THE MORDELL-LANG THEOREM FOR DRINFELD MODULES

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Abstract. We study the ring of quasi-endomorphisms for certain infinitely definable subgroups in separably closed fields. Based on the results we obtain, we are able to prove a Mordell-Lang theorem for Drinfeld modules of finite characteristic. Using specialization arguments we prove also a Mordell-Lang theorem for Drinfeld modules of generic characteristic.

1. Introduction

Faltings proved the Mordell-Lang Conjecture in the following form (see [6]).

Theorem 1.1 (Faltings). *Let G be an abelian variety defined over the field of complex numbers $\mathbb{C}$. Let $X \subset G$ be a closed subvariety and $\Gamma \subset G(\mathbb{C})$ a finitely generated subgroup of the group of $\mathbb{C}$-points on G. Then $X(\mathbb{C}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.*

If we try to formulate the Mordell-Lang Conjecture in the context of algebraic subvarieties contained in a power of the additive group scheme $\mathbb{G}_a$, the conclusion is either false (in the characteristic 0 case, as shown by the curve $y = x^2$ which has an infinite intersection with the finitely generated subgroup $\mathbb{Z} \times \mathbb{Z}$, without being itself an additive algebraic group) or it is trivially true (in the characteristic $p > 0$ case, because every finitely generated subgroup of a power of $\mathbb{G}_a$ is finite). In the fourth section we will present a nontrivial formulation of the Mordell-Lang conjecture for a power of the additive group in characteristic p in the context of Drinfeld modules. We will replace the *finitely generated subgroup* from the usual Mordell-Lang statement with a *finitely generated ϕ-submodule*, where ϕ is a Drinfeld module. We also strengthen the conclusion of the Mordell-Lang statement in our setting by asking that the *subgroups* whose cosets are contained in the intersection of the algebraic variety with the finitely generated ϕ-submodule be actually *ϕ-submodules*.

In order to obtain the results of the present paper we need first to analyze the ring of quasi-endomorphisms for certain infinitely definable subgroups in the theory of separably closed fields. In the next section we introduce the basic notation and results, while in the third section we prove the main result (Theorem 3.8) needed for the proof of Theorem 4.6 (the Mordell-Lang Theorem for Drinfeld modules of finite characteristic). Using specialization arguments we also prove a Mordell-Lang statement for Drinfeld modules of generic characteristic (Theorem 4.14).

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2. Infinitely definable groups in the theory of separably closed fields

Everywhere in this paper, for two sets A and B, the notation $A \subset B$ means that A is a subset, not necessarily proper, of B.

Let K be a finitely generated field of characteristic $p > 0$. Let τ_0 be the usual Frobenius, i.e. $\tau_0(x) = x^p$, for every x. We let $K\{\tau_0\}$ be the non-commutative ring of all polynomials in τ_0 with coefficients from K, where the addition is the usual one while the multiplication is the composition of functions. If $f, g \in K\{\tau_0\}$, fg will represent the composition of f and g.

Fix an algebraic closure K^{alg} of K. Let K^{sep} be the separable closure of K inside K^{alg}. Let $\mathbb{F}_p^{\mathrm{alg}}$ be the algebraic closure of $\mathbb{F}_p$ inside K^{sep}.

There exists a non-negative integer ν such that $[K : K^p] = [K^{\mathrm{sep}} : K^{\mathrm{sep}^p}] = p^\nu$. The number ν is called the Ersov invariant of K. When K is a finitely generated field, $\nu = \mathrm{trdeg}_{\mathbb{F}_p} K$.

Notation 2.1. Let k be a positive integer. We denote by $p^{(k)}$ the set of functions

$$f : \{1, \dots, k\} \to \{0, \dots, p-1\}.$$

Definition 2.2. A subset $B = \{b_1, \dots, b_\nu\} \subset K$ is called a p-basis of K, or equivalently, of K^{sep}, if the following set of monomials,

$$\left\{ m_i = \prod_{j=1}^{\nu} b_j^{i(j)} \mid i \in p^{(\nu)} \right\}$$

forms a basis for K/K^p, or equivalently for $K^{\mathrm{sep}}/K^{\mathrm{sep}^p}$.

For the rest of this paper we fix a p-basis B for K. There exists a unique collection of functions $\lambda_i : K^{\mathrm{sep}} \to K^{\mathrm{sep}}$ for $i \in p^{(\nu)}$, such that for every $x \in K^{\mathrm{sep}}$,

$$x = \sum_{i \in p^{(\nu)}} \lambda_i(x)^p m_i.$$

We call these functions λ_i the λ-functions of level 1. For every $k \geq 2$ and for every choice of $i_1, \dots, i_k \in p^{(\nu)}$,

$$\lambda_{i_1, i_2, \dots, i_k} = \lambda_{i_1} \circ \lambda_{i_2} \circ \cdots \circ \lambda_{i_k}$$

is called a λ-function of level k.

Definition 2.3. We let $\mathrm{SCF}_{\mathrm{p},\nu}$ be the theory of separably closed fields of characteristic p and Ersov invariant ν in the language

$$\mathfrak{L}_{p,\nu} = \{0, 1, +, -, \cdot\} \cup \{b_1, \dots, b_\nu\} \cup \{\lambda_i \mid i \in p^{(\nu)}\}.$$

From now on we consider a finitely generated field K of Ersov invariant ν and so, K^{sep} is a model of $\mathrm{SCF}_{\mathrm{p},\nu}$. We let L be an $\aleph_1$-saturated elementary extension of K^{sep}. Because L is an elementary extension of K^{sep}, $L \cap K^{\mathrm{alg}} = K^{\mathrm{sep}}$. We are interested in studying infinitely definable subgroups G of $(L, +)$, i.e. G is possibly an infinite intersection of definable subgroups of $(L, +)$. If $k \geq 1$ and G is an infinitely definable subgroup of $(L, +)$, then the *relatively* definable subsets of G^k (the cartesian product of G with itself k times) are the intersections of G^k with definable subsets of $(L, +)^k$. If there is no risk of ambiguity, we will say a definable subset of G^k, instead of relatively definable subset of G^k. The structure *induced* by L on G over a set S of parameters, is the set G together with all the relatively S-definable subsets of the cartesian powers of G. We will consider only the case when the set S of parameters equals K^{sep}. Thus, when we say a definable subset, we will mean a K^{sep}-definable subset. Also, we call the subgroups of $(L, +)$ *additive.* Finally, we observe that because the theory of separably closed fields is a stable theory (see Messmer's article from [11]), a definable subgroup of an infinitely definable group $G \subset L$ is the intersection of G with a definable subgroup of L.

Remark 2.4. In all of our arguments we will work with infinitely definable subgroups G of $(L, +)$. To interpret such a group G from a purely model theoretic point of view, we could do the following. We associate to G the (partial) type P with the property that the realizations of P in the model L of separably closed fields is G, i.e. $G = P(L)$. Thus in our results we will loosely interchange the notion of G as a subgroup of $(L, +)$ and G as the set of realizations of a (partial) type in the language of separably closed fields.

Definition 2.5. For every infinitely definable subgroup G, the connected component of G, denoted G^0, is the intersection of all definable subgroups of finite index in G.

Definition 2.6. The group G is connected if $G = G^0$.

The following result will be used in the proof of Theorem 4.6.

Lemma 2.7. *The cartesian product of a finite number of connected groups is connected.*

Proof. Using induction, it is enough to prove the product of two connected groups is connected. Therefore, we assume G_1 and G_2 are connected and $H \subset G_1 \times G_2$ is a definable subgroup of finite index. Let π_1 be the projection of $G_1 \times G_2$ on the first component. Because $[G_1 \times G_2 : H]$ is finite, $[G_1 : \pi_1(H)]$ is also finite. Because G_1 is connected and $\pi_1(H)$ is definable, we conclude $\pi_1(H) = G_1$. Let π_2 be the second projection of $G_1 \times G_2$. Then $H_2 := \pi_2(\text{Ker}(\pi_1|_H))$ is a definable subgroup of G_2. Because $[G_1 \times G_2 : H]$ is finite, $[G_2 : H_2]$ is also finite. Because G_2 is connected, we conclude $H_2 = G_2$. Hence $H = G_1 \times G_2$, which concludes the proof of Lemma 2.7. ☐

Definition 2.8. Let G be an infinitely definable additive subgroup of L. We denote by $\text{End}_{\text{K}^{\text{sep}}}(G)$ the set of K^{sep}-definable endomorphisms f of G.

The endomorphisms $f \in \text{End}_{\text{K}^{\text{sep}}}(G)$ that are both injective and surjective, form the group of K^{sep}-automorphisms of G, denoted $\text{Aut}_{\text{K}^{\text{sep}}}(G)$.

Remark 2.9. If G is a connected group, then the graph of f is a connected subgroup of $G \times G$.

From now on, "endomorphism of G" means "element of $\text{End}_{\text{K}^{\text{sep}}}(G)$" and "automorphism of G" means "element of $\text{Aut}_{\text{K}^{\text{sep}}}(G)$".

Definition 2.10. Let G and H be infinitely definable connected groups. We call the subgroup $\psi \subset G \times H$ a K^{sep}-quasi-morphism from G to H if the following three properties are satisfied

1) ψ is a connected, K^{sep}-definable subgroup of $G \times H$.
2) the first projection $\pi_1(\psi)$ equals G.
3) the set $\{x \in H \mid (0, x) \in \psi\}$ is finite.

The set of all K^{sep}-quasi-morphisms from G to H is denoted by $\text{QsM}_{\text{K}^{\text{sep}}}(G, H)$.

When $G = H$, we call ψ a K^{sep}-quasi-endomorphism of G. The set of all K^{sep}-quasi-endomorphisms of G is denoted by $\text{QsE}_{\text{K}^{\text{sep}}}(G)$.

For every infinitely definable connected subgroup G, a "quasi-endomorphism of G" will be an element of $\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G)$.

Let f be an endomorphism of the connected group G. We interpret f as a quasi-endomorphism of G by

$$f = \{(x, f(x)) \mid x \in G\} \in \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G).$$

Definition 2.11. Let G be an infinitely definable connected group. We define the following two operations that will induce a ring structure on $\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G)$.

1) *Addition.* For every $\psi_1, \psi_2 \in \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G)$, we let $\psi_1 + \psi_2$ be the connected component of the group

$$\{(x, y) \in G \times G \mid \exists y_1, y_2 \in G \text{ such that } (x, y_1) \in \psi_1 \text{ , } (x, y_2) \in \psi_2 \text{ and } y_1 + y_2 = y\}.$$

2) *Composition.* For every $\psi_1, \psi_2 \in \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G)$, we let $\psi_1\psi_2$ be the connected component of the group

$$\{(x, y) \in G \times G \mid \text{ there exists } z \in G \text{ such that } (x, z) \in \psi_2 \text{ and } (z, y) \in \psi_1\}.$$

See [2] for the proof that the above defined operations endow $\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G)$ with a ring structure.

Definition 2.12. Let G be an infinitely definable additive subgroup. Then G is c-minimal if it is infinite and every definable subgroup of G is either finite or has finite index.

Lemma 2.13. *If G is a c-minimal connected group, then for all $f \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(G) \setminus \{0\}$, $f(G) = G$.*

Proof. Because $f \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(G)$ and G is connected, $f(G)$ is a definable, connected subgroup of G. Thus, since $f \neq 0$, $f(G)$ cannot be finite. Then, because G is c-minimal, $f(G)$ has finite index in G. Because G is connected, we conclude that f is surjective. □

The next result is proved in a larger generality in Chapter 4.4 of [16]. Because for the case we are interested in we can give a simpler proof, we present our argument below.

Proposition 2.14. *If G is a c-minimal, connected group, then $\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G)$ is a division ring.*

Proof. Let $\psi \in \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G) \setminus \{0\}$. Let $\pi_2(\psi)$ be the projection of $\psi \subset G \times G$ on the second component. Then $\pi_2(\psi)$ is a definable subgroup of G. Because ψ is connected and $\psi \neq 0$, $\pi_2(\psi)$ is not finite. Then, because G is a c-minimal, connected group, $\pi_2(\psi) = G$.

Because $\pi_2(\psi) = G$ and G is c-minimal and $\psi \neq G \times G$, the set

$$\{x \in G \mid (x, 0) \in \psi\} \tag{1}$$

is finite. We define $\phi = \{(y, x) \in G \times G \mid (x, y) \in \psi\}$. Because ψ is a connected, K^{sep}-definable subgroup of $G \times G$, then also ϕ is a connected, K^{sep}-definable subgroup of $G \times G$. By construction, $\pi_1(\phi) = \pi_2(\psi) = G$. By construction of ϕ,

$$\{x \in G \mid (0, x) \in \phi\} = \{x \in G \mid (x, 0) \in \psi\}.$$

Using (1), we conclude that $\{x \in G \mid (0, x) \in \phi\}$ is finite. Thus condition 3) of Definition 2.10 holds and so, $\phi \in \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G)$. By definition of ϕ, $\psi\phi$ (as defined in Definition 2.11) is the identity function on G. Thus $\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(G)$ is a division ring ($1 \neq 0$ because G is infinite). □

Definition 2.15. Let $f \in K\{\tau_0\}\tau_0 \setminus \{0\}$. We define $f^\sharp = f^\sharp(L) = \bigcap_{n \geq 1} f^n(L)$.

In [2] (Lemma 4.23) and [14] the following result is proved.

