indeed, if $\dim(X) = M$, then the M-dimensional irreducible components of X are also irreducible components of Y. But the irreducible components of Y are torsion subvarieties by Lemma 4.11.

Removing the irreducible components of X which are also irreducible components of Y would make X have dimension strictly smaller than M and would not change the conclusion of Theorem 4.16, because we would remove only finitely many maximal torsion subvarieties of X. Hence, we may assume $\dim(X) < M$.

Secondly, we may replace X with

$$\bigcup_{\sigma \in \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})} \sigma(X)$$

and replace Y with

$$\bigcup_{\sigma \in \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})} \sigma(Y),$$

which remains an algebraic ϕ-module because the action of ϕ is invariant under $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$. If our Theorem 4.16 would fail for $X \subset Y$ in the first place, then it would also fail for the above varieties which replace them. The advantage of our reduction is that both Y and X are now invariant under $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$. Note that while making this reduction we do not change the dimension of X. So, we still have $\dim(X) < M$.

Assume X has infinitely many maximal torsion subvarieties and let $\{Y_i\}$ be a complete list of them. For distinct i and j, $Y_i \cap Y_j$ is a proper subvariety for both Y_i and Y_j (as both Y_i and Y_j are maximal torsion subvarieties of X). Moreover, because torsion subvarieties are irreducible by definition, we conclude

$$\dim(Y_i \cap Y_j) < \min\{\dim(Y_i), \dim(Y_j)\}. \tag{48}$$

Hence, using (48) we obtain that for each k,

$$\left(\cup_{i=1}^{k-1} Y_i\right) \cap Y_k$$

is a proper subvariety of Y_k. Because the torsion submodule ϕ^N_{tor} is Zariski dense in Y_k (see Corollary 4.14), we conclude that there exists a torsion point $x_k \in Y_k \setminus \left(\bigcup_{i=1}^{k-1} Y_i\right)$.

We will prove next that the sequence of torsion points $\{x_k\}$ is strict relative to Y. Let Z be a torsion subvariety of Y with $\dim(Z) < \dim(Y)$. We will show $X \cap Z$ contains finitely many x_k (note that by construction, $\{x_k\} \subset X$).

Because $\dim(Z) < M$, we can apply the induction hypothesis and conclude there are finitely many maximal torsion subvarieties of $X \cap Z$. Indeed, let $Z = \alpha + W$, where α is a torsion point and W is an irreducible algebraic ϕ-submodule. Then

$$X \cap Z = \alpha + ((-\alpha + X) \cap W). \tag{49}$$

Thus we apply the inductive hypothesis to $X' := (-\alpha + X) \cap W$ and derive that $X' \subset W$ contains finitely many maximal torsion subvarieties. Therefore, $X \cap Z = \alpha + X'$ contains finitely many maximal torsion subvarieties. Let $W_1, \dots, W_l \subset X \cap Z$ be a complete list of them. Because they are torsion subvarieties contained in X, for each $i \in \{1, \dots, l\}$, there exists j such that $W_i \subset Y_j$. But each Y_j contains only finitely many x_k, by the construction of $\{x_k\}$. Hence, each of W_i contains only finitely many of the x_k and thus, Z contains finitely many of $\{x_k\}$. This proves that the sequence $\{x_k\}$ is strict relative to Y.

Because $\dim(Y) = M$, there exists a suitable projection π of Y on M of the N coordinates of $\mathbb{G}_a^N$ such that π is a dominant morphism. At the expense of relabelling the coordinates of $\mathbb{G}_a^N$, we may assume $\pi : Y \to \mathbb{G}_a^M$. Moreover, because Y is an algebraic group, $\pi(Y)$ is also an algebraic group. Using $\dim(\pi(Y)) = M$ (because π is a dominant morphism), we conclude $\pi(Y) = \mathbb{G}_a^M$. Because $\dim(Y) = M = \dim(\pi(Y))$, we conclude each fiber of π is finite.

We claim the sequence $\{\pi(x_k)\} \subset \mathbb{G}_a^M(K^{\text{sep}})$ is strict. Assume there exists some proper torsion subvariety $Z := \alpha + W \subset \mathbb{G}_a^M$ which contains infinitely many $\pi(x_k)$ (α is a torsion point and W is a proper algebraic ϕ-submodule of $\mathbb{G}_a^M$). Let S be the finite orbit of α under the action of ϕ on $\mathbb{G}_a^M$. Let $Z_0 := \cup_{\beta\in S}(\beta + W)$. Clearly, Z_0 is a proper algebraic ϕ-submodule of $\mathbb{G}_a^M$ ($\dim(Z_0) = \dim(W) < M$). Moreover, by our assumption, Z_0 contains infinitely many $\pi(x_k)$.

