PORTRAITS OF PREPERIODIC POINTS FOR RATIONAL MAPS

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Abstract. Let K be a function field over an algebraically closed field k of characteristic 0, let $\varphi \in K(z)$ be a rational function of degree at least equal to 2 for which there is no point at which φ is totally ramified, and let $\alpha \in K$. We show that for all but finitely many pairs $(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ there exists a place $\mathfrak{p}$ of K such that the point α has preperiod m and minimum period n under the action of φ. This answers a conjecture made by Ingram-Silverman [13] and Faber-Granville [8]. We prove a similar result, under suitable modification, also when φ has points where it is totally ramified. We give several applications of our result, such as showing that for any tuple $(c_0, \dots, c_{d-2}) \in k^{n-1}$ and for almost all pairs $(m_i, n_i) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ for $i = 0, \dots, d-2$, there exists a polynomial $f \in k[z]$ of degree d in normal form such that for each $i = 0, \dots, d-2$, the point c_i has preperiod m_i and minimum period n_i under the action of f.

1. Introduction

Throughout this paper, let K be a finitely generated field of transcendence degree 1 over an algebraically closed field k of characteristic 0. By a place $\mathfrak{p}$ of K, we mean an equivalence class of nontrivial valuations on K that are trivial on k; the set of all such places is denoted by Ω_K. Each such place $\mathfrak{p}$ gives rise to a valuation ring $\mathfrak{o}_{\mathfrak{p}}$ and a maximal ideal denoted (by an abuse of notation) by $\mathfrak{p}$. The residue field is (canonically isomorphic to) k. We let $v_{\mathfrak{p}}$ and $|\cdot|_{\mathfrak{p}}$ respectively denote the corresponding additive and multiplicative valuations normalized by $v_{\mathfrak{p}}(K) = \mathbb{Z}$ and $|\alpha|_{\mathfrak{p}} = e^{-v_{\mathfrak{p}}(\alpha)}$ for every $\alpha \in K$. For a rational map $\varphi \in K(x)$, for all but finitely many places $\mathfrak{p}$ of K we can define the reduction of φ modulo $\mathfrak{p}$, see Section 3.1 or [20, Chapter 2]. We start with a definition which is central for our paper:

Definition 1.1. *Let F be any field. For a rational map $\varphi \in F(z)$ and for $\alpha \in F$, we say that $(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ is the preperiodicity portrait, or simply portrait, of α (with respect to φ) if $\varphi^m(\alpha)$ is periodic of minimum period n for φ, while m is the smallest nonnegative integer such that $\varphi^m(\alpha)$ is periodic (as always in dynamics, we denote by $\varphi^k = \varphi \circ \cdots \circ \varphi$ composed with itself k times). We call m the preperiod of α and call n the minimum period of α.*

Let $\varphi \in K(z)$, $\alpha \in \mathbb{P}^1(K)$, and $\mathfrak{p}$ a place of K such that the reduction of φ modulo $\mathfrak{p}$ is well-defined. Assume that α has portrait (m, n) under the induced reduction self-map on $\mathbb{P}^1(k)$. We call (m, n) the preperiodicity portrait of α (under the action of φ) *modulo* $\mathfrak{p}$. The existence of places $\mathfrak{p}$ of K for which α has portrait (m, n) under

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the action of φ modulo $\mathfrak{p}$ has been studied for more than 100 years (see the papers of Bang [2] and Zsigmondy [25]) and also more recently (see [18, 13, 8, 11, 16]). Some of the results are easier to obtain when the dynamical system induces a group action (see [2, 25], and also more recent results for Drinfeld modules [10, 12]). However, when there is no group action, the problem is much harder. Certain conjectures on this problem have been made by Ingram-Silverman [13, pp. 300–301] and modified by Faber-Granville [8, pp. 190], yet very few general results are known. Faber and Granville [8] have proved that if $\varphi \in \mathbb{Q}(z)$ satisfies $\deg \varphi > 1$ and $\alpha \in \mathbb{Q}$, then for all but finitely many $m \in \mathbb{Z}_{\geq 0}$ there exists a co-finite set $S_m \subseteq \mathbb{N}$ (i.e., $\mathbb{N} \setminus S_m$ is finite) such that for each $n \in S_m$, there exists a prime p such that the preperiod of α modulo p is m, while its minimum period *divides* n. On the other hand, Ingram-Silverman [13] and Faber-Granville [8] conjecture that one can obtain a similar conclusion this time for minimum period *equal* to n. In this paper, we are able to resolve their conjecture over function fields. First we define the set of exceptions from the conjecture of Ingram-Silverman [13] and Faber-Granville [8].

Definition 1.2. *For any rational function $\varphi \in K(z)$, let $X(\varphi)$ be the set of n such that φ is totally ramified at every point of minimum period n. For a rational function φ and a point α, we let $Y(\varphi, \alpha)$ be the set of positive integers m such that φ is totally ramified over $\varphi^m(\alpha)$ (i.e. φ is totally ramified at $\varphi^{m-1}(\alpha)$).*

Now we can state our main result; for more details on good reduction of a rational map φ and for the canonical height $\widehat{h}_\varphi$ associated to φ, see Section 3.

Theorem 1.3. *Let K be a function field. Let $\varphi \in K(z)$ have degree $d > 1$. Let $\tau, s > 0$ and let S be a set of places of K containing all the places of bad reduction for φ such that $\#S \leq s$. Then there is a finite set $\mathcal{Z}(\tau, s) \subset \mathbb{Z}_{\geq 0} \times \mathbb{N}$ depending only on φ, K, τ, and s with the following property: for any $\alpha \in \mathbb{P}^1(K)$ with $\widehat{h}_\varphi(\alpha) > \tau$ and any $(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ such that $(m, n) \notin \mathcal{Z}(\tau, s)$, $m \notin Y(\varphi, \alpha)$ and $n \notin X(\varphi)$, there is a place $\mathfrak{p} \notin S$ such that α has portrait (m, n) under the action of φ modulo $\mathfrak{p}$. Moreover, the set $\mathcal{Z}(\tau, s)$ is effectively computable.*

The next two remarks explain why the conditions in Theorem 1.3 are necessary.

Remark 1.4. Theorem 1.3 *without* the conditions that $m \notin Y(\varphi, \alpha)$ and $n \notin X(\varphi)$ was essentially conjectured by Ingram-Silverman [13]. It was Faber and Granville [8] who pointed out that the condition $n \notin X(\varphi)$ would be necessary. By Lemma 4.24, $n \in X(\varphi)$ if and only if either φ has no point of minimum period n (see Kisaka's classification [15] or [8, Appendix B] for a complete list of all rational maps which have no point of minimum period n), or $n = 2$ and $\varphi(z)$ is linearly conjugate to z^{-2}. Note that Ingram-Silverman made their conjectures over number fields while Faber-Granville even restricted further to the field of rational numbers. Theorem 1.3 answers completely the same question for function fields.

Remark 1.5. Theorem 1.3 *without* the condition that $m \notin Y(\varphi, \alpha)$ was essentially conjectured by Faber-Granville [8, pp. 190]. We briefly explain why this condition is also necessary. Suppose $M > 0$ such that φ is totally ramified over $\varphi^M(\alpha)$. Write $\beta = \varphi^M(\alpha)$. We may assume $\varphi^{M-1}(\alpha) = 0$ and $\beta \neq \infty$ by making a linear change of variables. Therefore φ has the form:

$$\varphi(z) = \frac{z^d}{\psi(z)} + \beta$$

where ψ is a polynomial of degree at most d satisfying $\psi(0) \neq 0$. For almost all primes $\mathfrak{p}$ of good reduction and for every $a \in \mathbb{P}^1(K)$, we have that $\varphi(a) \equiv \beta$ mod $\mathfrak{p}$ if and only if $a \equiv 0$ mod $\mathfrak{p}$. Hence if $\varphi^{M+n}(\alpha) \equiv \beta$ mod $\mathfrak{p}$ then $\varphi^{M+n-1}(\alpha) \equiv 0$ mod $\mathfrak{p}$. So, for almost all n, there does not exist $\mathfrak{p}$ such that α has portrait (M, n) under the action of φ mod $\mathfrak{p}$.

We also note that the hypothesis that the function field has characteristic 0 is used crucially in our proof because we employ in our arguments Mason's [17] and Stothers' [21] *abc*-theorem for function fields of characteristic 0 (see also [19]). It would be interesting to treat the question in positive characteristic as well, but it appears that new ideas or techniques may be required.

We have explained why the conditions $m \notin Y(\varphi, \alpha)$ and $n \notin X(\varphi)$ are necessary. Hence the Ingram-Silverman-Faber-Granville conjecture (which was originally made over number fields) should be modified accordingly. One may also adapt our proof in this paper to resolve their conjecture *assuming the abc-conjecture* in the context of number fields. We will treat this in a future paper.

The original question that motivates this paper is the "simultaneous portrait problem" (see Theorem 2.2) over function fields in several parameters; this problem has no obvious analog over number fields.

Since we are excluding places outside S, our conclusion involving the finiteness of $\mathcal{Z}(\tau, s)$ is *the best one can hope for.* We also note the remarkable *uniformity* obtained here: the set $\mathcal{Z}(\tau, s)$ only depends on φ, K, s, and the lower bound τ on the canonical height of α under φ rather than depending on α.

The following result is an immediate consequence of Theorem 1.3 in the case when $\varphi \in K[z]$ is a polynomial which is totally ramified at no point in K; in this case, $X(\varphi) = \emptyset$ and also $Y(\varphi, \alpha) = \emptyset$ for any $\alpha \in K$ (see Remark 1.4).