Theorem 2.16. *If $f \in K\{\tau_0\}\tau_0 \setminus \{0\}$, then $f^\sharp$ is c-minimal. In particular, $f^\sharp$ is infinite.*

The theory of separably closed fields is stable, as shown in [11]. Because [12] proves that every stable field is connected as an additive group, the following result holds.

Theorem 2.17. *The groups $(K^{\text{sep}}, +)$ and $(L, +)$ are connected.*

Because the image of a connected group through a definable map is also connected, we get the following result.

Corollary 2.18. *For every $f \in K\{\tau_0\}$, $f(K^{\text{sep}})$ is connected.*

Lemma 2.19. *Let $(H_n)_{n\geq 1}$ be a countable descending chain of connected definable subgroups of $(L, +)$. Then the infinitely definable subgroup $H = \bigcap_{n\geq 1} H_n$ is connected.*

Proof. It suffices to show that for every definable additive subgroup G of L, if G intersects H in a subgroup of finite index, then G contains H. So, let G be a definable additive subgroup of L such that $[H : G \cap H]$ is finite.

Assume that there exists $n \geq 1$ such that $[H_n : G \cap H_n]$ is finite. For such n, because H_n is connected (see Corollary 2.18), we conclude that $H_n = G \cap H_n$. So, $H_n \subset G$. Then, by the definition of H, we get that $H \subset G$.

Suppose that for all $n \geq 1$, $[H_n : G \cap H_n]$ is infinite. By compactness and the fact that the groups H_n form a descending sequence and the fact that L is $\aleph_1$-saturated, we conclude that also $[H : G \cap H]$ is infinite, which contradicts our assumption. For the reader's convenience, we provide the compactness argument.

Let the descending sequence of groups H_i be represented by formulas ϕ_i. Also, let the group G be represented by the formula ψ.

For each positive integer m and for each finite subset of indices $n_1 < \cdots < n_k$ let $F_{m,n_1,\dots,n_k}(x_1, \dots, x_m)$ be the formula which says:

$\phi_{n_i}(x_j)$ for every $1 \leq i \leq k$ and for every $1 \leq j \leq m$ (i.e. each x_j realizes each formula ϕ_{n_i}) and for distinct j and j' between 1 and m, $\neg\psi(x_j - x_{j'})$ (i.e. for distinct j and j', $x_j - x_{j'} \notin G$). So, the x_j are in all the groups H_{n_i} but they live in different cosets modulo G.

We know that each individual formula $F_{m,n_1,\dots,n_k}(x_1, \dots, x_m)$ has a realization in the model L (to see this, we recall the ϕ_{n_i} are descending and so, $F_{m,n_1,\dots,n_k}$ says that $[H_{n_k} : H_{n_k} \cap G]$ is at least m, because n_k is the largest index among $n_1, \dots, n_k$).

Then for every finite subset of formulas $F_{m,n_1,\dots,n_k}$, let M be the largest among the numbers m appearing as an index for the formulas F. We prove there exist $x_1, \dots, x_M$ realizing simultaneously all of the formulas $F_{m,n_1,\dots,n_k}$. Indeed, just replace all of the formulas $F_{m,n_1,\dots,n_k}$ with just one formula $F_{M,l_1,\dots,l_s}$ where the indices $l_1, \dots, l_s$ form a set containing all the indices $n_1, \dots, n_k$ from all the formulas F of the chosen finite subset of formulas. We know that $F_{M,l_1,\dots,l_s}$ is realizable and so, all of the finitely many formulas F from above are also realizable. Then we can use compactness and $\aleph_1$-saturation to conclude $G \cap \bigcap_{n\geq 1} H_n$ has infinite index in $\bigcap_{n\geq 1} H_n$. □

The following result is an immediate corollary to Lemma 2.19.

Lemma 2.20. *If $f \in K\{\tau_0\}\tau_0 \setminus \{0\}$, then $f^\sharp$ is connected.*

Proof. Because of Corollary 2.18 we can apply Lemma 2.19 to the collection of connected groups $f^n(L)$. $\square$

Corollary 2.21. *Let $f, g \in K\{\tau_0\}\tau_0 \setminus \{0\}$. If $g^\sharp \subset f^\sharp$, then $f^\sharp = g^\sharp$.*

Proof. By Theorem 2.16 and our hypothesis, $g^\sharp$ is an infinite subgroup of $f^\sharp$. Thus for every $n \geq 1$, $g^n(L) \cap f^\sharp$ is a definable infinite subgroup of $f^\sharp$. By Theorem 2.16 and Lemma 2.20, $f^\sharp \subset g^n(L)$. Because this last inclusion holds for all $n \geq 1$, we conclude that $f^\sharp \subset g^\sharp$. Thus $f^\sharp = g^\sharp$. $\square$

In [2] (see Proposition 3.1 and the *Remark* after the proof of Lemma 3.8) the following result is proved.

Proposition 2.22. *The following statements hold:*

(i) *The Frobenius τ_0, the λ-functions of level 1 and the elements of K^{sep} seen as scalar multiplication functions generate $\mathrm{End}_{\mathrm{K^{sep}}}(L, +)$ as a ring (i.e., with respect to the addition and the composition of functions). Each such element of $\mathrm{End}_{\mathrm{K^{sep}}}(L, +)$ will be called an (additive) λ-polynomial. (Because we will only deal with additive λ-polynomials, we will call them simply λ-polynomials.)*

(ii) *For every $\psi \in \mathrm{End}_{\mathrm{K^{sep}}}(L, +)$, there exists $n \geq 1$ such that for all $g \in K^{\mathrm{sep}}\{\tau_0\}\tau_0^n$, $\psi g \in K^{\mathrm{sep}}\{\tau_0\}$.*

(iii) *Let G be an infinitely definable subgroup of $(L, +)$. Then each endomorphism $f \in \mathrm{End}_{\mathrm{K^{sep}}}(G)$ extends to an element of $\mathrm{End}_{\mathrm{K^{sep}}}(L, +)$.*

3. Quasi-endomorphisms of minimal groups associated to Drinfeld modules

Let q be a power of p and let τ be the power of the Frobenius for which $\tau(x) = x^q$, for every x. Let K be a finitely generated field extension of $\mathbb{F}_q$ of positive transcendence degree. We let $K\{\tau\}$ be the ring of all polynomials in τ with coefficients from K. Let

$$f = \sum_{i=0}^{r} a_i \tau^i \in K\{\tau\},$$

with $a_r \neq 0$. The *order* $\mathrm{ord}_\tau f$ is defined as the smallest i such that $a_i \neq 0$. Thus, f is inseparable if and only if $\mathrm{ord}_\tau f > 0$.

Let C be a non-singular projective curve defined over $\mathbb{F}_q$. Let A be the $\mathbb{F}_q$-algebra of functions on C regular away from a fixed closed point of C. Then A is a Dedekind domain. Let $i : A \to K$ be a morphism. We call the morphism $\phi : A \to K\{\tau\}$ a Drinfeld module if for every $a \in A$, the coefficient of τ^0 in ϕ_a is $i(a)$, and if there exists $a \in A$ such that $\phi_a \neq i(a)\tau^0$. Following the definition from [8], we call ϕ a Drinfeld module of *generic* characteristic if $\ker(i) = \{0\}$. If $\ker(i) = \mathfrak{p} \neq \{0\}$, we call ϕ a Drinfeld module of *finite* characteristic $\mathfrak{p}$. If ϕ is a Drinfeld module of generic characteristic, we let $i : A \to K$ extend to an embedding of $\mathrm{Frac}(A) \subset K$.

For any field extension K' of K, $\phi(K')$ represents the field K' with the A-module structure induced by the action of the Drinfeld module ϕ.

As in Section 1, let L be an $\aleph_1$-saturated elementary extension of K^{sep}.

Definition 3.1. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic. We define

$$\phi^\sharp = \phi^\sharp(L) = \bigcap_{a \in A \setminus \{0\}} \phi_a(L).$$

Lemma 3.2. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic $\mathfrak{p}$. Let $t \in \mathfrak{p} \setminus \{0\}$. Then*

$$\phi^\sharp = \bigcap_{n\geq 1} \phi_{t^n}(L) = (\phi_t)^\sharp.$$

Proof. If $a \notin \mathfrak{p}$, then ϕ_a is a separable polynomial and $\phi_a(L) = L$. Thus

$$\phi^\sharp = \bigcap_{a\in\mathfrak{p}\setminus\{0\}} \phi_a(L). \tag{2}$$

Let $a \in \mathfrak{p} \setminus \{0\}$. Because $t \in \mathfrak{p} \setminus \{0\}$, there exist $n, m \geq 1$ and there exist $u, v \in A \setminus \mathfrak{p}$ such that $t^n v = a^m u$. Then ϕ_u and ϕ_v are separable and so,

$$\phi_{a^m}(L) = \phi_{a^m}(\phi_u(L)) = \phi_{a^m u}(L) = \phi_{t^n v}(L) = \phi_{t^n}(\phi_v(L)) = \phi_{t^n}(L). \tag{3}$$

So, $\phi_{t^n}(L) \subset \phi_a(L)$. Thus, using (2), we conclude that the result of Lemma 3.2 holds. □

The following result is an immediate consequence of Lemmas 3.2 and 2.20 and Theorem 2.16.

Corollary 3.3. *The group $\phi^\sharp$ is a c-minimal, connected additive group.*

Lemma 3.4. *Let ϕ be a Drinfeld module of finite characteristic. Let* $\mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi)$ *be the ring of endomorphisms of ϕ (defined as in* [8]*). Then each endomorphism of ϕ induces an endomorphism of $\phi^\sharp$, and this association defines injective ring homomorphisms, i.e.* $\mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi) \subset \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp) \subset \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$.

Proof. Let t be a uniformizer of the prime ideal of A which is the characteristic of ϕ. The inclusion $\mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp) \subset \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$ is clear. Let now $f \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi)$ and $x \in \phi^\sharp$. We need to show that $f(x) \in \phi^\sharp$. Because $x \in \phi^\sharp$, for all $n \geq 1$, there exists $x_n \in L$ such that $x = \phi_{t^n}(x_n)$. Because $f \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi)$, $f(x) = f(\phi_{t^n}(x_n)) = \phi_{t^n}(f(x_n)) \in \phi_{t^n}(L)$, for all $n \geq 1$. Thus indeed, $f(x) \in \phi^\sharp$ (see Lemma 3.2). Finally, the above defined association is injective because $\phi^\sharp$ is an infinite set and so, there is no nonzero endomorphism of ϕ which restricted to $\phi^\sharp$ is identically equal to 0. □

Corollary 3.5. *If ϕ is a finite characteristic Drinfeld module, then*

$$\phi^\sharp = \bigcap_{f\in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)\setminus\{0\}} f(L).$$

Proof. For every nonzero $a \in A$, $\phi_a \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi) \subset \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$. Thus

$$\bigcap_{f\in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)\setminus\{0\}} f(L) \subset \bigcap_{a\in A\setminus\{0\}} \phi_a(L) = \phi^\sharp.$$

But by Lemma 2.13 and Corollary 3.3, all the endomorphisms of $\phi^\sharp$ are surjective on $\phi^\sharp$. So, then indeed

$$\phi^\sharp = \bigcap_{f\in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)\setminus\{0\}} f(L).$$

□

Using Corollary 3.3 and Proposition 2.22, we get the following result.

Corollary 3.6. *Let $f \in \mathrm{End}_{\mathrm{K^{sep}}}(\phi^\sharp)$. Then f is a λ-polynomial. In particular, there exists $m \geq 1$ such that for all $h \in K^{\mathrm{sep}}\{\tau\}\tau^m$, $fh \in K^{\mathrm{sep}}\{\tau\}$.*

We define ϕ_{tor} as the set of all $x \in K^{\mathrm{alg}}$ for which there exists some nonzero $a \in A$ such that $\phi_a(x) = 0$. For every $a \in A \setminus \{0\}$, we let $\phi[a] = \{x \in K^{\mathrm{alg}} \mid \phi_a(x) = 0\}$. Then for $a \in A \setminus \{0\}$, we let $\phi[a^\infty] = \cup_{n \geq 1} \phi[a^n]$. If $\mathfrak{p}$ is any nontrivial prime ideal in A, then we define

$$\phi[\mathfrak{p}'] = \{x \in K^{\mathrm{alg}} \mid \text{ there exists } a \notin \mathfrak{p} \text{ such that } \phi_a(x) = 0\}.$$

We define $\phi^\sharp(K^{\mathrm{sep}}) = \phi^\sharp(L) \cap K^{\mathrm{sep}}$. We claim that this definition for $\phi^\sharp(K^{\mathrm{sep}})$ is equivalent to $\phi^\sharp(K^{\mathrm{sep}}) = \bigcap_{a \in A \setminus \{0\}} \phi_a(K^{\mathrm{sep}})$. Indeed, if $x \in \phi^\sharp(L) \cap K^{\mathrm{sep}}$, then for every $a \in A \setminus \{0\}$, there exists $x_a \in L$ such that $x = \phi_a(x_a)$. Because $\phi_a \in K^{\mathrm{sep}}\{\tau\}$ and $x \in K^{\mathrm{sep}}$, $x_a \in K^{\mathrm{alg}}$. Because $L \cap K^{\mathrm{alg}} = K^{\mathrm{sep}}$, $x_a \in K^{\mathrm{sep}}$. Moreover, a similar proof as in Lemma 3.2, shows that $\phi^\sharp(K^{\mathrm{sep}}) = \bigcap_{n \geq 1} \phi_{t^n}(K^{\mathrm{sep}})$, if ϕ_t is inseparable.

