THE MORDELL-LANG CONJECTURE FOR SEMIABELIAN VARIETIES DEFINED OVER FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. Let G be a semiabelian variety defined over an algebraically closed field K of prime characteristic. We describe the intersection of a subvariety X of G with a finitely generated subgroup of $G(K)$.

1. Introduction

The purpose of this note is to prove a variant of the Mordell-Lang conjecture for semiabelian varieties defined over fields of positive characteristic. More precisely, let G be a semiabelian variety defined over an algebraically closed field K, i.e., there exists a short exact sequence of algebraic groups defined over K:

$$1 \longrightarrow \mathbb{G}_m^N \longrightarrow G \longrightarrow A \longrightarrow 1, \tag{1.0.1}$$

where $N \geq 0$ is an integer and A is an abelian variety. Assuming K has characteristic $p > 0$, then for any subvariety $X \subseteq G$ defined over K and any finitely generated subgroup $\Gamma \subset G(K)$, we describe the intersection $X(K) \cap \Gamma$. In particular, we fix an error in the paper [Ghi08] of the first author where a simplified form of the aforementioned was claimed in the case G is defined over a finite subfield of K; we present several examples showing that the intersection $X(K) \cap \Gamma$ involves the more general F-sets appearing in Definition 1.5.

1.1. General background.

The Mordell-Lang conjecture for semiabelian varieties G defined over fields of characteristic 0 predicts that the intersection of a subvariety $X \subseteq G$ with a finitely generated subgroup Γ of G is a finite union of cosets of subgroups of Γ. This conjecture was proven by Laurent [Lau84] in the case of tori, Faltings [Fal91] in the case of abelian varieties, and by Vojta [Voj96] in the general case of semiabelian varieties. In particular, their results show that if X is an irreducible subvariety of G which intersects a finitely generated group in a Zariski dense subset, then X must be a translate of a semiabelian subvariety of G.

The picture for positive characteristic fields K is more complicated due to the existence of the Frobenius endomorphism for varieties defined over finite fields; in particular, it is no longer true that only translates of semiabelian subvarieties of G have the property that they intersect a finitely generated subgroup of G in a Zariski dense subset. Hrushovski [Hru96] obtained the right shape for the irreducible subvarieties X whose intersection with a finitely generated subgroup Γ is Zariski dense.

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Theorem 1.1 (Hrushovski [Hru96])**.** *Let G be a semiabelian variety defined over an algebraically closed field K of characteristic p. Let $\Gamma\subset G(K)$ be a finitely generated subgroup and let $X\subseteq G$ be an irreducible subvariety with the property that $X(K)\cap\Gamma$ is Zariski dense in X. Then there exists $\gamma\in G(K)$, there exists a semiabelian subvariety $G_0\subseteq G$ defined over K, there exists a semiabelian variety H along with a subvariety $X_0\subseteq H$ both defined over a finite subfield $\mathbb{F}_q$ of K, and there exists a surjective group homomorphism $h: G_0\longrightarrow H$ such that $X=\gamma+h^{-1}(X_0)$.*

However, [Hru96] left open the description of the actual intersection between the subvariety X and the group Γ; next, we will address exactly this issue.

1.2. **The case of semiabelian varieties defined over finite fields and of finitely generated subgroups invariant under the Frobenius endomorphism.** Essentially, Hrushovski's result (see Theorem 1.1) reduced the description of the intersection $X(K)\cap\Gamma$ to the case when the ambient semiabelian variety is defined over a finite field. Moosa and Scanlon [MS04, MS02] addressed precisely this problem under an additional assumption on the subgroup Γ; in order to state their main result, we introduce a little bit of notation.

Definition 1.2. *For a semiabelian variety G defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K of characteristic p, we define a* groupless F-set *any subset of $G(K)$ of the form:*

$$\left\{\alpha_0+\sum_{i=1}^r F^{kn_i}(\alpha_i)\colon n_i\in\mathbb{N}\right\}, \tag{1.0.2}$$

where $r\ge 0$, $\alpha_0,\alpha_1,\dots,\alpha_r\in G(K)$ and $k\in\mathbb{N}$, while F is the Frobenius endomorphism of G corresponding to the finite field $\mathbb{F}_q$.