Corollary 1.6. *Let K, φ, τ, s, and S be as in Theorem 1.3. Assume that $\varphi \in K[z]$ is a polynomial that is totally ramified at no point in K (equivalently, $\varphi(z)$ is not linearly conjugate over K to a polynomial of the form $z^d + c$). Then there exists a finite set $\mathcal{Z}(\tau, s) \subset \mathbb{Z}_{\geq 0} \times \mathbb{N}$ depending only on K, φ, τ, and s such that for every $\alpha \in K$ satisfying $\widehat{h}_\varphi(\alpha) > \tau$ and for every $(m, n) \in (\mathbb{Z}_{\geq 0} \times \mathbb{N}) \setminus \mathcal{Z}(\tau, s)$ there exists a place $\mathfrak{p} \notin S$ such that α has portrait (m, n) mod $\mathfrak{p}$.*

Theorem 1.3 allows us to prove a result (see Theorem 2.2) for simultaneous portraits of complex numbers realized by polynomials in normal form. Also, Theorem 1.3 allows us to prove a strong uniform result for realizing all possible portraits by almost any constant starting point (see Theorem 2.5). We will state in Section 2 these two results, together with other applications of our Theorem 1.3.

We sketch briefly the plan of our paper. We state in Section 2 applications of Theorem 1.3 (see Theorems 2.2, 2.3 and 2.5). In Section 3 we introduce the notation and the basic notions used in the paper. We prove Theorem 1.3 in Section 4 and then we prove its various applications in Section 5.

2. Applications

One might also ask for a "higher-parameter" version of Theorem 1.3. That is, rather than asking about portraits of a single point as one varies over a one-dimensional family, one might ask more generally about the n-tuples of portraits of a n-type of points as one varies an n-dimensional family. One natural place to explore this question is in the context of the family of polynomials of degree d in normal form.

We say that a polynomial $\varphi(z)$ of degree d is in *normal form* if it is monic and its coefficient of z^{d-1} equals 0. Note that each polynomial φ is linearly conjugate to a polynomial in normal form. Therefore, when discussing preperiodicity portraits in the family of polynomials of degree d, it makes sense to restrict the analysis to the case of polynomials in normal form.

Let $d \geq 2$ be an integer, let k be an algebraically closed field of characteristic 0, let $c_0, \dots, c_{d-2} \in k$, and let $(m_i, n_i) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ for $i = 0, \dots, d-2$. It is natural to ask whether there exists a polynomial $f \in k[z]$ in normal form and of degree d such that for each $i = 0, \dots, d-2$, the point c_i has preperiodicity portrait (m_i, n_i) for the action of $f(z)$.

Already Theorem 1.3 relates to the above question when $d = 2$. Indeed, one considers the polynomial $f(z) = z^2 + t \in K[z]$, where $K := k(t)$ and then Theorem 1.3 yields that if $c_0 \neq 0$, then at the expense of excluding finitely many portraits (note also that $X(f) = Y(f, c_0) = \emptyset$ by Remark 1.4), there exists a place $\mathfrak{p}$ of K such that c_0 has preperiodicity portrait (m_0, n_0) for the action of $f(z)$ modulo $\mathfrak{p}$. Reducing $f(z)$ modulo a place of K is equivalent with specializing t to a value in k, hence providing an answer to the above question if $d = 2$.

Using the geometry of dynatomic curves, John Doyle [7] was able to treat the case of $d = 2$ completely. In particular, he showed that if $c_0 \neq 0, \frac{\pm 1}{2}, \pm 1$, then for any $(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$, there is an $a_0 \in \mathbb{C}$ such that c_0 has preperiodicity portrait (m, n) under the action of $z^2 + a_0$. Doyle proved an even stronger result in the case of higher degree unicritical polynomials, proving that if $d \geq 3$ and $c_0 \neq 0$ then for any $(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$, there is an $a_0 \in \mathbb{C}$ such that c_0 has preperiodicity portrait (m, n) under the action of $z^d + a_0$.

We pose the following simultaneous portrait question for $d \geq 3$.

Question 2.1. *Let $d \geq 3$ be be a positive integer, let $c_0, \dots, c_{d-2}$ be distinct elements of $\mathbb{C}$, and let $(m_0, n_0), \dots (m_{d-2}, n_{d-2}) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$. Is it true that there must be a polynomial $f \in k[z]$ in normal form and of degree d such that for each $i = 0, \dots, d-2$, the point c_i has preperiodicity portrait (m_i, n_i) under the action of $f(z)$?*

We are not able to answer Question 2.1, but we are able to prove a partial result, which we now state. As a matter of notation, by a *co-finite set of portraits* we mean a subset of $\mathbb{Z}_{\geq 0} \times \mathbb{N}$ whose complement is finite.

Theorem 2.2. *Let k be an algebraically closed field of characteristic 0, let $d \geq 3$ be an integer, and let $c_0, \dots, c_{d-2} \in k$ be $d-1$ distinct elements. Then there exists a co-finite set of portraits $Z^{(0)}$ depending on k and d such that for each $(m_0, n_0) \in Z^{(0)}$, there exists a co-finite set of portraits $Z^{(1)} := Z^{(1)}(c_0, m_0, n_0)$ depending on k, d, c_0, m_0, and n_0 such that for each $(m_1, n_1) \in Z^{(1)}$, there exists a co-finite set of portraits $Z^{(2)} := Z^{(2)}(c_0, m_0, n_0, c_1, m_1, n_1)$ depending on k, d, c_0,..., n_1 such that for each $(m_2, n_2) \in Z^{(2)}$, and so on ..., there exists a co-finite set of portraits $Z^{(d-2)} := Z^{(d-2)}(c_0, ..., n_{d-3})$ depending on k, d, c_0,..., n_{d-3} such that for each $(m_{d-2}, n_{d-2}) \in Z^{(d-2)}$ there exist $a_0, \dots, a_{d-2} \in k$ such that the following holds. For $0 \leq i \leq d-2$, the point c_i has portrait (m_i, n_i) under $z^d + a_{d-2} z^{d-2} + \ldots + a_1 z + a_0$.*

We are interested next in the reverse situation from Theorem 2.2, i.e. given a set of $(d-1)$ *distinct* portraits (m_i, n_i), for which starting points $c_0, \dots, c_{d-2} \in k$ is there possible to find a polynomial $f \in k[z]$ of degree d and in normal form such that the preperiodicity portrait of c_i with respect to the action of $f(z)$ is (m_i, n_i) for each $i = 0, \dots, d-2$? In other words, Theorem 2.2 tells us how many portraits may be missed for a given set of starting points, while the next result gives information on how many tuples of starting points have to be excluded if a certain set of portraits is to be realized by those starting points.

Theorem 2.3. *Let k be an algebraically closed field of characteristic 0, let $d \geq 2$ be an integer, and let $(m_0, n_0), \dots, (m_{d-2}, n_{d-2})$ be distinct elements in $\mathbb{Z}_{\geq 0} \times \mathbb{N}$. Then there exists a co-finite subset $T^{(0)}$ of $\mathbb{P}^1(k)$ depending on k and d such that for each $c_0 \in T^{(0)}$, there exits a co-finite subset $T^{(1)} := T^{(1)}(m_0, n_0, c_0)$ of $\mathbb{P}^1(k)$ depending on k, d, m_0, n_0, and c_0 such that for each $c_1 \in T^{(1)}$, there exists a co-finite subset $T^{(2)} := T^{(2)}(m_0, n_0, c_0, m_1, n_1, c_1)$ depending on k, d, $m_0, \dots, c_1$ such that for each $c_2 \in T^{(2)}$, and so on ..., there exists a co-finite subset $T^{(d-2)} := T^{(d-2)}(m_0, \dots, c_{d-3})$ depending on k, d, $m_0, \dots, c_{d-3}$ such that for each $c_{d-2} \in T^{(d-2)}$ there exists $a_0, \dots, a_{d-2} \in k$ such that the following holds. For $0 \leq i \leq d-2$, the point c_i has portrait (m_i, n_i) under $z^d + a_{d-2}z^{d-2} + \dots + a_1 z + a_0$.*

Theorem 2.3 follows through an argument similar to the proof of Theorem 2.2 once we prove a strong uniform result for the set of possible exceptions of starting points which cannot realize a given portrait, i.e. the "dual" statement from Theorem 1.3. For this "dual" result we require first the definition of *isotrivial rational maps.*

Definition 2.4. *Let k be an algebraically closed field of characteristic* 0*, and let K be a finitely generated function field over k of transcendence degree equal to* 1*. Let $\varphi \in K(z)$ of degree $d \geq 2$. We say that $\varphi(z)$ is isotrivial if there is $\sigma \in \overline{K}(z)$ of degree* 1 *such that $\sigma^{-1} \circ \varphi \circ \sigma \in k(z)$.*

Let k, K, and φ be as in Definition 2.4. We let $W(\varphi)$ be the set of $(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ such that the set of points having portrait (m, n) with respect to φ is contained in $\mathbb{P}^1(k)$. If φ is not isotrivial, there are at most finitely many $x \in \mathbb{P}^1(k)$ which are preperiodic for φ (see [1]); hence, in this case, Theorem 1.3 implies that there are at most finitely many portraits $(m, n) \in W(\varphi)$ such that $n \notin X(\varphi)$.