Let $Z' := \pi^{-1}(Z_0) \subset Y$. Because Y is invariant under ϕ_t, then $\phi_t(Z') \subset Y$. Moreover,

$$\pi(\phi_t(Z')) = \phi_t(\pi(Z')) = \phi_t(Z_0) = Z_0.$$

Hence $\phi_t(Z') \subset Z'$. Using Lemma 4.11 we conclude Z' is a finite union of torsion subvarieties.

Because the kernel of π is finite, $\dim(Z') = \dim(Z_0) < M = \dim(Y)$. Moreover, because Z_0 contains infinitely many $\pi(x_k)$, Z' contains infinitely many x_k. Hence Z' is a finite union of torsion subvarieties of Y of dimension smaller than M, and Z' contains infinitely many x_k. This contradicts our hypothesis that $\{x_k\}$ is strict relative to Y. We conclude $\{\pi(x_k)\}$ is a strict sequence of torsion points in $\mathbb{G}_a^M(K^{\text{sep}})$.

Using Conjecture 4.4 for $g = M \leq N$ and for the strict sequence $\{\pi(x_k)\} \subset \phi_{\text{tor}}^M$, we conclude $\overline{\delta}_{\pi(x_k)} \xrightarrow{w} \nu^{(M)}$. By the second reduction step for our proof of Theorem 4.16, X is invariant under $\text{Gal}(\text{K}^{\text{sep}}/\text{K})$. Thus $\pi(X)$ is invariant under $\text{Gal}(\text{K}^{\text{sep}}/\text{K})$. Hence the measures $\overline{\delta}_{\pi(x_k)}$ are all supported on $\pi\left(X(K^{\text{sep}})\right)$. But $\pi(X(K^{\text{sep}})) \subset \pi(X(K^{\text{alg}}))$, which is a closed set in the v_∞-adic topology. Therefore, the weak limit $\nu^{(M)}$ is supported also on $\pi\left(X(K^{\text{alg}})\right)$. But, by construction, the support of $\nu^{(M)}$ is T^M, which contains the torsion submodule of $\mathbb{G}_a^M$. Therefore, $\pi(X)$ contains the torsion submodule of $\mathbb{G}_a^M$. As this torsion submodule is Zariski dense in $\mathbb{G}_a^M$ (see Corollary 4.13), we conclude $\pi(X) = \mathbb{G}_a^M$. Thus $\dim(X) = M = \dim(Y)$, which contradicts our first reduction step: $\dim(X) < M$. Therefore X has finitely many maximal torsion subvarieties. □

Theorem 4.6 for $g = N$ is an immediate corollary of Theorem 4.16.

Proof of Theorem 4.6. Assume X is not a torsion subvariety of $\mathbb{G}_a^N$. By Theorem 4.16 applied to $X \subset \mathbb{G}_a^N$, X is not the union Z of its maximal torsion subvarieties, because there are finitely many of them and each one has smaller dimension than X (here we use the irreduciblity of X). By construction, Z contains all the torsion points of X, which thus contradicts the hypothesis that the set of torsion points of $\mathbb{G}_a^N$ is dense in X. Therefore, X is indeed a torsion subvariety of $\mathbb{G}_a^N$. □

Assuming the validity of Conjecture 4.7 for all $g \leq N$, we prove the following generalization of Theorem 4.9 for $g = N$.

Theorem 4.17. *Let Y be an algebraic ϕ-submodule of $\mathbb{G}_a^N$. Let X be a K^{sep}-subvariety of Y and let Z be the union of all (finitely many) maximal torsion subvarieties of X. If $Z \neq X$, then there exists a positive constant C (depending on X) such that for each $x \in (X \setminus Z)(K^{\text{sep}})$, $\widehat{\text{h}}(x) \geq C$.*

We first note that because we assumed the validity of Conjecture 4.7 for all $g \leq N$, we also assume the validity of Conjecture 4.4 for all $g \leq N$, because Conjecture 4.4 is a particular case of Conjecture 4.7. Hence Theorem 4.16 holds and we *do* know that X has finitely many maximal torsion subvarieties.

Before proving Theorem 4.17, we sketch the proof of Theorem 4.9 using the result of Theorem 4.17 applied to $X \subset Y = \mathbb{G}_a^g$. If X is not a torsion subvariety, then it is not equal with the finite union Z of its maximal torsion subvarieties, because $\dim(Z) < \dim(X)$ (we also use here the fact that X is irreducible in Theorem 4.9). Hence, there exists $C > 0$ as in Theorem 4.17. Let n be a positive integer such that $\frac{1}{n} < C$. Then X_n (defined as in Theorem 4.9) is a subset of Z, which contradicts the hypothesis that X_n is dense in X. Therefore, our assumption was false and so, indeed X is a torsion subvariety.