For any finitely generated subgroup $\Gamma\subset G(K)$, we define a groupless F-set in Γ *as a groupless F-set contained in Γ. Also, an* F-set in Γ *is any set of the form $S+B$, where S is a groupless F-set in Γ and B is a subgroup of Γ (as always, for any two subsets B and C of G, we have that $C+B$ is simply the set of all $c+b$ where $b\in B$ and $c\in C$).*

Remark 1.3. In [MS04, Theorem B], Moosa and Scanlon allowed for the possibility that a groupless F-set involves sums of F-orbits as in equation (1.0.2) of the form

$$\alpha_0+\sum_{i=1}^r F^{k_in_i}(\alpha_i) \text{ (as } n_i \text{ vary in } \mathbb{N}), \tag{1.0.3}$$

for given, but potentially distinct, positive integers k_i. However, each F-set from equation (1.0.3) is a union of finitely many F-sets given as in Definition 1.2 (simply by working with k as the least common multiple of $k_1,\dots,k_r$).

Theorem 1.4 (Moosa-Scanlon [MS04])**.** *Let G be a semiabelian variety defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K and let $F: G\longrightarrow G$ be the Frobenius endomorphism associated to the finite field $\mathbb{F}_q$. Let $X\subseteq G$ be a subvariety defined over K and let $\Gamma\subset G(K)$ be a finitely generated subgroup. If Γ is invariant under F^ℓ for some $\ell\in\mathbb{N}$, then $X(K)\cap\Gamma$ is a finite union of F-sets in Γ.*

1.3. **The case of an arbitrary finitely generated subgroup.** It is natural to ask whether the above description from Theorem 1.4 of the intersection $X(K)\cap\Gamma$ remains valid also when Γ is no longer invariant under a power of the Frobenius endomorphism of G (but only assume Γ is finitely generated).

One could consider the $\mathbb{Z}[F]$-submodule $\tilde{\Gamma}\subset G(K)$ spanned by Γ and since F is integral over $\mathbb{Z}$ (seen as a subring of $\mathrm{End}(G)$), then $\tilde{\Gamma}$ is still finitely generated and so, Moosa-Scanlon's result (see Theorem 1.4) yields that $X(K)\cap\tilde{\Gamma}$ is a finite union of F-sets in $\tilde{\Gamma}$. So, the problem reduces to understanding the intersection of an F-set S in $\tilde{\Gamma}$ with Γ. The first author [Ghi08, Theorem 3.1] proved that when S is a *groupless* F-set in $\tilde{\Gamma}$, then its intersection with Γ is a finite union of groupless F-sets in Γ. Also, the first author analyzed in [Ghi08] the intersection with Γ of an arbitrary F-set in $\tilde{\Gamma}$; however, the final assertion from [Ghi08, Step 3, p. 3842] claiming that the general case of an F-set reduces to the groupless case is not valid, as shown by the constructions from Section 2 (see Examples 2.1 and 2.2 which were found by the second author). Essentially, the error from [Ghi08] was to claim that the pullback of a groupless F-set in $\tilde{\Gamma}$ through a group homomorphism restricted to Γ must be an F-set in Γ (as in Definition 1.2). Furthermore, Example 2.3 shows that when Γ is an arbitrary finitely generated subgroup, the intersection $X(K)\cap\Gamma$ can be quite wild; this motivates our Definition 1.5 which yields the right form of the sets appearing in the intersection of a subvariety of G with a finitely generated group.

Definition 1.5. *For a semiabelian variety G defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K of characteristic p and a finitely generated subgroup $\Gamma\subset G(K)$, we define a* generalized F-set in Γ *any subset of Γ of the form:*

$$(\pi|_{\Gamma})^{-1}(S), \tag{1.0.4}$$

where $\pi: G\longrightarrow H$ is a surjective group homomorphism of semiabelian varieties both defined over a finite subfield of K for which $\dim(\ker(\pi))>0$, $\pi|_{\Gamma}$ is its restriction to the subgroup Γ, and $S\subset H(K)$ is a groupless F-set in $\pi(\Gamma)$.