Theorem 2.5. *Let S be a finite set of places of K containing all places of bad reduction. Let $\varphi \in K(z)$ be non-isotrivial. Then there is a finite set $T(S) \subset \mathbb{P}^1(k)$ depending only on K, φ, and S satisfying the following property: for every $(m, n) \in (\mathbb{Z}_{\geq 0} \times \mathbb{N}) \setminus W(\varphi)$ and for every $\alpha \in \mathbb{P}^1(k) \setminus T(S)$, there is a place $\mathfrak{p} \in \Omega_K \setminus S$ such that α has portrait (m, n) under the action of φ modulo $\mathfrak{p}$.*

Remark 2.6. From Lemma 4.24 and Corollary 4.25, we have that $(m, n) \notin W(\varphi)$ if and only if $n \notin X(\varphi)$ and some point of portrait (m, n) is not constant. The first condition $n \notin X(\varphi)$ is necessary for Theorem 1.3 which will be used in the proof of Theorem 2.5. Now, if all points of portrait (m, n) are contained in $\mathbb{P}^1(k)$, then for some $\alpha \in \mathbb{P}^1(k)$ which is not a point with portrait (m, n) for φ, we cannot find a place $\mathfrak{p}$ of K such that the portrait of α for φ modulo $\mathfrak{p}$ is (m, n) because that would mean that α is in the same residue class modulo $\mathfrak{p}$ as another point in $\mathbb{P}^1(k)$ (which has portrait (m, n) for φ globally).

We expect Theorem 2.3 remains valid without the condition that the given portraits are distinct. In Theorem 2.3, we note that the co-finite set $T^{(i)}$ depends on the previously chosen points $c_0, \dots, c_{i-1}$ *together with* the portraits (m_0, n_0),..., (m_{i-1}, n_{i-1}). It is an interesting problem to relax such dependence on portraits, for which we present the following result for cubic polynomials in normal form. By a co-countable subset of a set, we mean a subset whose complement is countable.

Corollary 2.7. *Suppose that k is algebraically closed of characteristic 0. Then there exist a co-finite subset $U^{(0)}$ of k such that for every $c_0 \in U^{(0)}$, there exists a co-countable subset $U^{(1)}(c_0)$ of k depending on c_0 such that for every $c_1 \in U^{(1)}(c_0)$, the following holds. For every pair of portraits (m_0, n_0) and (m_1, n_1), there exist $a, b \in k$ such that for each $i = 0, 1$, c_i has portrait (m_i, n_i) under $z^3 + az + b$.*

As a consequence, when k is uncountable there exist uncountably many $(c_0, c_1) \in k^2$ such that for every pair of portraits (m_0, n_0) and (m_1, n_1), there exist $a, b \in k$ such that for each $i = 0, 1$, c_i has portrait (m_i, n_i) under $z^3 + az + b$.

One might also ask how much of Theorem 2.2, Theorem 2.3, and Corollary 2.7 holds in the more general context of nonconstant points in $\overline{k(a_0, \dots, a_{d-2})}$. One issue that arises here is the possibility that some c_i is an iterate of another c_j under the general degree d polynomial, something that cannot happen when all of the c_i are in k.

The simultaneous portrait problem studied here was inspired by work of Douady, Hubbard, and Thurston [6], who treated the problem of portraits of critical points of rational functions. Their work yielded not only existence results, but also information about finiteness (up to change of variables) and transversality (of intersections of hypersurfaces corresponding to portraits of marked critical points). We hope to treat constant point analogs of these results in future work.

3. Preliminaries

3.1. **Good reduction of rational maps.** If $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism defined over K, then (fixing a choice of homogeneous coordinates) there are relatively prime homogeneous polynomials $F, G \in K[X, Y]$ of the same degree $d = \deg \varphi$ such that $\varphi([X, Y]) = [F(X, Y) : G(X, Y)]$. In affine coordinates, $\varphi(z) = F(z, 1)/G(z, 1) \in K(z)$ is a rational function in one variable. Note that by our choice of coordinates, F and G are uniquely defined up to a nonzero constant multiple.

Let $\mathfrak{p}$ be a place of K with valuation ring $\mathfrak{o}_{\mathfrak{p}}$ and residue field k. We define as follows the reduction modulo $\mathfrak{p}$ of a point $P \in \mathbb{P}^1(K)$. We let $x, y \in \mathfrak{o}_v$ not both in the maximal ideal of $\mathfrak{o}_{\mathfrak{p}}$ such that $P = [x : y]$ and then the reduction of P modulo $\mathfrak{p}$ is defined to be $r_{\mathfrak{p}}(P) := [\overline{x} : \overline{y}]$, where $\bar{z} \in k$ is the reduction modulo $\mathfrak{p}$ of the element $z \in \mathfrak{o}_{\mathfrak{p}}$.

Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism over K, given by $\varphi([X, Y]) = [F(X, Y) : G(X, Y)]$, where $F, G \in \mathfrak{o}_{\mathfrak{p}}[X, Y]$ are relatively prime homogeneous polynomials of the same degree such that at least one coefficient of F or G is a $\mathfrak{p}$-adic unit. Let $\varphi_{\mathfrak{p}} := [F_{\mathfrak{p}} : G_{\mathfrak{p}}]$, where $F_{\mathfrak{p}}, G_{\mathfrak{p}} \in k[X, Y]$ are the reductions of F and G modulo $\mathfrak{p}$. We say that φ has *good reduction* at $\mathfrak{p}$ if $\varphi_{\mathfrak{p}} : \mathbb{P}^1(k) \longrightarrow \mathbb{P}^1(k)$ is a morphism of the same degree as φ. Equivalently, φ has good reduction at $\mathfrak{p}$ if φ extends as a morphism to the fibre of $\mathbb{P}^1_{\mathrm{Spec}(\mathfrak{o}_{\mathfrak{p}})}$ above $\mathfrak{p}$. For all but finitely many places $\mathfrak{p}$ of K, the map φ has good reduction at $\mathfrak{p}$ (for more details, see the comprehensive book of Silverman [20, Chapter 2]).

If $\varphi \in K[z]$ is a polynomial, we can give the following elementary criterion for good reduction: φ has good reduction at v if and only if all coefficients of φ are v-adic integers, and its leading coefficient is a v-adic unit. For simplicity, we will always use this criterion when we choose a place v of good reduction for a polynomial φ.

3.2. **Absolute values and heights in function fields.** For any finite extension L/K we let Ω_L be the set of places of L. For $\mathfrak{q} \in \Omega_L$ and $\mathfrak{p} \in \Omega_K$, if $\mathfrak{q}\mid_K = \mathfrak{p}$ then we write $\mathfrak{q} \mid \mathfrak{p}$ and let $v_\mathfrak{q}$ and $|\cdot|_\mathfrak{q}$ respectively denote the extension of $v_\mathfrak{p}$ and of $|\cdot|_\mathfrak{q}$ on L. For every $\mathfrak{q} \in \Omega_\mathfrak{q}$, we let $e(\mathfrak{q})$ be the ramification index for the extension of places $\mathfrak{q} \mid \mathfrak{p}$ where $\mathfrak{p} = \mathfrak{q}\mid_K$.

For each $x \in \overline{K}$ we define its Weil height as

$$h_K(x) = \frac{1}{[K(x):K]} \sum_{\mathfrak{p}\in\Omega_K} \sum_{\substack{\mathfrak{q}\in\Omega_{K(x)} \\ \mathfrak{q}\mid\mathfrak{p}}} e(\mathfrak{q}) \cdot \log^+ |x|_\mathfrak{q},$$

where always $\log^+(z) := \log\max\{1, z\}$ for any real number z. We prefer to use the notation h_K for the Weil height (normalized with respect to K) in order to emphasize the dependence on the ground field K for our definition of the height. For example, if L/K is a finite field extension, and $x \in \overline{K}$, then $h_L(x) = [L:K]\cdot h_K(x)$. We extend h_K on $\mathbb{P}^1(\overline{K})$ by $h_K(\infty) = 0$.

Let x and y be distinct elements of $\mathbb{P}^1(K)$, we have the following inequality:

$$\#\{\mathfrak{p} \in \Omega_K : \ r_\mathfrak{p}(x) = r_\mathfrak{p}(y)\} \leq 2(h_K(x) + h_K(y)) \tag{3.1}$$

To prove this, we assume that $x, y \in K$ since the case $x = \infty$ or $y = \infty$ is easy. The set in the left-hand side of (3.1) is contained in:

$$\{\mathfrak{p} \in \Omega_K : \ |x-y|_\mathfrak{p} < 1\} \cup \{\mathfrak{p} \in \Omega_\mathfrak{p} : \ |x|_\mathfrak{p} > 1 \text{ and } |y|_\mathfrak{p} > 1\}$$

whose cardinality is bounded above by:

$$h_K(x-y) + h_K(x) + h_K(y) \leq 2(h_K(x) + h_K(y)).$$

3.3. **Canonical heights for rational maps.** If $\varphi \in \overline{K}(z)$ is a rational map of degree $d \geq 2$, then for each point $x \in \mathbb{P}^1(\overline{K})$, following [5] we define the canonical height of x under the action of φ by:

$$\widehat{h}_\varphi(x) = \lim_{n\to\infty} \frac{h_K(\varphi^n(x))}{d^n}.$$

According to [5], there is a constant C_φ depending only on K and φ such that $|h_K(z) - \widehat{h}_\varphi(z)| < C_\varphi$ for all $z \in \mathbb{P}^1(\overline{K})$.

4. Proof of Theorem 1.3

Throughout this section, k is an algebraically closed field of characteristic 0, K is a finitely generated function field over k of transcendence degree equal to 1, and $\varphi \in K(z)$ is a rational function of degree $d > 1$. Throughout this section, unless stated otherwise all constants depend on K and φ. If a constant depends on other arguments, our notation will clearly indicate them. For example, $C_1, B_2, D_3, \ldots$ denote constants depending on K and φ only, while $C_4(\alpha, \beta, \gamma, \ldots)$ denotes a constant depending on $K, \varphi, \alpha, \beta, \gamma, \ldots$.

