LINEAR FAMILIES OF SMOOTH HYPERSURFACES OVER FINITELY GENERATED FIELDS

SHAMIL ASGARLI, DRAGOS GHIOCA, AND ZINOVY REICHSTEIN

ABSTRACT. Let K be a finitely generated field. We construct an n-dimensional linear system $\mathcal{L}$ of hypersurfaces of degree d in $\mathbb{P}^n$ defined over K such that each member of $\mathcal{L}$ defined over K is smooth, under the hypothesis that the characteristic p does not divide $\gcd(d, n+1)$ (in particular, there is no restriction when K has characteristic 0). Moreover, we exhibit a counterexample when p divides $\gcd(d, n+1)$.

1. INTRODUCTION

The study of hypersurfaces varying in a pencil, or more generally, in a linear system of arbitrary dimension, is an active research area. For instance, determining the number of reducible members in a pencil is already a challenging problem [Ste89], [Vis93], [PY08]. When the base field is a number field, the study of pencils has deep connections to Diophantine geometry; see, for example [DGH21]. Linear systems of hypersurfaces over finite fields have been studied by Ballico [Bal07], [Bal09].

Our primary goal in the present paper is to address the following question from a recent paper [AG22] by the first two authors. While the version stated in [AG22] was concerned with linear systems of hypersurfaces over finite fields, in this paper we will work over an arbitrary *finitely generated field*. Recall that a field K is called finitely generated if it is generated by a finite number of elements as a field (or equivalently, as a field extension of its prime subfield).

Question 1. Let K be a finitely generated field and $r \geqslant 1$, $n \geqslant 2$, $d \geqslant 2$ be integers. Do there exist $r+1$ linearly independent homogeneous polynomials $F_0, F_1, ..., F_r \in K[x_0, ..., x_n]$ of degree d such that the hypersurface

$$X_{[a_0:a_1:...:a_r]} = \{a_0F_0 + a_1F_1 + ... + a_rF_r = 0\} \subset \mathbb{P}^n$$

is smooth for every $[a_0 : a_1 : ... : a_r] \in \mathbb{P}^r(K)$?

Here, as usual, "smooth" means "smooth at every $\overline{K}$-point", not just at every K-point. Question 1 can be rephrased in geometric terms as follows. Consider the linear system $\mathcal{L} = \langle F_0, ..., F_r \rangle$ of (projective) dimension r spanned by $F_0, \ldots, F_r$. We say that $\mathcal{L}$ is K-*smooth* if for every $[a_0, \ldots, a_r] \in \mathbb{P}^r(K)$, the hypersurface cut out by $a_0F_0 + a_1F_1 + ... + a_rF_r$ is smooth in $\mathbb{P}^n$. In other words, Question 1 asks for existence of a K-smooth linear system $\mathcal{L}$ in $\mathbb{P}^n$ of prescribed degree and dimension.

We show that, under a mild assumption on the characteristic, the maximum value of r for which Question 1 has a positive answer is $r = n$.

2020 *Mathematics Subject Classification.* Primary 14N05; Secondary 14J70, 14G15.
Key words and phrases. linear system, hypersurface, finite fields, smoothness.

Theorem 2. *Let K be an arbitrary field.*

(1) *If $r \geqslant n+1$, then there does not exist a K-smooth linear system of (projective) dimension r (of any degree $d \geqslant 2$).*
(2) *Suppose K is a finitely generated field of characteristic $p \geqslant 0$. If $r \leqslant n$ and $p \nmid \gcd(d, n+1)$, then there exist homogeneous polynomials $F_0, \ldots, F_r$ in $x_0, \ldots, x_n$ of degree d such that $\mathcal{L} = \langle F_0, \ldots, F_r \rangle$ is a K-smooth linear system of (projective) dimension r.*

Note that the assumption $p \nmid \gcd(d, p+1)$ on the characteristic of K holds automatically when $\mathrm{char}(K) = 0$. On the other hand, we will show in Section 5 that this assumption *cannot* be dropped in general. More precisely, will show that no n-dimensional linear system of hypersurfaces in $\mathbb{P}^n$ can be K-smooth in the case where K is a field of characteristic 2, $d = 2$ and n is an arbitrary odd integer; see Theorem 6.