Note that H may be defined over another finite subfield $\mathbb{F}_{q'}\subset K$ and thus the set S from equation (1.0.4) is a groupless F-set in $\pi(\Gamma)$ where F stands for the Frobenius endomorphism of H associated to the finite field $\mathbb{F}_{q'}$.

1.4. **Our results.** Now we can state our main results, first for describing the intersection with a finitely generated group of a subvariety of a semiabelian variety defined over a finite field.

Theorem 1.6. *Let G be a semiabelian variety defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K of characteristic p. Let $X\subset G$ be a subvariety defined over K and let $\Gamma\subset G(K)$ be a finitely generated subgroup. Then the intersection $X(K)\cap\Gamma$ is a union of finitely many groupless F-sets in Γ along with finitely many generalized F-sets in Γ.*

Our Examples 2.1, 2.2 and 2.3 show that the sets appearing as intersections between a subvariety X of a semiabelian variety G defined over a finite field with a finitely generated subgroup can be quite complicated, well-beyond the world of F-sets from Definition 1.2. However, when X is a curve or G is a simple semiabelian variety (i.e., either a simple abelian variety or a 1-dimensional torus), then we can show that the intersection $X(K)\cap\Gamma$ is a finite union of F-sets in Γ.

Theorem 1.7. *Let G be a semiabelian variety defined over a finite subfield of an algebraically closed field K of prime characteristic, let $X \subseteq G$ be a subvariety defined over K and let $\Gamma \subset G(K)$ be a finitely generated subgroup. If either* $\dim(X) = 1$ *or G is a simple semiabelian variety, then $X(K) \cap \Gamma$ is a finite union of F-sets.*

Next, combining our Theorem 1.6 with Hrushovski's result (see Theorem 1.1), we obtain the description of the intersection of a subvariety of an arbitary semiabelian variety G defined over a field of prime characteristic with a finitely generated subgroup of G. For this end we introduce the notion of *pseudo-generalized F-sets.*

Definition 1.8. *Let G be a semiabelian variety defined over an algebraicaly closed field K of characteristic p and let $\Gamma \subset G(K)$ be a finitely generated subgroup. A* pseudo-generalized F-set in Γ *is a set of the form*

$$x_0 + \left(\pi|_{\Gamma_0}\right)^{-1}(S),$$

where $x_0 \in \Gamma$, $G_0 \subseteq G$ is a semiabelian subvariety, $\Gamma_0 = G_0(K) \cap \Gamma$, H is a semiabelian variety defined over a finite subfield $\mathbb{F}_q \subset K$, $\pi : G_0 \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties, and $S \subset H(K)$ is a groupless F-set in $\pi(\Gamma_0)$.

Remark 1.9. In Definition 1.8, if G is defined over a finite subfield of K, then the pseudo-generalized F-sets from Definition 1.8 cover both the groupless F-sets in Γ from Definition 1.2 and also the generalized F-sets in Γ from Definition 1.5, but they are a bit more general than those two types of sets.

Theorem 1.10. *Let G be a semiabelian variety defined over an algebraicaly closed field K of characteristic p, let $X \subseteq G$ be a subvariety and let $\Gamma \subset G(K)$ be a finitely generated group. Then $X(K) \cap \Gamma$ is a finite union of pseudo-generalized F-sets in Γ.*

1.5. **Plan for our paper.** In Section 2 we introduce three examples which progressively show the complexity of the sets appearing as intersections between a subvariety of a semiabelian variety G with a finitely generated group. Even though in our examples, G is defined over a finite field, each such example can be "embedded" as isotrivial semiabelian subvarieties of a semiabelian variety defined over an arbitrary field of positive characteristic, thus providing complex examples of pseudo-generalized F-sets. In Section 3 we prove Theorems 1.6 and 1.10. Also, we prove Theorem 1.7 as a consequence of two more precise results (see Propositions 3.1 and 3.2) regarding the structure of the intersection $X(K) \cap \Gamma$ when either X is a curve, or G is a simple semiabelian variety.