4.1. **A preliminary estimate.** At each place $\mathfrak{p}$ of K, we use the *chordal metric* $d_{\mathfrak{p}}(\cdot,\cdot)$ defined as

$$d_{\mathfrak{p}}([x:y],[a:b]) = \frac{|xb-ya|_{\mathfrak{p}}}{(\max(|x|_{\mathfrak{p}},|y|_{\mathfrak{p}}))\,(\max(|a|_{\mathfrak{p}},|b|_{\mathfrak{p}}))}.$$

We see then that for any place $\mathfrak{p}$, we have $r_{\mathfrak{p}}([x:y]) = r_{\mathfrak{p}}([a:b])$ if and only if $d_{\mathfrak{p}}([x:y],[a:b]) < 1$.

We will need the following technical result:

Proposition 4.1. *Let $\tau,\delta > 0$ be real numbers, let $i \geq 1$ be an integer, let $\alpha,\beta \in K$ such that $\widehat{h}_{\varphi}(\alpha) \geq \tau > 0$, and let $F(z)$ be a monic, separable polynomial with coefficients in K whose roots γ_j satisfy the following conditions:*

1. *$\varphi^i(\gamma_j) = \beta$ for each j;*
2. *each γ_j is not periodic; and*
3. *for each j and for each $\ell = 0,\dots,i-1$, $\varphi^{\ell}(\gamma_j) \neq \beta$.*

For each positive integer $n \geq i$, we let Z_n be the set of places $\mathfrak{p}$ of K such that either φ has bad reduction at $\mathfrak{p}$ or

$$\max(d_{\mathfrak{p}}(\varphi^m(\alpha),\beta), |F(\varphi^{n-i}(\alpha))|_{\mathfrak{p}}) < 1 \tag{4.2}$$

for some positive integer $m < n$. Then there are constants $C_1(\delta,i,\tau)$, $C_2(\delta,i,\tau)$, and $B(\delta,i,\tau)$ depending only on i, δ, τ, and φ such that for all positive integers $n > B(\delta,i,\tau)$, we have

$$\#Z_n \leq \delta h_K(\varphi^n(\alpha)) + (C_1(\delta,i,\tau)+2n)h_K(\beta) + C_2(\delta,i,\tau) \tag{4.3}$$

Informally, the set Z_n consists of all places $\mathfrak{p}$ such that $\varphi^{n-i}(\alpha)$ is in the same residue class modulo $\mathfrak{p}$ as one of the roots γ_j of F (see Remark 4.4) and β is in the same residue class modulo $\mathfrak{p}$ as an iterate $\varphi^m(\alpha)$ with $m < n$. In other words, we are looking at places $\mathfrak{p}$ such that $\varphi^m(\alpha)$, $\varphi^n(\alpha)$, and β have the same reduction modulo $\mathfrak{p}$. The conclusion of Proposition 4.1 is that $\#Z_n$ is bounded above by an explicit quantity whose major term (as n grows) is $\delta h_K(\varphi^n(\alpha))$.

Remark 4.4. Let the notation be as in Proposition 4.1. Let L be the splitting field of $F(z)$ over K. Let Γ be the set of places $\mathfrak{p}$ of K such that there is a place $\mathfrak{q} \mid \mathfrak{p}$ of L and a root γ_j of F such that $|\gamma_j|_{\mathfrak{q}} > 1$. We have:

$$\#\Gamma \leq \sum_j h_K(\gamma_j).$$

Using $\varphi^i(\gamma_j) = \beta$ so that $\deg(F) \leq d^i$ and $h_K(\gamma_j) = \frac{1}{d^i}h_K(\beta) + O(1)$, there exist constants $C_3(i)$ and $C_4(i)$ such that:

$$\#\Gamma \leq \sum_j h_K(\gamma_j) \leq C_3(i)h_K(\beta) + C_4(i).$$

For every place $\mathfrak{p}$ of K outside Γ, for every $x \in K$, the inequality $|F(x)|_{\mathfrak{p}} < 1$ is equivalent to the assertion that there is a root γ_j of F and a place $\mathfrak{q} \mid \mathfrak{p}$ of L such that $r_{\mathfrak{q}}(x) = r_{\mathfrak{q}}(\gamma_j)$.

Proof of Proposition 4.1. Recall that we have $\widehat{h}_{\varphi}(\varphi(z)) = d\widehat{h}_{\varphi}(z)$ for all $z \in K$ and that there is a constant C_{φ} such that $|h_K(z) - \widehat{h}_{\varphi}(z)| < C_{\varphi}$ for all $z \in K$. The strategy of our proof is to divide Z_n into sets denoted Y_0, Y_2, and Y_3 below in which the inequality (4.2) holds when $n-m$ is respectively large, small, and moderate.

Choose $B(\delta, i, \tau)$ such that the inequalities:

$$1/d^{B(\delta,i,\tau)+i} < \min\{1,\delta\}/8 \tag{4.5}$$

and

$$2(n+1)C_\varphi \leq \frac{\delta}{2}\left(d^n\tau - C_\varphi\right) \leq \frac{\delta}{2} h_K(\varphi^n(\alpha)) \tag{4.6}$$

hold for every $n > B(\delta, i, \tau)$. We note that the first inequality in (4.6) is possible since d^n dominates other terms when n grows, and that the second inequality in (4.6) is always true since $h_K(\varphi^n(\alpha)) + C_\varphi \geq \widehat{h}_\varphi(\varphi^n(\alpha)) \geq d^n\tau$.

For any $\alpha \in \mathbb{P}^1(K)$ and all $n > B(\delta, i, \tau)$, we have

$$\begin{aligned}
&\sum_{\ell=0}^{n-B(\delta,i,\tau)-i} \#\{\mathfrak{p}\colon r_\mathfrak{p}(\varphi^\ell(\alpha)) = r_\mathfrak{p}(\beta)\} \\
&\leq \sum_{\ell=0}^{n-B(\delta,i,\tau)-i} 2\left(h_K(\varphi^\ell(\alpha)) + h_K(\beta)\right) \quad \text{(by (3.1))} \\
&\leq 2n(C_\varphi + h_K(\beta)) + 2\sum_{\ell=0}^{n-B(\delta,i,\tau)-i} \widehat{h}_\varphi(\varphi^\ell(\alpha)) \\
&= 2n(C_\varphi + h_K(\beta)) + \frac{2}{d^{B(\delta,i,\tau)+i}} \sum_{r=0}^{n-B(\delta,i,\tau)-i} \frac{\widehat{h}_\varphi(\varphi^n(\alpha))}{d^r} \\
&\leq \left(\frac{2}{d^{B(\delta,i,\tau)+i}} \sum_{r=0}^{\infty} \frac{1}{d^r}\right) \widehat{h}_\varphi(\varphi^n(\alpha)) + 2n(C_\varphi + h_K(\beta)) \\
&\leq \frac{\min\{1,\delta\}}{2} \widehat{h}_\varphi(\varphi^n(\alpha)) + 2n(C_\varphi + h_K(\beta)) \quad \text{(by (4.5))} \\
&\leq \frac{\delta}{2} h_K(\varphi^n(\alpha)) + 2(n+1)(C_\varphi + h_K(\beta)) \\
&\leq \delta h_K(\varphi^n(\alpha)) + 2(n+1)h_K(\beta) \quad \text{(by (4.6))}
\end{aligned} \tag{4.7}$$

Thus, if Y_0 is the set of primes such that $d_\mathfrak{p}(\varphi^\ell(\alpha), \beta) < 1$ for some $\ell \leq n - B(\delta, i, \tau) - i$, then

$$\#Y_0 \leq \delta h_K(\varphi^n(\alpha)) + 2(n+1)h_K(\beta). \tag{4.8}$$

Let Y_1 be the set of primes of K for which φ does not have good reduction. Then clearly,

$$\#Y_1 \leq C_5 \tag{4.9}$$

where C_5 depends only on K and φ.

Now, let L be the splitting field for $F(z)$. Since $\varphi^i(\gamma_j) = \beta$, we have:

$$\widehat{h}_\varphi(\gamma_j) = \frac{1}{d^i} \widehat{h}_\varphi(\beta). \tag{4.10}$$

Let Y_2 be the set of primes $\mathfrak{p}$ outside Y_1 and the set Γ in Remark 4.4 such that

$$\max\left(d_\mathfrak{p}(\varphi^m(\alpha), \beta), |F(\varphi^{n-i}(\alpha))|_\mathfrak{p}\right) < 1$$

for $n - i \leq m < n$. For each such prime we have $r_\mathfrak{q}(\varphi^{m-(n-i)}(\gamma_j)) = r_\mathfrak{q}(\beta)$ for some root γ_j of F and some prime $\mathfrak{q}$ of L with $\mathfrak{q} \mid \mathfrak{p}$. From $m - (n-i) < i$, condition (3) and (4.10), we have $\varphi^{m-(n-i)}(\gamma_j) \neq \beta$ and $\widehat{h}_\varphi(\varphi^{m-(n-i)}(\gamma_j)) \leq \widehat{h}_\varphi(\beta)$. This latter inequality implies:

$$h_K(\varphi^{m-(n-i)}(\gamma_j)) \leq h_K(\beta) + 2C_\varphi. \tag{4.11}$$

Using (3.1) and (4.11) we have

$$\#Y_2 \leq i(4h_K(\beta) + 4C_\varphi). \tag{4.12}$$

Let Y_3 be the set of primes $\mathfrak{p}$ outside Y_1 and the set Γ in Remark 4.4 such that

$$\max\left(d_\mathfrak{p}(\varphi^m(\alpha), \beta), |F(\varphi^{n-i}(\alpha))|_\mathfrak{p}\right) < 1$$

for some positive integer m with $n - i > m > n - i - B(\delta, i, \tau)$. If $\mathfrak{p} \in Y_3$, then $\varphi^m(\alpha) \equiv \varphi^n(\alpha) \equiv \beta \pmod{\mathfrak{p}}$, so β modulo $\mathfrak{p}$ is in a cycle of period dividing $n-m$. There is a prime $\mathfrak{q} \mid \mathfrak{p}$ of L and a root γ_j of $F(z)$ such that $\gamma_j \equiv \varphi^{n-i}(\alpha) \equiv \varphi^{(n-i)-m}(\beta) \pmod{\mathfrak{p}}$, we see that γ_j is in the same cycle modulo $\mathfrak{q}$. This implies that γ_j modulo $\mathfrak{q}$ has period dividing $n-m$. From (4.11) and $n-m < B(\delta, i, \tau)+i$, we have:

$$\begin{aligned} h_K(\varphi^{n-m}(\gamma_j)) + h_K(\gamma_j) &\leq \widehat{h}_\varphi(\varphi^{n-m}(\gamma_j)) + \widehat{h}_\varphi(\gamma_j) + 2C_\varphi \\ &\leq \left(d^{B(\delta,i,\tau)} + \frac{1}{d^i}\right)\widehat{h}_\varphi(\beta) + 2C_\varphi \\ &\leq \left(d^{B(\delta,i,\tau)} + \frac{1}{d^i}\right)(h_K(\beta) + C_\varphi) + 2C_\varphi \end{aligned} \tag{4.13}$$