The case where $r = 1$, which corresponds to a pencil of hypersurfaces, is of particular interest. For any given n, the condition that $p \nmid \gcd(d, n+1)$ is satisfied for all but finitely many characteristics p. In particular, Theorem 2 tells us that for every value of $d \geqslant 1$ and every finitely generated field K there exists

- a K-smooth pencil of degree d in $\mathbb{P}^2$ if $\mathrm{char}(K) \neq 3$.
- a K-smooth pencil of degree d in $\mathbb{P}^3$ if $\mathrm{char}(K) \neq 2$.
- a K-smooth pencil of degree d in $\mathbb{P}^4$ if $\mathrm{char}(K) \neq 5$.
- a K-smooth pencil of degree d in $\mathbb{P}^5$ if $\mathrm{char}(K) \neq 2, 3$.

On the other hand, the main result [AG22, Theorem 1.3] proves the existence of a K-smooth pencil $\mathcal{L}$ of degree d hypersurfaces in $\mathbb{P}^n$ defined over the field $K = \mathbb{F}_q$ under a different hypothesis:

$$q > \left(\frac{1+\sqrt{2}}{2} \right)^2 ((n+1)(d-1)^n)^2 \left((n+1)(d-1)^n - 1\right)^2 \left((n+1)(d-1)^n - 2\right)^2 .$$

In particular, an $\mathbb{F}_q$ smooth pencil of degree d hypersurfaces exists in any characteristic as long as q is sufficiently large. It is reasonable to ask if smooth pencils exist over every finitely generated field.

Acknowledgements. In an earlier version of this paper our main result, Theorem 2(2), was only stated for finite fields. We are grateful to Angelo Vistoli for suggesting that it can be extended to finitely generated fields and contributing the inductive argument of Section 4.

The first author is supported by a postdoctoral research fellowship from the University of British Columbia and the NSERC PDF award. The second and third authors are supported by NSERC Discovery grants.

2. Proof of Theorem 2(1)

In this section K will denote an arbitrary field. We will denote by $K[x_0, \ldots, x_n]_d$ the space of homogeneous polynomials of degree d in $x_0, \ldots, x_n$ with coefficients in K. This is a K-vector space of dimension $N = \binom{n+d}{d}$. The projective space $\mathbb{P}(K[x_0, \ldots, x_n]_d)$ is naturally identified with degree d hypersurfaces in $\mathbb{P}^n$.

We now proceed with the proof of part Theorem 2(1). Assume the contrary: there exists a K-smooth linear system $\mathcal{L} \subset K[x_0, \ldots, x_n]_d$ of (affine) dimension $\geqslant n+2$.

Let $x_0^{d-1}K[x_0,\ldots,x_n]_1$ denote the $(n+1)$-dimensional K-vector space of degree d forms divisible by x_0^{d-1}. Any such form can be written as $x_0^{d-1}l(x_0,\ldots,x_n)$, where $l \in K[x_0,\ldots,x_n]_1$. Consider the K-linear map

$$\Psi\colon K[x_0,\ldots,x_n]_d \to x_0^{d-1}K[x_0,\ldots,x_n]_1$$

which removes from $F \in \mathcal{L}(K)$ all monomials which are not multiples of x_0^{d-1}. In other words, for any non-negative integers $i_0,\ldots,i_n$ satisfying $i_0+\ldots+i_n=d$,

$$\Psi(x_0^{i_0}x_1^{i_1}\ldots x_n^{i_n}) = \begin{cases} x_0^{i_0}x_1^{i_1}\ldots x_n^{i_n}, \text{ if } i_0 \geqslant d-1, \text{ and} \\ 0, \text{ otherwise.}\end{cases}$$

The kernel, $\mathrm{Ker}(\Psi)$, is precisely the set of polynomials $F \in K[x_0,\ldots,x_n]_d$ with the property that the associated hypersurface in $\mathbb{P}^n$ is singular at $P=[1:0:\ldots:0]$. Since the codimension of $\mathrm{Ker}(\Psi)$ in $K[x_0,\ldots,x_n]_d$ is at least $\dim(x_0^{d-1}K[x_0,\ldots,x_n]_1) = n+1$ and $\dim(\mathcal{L}) \geqslant n+2$, we see that $\mathcal{L}\cap\mathrm{Ker}(\Psi)$ must contain a non-zero K-point of $\mathcal{L}$. In other words, $\mathcal{L}(K)$ contains a hypersurface which is singular at P. This shows that $\mathcal{L}$ cannot be K-smooth. □

3. Proof of Theorem 2(2) in the case, where K is a finite field

We begin by exhibiting two families of smooth hypersurfaces.