We will continue to denote by $\phi^\sharp$ the group $\phi^\sharp(L)$ and by $\phi^\sharp(K^{\mathrm{sep}})$, its subgroup contained in K^{sep}.

Lemma 3.7. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic $\mathfrak{p}$. Then $\phi[\mathfrak{p}'] \subset \phi^\sharp(K^{\mathrm{sep}})$.*

Proof. Let $x \in \phi[\mathfrak{p}']$ and let $a \notin \mathfrak{p}$ such that $\phi_a(x) = 0$. Because ϕ_a is separable, $x \in K^{\mathrm{sep}}$. Let t be an element of $\mathfrak{p}$, coprime with a, i.e. t and a generate the unit ideal in A.

Let $n \geq 1$. Because t and a are coprime, so are t^n and a. Thus there exist $r, s \in A$ such that $t^n r + as = 1$. Applying this last equality to x gives $\phi_{t^n}(\phi_r(x)) = x$, which shows that $x \in \phi_{t^n}(K^{\mathrm{sep}})$. Because n was arbitrary, we conclude $x \in \phi^\sharp(K^{\mathrm{sep}})$. □

Theorem 3.8. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic $\mathfrak{p}$. Assume there exists a non-constant $t \in A$ such that $\phi[t^\infty] \cap K^{\mathrm{sep}}$ is finite. Then $\phi^\sharp(K^{\mathrm{sep}}) = \phi[\mathfrak{p}']$. Moreover, with the above hypothesis on ϕ_t, we have that for every $\psi \in \mathrm{QsE}_{\mathrm{K^{sep}}}(\phi^\sharp)$, there exists $n \geq 1$ such that $\psi\phi_{t^n} = \phi_{t^n}\psi$ in $\mathrm{QsE}_{\mathrm{K^{sep}}}(\phi^\sharp)$.*

Proof. Clearly, $t \in \mathfrak{p} \setminus \{0\}$, because for all $a \in A \setminus \mathfrak{p}$, ϕ_a is separable and so, $\phi[a^\infty] \subset K^{\mathrm{sep}}$. By Lemma 3.2, we know that

$$\phi^\sharp = \bigcap_{n \geq 1} \phi_{t^n}(L) \tag{4}$$

and $\phi^\sharp(K^{\mathrm{sep}}) = \bigcap_{n \geq 1} \phi_{t^n}(K^{\mathrm{sep}})$.

Because $\phi[t^\infty] \cap K^{\mathrm{sep}}$ is finite, let $N_0 \geq 1$ satisfy

$$\phi[t^\infty] \cap K^{\mathrm{sep}} \subset \phi[t^{N_0}]. \tag{5}$$

Thus

$$\phi[t^\infty] \cap \phi^\sharp = \{0\}. \tag{6}$$

We will prove Theorem 3.8 through a series of lemmas.

Lemma 3.9. *Under the hypothesis of Theorem 3.8, $\phi_t \in \mathrm{Aut}_{\mathrm{K^{sep}}}(\phi^\sharp)$.*

Proof of Lemma 3.9. By Lemma 3.4, we know that $\phi_t \in \mathrm{End}_{\mathrm{K^{sep}}}(\phi^\sharp)$. By the definition of $\phi^\sharp$, we know that ϕ_t is a surjective endomorphism of $\phi^\sharp$. By (6), we know that ϕ_t is an injective endomorphism of $\phi^\sharp$. □

Lemma 3.10. *Assume $x \in \phi^\sharp(K^{\mathrm{sep}})$. We can find a sequence $(x_n)_{n\geq 0} \subset \phi^\sharp(K^{\mathrm{sep}})$ such that $x_0 = x$ and for all $n \geq 0$, $\phi_t(x_{n+1}) = x_n$.*

Proof of Lemma 3.10. Let $x \in \phi^\sharp(K^{\mathrm{sep}})$. Let N be a positive integer. Because $x \in \phi^\sharp(K^{\mathrm{sep}})$, there exists $x_N \in K^{\mathrm{sep}}$ such that $x = \phi_{t^N}(x_N)$. For each $1 \leq n \leq N$ we let $x_{N-n} = \phi_{t^n}(x_N)$. Thus we constructed the sequence $(x_n)_{0\leq n\leq N} \subset K^{\mathrm{sep}}$ such that $x = x_0$ and for every $0 \leq n \leq N-1$, $x_n = \phi_t(x_{n+1})$. We repeat this construction for each positive integer N. By compactness, because L is $\aleph_1$-saturated, there exists an infinite coherent sequence $(x_n)_{n\geq 0} \subset L$ such that $x = x_0$ and for every $n \geq 0$, $x_n = \phi_t(x_{n+1})$. Because $x \in K^{\mathrm{sep}}$ and $\phi_t \in K\{\tau\}$, $(x_n)_{n\geq 0} \subset K^{\mathrm{alg}} \cap L = K^{\mathrm{sep}}$ (the intersection of the two fields being taken inside a fixed algebraic closure of L which contains K^{alg}). □

An immediate corollary of the above proof is the following result.

Corollary 3.11. *For an arbitrary Drinfeld module $\psi : A \to K\{\tau\}$ of positive characteristic and for $t \in A$ such that ψ_t is inseparable, the set $\psi[t^\infty](K^{\mathrm{sep}})$ is finite if and only if $\psi_t \in \mathrm{Aut}_{\mathrm{K^{sep}}}(\psi^\sharp)$.*

Proof of Corollary 3.11. If $\psi[t^\infty](K^{\mathrm{sep}})$ is finite, then clearly there is no t-power-torsion of ψ in $\psi^\sharp$ and so, ψ_t is injective on $\psi^\sharp$. Because all the endomorphisms of $\psi^\sharp$ are surjective ($\psi^\sharp$ is a c-minimal, connected group), then indeed, $\psi_t \in \mathrm{Aut}_{\mathrm{K^{sep}}}(\psi^\sharp)$.

If $\psi_t \in \mathrm{Aut}_{\mathrm{K^{sep}}}(\psi^\sharp)$, we claim there is only finite t-power-torsion of ψ in K^{sep}. Assume this is not the case. Then there are arbitrarily long sequences $(x_n)_{0\leq n\leq m} \in \psi[t^\infty](K^{\mathrm{sep}})$ such that

$$x_n = \psi_t(x_{n+1}) \text{ , for all } n \in \{0, \dots, m-1\} \text{ and } x_0 \neq 0.$$

Arguing as in the proof of Lemma 3.10, we conclude there exists an infinite coherent sequence $(x_n)_{n\geq 0} \in \psi[t^\infty](K^{\mathrm{sep}})$ such that

$$x_n = \psi_t(x_{n+1}) \text{ , for all } n \geq 0 \text{ and } x_0 \neq 0.$$

Hence $x_0 \in \psi^\sharp \cap \psi[t^\infty]$, which provides a contradiction with our assumption. This concludes the proof of Corollary 3.11. □

The result of Lemma 3.10 is instrumental in proving that $\phi^\sharp(K^{\mathrm{sep}}) \subset \phi_{\mathrm{tor}}$. Indeed, take $x \in \phi^\sharp(K^{\mathrm{sep}})$ and construct the associated sequence $(x_n)_{n\geq 0}$ as in (3.10).

Let $K' = K(x)$. We claim that $x_n \in K'$, for all $n \geq 1$.

Fix $n \geq 1$ and pick any $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K')$. Because $\phi_t \in K\{\tau\} \subset K'\{\tau\}$, for every $m \geq 1$, $\sigma(x_m) = \sigma(\phi_t(x_{m+1})) = \phi_t(\sigma(x_{m+1}))$. So, for every $m \geq 1$, $x_n - \sigma(x_n) = \phi_{t^m}(x_{n+m} - \sigma(x_{n+m}))$. Thus,

$$x_n - \sigma(x_n) \in \phi^\sharp. \tag{7}$$

But $\phi_{t^n}(x_n - \sigma(x_n)) = \phi_{t^n}(x_n) - \phi_{t^n}(\sigma(x_n)) = \phi_{t^n}(x_n) - \sigma(\phi_{t^n}(x_n)) = x - \sigma(x) = 0$, because $x \in K'$. Thus

$$x_n - \sigma(x_n) \in \phi[t^n]. \tag{8}$$

As shown by (6), there is no t-power torsion of ϕ in $\phi^\sharp$. Equations (8) and (7) yield

$$x_n - \sigma(x_n) = 0. \tag{9}$$

So, $x_n = \sigma(x_n)$, for all $n \geq 1$ and for all $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K')$. Thus, $x_n \in K'$, for all $n \geq 1$ as it was claimed. If $x \notin \phi_{\mathrm{tor}}$, then $x_n \notin \phi_{\mathrm{tor}}$ for all $n \geq 1$. This will give a contradiction to the structure theorem for $\phi(K')$.

In [13] (for fields of transcendence degree 1 over $\mathbb{F}_p$) and in [17] (for fields of arbitrary positive transcendence degree) it is established that a finitely generated field (such as K' in our setting) has the following ϕ-module structure: a direct sum of a finite torsion submodule and a free module of rank $\aleph_0$. In particular this means that there cannot be an infinitely t-divisible non-torsion element $x \in L$. So, $x \in \phi_{\mathrm{tor}}$ and we conclude that $\phi^\sharp(K^{\mathrm{sep}}) \subset \phi_{\mathrm{tor}}$.

By Lemma 3.7, we know that $\phi[\mathfrak{p}'] \subset \phi^\sharp$. We will prove next that under the hypothesis from Theorem 3.8 (see (5)), $\phi^\sharp(K^{\mathrm{sep}}) = \phi[\mathfrak{p}']$.

Suppose that there exists $x \in \phi^\sharp(K^{\mathrm{sep}}) \setminus \phi[\mathfrak{p}']$. Because we already proved that $\phi^\sharp(K^{\mathrm{sep}}) \subset \phi_{\mathrm{tor}}$, $x \in \phi_{\mathrm{tor}}$. Then there exists $a \in \mathfrak{p} \setminus \{0\}$ such that $\phi_a(x) = 0$. Because $t \in \mathfrak{p} \setminus \{0\}$, there exist $n, m \geq 1$ and $u, v \in A \setminus \mathfrak{p}$ such that $t^n v = a^m u$. Then

$$\phi_{t^n v}(x) = \phi_{a^m u}(x) = \phi_{a^{m-1}u}(\phi_a(x)) = 0.$$

So, $x \in \phi[t^n v]$. By our assumption, $x \notin \phi[\mathfrak{p}']$ and so, $y := \phi_v(x) \neq 0$. Thus

$$y \in \phi[t^n] \setminus \{0\}. \tag{10}$$

By Lemma 3.4, because $x \in \phi^\sharp(K^{\mathrm{sep}})$ and $\phi_v \in \mathrm{End}_{K^{\mathrm{sep}}}(\phi)$,

$$y = \phi_v(x) \in \phi^\sharp(K^{\mathrm{sep}}). \tag{11}$$

Equations (10) and (11) provide a contradiction to (6). So, indeed $\phi^\sharp(K^{\mathrm{sep}}) = \phi[\mathfrak{p}']$.

In order to prove the second part of our Theorem 3.8 regarding the quasi-endomorphisms of $\phi^\sharp$, we split the proof in two cases.

Case 1. The polynomial ϕ_t is purely inseparable.

Then $\phi_t = \alpha\tau^r$ for some $\alpha \in K$ and some $r \geq 1$. Let $\gamma \in K^{\mathrm{sep}}$ such that $\gamma^{q^r-1}\alpha = 1$.

Let $\phi^{(\gamma)}$ be the Drinfeld module defined by $\phi^{(\gamma)} = \gamma^{-1}\phi\gamma$. We call $\phi^{(\gamma)}$ the conjugate of ϕ by γ. Then $\phi_t^{(\gamma)} = \tau^r$. Moreover, because for all $a \in A$, $\phi^{(\gamma)} = \gamma^{-1}\phi_a\gamma$ and $\gamma \in K^{\mathrm{sep}}$, we conclude that

$$\phi^{(\gamma)^\sharp} = \gamma^{-1}\phi^\sharp \tag{12}$$

and

$$\mathrm{QsE}_{\mathrm{K^{sep}}}(\phi^\sharp) = \gamma\, \mathrm{QsE}_{\mathrm{K^{sep}}}\left(\phi^{(\gamma)^\sharp}\right)\gamma^{-1}. \tag{13}$$

Because $\phi_t^{(\gamma)} = \tau^r$, $\phi^{(\gamma)^\sharp} = \bigcap_{n\geq 1} L^{p^n} =: L^{p^\infty}$.

By [2] (Proposition 4.10), the ring $\mathrm{QsE}_{\mathrm{K^{sep}}}\left(L^{p^\infty}\right)$ is the division ring of fractions of the Ore ring $\mathbb{F}_p^{\mathrm{alg}}\{\tau_0, \tau_0^{-1}\}$, where τ_0 is the usual Frobenius (see [8] for constructing the division ring of fractions for an Ore ring). Then clearly, for all $\psi \in \mathrm{QsE}_{\mathrm{K^{sep}}}\left(\phi^{(\gamma)^\sharp}\right)$, there exists $n \geq 1$ such that

$$\phi_{t^n}^{(\gamma)} = \tau^{rn}$$

commutes with ψ in $\mathrm{QsE}_{\mathrm{K^{sep}}}(\phi^{(\gamma)^\sharp})$. By (13), we conclude that also for every $\psi \in \mathrm{QsE}_{\mathrm{K^{sep}}}(\phi^\sharp)$, there exists $n \geq 1$ such that $\psi\phi_{t^n} = \phi_{t^n}\psi$.