Note that each γ_j has degree at most d^i over K since $\varphi^i(\gamma_j) = \beta$ for each j. Using (3.1) and (4.13), we have:

$$\#Y_3 \leq B(\delta, i, \tau)d^i\left(2\left(d^{B(\delta,i,\tau)} + \frac{1}{d^i}\right)(h_K(\beta) + C_\varphi) + 4C_\varphi\right). \tag{4.14}$$

Since Z_n is contained in $Y_0 \cup Y_1 \cup \Gamma \cup Y_2 \cup Y_3$, we see that (4.3) is a consequence of Remark 4.4, (4.8), (4.9), (4.12), and (4.14). ☐

4.2. **A consequence of the *abc*-theorem for function fields.** The following result is crucial for the proof of Theorem 1.3 and it is a consequence of the *abc*-theorem for function fields by Mason-Stothers (see Silverman's formulation [19]). We will treat the number field case in a future work in which a similar result holds assuming the *abc*-conjecture.

Proposition 4.15. *Let K be a function field. Let $e \geq 3$ be a positive integer. Then for any monic $f(z) \in K[z]$ of degree e without repeated roots, and for any $\gamma \in K$ we have*

$$\#\{\text{primes } \mathfrak{p} \text{ of } K \mid v_\mathfrak{p}(f(\gamma)) > 0\} \geq h_K(\gamma) - \left(7e\sum_{i=1}^{e} h_K(\eta_i) + 2g_K\right) \tag{4.16}$$

where $\eta_1, \ldots, \eta_e$ are the roots of f in $\overline{K}$, and we denote by g_L the genus of any function field L.

Proof. The given inequality holds trivially if γ is a root of $f(z)$. We now assume γ is not a root of $f(z)$. Let $L = K(\eta_1, \ldots, \eta_e)$. Suppose that $v_\mathfrak{q}(\gamma - \eta_i) > 0$ for

$1 \le i \le 3$ and some prime $\mathfrak{q}$ of L. Then we have $v_{\mathfrak{p}}(f(\gamma)) > 0$, for $\mathfrak{p}$ in K such that $\mathfrak{q}$ lies over $\mathfrak{p}$, whenever $v_{\mathfrak{q}}(\eta_j) \ge 0$ for $j = 1, \dots, e$. Thus, we have

$$\begin{aligned} &[L:K]\#\{\text{primes } \mathfrak{p} \text{ of } K \mid v_{\mathfrak{p}}(f(\gamma)) > 0\} \\ &\qquad \ge \# \bigcup_{i=1}^{3} \{\text{primes } \mathfrak{q} \text{ of } L \mid v_{\mathfrak{q}}(\gamma - \eta_i) > 0\} - [L:K] \sum_{i=1}^{e} h_K(\eta_i) \end{aligned} \tag{4.17}$$

Now, when L ramifies over K at a prime $\mathfrak{p}$, we must have that either some η_i has a pole at a prime lying over $\mathfrak{p}$ for $1 \le i \le e$, or $\eta_i - \eta_j$ has a zero at a prime lying over $\mathfrak{p}$ for $1 \le i < j \le e$. Thus, the number of primes $\mathfrak{p}$ of K that are ramified in L/K is bounded by

$$\sum_{i=1}^{e} h_K(\eta_i) + \sum_{1 \le i < j \le e} h_K(\eta_i - \eta_j) \le e \sum_{i=1}^{e} h_K(\eta_i).$$

We now apply the Riemann-Hurwitz theorem for L/K. Note that for each $\mathfrak{p}$ of K where L/K is ramified, the total ramification contribution of primes of L lying above $\mathfrak{p}$ in the Riemann-Hurwitz formula is at most $[L:K] - 1$, hence:

$$2g_L - 2 \le \left(e \sum_{i=1}^{e} h_K(\eta_i) + 2g_K - 2 \right) [L:K]. \tag{4.18}$$

Now we construct a change of coordinates σ that takes η_1, η_2, η_3 to 0, 1, ∞, i.e.

$$\sigma(z) = \frac{\eta_2 - \eta_3}{\eta_2 - \eta_1} \cdot \frac{z - \eta_1}{z - \eta_3}.$$

Then for any z, we have

$$h_K(z) - 4 \sum_{i=1}^{3} h_K(\eta_i) \le h_K(\sigma(z)) \le h_K(z) + 4 \sum_{i=1}^{3} h_K(\eta_i). \tag{4.19}$$

Let B be the set of primes $\mathfrak{q}$ of L such that $v_{\mathfrak{q}}(\eta_i) \ne 0$ for some $1 \le i \le 3$ or $v_{\mathfrak{p}}(\eta_i - \eta_j) \ne 0$ for some $1 \le i < j \le 3$. Then we have

$$\#B \le 2 \left(\sum_{i=1}^{3} h_L(\eta_i) + \sum_{1 \le i < j \le 3} h_L(\eta_i - \eta_j) \right) \le 6[L:K] \sum_{i=1}^{3} h_K(\eta_i). \tag{4.20}$$

For all $\mathfrak{q} \notin B$, we see that σ and σ^{-1} are well defined modulo $\mathfrak{q}$. Thus, for any $\mathfrak{q}$ outside B, we have $v_{\mathfrak{q}}(\gamma - \eta_i) > 0$ for some $1 \le i \le 3$ if and only if $\sigma(\gamma)$ has the same reduction mod $\mathfrak{q}$ to 0, 1, or ∞. Equivalently: $v_{\mathfrak{q}}(\sigma(\gamma)) \ne 0$ or $v_{\mathfrak{q}}(\sigma(\gamma) - 1) > 0$. Now, applying the *abc*-theorem for function fields [17, 21] (especially Silverman's formulation [19]), we have

$$\begin{aligned} \#\{\text{primes } \mathfrak{q} \text{ of } L \mid v_{\mathfrak{q}}(\sigma(\gamma)) \ne 0 \text{ or } v_{\mathfrak{q}}(\sigma(\gamma) - 1) > 0\} \\ \ge h_L(\sigma(\gamma)) - (2g_L - 2) \\ = [L:K] h_K(\sigma(\gamma)) - (2g_L - 2), \end{aligned} \tag{4.21}$$

if $\sigma(\gamma) \notin \mathbb{P}^1(k)$, where we recall that k is the ground field of K. But the inequality (4.21) holds trivially for $\sigma(\gamma) \in k \setminus \{0, 1\}$ once we replace $2g_L - 2$ by $2g_L$. The fact that $\sigma(\gamma) \notin \{0, 1, \infty\}$ follows from the assumption that γ is not a root of f.

Since $e \ge 3$, combining equations (4.17), (4.18), (4.19), (4.20), and (4.21) gives the desired inequality (4.16). $\square$

4.3. **Proof of Theorem 1.3: small m or small n.** Assume the notation in Theorem 1.3, we prove the existence of $\mathfrak{p}$ such that $r_{\mathfrak{p}}(\alpha)$ has portrait (m, n) for almost all (m, n) where $n \notin X(\varphi)$, $m \notin Y(\varphi, \alpha)$, and either m or n is small. Note that the constants that appear here may depend on the finite set of places S. As before, we will always indicate such dependence. We will use the following very simple lemmas repeatedly.

Lemma 4.22. *Let $\mathfrak{p}$ be a prime of good reduction for φ. Suppose that $r_{\mathfrak{p}}(\gamma_1) \neq r_{\mathfrak{p}}(\gamma_2)$ but $r_{\mathfrak{p}}(\varphi(\gamma_1)) = r_{\mathfrak{p}}(\varphi(\gamma_2))$. If γ_1 is periodic for φ modulo $\mathfrak{p}$, then γ_2 is not periodic for φ modulo $\mathfrak{p}$.*

Proof. We write $r_{\mathfrak{p}}(\varphi^{n_1}(\gamma_1)) = r_{\mathfrak{p}}(\gamma_1)$ for some $n_1 > 0$. Suppose that γ_2 was also periodic modulo $\mathfrak{p}$; then we can write $r_{\mathfrak{p}}(\varphi^{n_2}(\gamma_2)) = r_{\mathfrak{p}}(\gamma_2)$ for some $n_2 > 0$. Since $r_{\mathfrak{p}}(\varphi(\gamma_1)) = r_{\mathfrak{p}}(\varphi(\gamma_2))$ we then must have

$$r_{\mathfrak{p}}(\gamma_1) = r_{\mathfrak{p}}(\varphi^{n_1 n_2}(\gamma_1)) = r_{\mathfrak{p}}(\varphi^{n_1 n_2}(\gamma_2)) = r_{\mathfrak{p}}(\gamma_2),$$

a contradiction. □

The next lemma is immediate since for any finite extension L/K, and for any place $\mathfrak{q}$ of L that lies above the place $\mathfrak{p}$ of K, two points of K have the same reduction modulo $\mathfrak{p}$ if and only if they have the same reduction modulo $\mathfrak{q}$.