Lemma 3. *Suppose $p \nmid d$. Set $F = c_0x_0^d + c_1x_1^d + \ldots + c_nx_n^d$. If $c_0, c_1, \ldots, c_n \neq 0$, then F cuts out a smooth hypersurface in $\mathbb{P}^n$.*

Proof. This is clear from the Jacobian criterion: the equations

$$\frac{\partial F}{\partial x_i} = dc_ix_i^{d-1} = 0 \quad (i=0,1,\ldots,n)$$

have no common solution in $\mathbb{P}^n$. □

Lemma 4. *Suppose $p \mid d$ but $p \nmid (n+1)$. Set $F = c_0x_0^{d-1}x_1 + c_1x_1^{d-1}x_2 + \ldots + c_nx_n^{d-1}x_0$. If $c_0, c_1, \ldots, c_n \neq 0$, then F cuts out a smooth hypersurface in $\mathbb{P}^n$.*

Proof. Assume the contrary: the hypersurface cut out by F in $\mathbb{P}^n$ is singular at some point $P = [u_0 : u_1 : \ldots : u_n] \in \mathbb{P}^n$. By symmetry we may assume without loss of generality that $u_1 \neq 0$. Using the Jacobian criterion, and remembering that $p \mid d$, we obtain:

$$\frac{\partial F}{\partial x_i}(P) = c_{i-1}u_{i-1}^{d-1} - c_iu_i^{d-2}u_{i+1} = 0 \tag{3.1}$$

for each $0 \leqslant i \leqslant n$, where the subscripts are taken modulo $n+1$. Multiplying both sides of (3.1) by u_i, we obtain

$$c_{i-1}u_{i-1}^{d-1}u_i = c_iu_i^{d-1}u_{i+1}. \tag{3.2}$$

Now recall that

$$F(P) = c_0u_0^{d-1}u_1 + c_1u_1^{d-1}u_2 + \ldots + c_nu_n^{d-1}u_0 = 0.$$

By (3.2), the n terms in this sum are all equal to each other. Hence,

$$0 = F(P) = \sum_{i=0}^{n} c_iu_i^{d-1}u_{i+1} = (n+1)c_0u_0^{d-1}u_1.$$

Since $p \nmid (n+1)$, $c_0 \neq 0$, and $u_1 \neq 0$, we conclude that $u_0 = 0$.

We will divide the remainder of the proof into two cases, according to whether $d = 2$ or $d \geqslant 2$. If $d \geqslant 2$, then (3.1) tells us that $u_i = 0$ implies $u_{i-1} = 0$ for any $i \in \mathbb{Z}/(n+1)\mathbb{Z}$. (Recall that the subscripts in (3.1) are viewed modulo $n+1$.) Using this implication recursively, starting from $u_0 = 0$, we see that $u_0 = u_n = u_{n-1} = \ldots = u_1 = 0$, a contradiction.

Now assume $d = 2$. In this case (3.1) tells us that $u_{i-1} = 0$ implies $u_{i+1} = 0$ for any $i \in \mathbb{Z}/(n+1)\mathbb{Z}$. Since we know that $u_0 = 0$, this tells us that $u_i = 0$ for every even i. Since $d = 2$, the assumption that p divides d tells us that $p = 2$ and the assumption that p does not divide $n + 1$ tells us that that $n = 2k$ is even. Thus, $2k + 2 \equiv 1$ modulo $n + 1$ and hence, $0 = u_{2k+2} = u_1 = 0$, a contradiction. □

We are now ready to prove Theorem 2(2) in the case, where $K = \mathbb{F}_q$ is a finite field. Since any K-linear subspace of a K-smooth linear system is again K-smooth, we may assume without loss of generality that $r = n$. Note also that $p \nmid \gcd(d, n+1)$ if and only if $p \nmid d$ or $p \nmid n + 1$. Thus we may consider two cases.