Case 2. The polynomial ϕ_t is not purely inseparable, i.e. $\phi[t] \neq \{0\}$.

Lemma 3.12. *For every $\psi \in \mathrm{QsE}_{\mathrm{K^{sep}}}(\phi^\sharp)$ there exists $a \in A \setminus \{0\}$ and $n \geq 1$ such that $\phi_a\psi\phi_{t^n} \in \mathrm{End}_{\mathrm{K^{sep}}}(\phi^\sharp) \cap K^{\mathrm{sep}}\{\tau\}$ (the intersection is taken inside $\mathrm{QsE}_{\mathrm{K^{sep}}}(L)$).*

Proof. Let $\psi \in \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$ and let $S = \{x \in \phi^\sharp | (0,x) \in \psi\}$. Thus, S is a finite, K^{sep}-definable subgroup of $\phi^\sharp$. Because L is an elementary extension of K^{sep}, $S \subset K^{\mathrm{sep}}$. Thus $S \subset \phi^\sharp(K^{\mathrm{sep}}) \subset \phi_{\mathrm{tor}}$. Hence there exists $a \in A \setminus \{0\}$ such that $S \subset \phi[a]$. By Lemma 3.4, $\phi_a\psi \in \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$ and the subgroup

$$\{x \in \phi^\sharp \mid (0,x) \in \phi_a\psi\}$$

is trivial by our choice for a. Thus, $\phi_a\psi$ is actually an endomorphism of $\phi^\sharp$. Also, according to Proposition 2.22, the endomorphisms of $\phi^\sharp$ are λ-polynomials. Thus, by Corollary 3.6, because ϕ_t is inseparable, there exists $n \geq 1$ such that $\phi_a\psi\phi_{t^n} \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp) \cap K^{\mathrm{sep}}\{\tau\}$. ☐

Proposition 3.13. *Let R be a domain, i.e. a unital (not necessarily commutative) ring with no nontrivial zero-divisors.*

a) Let $y \in R$ be nonzero and suppose that $g \in R$ commutes with y and xy for some $x \in R$. Then g also commutes with x.

b) Let $y \in R$ be nonzero and suppose that $g \in R$ commutes with y and yx for some $x \in R$. Then g also commutes with x.

Proof of Proposition 3.13. It suffices to prove a), because the proof of b) follows from a) applied to R^{op}.

Thus, for the proof of a), we know that

$$(gx)y = g(xy) = (xy)g = x(yg) = x(gy) = (xg)y. \tag{14}$$

Because $y \in R\setminus\{0\}$ and R is a domain, equation (14) concludes the proof of Proposition 3.13 a). ☐

We use Proposition 3.13 with $R = \mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$ because from Proposition 2.14, we know that $\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$ is a division ring. Then by Lemma 3.12 and Proposition 3.13, it suffices to prove Theorem 3.8 for $\psi =: f \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp) \cap K^{\mathrm{sep}}\{\tau\}$.

Let $f \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp) \cap K^{\mathrm{sep}}\{\tau\}$. By Lemma 3.9, $\phi_t^{-1} \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$ and so, $\phi_t^{-1}f \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$. Hence, $\phi_t^{-1}f$ is a λ-polynomial. By Proposition 3.6, there exists $m \geq 1$ such that for every polynomial $h \in K\{\tau\}\tau^m$,

$$\phi_t^{-1}h \in K^{\mathrm{sep}}\{\tau\}. \tag{15}$$

Because ϕ_t has inseparable degree at least 1 and $f \in K^{\mathrm{sep}}\{\tau\}$, equation (15) yields that $g_1 := \phi_t^{-1}f\phi_{t^m} \in K^{\mathrm{sep}}\{\tau\}$. Moreover, by Lemma 3.9, $g_1 \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp)$. This means that the equation

$$f\phi_{t^m} = \phi_t g_1, \tag{16}$$

which initially was true only on $\phi^\sharp$ is an identity in $K^{\mathrm{sep}}\{\tau\}$. Indeed, $\phi^\sharp$ is infinite (see Lemma 3.7) and so, (16) holds for infinitely many points of L. Thus, because $f\phi_{t^m}$ and $\phi_t g_1$ are polynomials, (16) holds identically in L.

Because in equation (16) all the functions are polynomials in τ, we can equate the order of τ in g_1. We obtain

$$\mathrm{ord}_\tau\, g_1 = \mathrm{ord}_\tau\, f + (m-1)\,\mathrm{ord}_\tau\, \phi_t \geq (m-1)\,\mathrm{ord}_\tau\, \phi_t \geq m-1. \tag{17}$$

Thus $\mathrm{ord}_\tau(g_1\phi_t) \geq m$ and using (15), we get that $\phi_t^{-1}g_1\phi_t \in \mathrm{End}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp) \cap K^{\mathrm{sep}}\{\tau\}$. So, denote by $g_2 = \phi_t^{-1}g_1\phi_t$. This means that the identity

$$\phi_t g_2 = g_1\phi_t, \tag{18}$$

which initially was true only on $\phi^\sharp$ is actually true everywhere. It is the same argument as above when we explained that equation (16) is an identity of polynomials from $K^{\text{sep}}\{\tau\}$.

We equate the order of τ of the polynomials from (18) and conclude that

$$\operatorname{ord}_\tau g_2 = \operatorname{ord}_\tau g_1 \geq m-1. \tag{19}$$

So, then again $\operatorname{ord}_\tau(g_2\phi_t) \geq m$ and we can apply (15) and find a polynomial

$$g_3 \in K^{\text{sep}}\{\tau\} \cap \operatorname{End}_{\mathrm{K}^{\text{sep}}}(\phi^\sharp) \text{ such that } \phi_t g_3 = g_2\phi_t.$$

Once again $\operatorname{ord}_\tau g_3 = \operatorname{ord}_\tau g_2$ and so the above process can continue and we construct an infinite sequence $(g_n)_{n\geq 1} \in K^{\text{sep}}\{\tau\} \cap \operatorname{End}_{\mathrm{K}^{\text{sep}}}(\phi^\sharp)$ such that for every $n \geq 1$,

$$\phi_t g_{n+1} = g_n\phi_t. \tag{20}$$

Let $g_0 = f\phi_{t^{m-1}}$. Then, using (16), we conclude that equation (20) holds also for $n = 0$.

An easy induction will show that for every $k \geq 1$ and for all $n \geq 0$,

$$\phi_{t^k} g_{n+k} = g_n\phi_{t^k}. \tag{21}$$

Indeed, case $k = 1$ is equation (20). So, we suppose that (21) holds for some $k \geq 1$ and for all $n \geq 0$ and we will prove it holds for $k+1$ and all $n \geq 0$. By equations (20) and (21) we have that

$$\phi_{t^{k+1}} g_{n+k+1} = \phi_t(\phi_{t^k} g_{n+1+k}) = \phi_t g_{n+1}\phi_{t^k} = g_n\phi_t\phi_{t^k} = g_n\phi_{t^{k+1}},$$

which proves the inductive step of our assertion.

Equation (21) shows that for every $k \geq 1$, g_{n+k} maps $\phi[t^k]$ into itself, for every $n \geq 0$. Equation (20) shows that all the polynomials g_n have the same degree, call it d. Because ϕ_t is not purely inseparable, we may choose $k_0 \geq 1$ such that

$$|\phi[t^{k_0}]| > d. \tag{22}$$

Because $\phi[t^{k_0}]$ is a finite set and our sequence of polynomials $(g_n)_{n\geq 0}$ is infinite, it means that there exist $n_2 > n_1 \geq 0$ such that

$$g_{n_1+k_0}|_{\phi[t^{k_0}]} = g_{n_2+k_0}|_{\phi[t^{k_0}]}. \tag{23}$$

By another application of the fact that all g_n are polynomials, equations (22) and (23) yield that

$$g_{n_1+k_0} = g_{n_2+k_0}. \tag{24}$$

But then, using (21) (with $k = n_2 - n_1$ and $n = n_1 + k_0$) we conclude that

$$\phi_{t^{n_2-n_1}} g_{n_2+k_0} = g_{n_1+k_0}\phi_{t^{n_2-n_1}}. \tag{25}$$

If we denote by g the polynomial represented by both $g_{n_2+k_0}$ and $g_{n_1+k_0}$ (according to (24)), equation (25) shows that g commutes with $\phi_{t^{n_2-n_1}}$. We let $n_0 = n_2 - n_1 \geq 1$ and so,

$$g\phi_{t^{n_0}} = \phi_{t^{n_0}} g. \tag{26}$$

The definition of $g = g_{n_1+k_0}$ and equation (21) (with $k = n_1 + k_0$ and $n = 0$) give

$$\phi_{t^{n_1+k_0}} g = g_0\phi_{t^{n_1+k_0}}. \tag{27}$$

Equation (26) shows that $\phi_{t^{n_0}}$ commutes with $\phi_{t^{n_1+k_0}} g$. Thus, by equation (27), $\phi_{t^{n_0}}$ commutes also with $g_0\phi_{t^{n_1+k_0}}$. We apply now Proposition 3.13 a) to conclude that $\phi_{t^{n_0}}$ commutes with g_0. Because $g_0 = f\phi_{t^{m-1}}$, another application of the above mentioned proposition gives us

$$\phi_{t^{n_0}} f = f\phi_{t^{n_0}}$$

and ends the proof of Theorem 3.8. □

Theorem 3.14. *Let ϕ be a Drinfeld module of finite characteristic $\mathfrak{p}$. Assume that there exists $f \in \mathrm{Aut}_{K^{\mathrm{sep}}}(\phi^\sharp) \cap K^{\mathrm{sep}}\{\tau\}\tau$. Then $\phi^\sharp(K^{\mathrm{sep}}) \subset \phi_{\mathrm{tor}}$ and for all $\psi \in \mathrm{QsE}_{K^{\mathrm{sep}}}(\phi^\sharp)$, there exists $n \geq 1$ such that $\psi f^n = f^n\psi$ (the identity being seen in $\mathrm{QsE}_{K^{\mathrm{sep}}}(\phi^\sharp)$).*

Proof. Construct another Drinfeld module $\Phi : \mathbb{F}_q[t] \to K^{\mathrm{sep}}\{\tau\}$ by $\Phi_t = f$. By Lemma 3.2, $\Phi^\sharp = f^\sharp$. Using Corollary 3.5 and $f \in \mathrm{End}_{K^{\mathrm{sep}}}(\phi^\sharp)$, we get that

$$\phi^\sharp \subset \Phi^\sharp. \tag{28}$$

Because both $\phi^\sharp$ and $\Phi^\sharp$ are connected, c-minimal groups (see Corollary 3.3), applying Corollary 2.21, we conclude that they are equal.

Because $\Phi_t \in \mathrm{Aut}_{K^{\mathrm{sep}}}(\phi^\sharp) = \mathrm{Aut}_{K^{\mathrm{sep}}}(\Phi^\sharp)$, $\Phi[t^\infty] \cap K^{\mathrm{sep}}$ is finite (or otherwise we would have t-power-torsion of Φ in $\Phi^\sharp$, as shown by Corollary 3.11). Hence, we are in the hypothesis of Theorem 3.8 with Φ and t. Thus, we conclude that

$$\Phi^\sharp(K^{\mathrm{sep}}) = \Phi[(t)'], \tag{29}$$

where by $\Phi[(t)']$ we denoted the prime-to-t-torsion of Φ.

Because for all $a \in A$, $\phi_a \in \mathrm{End}_{K^{\mathrm{sep}}}(\phi) \subset \mathrm{End}_{K^{\mathrm{sep}}}(\phi^\sharp) = \mathrm{End}_{K^{\mathrm{sep}}}(\Phi^\sharp)$, there exists $n_a \geq 1$ such that $\phi_a f^{n_a} = f^{n_a}\phi_a$, by Theorem 3.8. Because A is finitely generated as an $\mathbb{F}_p$-algebra, we can find $n_0 \geq 1$ such that for all $a \in A$, $\phi_a f^{n_0} = f^{n_0}\phi_a$, i.e. $f^{n_0} \in \mathrm{End}_{K^{\mathrm{sep}}}(\phi)$.

Claim 3.15. Let $c(t) \in \mathbb{F}_q[t] \setminus \{0\}$ and let $m \geq 1$. Then there exists $d(t) \in \mathbb{F}_q[t^m] \setminus \{0\}$ such that $c(t)$ divides $d(t)$.

Proof of Claim 3.15. Because $\mathbb{F}_q[t]/(c(t))$ is finite and because $\mathbb{F}_q[t^m]$ is infinite, there exist $d_1(t) \neq d_2(t)$, both polynomials in $\mathbb{F}_q[t^m]$, such that $c(t)$ divides $d(t) = d_1(t) - d_2(t)$. □

Let $x \in \Phi_{\mathrm{tor}}$ and let $c(t) \in \mathbb{F}_q[t] \setminus \{0\}$ such that $\Phi_{c(t)}(x) = 0$. By Claim 3.15, we may assume that $c(t) \in \mathbb{F}_q[t^{n_0}]$. Because $\Phi_{t^{n_0}} = f^{n_0} \in \mathrm{End}_{K^{\mathrm{sep}}}(\phi)$, $\Phi_{c(t)} \in \mathrm{End}_{K^{\mathrm{sep}}}(\phi)$.