Lemma 4.23. *Let L be an algebraic extension of K. Let $\mathfrak{p}$ be a prime of good reduction for φ and suppose that α has portrait (m, n) modulo $\mathfrak{q}$ for some $\mathfrak{q} \mid \mathfrak{p}$. Then α has portrait (m, n) modulo $\mathfrak{p}$.*

We begin by considering the case where n is small. First, another lemma.

Lemma 4.24. *Assume that φ has a point of minimum period n and assume one of the following two conditions:*

(i) *$\varphi(z)$ is not linearly conjugate to z^{-2}; or*
(ii) *$\varphi(z)$ is linearly conjugate to z^{-2} and $n \neq 2$.*

Then there exists a point β of minimum period n such that $\varphi^{-1}(\beta)$ contains a point that is not periodic.

Proof. Note that if $\varphi^{-1}(\beta)$ contains only periodic points then $\varphi^{-1}(\beta)$ is a single point and thus φ ramifies completely at $\varphi^{-1}(\beta)$. Since φ has at most two totally ramified points, we see then that if $n \geq 3$, then φ ramifies completely over at most two points in any cycle of size n, and we are done. If $n = 1$, then after change of coordinates, φ is a polynomial, which we call it f. Then the fixed points of f are solutions to $f(x) - x = 0$, which has at least one solution (note that $\deg(f) = \deg(\varphi) = d > 1$). If f is totally ramified at one of these solutions then f is conjugate to $f(z) = z^d$, and the fixed point 1 has a non-periodic image ξ_d of 1, where ξ_d is a primitive d-th root of unity, and we are done. Similarly, if $n = 2$ and we have a point γ of period 2 such that φ ramifies completely at both γ and $\varphi(\gamma)$ then $\varphi(z)$ is conjugate to z^{-d}. The given condition implies that $d > 2$. Since $\varphi^2(x) - x = x^{d^2} - x$, we see that φ has exactly $d^2 - d > 2$ points of minimum period 2, so φ cannot ramify completely over all of them, so at least one of them has non-periodic inverse. □

Lemma 4.24 together with the Kisaka's classification [15] gives the complete description of $X(\varphi)$:

Corollary 4.25. *One of the following holds:*

(i) $X(\varphi) = \{n\}$, *where* n *equals 2 or 3, and* φ *does not have a point of minimum period* n.
(ii) $\varphi(z)$ *is linearly conjugate to* z^{-2} *and* $X(\varphi) = \{2\}$.
(iii) $X(\varphi) = \emptyset$.

Our next result yields the conclusion of Theorem 1.3 when the period n is small.

Proposition 4.26. *Let* τ, s, *and* S *be as in Theorem 1.3. Fix a positive integer* $n \notin X(\varphi)$. *Then there is a constant* $M(n, \tau, s)$ *depending on* K, φ, n, S, *and* τ *such that for any* $\alpha \in K$ *with* $\widehat{h}_\varphi(\alpha) > \tau$ *and any* $m > M(n, \tau, s)$, *there is a prime* $\mathfrak{p} \notin S$ *such that* α *has portrait* (m, n) *under the action of* φ *modulo* $\mathfrak{p}$.

Proof. By Lemma 4.24 and Corollary 4.25, there is $\beta \in \mathbb{P}^1(\overline{K})$ of minimum period n and non-periodic $\zeta \in \mathbb{P}^1(\overline{K})$ such that $\varphi(\zeta) = \beta$. Let $L = K(\zeta)$ and note that $\beta \in \mathbb{P}^1(L)$. We will occasionally apply previous results for L in place of K. Since β, ζ, and L depend on K, φ, and n, constants depending on L and φ will ultimately depend on K, φ, and n. Let S_L be the places of L lying above those in S.

We see that $\varphi^{-4}(\zeta)$ contains at least four points (see [20, pp. 142]), none of which are periodic. Thus, there is a monic separable polynomial F of degree greater than 2 with coefficients in L such such that for every root γ of F, we have $\varphi^4(\gamma) = \zeta$ and γ is not periodic. Because ζ is not periodic, then $\varphi^\ell(\gamma) \neq \zeta$ for each root γ of F, and for each $\ell = 0, \ldots, 3$.

Let $\alpha \in \mathbb{P}^1(K)$ such that $\widehat{h}_\varphi(\alpha) > \tau$. There is a constant $C_6(\tau) \geq 4$ such that $\varphi^{m-4}(\alpha) \neq \infty$ for $m > C_6(\tau)$ since we can simply require $d^{C_6(\tau)-4}\tau > \widehat{h}_\varphi(\infty)$. By Proposition 4.15, there is a constant $C_7(n)$ such that:

$$\#\{\mathfrak{q} \in \Omega_L : \ v_\mathfrak{q}(F(\varphi^{m-4}(\alpha))) > 0\} \geq h_L(\varphi^{m-4}(\alpha)) - C_7(n) \tag{4.27}$$

for every $m > C_6(\tau)$.

Let $\beta_j = \varphi^j(\beta)$ for $0 \leq j \leq n-1$ be elements in the periodic cycle containing β. Let $\mathcal{E}$ be the set of primes $\mathfrak{q} \in \Omega_L$ of good reduction satisfying one of the following two conditions:

$$\text{There are } 0 \leq i < j \leq n-1 \text{ such that } r_\mathfrak{q}(\beta_i) = r_\mathfrak{q}(\beta_j). \tag{4.28}$$

$$r_\mathfrak{q}(\zeta) = r_\mathfrak{q}(\beta_{n-1}) \tag{4.29}$$

By (3.1), there is a constant $C_8(n)$ such that $\#\mathcal{E} \leq C_8(n)$. This last inequality together with (4.27) and Remark 4.4 imply the existence of a constant $C_9(n, \tau, s)$ such that for every $m > C_9(n, \tau, s)$ the following holds. There exists a place $\mathfrak{q} \in \Omega_L \setminus (\mathcal{E} \cup S_L)$ such that $\varphi^{m-4}(\alpha)$ and some root γ of F have the same reduction modulo a place of $L(\gamma)$ lying above $\mathfrak{q}$. Therefore $\varphi^m(\alpha)$ and ζ have the same reduction modulo $\mathfrak{q}$. Since $\mathfrak{q} \notin \mathcal{E}$, conditions (4.28) and (4.29) together with Lemma 4.22 imply that $\varphi^{m+1}(\alpha) \equiv \beta$ is periodic of minimum period n and that $\varphi^m(\alpha) \equiv \zeta$ is not periodic modulo $\mathfrak{q}$. Therefore α has portrait $(m+1, n)$ modulo $\mathfrak{q}$. Lemma 4.23 finishes □

We now consider small values of m:

Proposition 4.30. *Let* τ, s, *and* S *be as in Theorem 1.3. Let* $m \geq 0$ *be an integer.*

(1) *If $m=0$, then there is a constant $N(\tau,s)$ such that for any $n>N(\tau)$ and any α with $\widehat{h}_\varphi(\alpha)\geq\tau$, there is a prime $\mathfrak{p}\notin S$ such that α has portrait (m,n) modulo $\mathfrak{p}$.*

(2) *If $m>0$, then there is a constant $N(m,\tau,s)$ such that for any $n>N(m,\tau,s)$ and any α satisfying $\widehat{h}_\varphi(\alpha)\geq\tau$ and φ is not totally ramified at $\varphi^{m-1}(\alpha)$, there is a prime $\mathfrak{p}\notin S$ such that α has portrait (m,n) under the action of φ modulo $\mathfrak{p}$.*

Proof. We may assume that $\varphi^m(\alpha)\neq\infty$. Otherwise, we can make the change of variables $z\mapsto\frac{1}{z}$.

If $m=0$, then let F be a monic separable polynomial of degree greater than two such that every root γ of F satisfies $\varphi^5(\gamma)=\alpha$. Since α is not preperiodic (because it has positive canonical height), then also each root γ of F is not periodic, and moreover, $F^\ell(\gamma)\neq\alpha$ for $\ell=0,\dots,4$. If $m>0$, and φ is not totally ramified at $\varphi^{m-1}(\alpha)$, then let F be a monic separable polynomial of degree greater than two such that each root γ of F satisfies:

(i) $\varphi^5(\gamma)=\varphi^m(\alpha)$ and $\varphi^\ell(\gamma)\neq\varphi^m(\alpha)$ for $\ell=0,\dots,4$ (also, γ is not periodic because $\varphi^m(\alpha)$ is not periodic); and

(ii) $\varphi^4(\gamma)\neq\varphi^{m-1}(\alpha)$.

The fact that $\deg(F)>2$ (or even $\deg(F)\geq 4$) follows from [20, pp. 142].

As before, there is a constant $C_{10}(\tau)\geq 5$ such that for every $n>C_{10}(\tau)$, we have $\varphi^{m+n-5}(\alpha)\neq\infty$. Then, by Proposition 4.15 there is a constant $C_{11}(m)$ such that:

$$\#\{\mathfrak{p}\colon v_\mathfrak{p}(F(\varphi^{m+n-5}(\alpha)))>0\}\geq h_K(\varphi^{m+n-5}(\alpha))-C_{11}(m) \tag{4.31}$$

for every $n>C_{10}(\tau)$.