Case 1: $p \nmid d$. We will explicitly construct a linear system $\mathcal{L}$ of dimension $r = n$ with the desired property. By the normal basis theorem, we can find an element $\alpha \in \mathbb{F}_{q^{n+1}}$ such that $\alpha, \alpha^q, \alpha^{q^2}, ..., \alpha^{q^n}$ form an $\mathbb{F}_q$-basis for the $(n + 1)$-dimensional vector space $\mathbb{F}_{q^{n+1}}$. Let

$$\begin{aligned}
F_0 &= (\alpha x_0 + \alpha^q x_1 + \alpha^{q^2} x_2 + ... + \alpha^{q^i} x_i + ... + \alpha^{q^n} x_n)^d \\
F_1 &= (\alpha^q x_0 + \alpha^{q^2} x_1 + \alpha^{q^3} x_2 + ... + \alpha^{q^{i+1}} x_i + ... + \alpha x_n)^d \\
F_2 &= (\alpha^{q^2} x_0 + \alpha^{q^3} x_1 + \alpha^{q^4} x_2 + ... + \alpha^{q^{i+2}} x_i + ... + \alpha^q x_n)^d \\
&\vdots \\
F_n &= (\alpha^{q^n} x_0 + \alpha^q x_1 + \alpha^{q^2} x_2 + ... + \alpha^{q^{i+n}} x_i + ... + \alpha^{q^{n-1}} x_n)^d
\end{aligned}$$

Note that the polynomials F_i are not defined over $\mathbb{F}_q$. However, the set $\{F_0, F_1, ..., F_n\}$ is invariant under the action of the q-th power Frobenius map. Thus, the linear system $\mathcal{L} = \langle F_0, ..., F_n\rangle$ is defined over $\mathbb{F}_q$, that is, one can find a set of new generators $G_0, G_1, ..., G_n$ for $\mathcal{L}$ where *each* G_i is defined over $\mathbb{F}_q$. Furthermore, we claim that $F_0, F_1, ..., F_n$ are linearly independent. To see this, let

$$y_j = \alpha^{q^j} x_0 + \alpha^{q^{j+1}} x_1 + \alpha^{q^{j+2}} x_2 + ... + \alpha^{q^{j+i}} x_i + ... + \alpha^{q^{j+n}} x_n \tag{3.3}$$

for each $0 \leqslant j \leqslant n$, and observe that $F_i = y_i^d$. The linear map $x_i \mapsto y_i$ is a linear automorphism of $\mathbb{P}^n$. Indeed, the matrix of this linear transformation, known as a *Moore matrix*, is known to be non-singular. Thus, y_i are algebraically independent, which in particular implies that $\{F_0, F_1, ..., F_n\}$ is a linearly independent set. Hence, $\mathcal{L} = \langle F_0, F_1, ..., F_n\rangle$ has (projective) dimension $r = n$.

We claim that each $\mathbb{F}_q$-member of $\mathcal{L}$ defines a smooth hypersurface. Suppose that we have a singular hypersurface X which belongs to $\mathcal{L}$:

$$X = \{c_0 F_0 + c_1 F_1 + ... + c_n F_n = 0\} \tag{3.4}$$

for some $c_i \in \overline{\mathbb{F}_q}$ where not all c_i are zero. In the new coordinates y_i, we can express (3.4) as:

$$X = \{c_0 y_0^d + c_1 y_1^d + ... + c_n y_n^d = 0\}$$

Since X is singular, we can apply Lemma 3 to deduce that $c_i = 0$ for some i. Without loss of generality, we assume $c_0 = 0$. By applying the Frobenius map, we see that X is sent to:

$$X^{\sigma} = \{c_1^q F_2 + ... + c_n^q F_0 = 0\}$$

We claim that X and X^{σ} are distinct. Indeed, their defining equations are not multiples of one another: otherwise, there would exist a constant b such that $c_i^q = b \cdot c_{i+1}$ for each $0 \leqslant i \leqslant n$ taken modulo $n+1$. As $c_0 = 0$, this would force $c_i = 0$ for each $0 \leqslant i \leqslant n$, which is a contradiction. Thus, X is not defined over $\mathbb{F}_q$. We have shown that each singular member of $\mathcal{L}$ is not defined over $\mathbb{F}_q$ giving us the desired conclusion.