Proof of Theorem 4.17. We proceed by induction on $\dim(Y)$. If $\dim(Y) = 0$, then both Y and X are finite unions of torsion points. Hence the theorem is vacuously true. We assume Theorem 4.17 holds for $\dim(Y) < M \leq N$ and we prove it also holds for $\dim(Y) = M$.

We may assume without loss of generality that $\dim(X) < M$. Otherwise, the irreducible components of X of dimension M are also irreducible components of Y and so, by Lemma 4.11, they are torsion subvarieties. Therefore, they are contained in Z. So, removing them will not change $X \setminus Z$.

At the expense of replacing Y and X by the respective finite unions of their orbits under the action of $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$, we may assume that both Y and X are invariant under $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$. Note that replacing X with $\cup_{\sigma\in\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})}\sigma(X)$, replaces $X \setminus Z$ with $\cup_{\sigma\in\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})}\sigma\left(X \setminus Z\right)$, which contains $X \setminus Z$.

Assume Theorem 4.17 does not hold for $X \subset Y$. Then we can find a sequence $\{x_k\} \subset (X \setminus Z)\,(K^{\mathrm{sep}})$ such that $\widehat{\mathrm{h}}(x_k) < \frac{1}{k}$ for each positive integer k. We prove next that the sequence $\{x_k\}$ is strict relative to Y.

Let Y' be a torsion subvariety of Y of dimension smaller than M and assume Y' contains infinitely many x_k. Let $Y' = \alpha + W$, where α is a torsion point and W is an (irreducible) algebraic ϕ-submodule of Y of dimension smaller than M. Then, as in (49),

$$X \cap Y' = \alpha + ((-\alpha + X) \cap W)$$

and so, we can apply the inductive hypothesis to $X \cap Y'$ (because $\dim(W) = \dim(Y') < M$). Note that the height function is not changed under translations by torsion points (this allows us to pass the inductive hypothesis from $(-\alpha + X) \cap W$ to $X \cap Y'$).

We conclude that either $X \cap Y'$ equals the finite union Z' of its maximal torsion subvarieties, or there exists a constant $C' > 0$ such that for every

$$x' \in ((X \cap Y') \setminus Z')\,(K^{\mathrm{sep}}),$$

$\widehat{\mathrm{h}}(x') \geq C'$. But the maximal torsion subvarieties of $X \cap Y'$ are contained in the maximal torsion subvarieties of X, which means that Z' contains no points from the sequence $\{x_k\} \subset (X \setminus Z)\,(K^{\mathrm{sep}})$. On the other hand,

$$\widehat{\mathrm{h}}(x_k) < \frac{1}{k} < C',$$

for every positive integer $k > \frac{1}{C'}$. So, there are finitely many points of the sequence $\{x_k\}$ contained in $((X \cap Y') \setminus Z')\,(K^{\mathrm{sep}})$. Therefore, Y' contains only finitely many points of the sequence $\{x_k\} \subset X$, which shows that indeed, $\{x_k\}$ is a strict sequence relative to Y.

We can redo now the argument from the last part of the proof of Theorem 4.16. Indeed, we find again a suitable projection $\pi : Y \to \mathbb{G}_a^M$ and prove as we did before, that $\{\pi(x_k)\}$ is a strict sequence (using that $\{x_k\}$ is a relative strict sequence). Because the height of a point in the affine space is the sum of the heights of each of its coordinates, $\widehat{\mathrm{h}}(\pi(x_k)) \le \widehat{\mathrm{h}}(x_k)$, for each k. Hence $\lim_{k\to\infty} \widehat{\mathrm{h}}(\pi(x_k)) = 0$. Thus, we can apply the conclusion of Conjecture 4.7 for the strict sequence $\{\pi(x_k)\} \subset \mathbb{G}_a^M(K^{\mathrm{sep}})$ of small points and conclude that $\overline{\delta}_{\pi(x_k)} \xrightarrow{w} \nu^{(M)}$. This shows that the support T^M of $\nu^{(M)}$ is contained in $\pi(X(K^{\mathrm{alg}}))$. Hence $\pi(X) = \mathbb{G}_a^M$ and so, $\dim(X) = \dim(Y)$, contradicting our assumption that $\dim(X) < M$. Therefore there exists a uniform positive lower bound C for the height of the points in $(X \setminus Z)\,(K^{\mathrm{sep}})$. $\square$

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