2. Examples

Our first example already shows that $X(K) \cap \Gamma$ is not always an F-set in Γ (when Γ is not invariant under the Frobenius endomorphism of G).

Example 2.1. We let $G = \mathbb{G}_m^2 \times E$, where E is a supersingular elliptic curve defined over $\mathbb{F}_p$; for example, we can take E be the elliptic curve given by the equation in affine coordinates $y^2 = x^3 + 1$ when $p = 5$, in which case, we have that the square F^2 of the usual Frobenius endomorphism of E corresponding to $\mathbb{F}_5$ equals the multiplication map $[-5]$ on E. We let $C \subset \mathbb{G}_m^2$ be the line given by the equation $x_2 = x_1 + 1$ and then let $X = C \times E$. We let

$K = \overline{\mathbb{F}_p(t)}$ and let $P \in E(K)$ be a nontorsion point. Finally, we let $\Gamma \subset G(K)$ be the cyclic group spanned by $Q := (t, t+1, P) \subset G(K)$. Then

$$X(K) \cap \Gamma = \{p^n Q \colon n \geq 0\}\,. \tag{2.0.1}$$

Furthermore, the set from (2.0.1) cannot be expressed as a groupless F-set; the closest it comes to being an F-set is expressing it as the following slight twist of groupless F-sets. We let $Q_1 := (t, t+1, 0) \in G(K)$ and $Q_2 := (1, 1, P) \in G(K)$ and then the set from (2.0.1) is the union of the following two sets:

$$\left\{F^{2n}(Q_1) + F^{4n}(Q_2) \colon n \geq 0\right\} \text{ and } \left\{F^{2n+1}(Q_1) - F^{4n+2}(Q_2) \colon n \geq 0\right\}. \tag{2.0.2}$$

Now, comparing the sets from (2.0.2) with the actual (groupless) F-sets, the difference seems quite small and so, one might think that perhaps slightly extending the definition of F-sets as in equation (2.0.2) would be enough. The main issue in Example 2.1 comes from the fact that the Frobenius endomorphism has "different weights" on the abelian, respectively affine part of G; so, it might be reasonable for one to think that allowing different weights also in the definition of a groupless F-set by considering sets of the form:

$$\left\{\sum_{i=1}^{r}\sum_{j=1}^{s} F^{k_{i,j}\cdot n_j}(\alpha_j) \colon n_j \geq 0 \text{ for } j = 1, \dots, s\right\}$$

would suffice for describing $X(K) \cap \Gamma$. However, the next example shows that no simple extension of the definition of F-sets would work.

Example 2.2. We still work with $G = \mathbb{G}_m^2 \times E$, but this time the elliptic curve E is ordinary; for example, we could take $p = 5$ and let E be the elliptic curve given by the equation in affine coordinates $y^2 = x^3 + x$. One can check that the Frobenius endomorphism corresponding to $\mathbb{F}_5$ satisfies the integral equation $F^2 - 2F + 5 = 0$ on E. We let as before $K = \overline{\mathbb{F}_p(t)}$ and we work with the cyclic group Γ spanned by $Q := (t, t+1, P) \in G(K)$ for some non-torsion point $P \in E(K)$. Then letting $X = C \times E$, where $C \subset \mathbb{G}_m^2$ is the line $x_2 = x_1 + 1$, we get that

$$X(K) \cap \Gamma = \{p^n Q \colon n \geq 0\}\,. \tag{2.0.3}$$

However, one can show that the set from (2.0.3) *cannot* be split into finitely many sets of the form:

$$\left\{\sum_{i=1}^{r}\sum_{j=1}^{s} F^{k_{i,j}n_j}(Q_j) \colon n_j \geq 0 \text{ for } j = 1, \dots, s\right\}, \tag{2.0.4}$$

for any given $r, s \in \mathbb{N}$ and any choice of non-negative integers $k_{i,j}$ and any choice of given points $Q_j \in G(K)$. In other words, even the most complex definition of a groupless F-set as in equation (2.0.4) would still not cover a possible intersection $X(K) \cap \Gamma$.