Case 2: $p \mid d$ but $p \nmid (n+1)$. Define $y_0, \ldots, y_n$ by the formula (3.3), and set $F_i = y_i^q y_{i+1}$ for $0 \leqslant i \leqslant n-1$ and $F_n = y_n^q y_0$. Arguing as in Case 1, one readily checks that $\mathcal{L} = \langle F_0, F_1, ..., F_n \rangle$ is a linear subspace of (projective) dimension n defined over $\mathbb{F}_q$. Moreover, the same argument as in Case 1, with Lemma 4 used in place of Lemma 3, shows that $\mathcal{L}$ is $\mathbb{F}_q$-smooth.

This completes the proof of Theorem 2(2) in the case, where $K = \mathbb{F}_q$ is a finite field. □

4. Conclusion of the proof of Theorem 2(2)

Given a finitely generated field K, we define its dimension $\dim(K)$ to be Krull the dimension of any finitely generated $\mathbb{Z}$-algebra whose fraction field is K. In other words, $\dim(K) = \operatorname{trdeg}_{\mathbb{F}_p}(K)$ if $\operatorname{char}(K) = p > 0$ and $\dim(K) = 1 + \operatorname{trdeg}_{\mathbb{Q}}(K)$ if $\operatorname{char}(K) = 0$. In this section we will prove Theorem 2(2) over an arbitrary finitely generated field K by induction on $\dim(K)$. The inductive step will be based on the following lemma.

Lemma 5. *Let R be discrete valuation ring with fraction field K and residue field L, and let $F_0, \ldots, F_r \in L[x_0, ..., x_n]$ be linearly independent homogeneous polynomials of degree d. Denote their liftings to R by $\overline{F_0}, \ldots \overline{F_r} \in R[x_0, ..., x_n] \subset K[x_0, ..., x_n]$, respectively. If the linear system $\langle F_0, \ldots, F_r \rangle$ is L-smooth, then the linear system $\langle \overline{F_0}, \ldots \overline{F_r} \rangle$ is K-smooth.*

Proof. Let $(a_0, \ldots, a_r)$ be in $K^{r+1} \backslash \{(0, \ldots, 0)\}$. We will show that the hypersurface in $\mathbb{P}^n_K$ defined by the form $a_0\overline{F_0} + \ldots + a_r\overline{F_r}$ is smooth. By scaling the a_i, we may assume that $a_i \in R$ for all i and a_i is invertible in R for at least one i. Consider the hypersurface $X \subset \mathbb{P}^n_K$ defined by $a_0\overline{F_0} + \cdots + a_r\overline{F_r} = 0$. Then X is flat over $\operatorname{Spec}(R)$ and its fiber over $\mathcal{L}$ is smooth by hypothesis. Since the smooth locus of the projection $X \to \operatorname{Spec}(R)$ is open in X, its complement must be empty. It follows that the fiber over the generic point of $\operatorname{Spec}(R)$ is smooth, as desired. □

We are now ready to finish the proof of Theorem 2(2) by induction on the dimension of the finitely generated field K. If $\dim(K) = 0$, then K is a finite field. In this case Theorem 2(2) is proved in Section 3. If $\dim(K) > 0$, then it is easy to see that K admits a discrete valuation with finitely generated residue field L such that $\dim(L) = \dim(K) - 1$. Furthermore, if $\operatorname{char}(K) = 0$, then this valuation can be chosen so that $\operatorname{char}(L)$ is positive and arbitrarily large. By applying Lemma 5, we can lift an L-smooth linear system of hypersurfaces in $\mathbb{P}^n$ to a K-smooth linear system of hypersurfaces in $\mathbb{P}^n$ of the same degree degree and the same dimension. □

5. Quadrics in characteristic 2

In this section, we will show that the hypothesis $p \nmid \gcd(d, n+1)$ in our main theorem cannot be removed in general. We will focus on the case, where $p = d = 2$ and n is odd. Our goal is to prove the following result.

Theorem 6. *Suppose n is an odd positive integer, and K be a field of characteristic 2 (not necessarily finitely generated). Then for any $d \geqslant 2$ there does* ***not*** *exist a linear system $\mathcal{L} = \langle F_0, \dots, F_n \rangle \subset K[x_0, \dots, x_n]_2$ of (projective) dimension n over K such that each K-member of $\mathcal{L}$ is a smooth quadric hypersurface in $\mathbb{P}^n$.*

In particular, Question 1 has a negative answer when $q = 2^s$, $d = 2$ and $r = n$ is an odd integer.

We begin with the following lemma.