For $n>C_{10}(\tau)$, we let W_n be the set of primes $\mathfrak{p}$ of K such that either φ has bad reduction at $\mathfrak{p}$ or

$$\max(d_\mathfrak{p}(\varphi^\ell(\varphi^m(\alpha)),\varphi^m(\alpha)),|F(\varphi^{n-5}(\varphi^m(\alpha)))|_\mathfrak{p})<1$$

for some positive integer $\ell<n$. Proposition 4.1 (for $\beta=\varphi^m(\alpha)$, $i=5$, and $\delta=\frac{1}{2d^5}$) shows that there are constants $C_{12}(m,\tau)>C_{10}(\tau)$ and $C_{13}(m,\tau)$ such that for all $n>C_{12}(m,\tau)$, we have

$$\#W_n\leq\frac{1}{2d^5}h_K(\varphi^{m+n}(\alpha))+nC_{13}(m,\tau)\left(1+h_K(\varphi^m(\alpha))\right). \tag{4.32}$$

If $m>0$, let $\mathcal{E}$ be the set places $\mathfrak{p}\in\Omega_K$ of good reduction such that $r_\mathfrak{q}(\varphi^4(\gamma))=r_\mathfrak{q}(\varphi^{m-1}(\alpha))$ for some root γ of F and some place $\mathfrak{q}$ of $K(\gamma)$ above $\mathfrak{p}$. If $m=0$, we let $\mathcal{E}$ be the empty set. There is a constant $C_{14}(m,\tau)$ such that

$$\#\mathcal{E}\leq C_{14}(m,\tau)\left(1+h_K(\varphi^m(\alpha))\right). \tag{4.33}$$

Note that $h_K(\varphi^r(\alpha))=d^r\widehat{h}_\varphi(\alpha)+O(1)$ for every integer $r\geq 0$ and every $\alpha\in\mathbb{P}^1(\overline{K})$, where $O(1)$ only depends on K and φ. From the right-hand sides of (4.31), (4.32), and (4.33) together with Remark 4.4, there is a constant $C_{15}(m,\tau,s)>C_{12}(m,\tau)$ such that for every $n>C_{15}(m,\tau,s)$ and every $\alpha\in\mathbb{P}^1(K)$ satisfying $\widehat{h}_\varphi(\alpha)\geq\tau$, there is a place $\mathfrak{p}\in\Omega_K$ satisfying the following conditions:

(I) $\mathfrak{p}\notin W_n\cup\mathcal{E}\cup S$.

(II) $|F(\varphi^{m+n-5}(\alpha))|_\mathfrak{p}<1$ and there are a root γ of F and a place $\mathfrak{q}\mid\mathfrak{p}$ of $K(\gamma)$ such that $r_\mathfrak{q}(\varphi^{m+n-5}(\alpha))=r_\mathfrak{q}(\gamma)$.

Now the condition (I) implies $r_{\mathfrak{p}}(\varphi^{m+n}(\alpha)) = r_{\mathfrak{q}}(\varphi^m(\alpha))$. The conditions $\mathfrak{p} \notin W_n$ and $|F(\varphi^{m+n-5}(\alpha))|_{\mathfrak{p}} < 1$ imply $d_{\mathfrak{p}}(\varphi^{m+\ell}(\alpha), \varphi^m(\alpha)) \geq 1$ for every positive integer $\ell < n$. Hence $r_{\mathfrak{p}}(\varphi^m(\alpha))$ has minimum period n. Finally when $m > 0$, by the definition of $\mathcal{E}$ and Lemma 4.22, the condition $\mathfrak{p} \notin \mathcal{E}$ implies that $\varphi^{m-1}(\alpha)$ is not periodic modulo $\mathfrak{p}$. Therefore α modulo $\mathfrak{p}$ has portrait (m, n). $\square$

4.4. **Proof of Theorem 1.3: large m and n.** Let τ, s, and S be as in Theorem 1.3. We now show that there are constants $C_{16}(\tau, s)$ and $C_{17}(\tau, s)$ such that for every $m \geq C_{16}(\tau, s)$, $n \geq C_{17}(\tau, s)$, and $\alpha \in \mathbb{P}^1(K)$ satisfying $\widehat{h}_{\varphi}(\alpha) \geq \tau$, there is a place $\mathfrak{p} \in \Omega \setminus S$ such that α has portrait (m, n) modulo $\mathfrak{p}$. Combining this with Propositions 4.26 and 4.30, we finish the proof of Theorem 1.3.

There is a constant $C_{18}(\tau) \geq 2$ such that for every $m \geq C_{18}(\tau)$ and every $\alpha \in \mathbb{P}^1(K)$ satisfying $\widehat{h}_{\varphi}(\alpha) \geq \tau$, we have $\varphi^m(\alpha) \neq \infty$ and:

$$\varphi^{-2}(\varphi^m(\alpha)) \text{ contains neither } \infty \text{ nor any ramification points of } \varphi. \tag{4.34}$$

Note that the above conditions are satisfied when $d^{m-2}\tau$ is greater than the canonical heights of ∞ and of the ramification points of φ.

Fix any $m \geq C_{18}(\tau)$ and $\alpha \in \mathbb{P}^1(K)$ satisfying $\widehat{h}_{\varphi}(\alpha) \geq \tau$. Now, there are $d^2 - 1 \geq 3$ distinct points $\eta_i \in \overline{K}$ such that $\eta_i \neq \varphi^{m-2}(\alpha)$ and $\varphi^2(\eta_i) = \varphi^m(\alpha)$ for $i = 1, \ldots, d^2-1$. Let $e = d^2-1$ and $F(z) = \prod_{i=1}^{e}(z-\eta_i)$. Applying Proposition 4.15 and using the fact that $|h_K - \widehat{h}_{\varphi}|$ is uniformly bounded, we have that there exist constants C_{19} and C_{20} such that the following inequality holds for every positive integer n:

$$\begin{aligned} \#\{\mathfrak{p}\colon v_{\mathfrak{p}}(F(\varphi^{m+n-2}(\alpha))) > 0\} &\geq \widehat{h}_{\varphi}(\varphi^{m+n-2}(\alpha)) - C_{19}\sum_{i=1}^{e}\widehat{h}_{\varphi}(\eta_i) - C_{20} \\ &= \widehat{h}(\alpha)\left(d^{m+n-2} - eC_{19}d^{m-2}\right) - C_{20}. \end{aligned} \tag{4.35}$$

Hence there is a constant $C_{21}(\tau, s)$ such that for every $n > C_{21}(\tau, s)$, we have:

$$\#\{\mathfrak{p} \in \Omega_K \setminus S:\ v_{\mathfrak{p}}(F(\varphi^{m+n-2}(\alpha))) > 0\} \geq \frac{3}{4}\widehat{h}_{\varphi}(\alpha)d^{m+n-2} \tag{4.36}$$

Let L be the splitting field of F. We now argue as in Remark 4.4 as follows. Let Γ be the set of places $\mathfrak{p} \in \Omega_K$ such that there are some $\mathfrak{q} \mid \mathfrak{p}$ in Ω_L and some root η_i of F satisfying $|\eta_i|_{\mathfrak{q}} > 1$. Then we have:

$$\#\Gamma \leq \sum_{i=1}^{e} h_K(\eta_i) \leq e\widehat{h}_{\varphi}(\alpha)d^{m-2} + O(1) \tag{4.37}$$

where $O(1)$ only depends on K and φ. As in Remark 4.4, if $\mathfrak{p} \notin \Gamma$, φ has good reduction at $\mathfrak{p}$, and if $|F(\varphi^{m+n-2}(\alpha))|_{\mathfrak{p}} < 1$ then there is a root η_i of F, and a place $\mathfrak{q}$ of L lying above $\mathfrak{p}$ such that $r_{\mathfrak{q}}(\eta_i) = r_{\mathfrak{q}}(\varphi^{m+n-2}(\alpha))$. This implies $r_{\mathfrak{p}}(\varphi^m(\alpha)) = r_{\mathfrak{p}}(\varphi^{m+n}(\alpha))$. Therefore, from (4.36) and (4.37) there is a constant $C_{22}(\tau, s) > C_{21}(\tau, s)$ such that for every $n > C_{22}(\tau, s)$, we have:

$$\#\{\mathfrak{p} \in \Omega_K \setminus S:\ r_{\mathfrak{p}}(\varphi^{m+n}(\alpha)) = r_{\mathfrak{p}}(\varphi^m(\alpha))\} \geq \frac{1}{2}\widehat{h}_{\varphi}(\alpha)d^{m+n-2} \tag{4.38}$$

Let $\mathcal{E}_1$ be the set of primes $\mathfrak{p} \in \Omega_K$ such that there are a prime $\mathfrak{q} \mid \mathfrak{p}$ of L and a root η_i of F satisfying $r_{\mathfrak{q}}(\varphi(\eta_i)) = r_{\mathfrak{q}}(\varphi^{m-1}(\alpha))$. If $r_{\mathfrak{q}}(\varphi(\eta_i)) = r_{\mathfrak{q}}(\varphi^{m-1}(\alpha))$ then we have either $v_{\mathfrak{q}}(\varphi^{m-1}(\alpha)) < 0$ or $v_{\mathfrak{q}}(\varphi(\eta_i) - \varphi^{m-1}(\alpha)) > 0$. Therefore:

$$\begin{aligned}\#\mathcal{E}_1 &\leq h_K(\varphi^{m-1}(\alpha)) + \sum_{i=1}^{e} h_K(\varphi(\eta_i) - \varphi^{m-1}(\alpha)) \\ &\leq h_K(\varphi^{m-1}(\alpha)) + \sum_{i=1}^{e} (h_K(\varphi(\eta_i)) + h_K(\varphi^{m-1}(\alpha))) \\ &= (e+1)\widehat{h}_\varphi(\alpha)d^{m-1} + e\widehat{h}_\varphi(\alpha)d^{m-2} + O(1)\end{aligned} \tag{4.39}$$

where $O(1)$ depends only on K and φ.