Let a be a non-constant element of A. Then for all $y \in \Phi[c(t)]$,

$$\Phi_{c(t)}(\phi_a(y)) = \phi_a(\Phi_{c(t)}(y)) = 0.$$

Thus $\phi_a(y) \in \Phi[c(t)]$ for all $y \in \Phi[c(t)]$. Similarly, ϕ_{a^m} maps $\Phi[c(t)]$ into itself for every $m \geq 1$. Because $\Phi[c(t)]$ is a finite set and $x \in \Phi[c(t)]$, there exist $m_2 > m_1 \geq 1$ such that $\phi_{a^{m_2}}(x) = \phi_{a^{m_1}}(x)$. Thus $x \in \phi[a^{m_2} - a^{m_1}]$ and $a^{m_2} - a^{m_1} \neq 0$ (a is not constant). This shows that $x \in \phi_{\mathrm{tor}}$ and because x was an arbitrary torsion point of Φ, then $\Phi_{\mathrm{tor}} \subset \phi_{\mathrm{tor}}$. Actually, because the above argument can be used reversely by starting with an arbitrary torsion point x of ϕ and concluding that $x \in \Phi_{\mathrm{tor}}$, we have $\phi_{\mathrm{tor}} = \Phi_{\mathrm{tor}}$. In any case, the inclusion $\Phi_{\mathrm{tor}} \subset \phi_{\mathrm{tor}}$ is sufficient to conclude that

$$\phi^\sharp(K^{\mathrm{sep}}) = \Phi^\sharp(K^{\mathrm{sep}}) \subset \Phi_{\mathrm{tor}} \subset \phi_{\mathrm{tor}}.$$

Also, Theorem 3.8 applied to Φ and $f = \Phi_t$ shows that for all

$$\psi \in \mathrm{QsE}_{K^{\mathrm{sep}}}(\Phi^\sharp) = \mathrm{QsE}_{K^{\mathrm{sep}}}(\phi^\sharp),$$

there exists $n \geq 1$ such that $\psi f^n = f^n\psi$ (in $\mathrm{QsE}_{K^{\mathrm{sep}}}(\phi^\sharp)$). □

The following example shows that one possible way of strengthening Theorem 3.8 does not hold and also shows how Theorem 3.14 applies when we do not have the hypothesis of (3.8).

Example 3.16. Assume $p > 2$. Let t be an indeterminate and $K = \mathbb{F}_q(t)$. Let $f = t\tau + \tau^3$. Then, for all $\lambda \in \mathbb{F}_{q^2}$,

$$f\lambda = \lambda^q f \tag{30}$$

where λ is seen as the operator $\lambda\tau^0$.

Define $\phi : \mathbb{F}_q[t] \to \mathbb{F}_q(t)\{\tau\}$ by $\phi_t = f(\tau^0 + f)$. We claim that

$$\phi[t^\infty] \cap K^{\text{sep}} \text{ is infinite.} \tag{31}$$

Because for all $n \geq 1$, $\phi_{t^n} = f^n(\tau^0 + f)^n$, $\operatorname{Ker}((\tau^0 + f)^n) \subset \operatorname{Ker}\phi_{t^n}$. Because $\tau^0 + f$ is a separable polynomial, all the roots of $(\tau^0 + f)^n$ are distinct and separable over K. So, indeed, (31) holds.

statement (31) shows that the hypothesis of Theorem 3.8 fails for ϕ and t. We will prove the conclusion of Theorem 3.8 regarding the quasi-endomorphisms of $\phi^\sharp$ fails, i.e. there exists a quasi-endomorphism of $\phi^\sharp$ that does not commute with any power of ϕ_t.

Let $\lambda \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Applying Lemma 3.2, we get that $\phi^\sharp = (\phi_t)^\sharp$. Applying Corollary 2.21 to ϕ_t and f^2 we conclude that $\phi^\sharp = (f^2)^\sharp$ (because f^2 is an endomorphism of ϕ and so, by Corollary 3.5, $(\phi_t)^\sharp = \phi^\sharp \subset (f^2)^\sharp$). But

$$f^2\lambda = \lambda f^2 \text{ (apply equation (30) twice).} \tag{32}$$

Thus, with the help of Lemma 3.4 applied to the Drinfeld module $\psi : \mathbb{F}_q[t] \to K\{\tau\}$ given by $\psi_t = f^2$, we get that

$$\lambda \in \operatorname{End}_{\mathrm{K}^{\text{sep}}}(\psi^\sharp) = \operatorname{End}_{\mathrm{K}^{\text{sep}}}\left((f^2)^\sharp\right) = \operatorname{End}_{\mathrm{K}^{\text{sep}}}(\phi^\sharp).$$

Suppose that there exists $n \geq 1$ such that $\phi_{t^n}\lambda = \lambda\phi_{t^n}$ on $\phi^\sharp$. Because $\phi^\sharp$ is infinite, $\phi_{t^n}\lambda = \lambda\phi_{t^n}$, as polynomials. Then also $\phi_{t^{2n}}\lambda = \lambda\phi_{t^{2n}}$. But $\phi_{t^{2n}} = f^{2n}(\tau^0 + f)^{2n}$ and using (32) and Proposition 3.13 applied to the domain $K\{\tau\}$, we get

$$(\tau^0 + f)^{2n}\lambda = \lambda(\tau^0 + f)^{2n}. \tag{33}$$

We will prove that (33) is impossible. Because of the skew commutation of f and λ as shown in equation (30), the only way for equation (33) to hold is if in the expansion of $(\tau^0 + f)^{2n}$, all the nonzero terms are even powers of f. Let p^l be the largest power of p that is less than or equal to $2n$. Then $\binom{2n}{p^l} \neq 0$ (in $\mathbb{F}_p$) and its corresponding power of f is odd. This shows that indeed, (33) cannot hold when $p > 2$.

On the other hand, $f \in \operatorname{End}_{\mathrm{K}^{\text{sep}}}(\phi)$ and the hypothesis of Theorem 3.14 is verified for ϕ and f. Indeed, $f \in \operatorname{End}_{\mathrm{K}^{\text{sep}}}(\phi^\sharp)$ and $\operatorname{Ker}(f) \cap K^{\text{sep}} = \{0\}$; thus $f \in \operatorname{Aut}_{\mathrm{K}^{\text{sep}}}(\phi^\sharp)$. As we can see from equation (32), also the conclusion of (3.14) regarding the commutation of the quasi-endomorphism λ of ϕ (i.e. the scalar multiplication function associated to λ) with a power of f holds with the power being f^2.

For the case $p = 2$ we can construct a similar example by taking $f = t\tau + \tau^4$ and defining the Drinfeld module $\phi : \mathbb{F}_q[t] \to \mathbb{F}_q(t)\{\tau\}$ by $\phi_t = f(\tau^0 + f)$. In this case, $\lambda \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$ will play the role of the endomorphism of $\phi^\sharp$ that commutes with a power of an endomorphism of ϕ, i.e. it commutes with f^3, but it does not commute with any power of ϕ_t.

4. Mordell-Lang conjecture for Drinfeld modules

We first note that in all of our arguments that will follow, "subvariety" means "closed subvariety".

Let K be a finitely generated field of positive transcendence degree over $\mathbb{F}_p$.

In [4], Laurent Denis formulated an analogue of the Mordell-Lang Conjecture in the context of Drinfeld modules. Even though the formulation from [4] is for Drinfeld modules of generic characteristic, we can ask the same question for Drinfeld modules of finite characteristic. Thus, our Statement 4.2 will cover both cases. Before stating (4.2) we need a definition.

Definition 4.1. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. For $g \geq 0$ we consider ϕ acting diagonally on $\mathbb{G}_a^g$. An algebraic ϕ-submodule of $\mathbb{G}_a^g$ is a connected algebraic subgroup of $\mathbb{G}_a^g$ which is stable under the action of ϕ.

Statement 4.2 (Mordell-Lang statement for ϕ)**.** Let ϕ be a Drinfeld module. If Γ is a finitely generated ϕ-submodule of $\mathbb{G}_a^g(K^{\text{alg}})$ for some $g \geq 0$ and if X is an algebraic subvariety of $\mathbb{G}_a^g$, then there are finitely many algebraic ϕ-submodules $B_1, \dots, B_s$ and there are finitely many elements $\gamma_1, \dots, \gamma_s$ of $\mathbb{G}_a^g(K^{\text{alg}})$ such that $X(K^{\text{alg}}) \cap \Gamma = \cup_{1 \leq i \leq s}(\gamma_i + B_i(K^{\text{alg}}) \cap \Gamma)$.

The first result towards Statement 4.2 was obtained by Thomas Scanlon in [14]. Before stating the theorem from [14], we need to introduce two definitions.

Definition 4.3. For a Drinfeld module $\phi : A \to K\{\tau\}$, its field of definition is the smallest subfield of K containing all the coefficients of ϕ_a, for every $a \in A$.

Definition 4.4. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. The modular transcendence degree of ϕ is the minimum transcendence degree over $\mathbb{F}_p$ of the field of definition for $\phi^{(\gamma)}$ (we recall $\phi_a^{(\gamma)} = \gamma^{-1}\phi_a\gamma$ for every $a \in A$), where the minimum is taken over all $\gamma \in K^{\text{alg}} \setminus \{0\}$.

In [7] (see Lemmas 7.0.42 and 7.0.43) we proved that if there exists a non-constant $t \in A$ such that $\phi_t = \sum_{i=0}^r a_i\tau^i$ is monic, then the modular transcendence degree of ϕ is $\text{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(a_0, \dots, a_{r-1})$.

Theorem 4.5 (Thomas Scanlon). *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic and modular transcendence degree at least* 1*. Let Γ be a finitely generated ϕ-submodule of $\mathbb{G}_a^g(K^{\text{alg}})$ and X be a K^{alg}-subvariety of $\mathbb{G}_a^g$. Then $X(K^{\text{alg}}) \cap \Gamma$ is a finite union of translates of subgroups of Γ.*

Using Theorem 3.8, we are able to strengthen the conclusion of (4.5) by showing that one could replace *subgroups* by *ϕ-submodules*.

Theorem 4.6. *If X is a K^{alg}-subvariety of $\mathbb{G}_a^g$ and $\phi : A \to K\{\tau\}$ is a Drinfeld module of positive modular transcendence degree for which there exists a non-constant $t \in A$ such that $\phi[t^\infty](K^{\text{sep}})$ is finite, then for every finitely generated ϕ-submodule Γ of $\mathbb{G}_a^g(K^{\text{alg}})$, there exists $n \geq 1$ such that $X(K^{\text{alg}}) \cap \Gamma$ is a finite union of translates of $\mathbb{F}_q[t^n]$-submodules of Γ.*

Before proving Theorem 4.6, we need to prove a technical lemma regarding groups of U-rank 1. For a definition and basic properties of the U-rank (also called, the Lascar rank) we refer the reader to Delon's article in [1]. We also mention that Lemma 4.8 is true in a larger generality; for example it is true if the U-rank is replaced by the Morley rank (for the definition of the Morley rank, see Ziegler's article in [1]) and so, it holds in the context of classical algebraic geometry. We denote by rk the U-rank. For the reader's convenience we recall here the properties of the U-rank that we will use in Lemma 4.8.

Proposition 4.7. *Let G be an infinitely definable group for which the U-rank is defined.*

1) *The U-rank of G is 0 if and only if G is finite.*

2) *If H is a definable subgroup of G, then H has U-rank. Moreover,* $\mathrm{rk}(H) \leq \mathrm{rk}(G)$, *with equality if and only if $[G : H]$ is finite.*

3) *If H is another group for which the U-rank is defined and $f : G \to H$ is a definable map, then both* $\mathrm{im}(f)$ *and* $\mathrm{Ker}(f)$ *have U-rank and* $\mathrm{rk}(G) = \mathrm{rk}(\mathrm{im}(f)) + \mathrm{rk}(\mathrm{Ker}(f))$.

4) *For each $n \geq 0$, the cartesian product G^n is a group for which the U-rank is defined. (By convention, the zeroth cartesian power of G is the trivial group.)*

Lemma 4.8. *Let G be a connected, infinitely definable subgroup of L of U-rank 1 over K^{sep}. Let n be a non-negative integer and let H be a definable subgroup of G^n of U-rank d. There exists a projection π of G^n to some d coordinates of G^n such that $\pi(H) = \pi(G^n) = G^d$ and the fibers of $\pi|_H$ are finite.*

Proof. Our proof is by induction on n. If $n = 0$, then the conclusion of our lemma holds trivially (the projection being the zero map).

Assume Lemma 4.8 holds for $n-1$, for some $n \geq 1$ and we prove it holds also for n. Let π_1 be the projection of G^n on the first $(n-1)$ coordinates. By property 3) of Proposition 4.7, $\mathrm{Ker}(\pi_1|_H)$ is a subgroup of G of U-rank equal either 0 or 1.