Lemma 7. *Let K be a field of characteristic 2 and $n \geqslant 1$ be an odd integer. Consider a quadric hypersurface $X \subset \mathbb{P}^n$ cut out by*

$$F(x_0, \dots, x_n) = x_0^2 + G(x_1, x_2, ..., x_n)$$

where $G \in K[x_1, \dots, x_n]$ is a homogeneous polynomial of degree 2. Then X is singular.

Proof. The Jacobian criterion gives rise to a homogeneous system

$$\frac{\partial G}{\partial x_1} = \dots = \frac{\partial G}{\partial x_n} = 0$$

of n linear equations in $x_1, \dots, x_n$. (Note x_0 never appears in this system.) We claim that this homogeneous linear system has a nontrivial solution. To prove the claim, it suffices to show that the matrix M of this linear system is singular. Note that M is the Hessian matrix of G and hence, is symmetric. (Since G is a quadratic polynomial, the entries of the Hessian matrix are constant.) Because we are in characteristic 2, M is also skew-symmetric. It remains to show that a skew-symmetric square $n \times n$ matrix M over any commutative ring has zero determinant, when n is odd.

Indeed, consider the universal skew-symmetric matrix $n \times n$ matrix A over the polynomial ring $R = \mathbb{Z}[x_{ij} | 1 \leqslant i < j \leqslant n]$. By definition, the (i, j)-th entry of A is x_{ij} if $i < j$, 0 if $i = j$ and $-x_{ij}$ if $i > j$. Taking the determinant on both sides of $A^T = -A$, and remembering that n is odd, we obtain $\det(A) = -\det(A)$ in R. Since R is an integral domain of characteristic 0, this implies that $\det(A) = 0$. A simple specialization argument (specializing x_{ij} to the (i, j)-th entry of M) now shows that $\det(M) = 0$, as desired.

Thus, we have found $(0, \dots, 0) \neq (t_1, \dots, t_n) \in K^n$ such that for any point $P \in \mathbb{P}^n$ of the form $P = [t_0 : t_1 : t_2 : ... : t_n]$, we have

$$\frac{\partial F}{\partial x_0}(P) = \dots = \frac{\partial F}{\partial x_n} = 0 \tag{5.1}$$

Note that since n is odd and we are in characteristic 2, conditions (5.1) do not guarantee that $F(P) = 0$. On the other hand, the partial derivatives of $F(x_0, \dots, x_n)$ depend only on $x_1, \dots, x_n$ and not on x_0. We thus want to choose t_0 so that the resulting point $P = [t_0 : \dots : t_n]$ lies on the hypersurface X cut out by F. To achieve this goal, we choose $t_0 \in \overline{K}$ so that

$$t_0^2 = -G(t_1, t_2, ..., t_n).$$

Then $P = [t_0 : t_1 : ... : t_n] \in \mathbb{P}^n(\overline{K})$ satisfies both (5.1) and $F(P) = 0$. In other words, X is singular at P. □

Remark 8. If K is a finite field in characteristic 2, then the above construction gives rise to a singular point $P = [t_0, \dots, t_n]$ of X defined over K. This is because K is closed under taking square roots, so we can always choose $t_0 \in K$ in the last step.

Remark 9. The conclusion of Lemma 7 is false when $n = 2k$ is even. Indeed, the quadric hypersurface in $\mathbb{P}^n$ defined by the polynomial

$$x_0^2 + x_1x_2 + x_3x_4 + ... + x_{2k-1}x_{2k} = 0$$

is smooth.

We now proceed with a proof of Theorem 6.

Proof of Theorem 6. Suppose, to the contrary, that $\mathcal{L} = \langle F_0, ..., F_n \rangle$ is a K-smooth linear system of quadric hypersurfaces of (projective) dimension n. Let $\mathcal{L}(K)$ denote the set of K-members of the system.