Now, Examples 2.1 and 2.2 may still suggest that the intersection $X(K) \cap \Gamma$ could be expressed using more general (groupless) F-sets in which one would allow also the multiplication-by-p map on G playing a similar role as the Frobenius endomorphism. However, the next example shows that $X(K) \cap \Gamma$ may have a very complex structure.

Example 2.3. We let A and B be semiabelian varieties defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K, let $G = A \times B$, and let F be the corresponding Frobenius endomorphism associated to $\mathbb{F}_q$. We let h be the minimal (monic) polynomial with integer coefficients for which $h(F) = 0$ on B. Depending on the abelian part of the semiabelian variety B, the degree m of the polynomial h may be arbitrarily large.

We let $C \subset B$ be a curve defined over $\mathbb{F}_q$ with trivial stabilizer in B and let $P \in C(K)$ be a non-torsion point; one can even choose C and P so that $C(K)$ intersects the cyclic $\mathbb{Z}[F]$-module Γ_1 spanned by P precisely in the orbit of P under the Frobenius endomorphism F. We also let $Q_1, \dots, Q_m \in A(K)$ be linearly independent points (note that $A(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an infinite dimensional $\mathbb{Q}$-vector space). Then we consider $X := A \times C$ and also, consider the group $\Gamma \subset G(K)$ spanned by the points:

$$R_1 := (Q_1, P);\ R_2 := (Q_2, F(P));\ R_3 := \left(Q_3, F^2(P)\right);\ \cdots;\ R_m := \left(Q_m, F^{m-1}(P)\right).$$

Then letting $\pi_2 : G \longrightarrow B$ be the projection of $G = A \times B$ on the second coordinate, we have that $\pi_2(\Gamma) = \Gamma_1$ because Γ_1 is spanned by the points

$$P,\ F(P),\ F^2(P), \cdots, F^{m-1}(P) \in B(K)$$

since Γ_1 is the cyclic $\mathbb{Z}[F]$-module spanned by P and $h(F)(P) = 0$. So, we can find m sequences $\left\{a_n^{(i)}\right\}_{n\ge 0}$ of integers (for $i = 0, \dots, m-1$) such that for any $n \ge 0$, we have that

$$F^n(P) = \sum_{i=0}^{m-1} a_n^{(i)} \cdot F^i(P). \tag{2.0.5}$$

Equation (2.0.5) yields that $X(K) \cap \Gamma$ is the set:

$$\left\{\sum_{i=1}^{m} a_n^{(i-1)} \cdot R_i \colon n \ge 0\right\}. \tag{2.0.6}$$

So, due to the potential complexity of the coefficients of the polynomial h satisfied by the Frobenius endomorphism (on the semiabelian variety B), the sequences $\left\{a_n^{(i)}\right\}_{n\ge 0}$ may be quite complicated.

3. Proofs of our main results

Proof of Theorem 1.6. We proceed by induction on $\dim(X)$; the case when $\dim(X) = 0$ is obvious since then $X(K) \cap \Gamma$ is a finite set and so, each of the groupless F-sets from our intersection are singletons (corresponding to $r = 0$ in equation (1.0.2)).

Clearly, it suffices to assume X is irreducible. Also, we may assume $X(K) \cap \Gamma$ is Zariski dense in X since otherwise we could replace X by the Zariski closure of $X(K) \cap \Gamma$ and be done by the inductive hypothesis.

We let $U := \mathrm{Stab}_G(X)$ be the stabilizer of X in G. We have two possibilities depending on whether U is finite, or not.

Case 1. $\dim(U) > 0$.

In this case, we let $\pi_0 : G \longrightarrow G/U$ be the natural group homomorphism; in particular, $G_0 := G/U$ is a semiabelian variety defined over a finite field since U is defined over a finite extension of $\mathbb{F}_q$. We let $\Gamma_0 := \pi_0(\Gamma)$ and also, let $X_0 := \pi_0(X)$.