If the former case holds, i.e. $\mathrm{rk}(\mathrm{Ker}(\pi_1|_H)) = 0$, then $\mathrm{Ker}(\pi_1|H)$ is finite, by property 1) of Proposition 4.7. Also, $\mathrm{rk}(\pi_1(H)) = d$, by property 3) of Proposition 4.7. We can apply the induction hypothesis to $\pi_1(H) \subset G^{n-1}$ and conclude there exists a suitable projection map π_2 such that $\pi_2(\pi_1(H)) = G^d$ and $\mathrm{Ker}(\pi_2|_{\pi_1(H)})$ is finite. Hence the projection map $\pi_2 \circ \pi_1$ satisfies the conclusion of Lemma 4.8 with respect to $H \subset G^n$.

If the latter case holds, i.e. $\mathrm{rk}(\mathrm{Ker}(\pi_1|_H)) = 1$, then $\mathrm{Ker}(\pi_1|_H) = G$, because of property 2) of Proposition 4.7 and the fact that G is connected. Thus $H = \pi_1(H) \times G$. We apply the induction hypothesis to $\pi_1(H) \subset G^{n-1}$ and conclude there exists a suitable projection map $\pi_2 : G^{n-1} \to G^{d-1}$ such that $\pi_2|_{\pi_1(H)}$ is surjective and $\mathrm{Ker}(\pi_2|_{\pi_1(H)})$ is finite. Considering the projection map $\pi_3 : G^n \to G^{d-1} \times G$ defined as $(\pi_2 \circ \pi_1, \pi_n)$ (where π_n is the projection of G^n on the last coordinate) and using the fact that $H = \pi_1(H) \times G$, we obtain the conclusion of Lemma 4.8. ☐

Proof of Theorem 4.6. First we prove the following

Claim 4.9. Let K_1 be a finite extension of K. Then $\phi[t^\infty](K_1^{\mathrm{sep}})$ is finite.

Proof of Claim 4.9. Let p^k be the inseparable degree of the finite extension K_1/K. Then $K_1^{\mathrm{sep}} \subset K^{\mathrm{sep}^{1/p^k}}$.

If we assume the set $\phi[t^\infty](K_1^{\mathrm{sep}})$ is infinite then, as shown in the proof of Corollary 3.11, there exists an infinite coherent sequence $(x_n)_{n\geq 0} \in \phi[t^\infty](K_1^{\mathrm{sep}})$ such that

$$x_n = \phi_t(x_{n+1}) \text{ , for all } n \geq 0 \text{ and } x_0 \neq 0.$$

Thus we know that for every $n \geq 0$, $x_n = \phi_{t^k}(x_{n+k})$. Because $\phi_{t^k} \in K\{\tau\}\tau^k$ and $x_{n+k} \in K_1^{\mathrm{sep}} \subset K^{\mathrm{sep}^{1/p^k}}$, we conclude that $x_n \in K^{\mathrm{sep}}$, for every $n \geq 0$. This contradicts our hypothesis that $\phi[t^\infty](K^{\mathrm{sep}})$ is finite and concludes the proof of Claim 4.9. ☐

Using Claim 4.9, it suffices to prove Theorem 4.6 under the hypothesis that both X is defined over K and $\Gamma \subset \mathbb{G}_a^g(K)$. Then $X(K^{\text{alg}}) \cap \Gamma = X(K) \cap \Gamma$.

As in Theorem 4.5, let H be an irreducible algebraic subgroup of $\mathbb{G}_a^g$ such that for some $\gamma \in \mathbb{G}_a^g(K^{\text{alg}})$,

$$\gamma + \big(H(K^{\text{alg}}) \cap \Gamma\big) \subset X(K^{\text{alg}}) \cap \Gamma. \tag{34}$$

If H is finite, then because H is irreducible, we conclude $H = \{0\}$ (H is a group). In this case, clearly H is invariant under the action of ϕ. Thus, from now on, we may assume H is an infinite irreducible algebraic group.

At the expense of replacing K by a finite extension, we may assume H is defined over K (note that replacing K by a finite extension does not change $X(K^{\text{alg}})\cap\Gamma$ because $\Gamma \subset \mathbb{G}_a^g(K)$).

We may assume that $H(K) \cap \Gamma$ is dense in H (otherwise we replace H by an irreducible component of the Zariski closure of $H(K) \cap \Gamma$ and again replace K by a finite extension so that H is defined over K). Hence, H is an irreducible algebraic group, whose translate by γ is contained in X (we use (34) and the fact that $H(K) \cap \Gamma$ is dense in H).

From this point on in this proof, the setting is that H is an infinite, irreducible algebraic subgroup defined over K, which appears in the conclusion of Theorem 4.5. Also, X is defined over K and $\Gamma \subset \mathbb{G}_a^g(K)$. In order to prove Theorem 4.6, we will prove H is invariant under a power of ϕ_t.

Let L be an $\aleph_1$-saturated elementary extension of K^{sep}. We apply Lemma 4.8 to the definable subgroup $H(L)\cap\phi^\sharp(L)^g$ of the infinitely definable group $\phi^\sharp(L)^g$ ($\phi^\sharp(L)$ is connected by Corollary 3.3 and $\phi^\sharp(L)$ has U-rank 1 as proved in [14]). We conclude there exists a projection map π satisfying the conclusions of the above mentioned lemma. We identify $\pi(\phi^\sharp(L)^g)$ with $\phi^\sharp(L)^d$, where d is the U-rank of $H(L) \cap \phi^\sharp(L)^g$. Thus for every point

$$(x_1, \dots, x_d) \in \phi^\sharp(L)^d$$

there is one and at most finitely many points

$$(x_{d+1}, \dots, x_g) \in \phi^\sharp(L)^{g-d}$$

such that

$$(x_1, \dots, x_g) \in H(L) \cap \phi^\sharp(L)^g.$$

Hence, we may identify π with the corresponding quasi-morphism from $\phi^\sharp(L)^d$ to $\phi^\sharp(L)^{g-d}$ (the above defined correspondence is additive because H is a group and $\phi^\sharp(L)^d$ and $\phi^\sharp(L)^{g-d}$ are connected, according to Lemma 2.7). More exactly, the connected component C of $H(L) \cap \phi^\sharp(L)^g$ is the graph of this quasi-morphism between $\phi^\sharp(L)^d$ and $\phi^\sharp(L)^{g-d}$. The following result is crucial for our argument.

Claim 4.10. C is Zariski dense in H.

Proof of Claim 4.10. We will first prove that $H(L) \cap \phi^\sharp(L)^g$ is Zariski dense in H.

We know that $H(L) \cap \Gamma$ is Zariski dense in H. Let $n \geq 1$. Because Γ is a finitely generated ϕ-submodule, the quotient $\Gamma/\phi_{t^n}(\Gamma)$ is finite. Thus there exists $\gamma \in \Gamma$ such that $H(L) \cap (\gamma + \phi_{t^n}(\Gamma))$ is Zariski dense in H. In particular, $H(L) \cap (\gamma + \phi_{t^n}(L)^g)$ is Zariski dense in H.

Let $y \in H(L) \cap (\gamma + \phi_{t^n}(L)^g)$. Because

$$H(L) \cap (\gamma + \phi_{t^n}(L)^g) = y + H(L) \cap \phi_{t^n}(L)^g,$$

we conclude

$$H(L) \cap \phi_{t^n}(L)^g \text{ is Zariski dense in } H. \tag{35}$$

Because (35) holds for every $n \geq 1$, using compactness and $\aleph_1$-saturation of L we conclude $H(L) \cap \phi^\sharp(L)^g$ is Zariski dense in H.

Because the theory of separably closed fields is stable (see Delon's article in [1]), the connected component of an infinitely definable group, such as $H(L) \cap \phi^\sharp(L)^g$, has finite index in the group (see chapter 5 of [12]). Hence C has finite index in $H(L) \cap \phi^\sharp(L)^g$. Moreover, the Zariski closure of C has finite index in H (because C has finite index in $H(L) \cap \phi^\sharp(L)^g$, which is dense in H). Because H is connected, we conclude C is Zariski dense in H, as desired. $\square$

By *Lemme* 3.5.3 of [3], $\mathrm{QsM}_{\mathrm{K}^{\mathrm{sep}}}\left(\phi^\sharp(L)^d, \phi^\sharp(L)^{g-d}\right)$ is naturally isomorphic to the ring of matrices $M_{g-d,d}\left(\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp(L))\right)$, where by $M_{g-d,d}\left(\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp(L))\right)$ we denote the ring of $(g-d)\times d$ matrices over the ring $\mathrm{QsE}_{\mathrm{K}^{\mathrm{sep}}}(\phi^\sharp(L))$. The image of π in $\mathrm{QsM}_{\mathrm{K}^{\mathrm{sep}}}\left(\phi^\sharp(L)^d, \phi^\sharp(L)^{g-d}\right)$ commutes with a power of ϕ_t (by Theorem 3.8). Let ϕ_{t^n} be this power for some $n \geq 1$.

For each $\overline{\mathrm{x}} = (x_1, \dots, x_d) \in \phi^\sharp(L)^d$, let

$$C_{\overline{\mathrm{x}}} = \{(y_1, \dots, y_{g-d}) \in \phi^\sharp(L)^{g-d} \mid (x_1, \dots, x_d, y_1, \dots, y_{g-d}) \in C\}.$$

Because π commutes with ϕ_{t^n}, for each $\overline{\mathrm{x}} \in \phi^\sharp(L)^d$, $\phi_{t^n} C_{\overline{\mathrm{x}}} \subset C_{\phi_{t^n}(\overline{\mathrm{x}})}$. Thus

$$\phi_{t^n}(C) \subset C. \tag{36}$$

Using (36) and the fact that C is Zariski dense in H (as proved by Claim 4.10), we conclude H is invariant under ϕ_{t^n}, as desired. $\square$

Remark 4.11. The result of Theorem 4.6 is sharp in the sense that its conclusion can fail for $n = 1$. For example, let the Drinfeld module $\phi : \mathbb{F}_q[t] \to \mathbb{F}_q(t)\{\tau\}$ be defined by $\phi_t = \tau + t\tau^3$ and $\lambda \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Let $X \subset \mathbb{G}_a^2$ be the curve $y = \lambda x$ and let Γ be the cyclic ϕ-submodule of $\mathbb{G}_a^2(\mathbb{F}_{q^2}(t))$ generated by $(1, \lambda)$. As shown in Example 3.16, $\phi_{t^2}\lambda = \lambda\phi_{t^2}$. Thus for every $n \geq 1$, $(\phi_{t^{2n}}(1), \phi_{t^{2n}}(\lambda)) \in X(\mathbb{F}_{q^2}(t))$. So, $X(\mathbb{F}_q(t)^{\mathrm{alg}}) \cap \Gamma$ is Zariski dense in X. But X is not invariant under ϕ_t; X is invariant under ϕ_{t^2}. Hence in this example (i.e. for this particular X and ϕ), Theorem 4.6 holds with $n = 2$.

Remark 4.12. If we drop the hypothesis on ϕ_t from Theorem 4.6 (i.e. allow $\phi[t^\infty](K^{\mathrm{sep}})$ be infinite) we may lose the conclusion, as is shown by the following example.

Let $p > 2$ and let ϕ, λ, X and Γ be as in Remark 4.11. Let $u = t + t^2$. As shown in Example 3.16, $\phi[u^\infty] \cap \mathbb{F}_p(t)^{\mathrm{sep}}$ is infinite and X is not invariant under any power of ϕ_u. But, as shown in Remark 4.11, $X(\mathbb{F}_p(t)^{\mathrm{alg}}) \cap \Gamma$ is infinite.

The above two remarks 4.11 and 4.12 show that the result of Theorem 4.6 is the most we can hope towards Statement 4.2 for Drinfeld modules of finite characteristic.

In order to prove the last result of this paper we need one more definition.

Definition 4.13. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. Let K_0 be any subfield of K. Then the relative modular transcendence degree of ϕ over K_0 is the minimum transcendence degree over K_0 of the compositum field of K_0 and the field of definition of $\phi^{(\gamma)}$, minimum being taken over all $\gamma \in K^{\mathrm{alg}} \setminus \{0\}$.

Theorem 4.14. *Let K be a field of characteristic p. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic and of relative modular transcendence degree at least 1 over $F = \mathrm{Frac}(A)$. Let $g \geq 0$ and X be a K^{alg}-subvariety of $\mathbb{G}_a^g$. Assume that X does not contain a translate of a nontrivial connected algebraic subgroup of $\mathbb{G}_a^g$. Then for every finitely generated ϕ-submodule Γ of $\mathbb{G}_a^g(K^{\mathrm{alg}})$, we have that $X(K^{\mathrm{alg}}) \cap \Gamma$ is finite.*

Proof. First we replace K by a finitely generated field L, which satisfies the following conditions:
1) ϕ is defined over L;
2) $\Gamma \subset \mathbb{G}_a^g(L)$;
3) X is defined over L.

If we prove Theorem 4.14 for L, then the result follows automatically for K. Hence, from now on, K has the properties $1) - 3)$.

We let F^{alg} be the algebraic closure of F inside K^{alg}. For any two subextensions of K^{alg}, their compositum is taken inside K^{alg}. We may replace K by any finite extension of it and prove the result for the larger field and then the result will also hold for K. Also, during this proof we will replace F by a finite extension of it contained in K.

In the beginning we will prove several reduction steps.