Consider the K-linear map

$$\Psi : K[x_0, \dots, x_n]_2 \to x_0K[x_0, \dots, x_n]_1$$

introduced in Section 2 (with $d = 2$). Recall that $x_0K[x_0, \dots, x_n]_1$ denotes the $(n+1)$-dimensional K-vector space of quadratic forms in $x_0, \dots, x_n$ divisible by x_0 and that Ψ removes from $F \in K[x_0, \dots, x_n]$ all monomials which are not multiples of x_0. When $d = 2$, Ψ is given by the simple formula

$$(\Psi F)(x_0, \dots, x_n) = F(x_0, x_1, \dots, x_n) - F(0, x_1, \dots, x_n).$$

As we noted in Section 2, F lies in the kernel of Ψ if and only if the hypersurface in $\mathbb{P}^n$ cut out by F is singular at the point $[1 : 0 : \dots : 0]$. Since the linear system $\mathcal{L}$ is K-smooth, this tells us that the restricted map

$$\Psi : \mathcal{L}(K) \to x_0K[x_0, \dots, x_n]_1$$

is injective. Since the vector spaces $\mathcal{L}(K)$ and $x_0K[x_0, \dots, x_n]_1$ are of the same dimension $n + 1$, we conclude that Ψ must also be surjective. In particular, there exists some $F \in \mathcal{L}(K)$ whose image under Ψ is x_0^2. In other words,

$$F(x_0, \dots, x_n) = x_0^2 + G(x_1, ..., x_n)$$

for some quadratic form G in $x_1, \dots, x_n$. By Lemma 7, F cuts out a singular quadric hypersurface. This contradicts the assumption that each K-member of $\mathcal{L}$ is smooth. We conclude that a K-smooth linear system $\mathcal{L}$ of quadric hypersurfaces in $\mathbb{P}^n$ of dimension n does not exist. □

We have shown that the hypothesis $p \nmid \gcd(d, n + 1)$ of Theorem 2(2) cannot be removed in the case $p = 2$. We do not know whether this assumption can be dropped for other primes p. We finish the paper with an example, which shows that it can be for one particular choice of K, p, d, and n.

Example 10. Set $d = 3$ and $n = 2$ and consider the following cubic homogeneous polynomials with coefficients in $K = \mathbb{F}_3$:

$$F_0 = x^3 + x^2y - xy^2 + y^3 + x^2z + xyz + y^2z - xz^2 + z^3$$
$$F_1 = x^3 + x^2y - x^2z - xyz + y^2z + z^3$$
$$F_2 = x^3 - x^2y + xy^2 + y^3 + x^2z + xyz + y^2z - yz^2$$

A computer calculation shows that $aF_0 + bF_1 + cF_2 = 0$ defines a smooth plane curve for each of the possible $q^2 + q + 1 = 13$ choices $[a : b : c] \in \mathbb{P}^2(\mathbb{F}_3)$. In other words, $\langle F_0, F_1, F_2 \rangle$ is a $\mathbb{F}_3$-smooth linear system of (projective) dimension $n = 2$. Thus the conclusion of Theorem 2(2) holds in this example, even though p divides $\gcd(d, n + 1)$.

References

[AG22] Shamil Asgarli and Dragos Ghioca, *Smoothness in pencils of hypersurfaces over finite fields*, Bull. Aust. Math. Soc. (to appear), arXiv e-prints (2022), available at `https://arxiv.org/abs/2203.07169`.

[Bal07] E. Ballico, *Bertini's theorem over a finite field for linear systems of quadrics*, Int. J. Pure Appl. Math. **35** (2007), no. 4, 453–455.

[Bal09] ———, *Vanishings and non-vanishings of homogeneous forms over a finite field*, Int. J. Pure Appl. Math. **57** (2009), no. 2, 219–224.

[DGH21] Vesselin Dimitrov, Ziyang Gao, and Philipp Habegger, *Uniform bound for the number of rational points on a pencil of curves*, Int. Math. Res. Not. IMRN **2** (2021), 1138–1159, DOI 10.1093/imrn/rnz248.

[PY08] Jorge Vitório Pereira and Sergey Yuzvinsky, *Completely reducible hypersurfaces in a pencil*, Adv. Math. **219** (2008), no. 2, 672–688, DOI 10.1016/j.aim.2008.05.014.

[Ste89] Yosef Stein, *The total reducibility order of a polynomial in two variables*, Israel J. Math. **68** (1989), no. 1, 109–122, DOI 10.1007/BF02764973.

[Vis93] Angelo Vistoli, *The number of reducible hypersurfaces in a pencil*, Invent. Math. **112** (1993), no. 2, 247–262, DOI 10.1007/BF01232434.

Department of Mathematics and Computer Science, Santa Clara University, 500 El Camino Real, USA 95053

Email address: `sasgarli@scu.edu`

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2

Email address: `dghioca@math.ubc.ca`

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2

Email address: `reichst@math.ubc.ca`