Since $\dim(U) > 0$, then $\dim(X_0) < \dim(X)$ and so, by the inductive hypothesis, we have that $X_0(K)\cap\Gamma_0$ is a union of finitely many groupless F-sets B_i in Γ_0 along with finitely many generalized F-sets C_i in Γ_0. We have that

$$X(K)\cap\Gamma = \pi_0^{-1}\left(X_0(K)\cap\Gamma_0\right)\cap\Gamma = \left(\pi_0|_\Gamma\right)^{-1}\left(X_0(K)\cap\Gamma_0\right). \tag{3.0.1}$$

Clearly, each $(\pi_0|_\Gamma)^{-1}(B_i)$ is a generalized F-set in Γ as in Definition 1.5. Now, each C_i is a set of the form

$$\left(f|_{\Gamma_0}\right)^{-1}(S_0) = f^{-1}(S_0)\cap\Gamma_0,$$

where $f : G_0 \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties over K in which $\dim(\ker(f)) > 0$ and H is defined over a finite extension of $\mathbb{F}_q$, and S_0 is a groupless F-set in $f(\Gamma_0) \subset H(K)$ as in Definition 1.2. So, using (3.0.1), along with the fact that

$$\pi_0^{-1}\left(f^{-1}(S_0)\cap\Gamma_0\right)\cap\Gamma = \left(f\circ\pi_0\right)^{-1}(S_0)\cap\Gamma, \tag{3.0.2}$$

then we obtain that $X(K)\cap\Gamma$ has the desired form as in the conclusion of Theorem 1.6.

Case 2. U is finite.

In this case, we let $\tilde{\Gamma}$ be the $\mathbb{Z}[F]$-submodule spanned by Γ inside $G(K)$; since F is integral over $\mathbb{Z}$ (inside $\mathrm{End}(G)$), then $\tilde{\Gamma}$ is also a finitely generated subgroup of $G(K)$. According to [MS04] (see Theorem 1.4), we have that

$$X(K)\cap\tilde{\Gamma} = \bigcup_{i=1}^{\ell}\left(S_i+\Gamma_i\right), \tag{3.0.3}$$

where each $S_i \subset \tilde{\Gamma}$ is a groupless F-set as in Definition 1.2, while each Γ_i is a subgroup of $\tilde{\Gamma}$. Now, since

$$X(K)\cap\Gamma = \left(X(K)\cap\tilde{\Gamma}\right)\cap\Gamma, \tag{3.0.4}$$

it suffices to prove that for each $i = 1,\dots,\ell$, there exists a subset $A_i \subseteq X(K)\cap\Gamma$ which is a union of finitely many groupless F-sets in Γ along with finitely many generalized F-sets in Γ such that

$$(S_i+\Gamma_i)\cap\Gamma \subseteq A_i; \tag{3.0.5}$$

then combining equations (3.0.3), (3.0.4) and (3.0.5), we would get that

$$X(K)\cap\Gamma = \bigcup_{i=1}^{\ell}(S_i+\Gamma_i)\cap\Gamma = \bigcup_{i=1}^{\ell}A_i$$

is indeed a finite union of groupless F-sets in Γ along with finitely many generalized F-sets in Γ, as claimed in the conclusion of Theorem 1.6.

In order to prove the existence of a set A_i (for each $i = 1,\dots,\ell$) as in equation (3.0.5), we deal with two additional cases.

Case 2a. Γ_i is an infinite subgroup.

In this case, we let X_i be the Zariski closure of $S_i+\Gamma_i$; clearly, $X_i \subseteq X$. We claim that X_i is a proper subvariety of X. Indeed, by construction, $\Gamma_i \subseteq \mathrm{Stab}_G(X_i)$ and since Γ_i is infinite, then we cannot have that $X_i = X$ because $\mathrm{Stab}_G(X)$ is finite. So, $\dim(X_i) < \dim(X)$ and by our inductive hypothesis, we have that that $A_i := X_i(K) \cap \Gamma$ satisfies the conclusion from Theorem 1.6; therefore

$$(S_i + \Gamma_i) \cap \Gamma \subseteq A_i,$$

where A_i is a union of finitely many groupless F-sets along with finitely many generalized F-sets, as desired for (3.0.5).