Step 1. It suffices to prove Theorem 4.14 for Γ of the form Γ_0^g where Γ_0 is a finitely generated ϕ-submodule of $\mathbb{G}_a(K^{\mathrm{alg}})$. Indeed, if we let Γ_0 be the finitely generated ϕ-submodule of K^{alg} generated by all the g coordinate projections of Γ, then clearly, $\Gamma \subset \Gamma_0^g$. So, we suppose that Γ has the form Γ_0^g. To simplify the notation we work with a finitely generated ϕ-submodule Γ of $\mathbb{G}_a(K^{\mathrm{alg}})$ and prove that $X(K^{\mathrm{alg}}) \cap \Gamma^g$ is finite.

Step 2. Let t be a non-constant element of A. Let $\gamma \in K^{\mathrm{alg}}$ such that for the Drinfeld module $\phi^{(\gamma)} = \gamma^{-1}\phi\gamma$, $\phi_t^{(\gamma)}$ is monic. We let $\gamma^{-1}X$ be the variety whose vanishing ideal is composed of functions of the form $f \circ \gamma$, where f is in the vanishing ideal of X and γ is interpreted as the multiplication-by-γ-map on each component of $\mathbb{G}_a^g$. The conclusion of Theorem 4.14 is equivalent to showing that

$$(\gamma^{-1}X)(K^{\mathrm{alg}}) \cap (\gamma^{-1}\Gamma)^g \text{ is finite.}$$

The variety $\gamma^{-1}X$ has the same property as X: it does not contain a translate of a non-trivial connected algebraic subgroup of $\mathbb{G}_a^g$. The group $\gamma^{-1}\Gamma$ is a finitely generated $\phi^{(\gamma)}$-submodule. So, it suffices to prove Theorem 4.14 under the extra hypothesis that ϕ_t is monic. From now on, let

$$\phi_t = \tau^r + a_{r-1}\tau^{r-1} + \cdots + a_0\tau^0.$$

Step 3. Let W be a variety defined over F whose function field is K. At the expense of replacing K by a finite extension and replacing F by a finite extension contained in K, we may assume that W is a projective, smooth, geometrically irreducible F-variety (see Remark 4.2 from [10]). We let C be a smooth projective curve defined over a finite field, whose function field is F. We spread out W over an open, dense subset $C_0 \subset C$ and obtain a projective, smooth variety $V_0 \subset \mathbb{P}^n_{C_0}$ (for some n) (we can always do this because there are finitely many polynomials defining the variety W and so, there are finitely many places of C that lie below poles of the coefficients of these polynomials). We let V be the projective closure of V_0 in $\mathbb{P}^n_C$. We let $\pi : V \to C$ be the natural morphism. The generic fiber of π is smooth and geometrically irreducible, because this is how we chose W. Finally, we replace

V by its normalization. By doing this last step, the generic fiber of π remains smooth and geometrically irreducible because V_0 is isomorphic with its preimage in the normalization.

The irreducible divisors $\mathfrak{P}$ of V are of two types:

(i) *vertical*, which means that $\pi(\mathfrak{P}) = \mathfrak{p}$ is a closed point of C.

(ii) *horizontal*, which means that $\pi|_{\mathfrak{P}} : \mathfrak{P} \to C$ is surjective.

We call *horizontal* a divisor of V that has at least one irreducible component that is horizontal.

Let S be the finite set of horizontal divisors of V that are irreducible components of the poles of the coefficients a_i of ϕ_t. According to Lemma 5.2.2 of [7], the set S is the set of horizontal irreducible divisors of V that are places of *bad reduction* for ϕ.

At the expense of replacing F by a finite extension F' and replacing K by $F'K$ and replacing V and W by their respective normalizations in $F'K$, we may assume that for each $\gamma \in S$, the generic fiber of $\gamma \to C$ is geometrically irreducible (we can do this because for each $\gamma \in S$, there exists a finite extension of F such that after the base extension, γ splits into finitely many divisors whose generic fibers are geometrically irreducible). Also, the properties of being smooth and geometrically irreducible (for the generic fiber of π) are unaffected by a base extension. So, from now on we work under the following assumptions for the projective, normal varieties V and C:

(37) the generic fiber of the morphism $\pi : V \to C$ is smooth and geometrically irreducible

(38) for each $\gamma \in S$, the generic fiber of γ is geometrically irreducible.

Step 4. We define the *division hull* of Γ, by

$$\overline{\Gamma} = \{\gamma \in K^{\mathrm{alg}} \mid \text{there exists } a \in A \setminus \{0\} \text{ such that } \phi_a(\gamma) \in \Gamma\}.$$

In [7] we proved the following result (Theorem 7.0.44).

Theorem 4.15. *Let F be a countable field of characteristic p and let K be a finitely generated field over F. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of positive relative modular transcendence degree over F. Then for every finite extension L of K, $\phi(L)$ is a direct sum of a finite torsion submodule and a free submodule of rank $\aleph_0$.*

Using the result of Theorem 4.15 for F^{alg}, which is countable, and for $F^{\mathrm{alg}}K$, which is finitely generated over F^{alg}, and for ϕ, which has positive relative modular transcendence degree over F^{alg}, we conclude that $\phi(F^{\mathrm{alg}}K)$ is the direct sum of a finite torsion submodule and a free module of rank $\aleph_0$. Thus, because $\overline{\Gamma}$ has finite rank, $\overline{\Gamma} \cap F^{\mathrm{alg}}K$ is finitely generated. At the expense of replacing K by a finite extension of the form $F'K$, where F' is a finite extension of F and replacing W and V by their normalizations in $F'K$, we may assume that $\overline{\Gamma} \cap F^{\mathrm{alg}}K \subset K$.

Step 5. We may replace Γ by $\overline{\Gamma} \cap K$, which is also a finitely generated ϕ-submodule that contains Γ.

From now on, we assume all of the above reductions made.

For each irreducible divisor $\mathfrak{P}$ of V, we let $K_{\mathfrak{P}}$ be the residue field of K at $\mathfrak{P}$. For any element x in the valuation ring $R_{\mathfrak{P}}$ of $\mathfrak{P}$, we let $x_{\mathfrak{P}}$ be the reduction of x at $\mathfrak{P}$. Also, we denote by $r_{\mathfrak{P}}$ the reduction map at $\mathfrak{P}$. If all the elements of Γ are integral at $\mathfrak{P}$, we let

$$\Gamma_{\mathfrak{P}} = \{x_{\mathfrak{P}} \mid x \in \Gamma\}.$$

If ϕ has good reduction at $\mathfrak{P}$, then we denote by $\phi^{\mathfrak{P}}$ the corresponding reduction.

The following two results are standard (see Theorem 2.10 (i) of [5], which proves that for an algebraic variety the property of being geometrically irreducible is a first order definable property).

Lemma 4.16. *Because the generic fiber of $\pi : V \to C$ is geometrically irreducible, for all but finitely many closed points $\mathfrak{p} \in C$, $\pi^{-1}(\mathfrak{p})$ is geometrically irreducible.*

Lemma 4.17. *Let $\gamma \in S$. Because the generic fiber of $\gamma \to C$ is geometrically irreducible, for all but finitely many closed points $\mathfrak{p} \in C$, $\gamma \cap \pi^{-1}(\mathfrak{p})$ is geometrically irreducible.*

Lemma 4.18. *Let T be the set of vertical irreducible divisors $\mathfrak{P}$ of V which satisfy the following properties:*
a) ϕ has good reduction at $\mathfrak{P}$.
b) $\phi^{\mathfrak{P}}$ is a finite characteristic Drinfeld module of positive modular transcendence degree.
c) the projective variety $\mathfrak{P}$ is smooth and $\pi^{-1}(\pi(\mathfrak{P}))$ is geometrically irreducible.
d) for each $\gamma \in S$, $\gamma_{\mathfrak{P}} := \gamma \cap \beta$ is geometrically irreducible.
e) for all $x \in \Gamma$, x is integral at $\mathfrak{P}$.
Then the set T is cofinite in the set of all vertical irreducible divisors of V.

Proof of Lemma 4.18. We will show that each of the conditions a)-e) is verified by all but finitely many vertical irreducible divisors of V.

a) There are only finitely many irreducible divisors of V that are places of bad reduction for ϕ. So, in particular, there are only finitely many irreducible vertical divisors of V that do not satisfy a).

b) By the definition of reduction at $\mathfrak{P}$ (which is a place sitting above a prime divisor of A), $\phi^{\mathfrak{P}}$ is a finite characteristic Drinfeld module.

Because ϕ has positive relative modular transcendence degree over F, there exists $a \in A$ and a coefficient c of ϕ_a such that $c \notin F^{\text{alg}}$. We view c as a dominant rational map from the generic fiber W of V to $\mathbb{P}^1_F$. We spread out c to a rational map $\tilde{c} : V \to \mathbb{P}^1_C$, whose generic fiber is c. Because c is dominant, $\tilde{c}$ is dominant. For all but finitely many closed points $\mathfrak{p} \in C$, the fiber $\tilde{c_{\mathfrak{p}}}$ is not constant. According to the result of Lemma 4.16, for all but finitely many $\mathfrak{p}$, $\pi^{-1}(\mathfrak{p}) = \mathfrak{P}$ is geometrically irreducible. For such $\mathfrak{P}$, we identify $\tilde{c_{\mathfrak{p}}}$ with the reduction of c at the place $\mathfrak{P}$, denoted $c_{\mathfrak{P}}$. Thus for all but finitely many irreducible vertical divisors $\mathfrak{P}$, $c_{\mathfrak{P}} \notin \mathbb{F}_p^{\text{alg}}$. So, for these divisors $\mathfrak{P}$, $\phi^{\mathfrak{P}}$ has positive modular transcendence degree (remember that $\phi_t^{\mathfrak{P}}$ is still monic because ϕ_t is monic).

c) Since V is projective, all the irreducible divisors of V are projective varieties.

Because the generic fiber of π is smooth and geometrically irreducible, for all but finitely many $\mathfrak{p} \in C$, $\pi^{-1}(\mathfrak{p})$ is also smooth and geometrically irreducible.

d) This is proved by Lemma 4.17.

e) Because Γ is finitely generated as a ϕ-module and ϕ has good reduction at all but finitely many irreducible divisors, the elements of Γ are integral at all but finitely many irreducible divisors of V. □

Lemma 4.19. *The set S is nonempty.*

Proof of Lemma 4.19. Assume all poles of all coefficients of ϕ_t are vertical. Because there are infinitely many $\mathfrak{P} \in T$, we can find $\mathfrak{P} \in T$ such that $\mathfrak{P}$ is disjoint from all the poles of the coefficients of ϕ_t (we can achieve this because they are finitely many and they are all vertical). Then the reduction of ϕ at $\mathfrak{P}$ is a Drinfeld module of modular transcendence degree at least 1 (by condition b) of Lemma 4.18. But on the other hand, because all the poles of the coefficients of ϕ_t are vertical and disjoint from $\mathfrak{P}$, the coefficients of $\phi_t^{\mathfrak{P}}$ are integral on $\mathfrak{P}$. Because $\mathfrak{P}$ is a projective variety, then the coefficients of $\phi_t^{\mathfrak{P}}$ are constant. This is a contradiction with the modular transcendence degree of $\phi^{\mathfrak{P}}$. □

For each $\mathfrak{P} \in T$, we let $S_{\mathfrak{P}}$ be the set of all the irreducible divisors of $\mathfrak{P}$ which are poles of the coefficients of $\phi_t^{\mathfrak{P}}$. As explained in Lemma 5.2.2 of [7], the places from $S_{\mathfrak{P}}$ are all the places of bad reduction for $\phi^{\mathfrak{P}}$.