Case 2b. Γ_i is finite.

In this case, letting $s := \#\Gamma_i$, we have that $S_i + \Gamma_i$ is a union of s groupless F-sets as in Definition 1.2. Now, [Ghi08, Theorem 3.1] shows that the intersection of a groupless F-set with a finitely generated group is a itself a finite union of groupless F-sets; so,

$$A_i := (S_i + \Gamma_i) \cap \Gamma$$

is a finite union of groupless F-sets in Γ as desired for (3.0.5).

This concludes our proof of Theorem 1.6. □

Theorem 1.7 is an immediate corollary of our next two results which provide a more precise form of the intersection between a subvariety X of G with a finitely generated subgroup of $G(K)$ when X is a curve, respectively when G is a simple semiabelian variety.

Proposition 3.1. *Let G be a semiabelian variety defined over a finite subfield of an algebraically closed field K, let $\Gamma \subset G(K)$ be a finitely generated subgroup, and let $X \subseteq G$ be an irreducible curve.*

(i) *If $\dim(\mathrm{Stab}_G(X)) > 0$, then $X(K) \cap \Gamma$ is a coset of a subgroup of Γ.*
(ii) *If $\mathrm{Stab}_G(X)$ is finite, then $X(K) \cap \Gamma$ is a finite union of groupless F-sets.*

Proof. The proof of part (i) is immediate since then $X = \gamma + G_1$, for some point $\gamma \in G(K)$ and some 1-dimensional connected algebraic subgroup $G_1 \subseteq G$. So, then the intersection $X(K) \cap \Gamma$ is simply a coset of the subgroup $G_1(K) \cap \Gamma$ of Γ.

Now, we assume $\mathrm{Stab}_G(X)$ is finite. Then we let $\tilde{\Gamma}$ be the $\mathbb{Z}[F]$-submodule of $G(K)$ spanned by Γ; by Theorem 1.4, we have that $\tilde{\Gamma}$ intersects $X(K)$ in a finite union of F-sets S_i in $\tilde{\Gamma}$. But then at the expense of replacing each S_i with finitely many other F-sets, we may assume that each such F-set is groupless (see also the proof of **Case 2b** in Theorem 1.6). Finally, another application of [Ghi08, Theorem 3.1] yields that each $S_i \cap \Gamma$ is a finite union of groupless F-sets in Γ, as desired. □

Proposition 3.2. *Let G be a simple semiabelian variety defined over a finite subfield of an algebraically closed field K, let $\Gamma \subset G(K)$ be a finitely generated group, and let $X \subset G$ be a proper subvariety. Then $X(K) \cap \Gamma$ is a finite union of groupless F-sets in Γ.*

Proof. First of all, we note that if Γ is a finite group, then clearly $X(K) \cap \Gamma$ is a finite set and thus a finite union of groupless F-sets, as desired.

So, from now on, we assume that Γ is infinite. According to our Theorem 1.6, we know that $X(K) \cap \Gamma$ is a finite union of groupless F-sets in Γ along with (possibly) finitely many generalized F-sets in Γ. Now, for any such generalized F-set in Γ (call it S), we have that

$$S = (\pi|_\Gamma)^{-1}(S_0),$$

where $\pi : G \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties defined over a finite subfield of K, S_0 is a groupless F-set in $\pi(\Gamma) \subset H(K)$ and moreover, $\dim(\ker(\pi)) > 0$. But since G is a simple semiabelian variety, this means that $\ker(\pi) = G$, i.e., H is the trivial group variety and so, S would have to be the entire subgroup Γ. But then its Zariski closure in G is an infinite algebraic subgroup of G (note that Γ is assumed now to be infinite) and so, once again because G is simple, we would conclude that Γ is Zariski dense in G. But then because $S = \Gamma$ is contained in X, we would have that $X = G$, contradicting the fact that X is a proper subvariety of G. Therefore, we have no generalized F-sets in Γ contained in the intersection $X(K) \cap \Gamma$.