Lemma 4.20. *For each $\mathfrak{P} \in T$, $1 \leq |S_{\mathfrak{P}}| \leq |S|$.*

Proof of Lemma 4.20. Fix $\mathfrak{P} \in T$. Let c be a coefficient of ϕ_t. We view c as a rational map from V to $\mathbb{P}^1_F$. The divisor of the pole of c is the pullback of $\infty \in \mathbb{P}^1_F$. Thus the poles of $c_{\mathfrak{P}}$ are the intersections of this divisor of poles with the vertical divisor $\mathfrak{P}$ (also remember that $\mathfrak{P}$ is a place of good reduction for ϕ and so, $\mathfrak{P}$ is not part of the pole of c). Using Lemma 4.18 d), the divisors of $\mathfrak{P}$ which are irreducible components of the divisor of poles of $c_{\mathfrak{P}}$ are of the form $\gamma_{\mathfrak{P}}$ for $\gamma \in S$. Thus, because S is nonempty (see Lemma 4.19), $1 \leq |S_{\mathfrak{P}}| \leq |S|$ (the second inequality might be strict because it is possible for two horizontal divisors from S have the same intersection with the vertical divisor $\mathfrak{P}$). □

Lemma 4.21. *For all but finitely many $\mathfrak{P} \in T$, the reduction map $r_{\mathfrak{P}}$ is injective on Γ_{tor}.*

Proof of Lemma 4.21. Because Γ_{tor} is finite (Γ is finitely generated), only finitely many $\mathfrak{P}$ from T appear as irreducible components of the divisor of zeros for some torsion element of Γ. □

Lemma 4.22. *There exists a non-constant $a \in A$ such that for all $\mathfrak{P} \in T$, $\Gamma_{\mathfrak{P}} \cap \phi^{\mathfrak{P}}[a] = \{0\}$.*

Proof of Lemma 4.22. Let $\mathfrak{P} \in T$. We note that $\mathfrak{P}$ is regular in codimension 1 (according to condition c) of Lemma 4.18) and so, the valuations associated to its irreducible divisors form a good set of valuations on the finitely generated field $K_{\mathfrak{P}}$ (see Remark 4.2.2 of [7]). Hence, using Lemma 4.20 and using Corollary 6.0.38 of [7] we conclude that for all $x \in \phi^{\mathfrak{P}}_{\text{tor}}(K_{\mathfrak{P}})$, there exists a polynomial $b(t) \in \mathbb{F}_q[t]$ of degree at most $(r^2+r)|S|$ such that $\phi^{\mathfrak{P}}_{b(t)}(x) = 0$. Because $\Gamma_{\mathfrak{P}} \subset K_{\mathfrak{P}}$, Lemma 4.22 holds with $a \in \mathbb{F}_q[t]$ being any irreducible polynomial of degree greater than $(r^2+r)|S|$. □

Lemma 4.23. *Let a be a non-constant element of A. For almost all $\mathfrak{P} \in T$, $r_{\mathfrak{P}} : \Gamma/\phi_a(\Gamma) \to \Gamma_{\mathfrak{P}}/\phi_a^{\mathfrak{P}}(\Gamma_{\mathfrak{P}})$ is injective.*

Proof of Lemma 4.23. We know that all the divisors $\mathfrak{P} \in T$ have the property that if $\mathfrak{p} = \pi(\mathfrak{P})$, then $\pi^{-1}(\mathfrak{p})$ is geometrically irreducible (this was part of condition c) from Lemma 4.18). Thus, specifying $\mathfrak{p}$ determines uniquely $\mathfrak{P}$ and so, just to simplify the notation in this lemma, we will use the convention that if $\mathfrak{P}$ is the only irreducible divisor lying above a closed point $\mathfrak{p} \in C$, then $K_{\mathfrak{p}}$ is the residue field of K at $\mathfrak{P}$ and "reducing $x \in K$ at $\mathfrak{p}$" is "reducing $x \in K$ at $\mathfrak{P}$". Also, we will identify T with the set of closed points $\mathfrak{p} \in C$ lying below the vertical divisors $\mathfrak{P} \in T$.

Suppose there are infinitely many irreducible divisors $\mathfrak{P}$ for which the map in (4.23) is not injective. Because $\Gamma/\phi_a(\Gamma)$ is finite, there exists $x \in \Gamma \setminus \phi_a(\Gamma)$ and there exists an infinite subset U of $T \subset C$ such that for every $\mathfrak{p}$ in this infinite subset, $x_{\mathfrak{p}} \in \phi_a^{\mathfrak{p}}(\Gamma_{\mathfrak{p}})$. For each such

$\mathfrak{p}$, let $z(\mathfrak{p}) \in \Gamma_{\mathfrak{p}} \subset K_{\mathfrak{p}}$ be such that

$$x_{\mathfrak{p}} = \phi_a^{\mathfrak{p}}(z(\mathfrak{p})). \tag{39}$$

Let L be the finite extension of K generated by all the roots $z_1, \dots, z_s \in K^{\mathrm{alg}}$ of the equation (in z) $\phi_a(z) = x$. For each $\mathfrak{p} \in U$ choose a place $\mathfrak{p}_1$ of L lying above $\mathfrak{p}$.

Fix now $\mathfrak{p} \in U$. Because $\mathfrak{p} \in T$ (and so, $\mathfrak{p}_1$) is a place of good reduction for ϕ, $z_1, \dots, z_s$ are integral at $\mathfrak{p}_1$ and their reductions at $\mathfrak{p}_1$, called $z_{1,\mathfrak{p}_1}, \dots, z_{s,\mathfrak{p}_1}$ are all the roots of the equation (in z) $\phi_a^{\mathfrak{p}}(z) = x_{\mathfrak{p}}$. Using (39), we conclude there exists $i \in \{1, \dots, s\}$ such that

$$z_{i,\mathfrak{p}_1} = z(\mathfrak{p}). \tag{40}$$

We apply the above argument for each $\mathfrak{p} \in U$ (and for the corresponding $\mathfrak{p}_1$) and so, conclude that for each $\mathfrak{p} \in U$, there exists some $i \in \{1, \dots, s\}$ such that (40) holds. Because U is infinite, there exists an infinite subset $U_1 \subset U$ and there exists $z \in \{z_1, \dots, z_s\}$ such that for each $\mathfrak{p} \in U_1$,

$$z_{\mathfrak{p}_1} = z(\mathfrak{p}) \in K_{\mathfrak{p}}, \tag{41}$$

because $z(\mathfrak{p}) \in K_{\mathfrak{p}}$. Let $K' = K(z)$. Because $z \in \overline{\Gamma} \setminus \Gamma$ (because $x \notin \phi_a(\Gamma)$) and $\overline{\Gamma} \cap F^{\mathrm{alg}}K = \Gamma$, K' is not contained in $F^{\mathrm{alg}}K$. So, if we let F' be the algebraic closure of F in K', then

$$l := [K' : F'K] > 1. \tag{42}$$

Let C' be the normalization of C in F'. Let V' be the normalization of V in $F'K$ and let V_1' be the normalization of V in K'. Let $\pi' : V' \to C'$ and $\pi_1' : V_1' \to C'$ be the induced morphisms. Thus the generic fibers W' and W_1' of π' and π_1', respectively, are geometrically irreducible. Let $f : V_1' \to V'$ be the induced finite morphism.

Because ϕ is a generic characteristic Drinfeld module, ϕ_a is a separable polynomial and so, K'/K is a separable extension. Thus f is ramified for finitely many irreducible divisors of V'. Also, let P be the minimal polynomial for z over $F'K$.

Let U_1' be the set of closed points of C' satisfying the following properties:

1) each $\mathfrak{p}' \in U_1'$ lies above some $\mathfrak{p} \in U_1$,

2) for each $\mathfrak{p}' \in U_1'$, the vertical divisor $\mathfrak{P}_1' := \pi_1'^{-1}(\mathfrak{p}')$ of V_1' is geometrically irreducible,

3) for each $\mathfrak{p}' \in U_1'$, f is not ramified at the divisor $\mathfrak{P}' := \pi'^{-1}(\mathfrak{p}')$ of V' (note that $\mathfrak{P}'$ is geometrically irreducible, once 2) holds),

4) for each $\mathfrak{p}' \in U_1'$, all the coefficients of P are integral at the corresponding $\mathfrak{P}'$ (and implicitly, at $\mathfrak{P}_1'$). Moreover, $\mathfrak{p}'$ is not an irreducible component of the divisor of zeros of $P'(z)$.

In all that will follow next in our argument, "condition i)" for $i \in \{1, \dots, 4\}$ is one of the above 4 conditions.

Because U_1 is infinite, condition 1) is satisfied by infinitely many $\mathfrak{p}' \in C'$. Condition 2) is satisfied by all but finitely many $\mathfrak{p}' \in C'$ because the generic fiber of π_1' is geometrically irreducible. Condition 3) is satisfied because f ramifies at finitely many irreducible divisors of V'. The first part of condition 4) is satisfied because there are finitely many divisors of V' (or V_1') which are irreducible components for the divisors of poles of the coefficients of P. The second part of condition 4) is satisfied because $P'(z) \neq 0$ (P is a separable polynomial because it divides ϕ_a, which is a separable polynomial). So, we conclude U_1' is infinite.

Let $\mathfrak{p}' \in U_1'$ and let $\mathfrak{P}'$ and $\mathfrak{P}_1'$ be the corresponding vertical divisors of V' and V_1', respectively. Because $\mathfrak{P}_1'$ is the only place of K' lying above the place $\mathfrak{p}'$ of C' (see condition

2)), (41) yields $z_{\mathfrak{P}'_1} \in (F'K)_{\mathfrak{P}'}$. Also by condition 2), $\mathfrak{P}'_1$ is the only place of K' lying above the place $\mathfrak{P}'$ of $F'K$.

Let R be the valuation ring of $F'K$ at $\mathfrak{P}'$ and let R' be the integral closure of R in K'. Because K' is not ramified above $\mathfrak{P}'$, the different of R'/R is the unit ideal in R' (see Theorem 1, page 53, [15]). By condition 4), $P'(z)$ is also a unit in R'. By Corollary 2 (page 56) of [15], $R' = R[z]$. Because P is defined over R (see condition 4)), Lemma 4 (page 18) of [15] yields the relative residue degree $f(\mathfrak{P}'_1|\mathfrak{P}')$ between the place $\mathfrak{P}'_1$ of K' and the place $\mathfrak{P}'$ of $F'K$ is 1. Using condition 3), we conclude that also the ramification index $e(\mathfrak{P}'_1|\mathfrak{P}')$ of $\mathfrak{P}'_1$ over $\mathfrak{P}'$ is 1. As explained in Remark 4.2.2 of [7], the valuations associated to irreducible divisors of a projective variety defined over a field are defectless and so, because $e(\mathfrak{P}'_1|\mathfrak{P}') = f(\mathfrak{P}'_1|\mathfrak{P}') = 1$ and $\mathfrak{P}'_1$ is the only place of K' lying above the place $\mathfrak{P}'$ of $F'K$, we conclude $[K' : F'K] = 1$. This contradicts (42). This contradiction comes from our assumption that there are infinitely many primes $\mathfrak{P}$ for which Lemma 4.23 is false. So, for all but finitely many $\mathfrak{P} \in T$, the conclusion of Lemma 4.23 holds, as desired. □

Using Lemmas 4.21, 4.22 and 4.23 we prove the following key result.

Lemma 4.24. *For all but finitely many $\mathfrak{P} \in T$, the reduction $\Gamma \to \Gamma_{\mathfrak{P}}$ is injective.*

Proof of Lemma 4.24. Shrink T so that all of the three lemmas 4.21, 4.22 and 4.23 hold for $\mathfrak{P} \in T$. Also, let a be as in Lemma 4.22.

If $x \in \Gamma \cap \operatorname{Ker}(r_{\mathfrak{P}})$, then by Lemma 4.23, $x \in \phi_a(\Gamma)$. This means that there exists $x_1 \in \Gamma$ such that $\phi_a(x_1) = x$. Reducing at $\mathfrak{P}$, we get $\phi_a^{\mathfrak{P}}(x_{1_{\mathfrak{P}}}) = 0$ which by Lemma 4.22 implies that $x_{1_{\mathfrak{P}}} = 0$. But then applying again 4.23, this time to x_1, we conclude $x_1 \in \phi_a(\Gamma)$; i.e. there exists $x_2 \in \Gamma$ such that $x_1 = \phi_a(x_2)$.

So, repeating the above process, an easy induction shows that

$$x \in \bigcap_{n \geq 1} \phi_{a^n}(\Gamma) = \Gamma_{\mathrm{tor}},$$

because Γ is finitely generated. But, by Lemma 4.21, Γ_{tor} injects through the reduction at $\mathfrak{P}$. Thus $x = 0$ and so the proof of Lemma 4.24 ends. □

Now, the property $\mathcal{P}$: *"X does not contain any translate of a nontrivial connected algebraic subgroup of $\mathbb{G}_a^g$"* is a definable property as shown in Lemma 11 (page 203) of [1] (there it is proved that the set of connected algebraic subgroups of an algebraic group G that are maximal under the property that one of their translates lies inside a given algebraic variety $X \subset G$ is definable). This means that property $\mathcal{P}$ is inherited by all but finitely many of the reductions of X. Coupling this result with Lemma 4.24, we see that for all but finitely many irreducible vertical divisors $\mathfrak{P}$ of V, the reduction of X, called $X_{\mathfrak{P}}$, is also a variety that satisfies the same hypothesis as X and moreover, Γ injects through such reduction. This means that

$$|X(K) \cap \Gamma^g| \leq |X_{\mathfrak{P}}(K_{\mathfrak{P}}) \cap \Gamma_{\mathfrak{P}}^g|. \tag{43}$$

According to condition b) of Lemma 4.18, for all $\mathfrak{P} \in T$, $\phi^{\mathfrak{P}}$ satisfies the hypotheses of Theorem 4.5. Thus, applying Theorem 4.5, $X_{\mathfrak{P}} \cap \Gamma_{\mathfrak{P}}^g$ is a finite union of translates of cosets of subgroups of $\Gamma_{\mathfrak{P}}^g$. Suppose that one of these subgroups of $\Gamma_{\mathfrak{P}}^g$ is infinite. Then $X_{\mathfrak{P}}$ contains the Zariski closure of the corresponding coset, which is a translate of a positive dimensional algebraic subgroup of $\mathbb{G}_a^g$. This would contradict the property inherited by $X_{\mathfrak{P}}$ from X. Thus $X_{\mathfrak{P}}(K_{\mathfrak{P}}) \cap \Gamma_{\mathfrak{P}}^g$ is finite. Using (43), we conclude that $X(K) \cap \Gamma^g$ is finite. □

Remark 4.25. Theorem 4.14 is a special case of Statement 4.2 because if we assume (4.2) and we work with the hypothesis on X from Theorem 4.14, then, with the notations from (4.2), the intersection of X with any translate of B_i is finite. Otherwise, the Zariski closure of $X \cap (\gamma_i + B_i)$ would be a translate of a positive dimensional algebraic subgroup of $\mathbb{G}_a^g$, and it would be contained in X.

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