This concludes our proof for Proposition 3.2. □

Theorem 1.10 follows easily from our Theorem 1.6 combined with Theorem 1.1.

Proof of Theorem 1.10. Clearly, as argued in the proof of Theorem 1.6, it suffices to prove Theorem 1.10 assuming that X is an irreducible subvariety of G and $X(K) \cap \Gamma$ is Zariski dense in X. Then Theorem 1.1 yields that

$$X = \gamma + \pi^{-1}(X_0),$$

where $\pi : G_0 \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties, while G_0 is a semiabelian subvariety of G and $\gamma \in G(K)$; moreover, H and the subvariety $X_0 \subseteq H$ are defined over over a finite subfield $\mathbb{F}_q \subset K$. Then for $x \in G(K)$, we have $x \in X(K)$ if and only if "$x - \gamma \in G_0(K)$ and $\pi(x - \gamma) \in X_0(K)$". We denote $\Gamma_0 = G_0(K) \cap \Gamma$.

Pick $x_0 \in X(K) \cap \Gamma$. Let $g_0 = x_0 - \gamma \in G_0(K)$. We have $x_0 + \Gamma_0 = (\gamma + G_0(K)) \cap \Gamma$. As a result, for any $x \in \Gamma$, we have $x - \gamma \in G_0(K)$ if and only if there exists $\gamma_0 \in \Gamma_0$ such that $x = x_0 + \gamma_0$. Thus $x - \gamma = g_0 + \gamma_0$ and so, $\pi(x - \gamma) \in X_0(K)$ yields $\pi(\gamma_0) \in -\pi(g_0) + X_0(K)$.

Let $X_0' = -\pi(g_0) + X_0$ which is a subvariety of H. The discussion above implies that $X(K) \cap \Gamma = x_0 + (\pi|_{\Gamma_0})^{-1}(X_0'(K) \cap \pi(\Gamma_0))$. So, considering the subvariety $X_0' \subseteq H$, along with the finitely generated subgroup $\pi(\Gamma_0)$ of $H(K)$, then we apply Theorem 1.6 to conclude that the intersection $X_0'(K) \cap \pi(\Gamma_0)$ is a finite union of generalized F-sets in $\pi(\Gamma_0)$ along with finitely many groupless F-sets in $\pi(\Gamma_0)$. But whether S is a generalized F-set in $\pi(\Gamma_0)$ or a groupless F-set in $\pi(\Gamma_0)$, $x_0 + (\pi|_{\Gamma_0})^{-1}(S)$ will always be a pseudo-generalized F-set in Γ (see also Remark 1.9). This shows that $X(K) \cap \Gamma$ is a finite union of pseudo-generalized F-sets in Γ, as desired. □

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References

[Fal91] G. Faltings, *The general case of S. Lang's conjecture.* Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), 175-182, Perspect. Math., 15, Academic Press, San Diego, CA, 1994.

[Ghi08] D. Ghioca, *The isotrivial case in the Mordell-Lang theorem*, Trans. Amer. Math. Soc. **360** (2008), no. 7, 3839–3856.

[Hru96] E. Hrushovski, *The Mordell-Lang conjecture for function fields.* J. Amer. Math. Soc. **9** (1996), no. 3, 667–690.

[Lau84] M. Laurent, *Équations diophantiennes exponentielles*, Invent. Math. **78** (1984), 299–327.

[MS02] R. Moosa and T. Scanlon, *The Mordell-Lang conjecture in positive characteristic revisited*, Model theory and applications, 273–296, Quad. Mat., **11**, Aracne, Rome, 2002.

[MS04] R. Moosa and T. Scanlon, *F-structures and integral points on semiabelian varieties over finite fields*, Amer. J. Math. **126** (2004), 473–522.

[Voj96] P. Vojta, *Integral points on subvarieties of semiabelian varieties. I*, Invent. Math. **126** (1996), no. 1, 133–181.

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