Hence there is $C_{23}(\tau, s)$ such that for every $n > C_{23}(\tau, s)$,we have:

$$\#\mathcal{E}_1 \leq \frac{1}{8}\widehat{h}_\varphi(\alpha)d^{m+n-2}. \tag{4.40}$$

Let $\mathcal{E}_2$ be the set of primes $\mathfrak{p}$ of good reduction such that $r_{\mathfrak{p}}(\varphi^{n'}(\varphi^m(\alpha))) = r_{\mathfrak{p}}(\varphi^m(\alpha))$ for n' a proper divisor of n. As before, we either have $v_{\mathfrak{p}}(\varphi^m(\alpha)) < 0$ or $v_{\mathfrak{p}}(\varphi^{m+n'}(\alpha) - \varphi^m(\alpha)) > 0$. Hence, we have

$$\#\mathcal{E}_2 \leq h_K(\varphi^m(\alpha)) + \sum_{p|n} h_K\left(\varphi^{n/p}(\varphi^m(\alpha)) - \varphi^m(\alpha)\right) \tag{4.41}$$

where p ranges over the distinct prime factors of n. The number of prime factors of n is at most $\log_2 n$. Thus, we have

$$\begin{aligned}\#\mathcal{E}_2 &\leq \widehat{h}_\varphi(\varphi^m(\alpha)) + \sum_{p|n}\left(\widehat{h}_\varphi\left(\varphi^{n/p}(\varphi^m(\alpha))\right) + \widehat{h}_\varphi(\varphi^m(\alpha))\right) + C_{25}\log(n) \\ &\leq (\log n + 1)\widehat{h}_\varphi(\alpha)d^m + (\log n)\widehat{h}_\varphi(\alpha)d^{m+\frac{n}{2}} + C_{25}\log n\end{aligned} \tag{4.42}$$

where C_{25} depends only on K and φ.

Since d^n dominates both $(\log(n))d^{n/2}$ and $(\log n + 1)$ when n grows sufficiently large (and independently from m), there exists a constant $C_{26} > C_{24}$ depending only on K and φ such that for $n > C_{26}$, we have:

$$\#\mathcal{E}_2 \leq \frac{1}{8}\widehat{h}_\varphi(\alpha)d^{m+n-2} \tag{4.43}$$

For $m > C_{18}(\tau)$ and for $n > \max\{C_{22}(\tau, s), C_{23}(\tau, s), C_{26}\}$, from (4.38), (4.40), and (4.43) there exist at least $\frac{1}{4}\widehat{h}_\varphi(\alpha)d^{m+n-2}$ many primes $\mathfrak{p} \in \Omega_K \setminus S$ such that the following conditions hold:

(I) $r_{\mathfrak{p}}(\varphi^{m+n}(\alpha)) = r_{\mathfrak{p}}(\varphi^m(\alpha))$.

(II) $\mathfrak{p} \notin \mathcal{E}_1 \cup \mathcal{E}_2$.

Condition (I) together with $\mathfrak{p} \notin \mathcal{E}_2$ imply that $\varphi^m(\alpha)$ has minimum period n under the action of φ modulo $\mathfrak{p}$. Condition $\mathfrak{p} \notin \mathcal{E}_1$ together with Lemma 4.22 imply that $\varphi^{m-1}(\alpha)$ is not periodic modulo $\mathfrak{p}$. Hence α has portrait (m, n) modulo $\mathfrak{p}$, which finishes the proof of Theorem 1.3.

5. Proof of the applications of Theorem 1.3

Using Theorem 1.3 we can prove now its applications. First we prove Theorem 2.2, and then we will prove Theorems 2.5 and 2.3.

5.1. **Simultaneous multiple portraits.** We begin with a few simple lemmas.

Lemma 5.1. *Let $\varphi(z) \in K(z)$ be a rational function of degree $d > 1$ and let $\alpha \in \mathbb{P}^1(K)$ be a preperiodic point with portrait (m, n). Then for all but finitely many places $\mathfrak{p} \in \Omega_K$ of good reduction, α modulo $\mathfrak{p}$ has portrait (m, n).*

Proof. Let $\mathcal{E}$ be the set of places $\mathfrak{p} \in \Omega_K$ of good reduction such that the following two conditions hold:

(i) If $m > 0$, we have $r_\mathfrak{p}(\varphi^{m-1}(\alpha)) = r_\mathfrak{p}(\varphi^{m+n-1}(\alpha))$.
(ii) For some prime divisor ℓ of n, we have $r_\mathfrak{p}(\varphi^{m+\frac{n}{\ell}}(\alpha)) = r_\mathfrak{p}(\varphi^m(\alpha))$.

By (3.1), $\mathcal{E}$ is finite. By Lemma 4.22, for every place $\mathfrak{p} \in \Omega_K \setminus \mathcal{E}$, we have α modulo $\mathfrak{p}$ has portrait (m, n). □

We have the following lemma for determining when polynomials in normal form are isotrivial.

Lemma 5.2. *Let k be an algebraically closed field of characteristic 0, and let K be a finitely generated function field over k of transcendence degree equal to 1. Let $\varphi(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0 \in K[z]$ where $d \geq 2$. Then φ is isotrivial if and only if $\varphi \in k[z]$.*

Proof. Suppose that $\sigma^{-1} \circ \varphi \circ \sigma \in k(z)$. Let $\tilde{\varphi}$ denote $\sigma^{-1} \circ \varphi \circ \sigma$. Let $\beta = \sigma^{-1}(\infty)$. Then $\tilde{\varphi}(\beta) = \beta$, so $\beta \in \mathbb{P}^1(k)$. Hence, after composing σ with a degree one element of $k(z)$, we may suppose that $\sigma(\infty) = \infty$, which means that σ is a polynomial $b_1 z + b_0 \in \overline{K}[z]$. Since

$$\sigma^{-1} \circ \varphi \circ \sigma \in k(z) = b_1^{d-1} z^d + d b_1^{d-2} b_0 z^{d-1} + \text{lower order terms},$$

we see that $b_1 \in k$ and that b_0 must therefore be in k as well. Thus, $\sigma \in k[z]$, so $\varphi \in k[z]$. □

The following lemma is crucial for the proof of Theorem 2.2.

Lemma 5.3. *Let k be an algebraically closed field of characteristic 0, let K be a finitely generated function field over k of transcendence degree equal to 1, let $d \geq 2$ be an integer and let m be an integer such that $0 \leq m \leq d-2$. Let*

$$f(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$$

where

1. *$a_i \in K$ for all i;*
2. *$a_i \in k$ for $i > m$; and*
3. *there is some $j \leq m$ such that $a_j \in K \setminus k$.*

Then there are at most m distinct constants $x \in k$ such that $\widehat{h}_f(x) < \frac{1}{d}$.

Proof. By (3), there is some place $\mathfrak{p}$ of K such that $|a_j|_\mathfrak{p} > 1$ for some $j \leq m$; fix this $\mathfrak{p}$. Take any $x \in K$ such that $|x|_\mathfrak{p} \geq \max_i |a_i|_\mathfrak{p}$. Then, for all $i \leq d-2$, we have $|a_i x^i|_\mathfrak{p} \leq |x^{i+1}|_\mathfrak{p} < |x^d|_\mathfrak{p}$, so $|f(x)|_\mathfrak{p} = |x|_\mathfrak{p}^d$. By induction, we then have $|f^n(x)|_\mathfrak{p} = |x|_\mathfrak{p}^{d^n}$ for all n. Thus, in particular for any $\alpha \in k$ such that $|f(\alpha)|_\mathfrak{p} \geq \max_i |a_i|_\mathfrak{p}$, we have

$$\widehat{h}_f(\alpha) = \frac{1}{d}\widehat{h}_f(f(\alpha)) \geq \frac{1}{d}.$$

Thus, it suffices to show that there are at most m constants $x \in k$ such that $|f(x)|_\mathfrak{p} < \max_i |a_i|_\mathfrak{p}$. Let $N = -\min_i v_\mathfrak{p}(a_i)$; then $N > 0$. Let $\pi \in K$ be a generator for the maximal ideal $\mathfrak{p}$. Then it suffices to show that at there are at most m constants $x \in k$ such that $|\pi^N f(x)|_{\mathfrak{p}} < 1$. Now, for each a_i, we have that $\pi^N a_i$ is in the local ring at $\mathfrak{p}$. We let b_i denote the image of $\pi^N a_i$ in the residue field $k_{\mathfrak{p}}$ of $\mathfrak{p}$ which is canonically isomorphic to k.

If $|\pi^N f(x)|_{\mathfrak{p}} < 1$ for $x \in k$, then we have

$$b_m x^m + \cdots + b_0 = 0 \tag{5.4}$$

since $b_i = 0$ for all $i > m$ by (2) (note that $N \geq 1$). Because $b_i \neq 0$ for some $i \leq m$, we see that (5.4) has at most m solutions x, and our proof is complete. □

Proof of Theorem 2.2. When $d = 2$, we have $f(z) = z^2 + a_0$ defined over the function field $k(a_0)$. By Corollary 4.25 and the Kisaka's classification [15], we have that $X(f) = \emptyset$. For every constant $c_0 \in k$, there does not exist a *positive* integer m such that $f^m(c_0) = 0$, hence the set $Y(f, c_0)$ is empty. By Lemma 5.3, $\widehat{h}_f(c_0) \geq 1/2$. We apply Theorem 1.3 to get the desired conclusion.

From now on, assume $d \geq 3$. Every polynomial of degree d in normal form whose coefficient a_{d-2} is nonzero is not totally ramified at any point (other than ∞). We will repeatedly use this observation and apply Theorem 1.3 (see also Corollary 1.6). We prove Theorem 2.2 by showing inductively the existence of the a_i's realizing the portraits for the c_i's. The a_i's will be first independent variables, and then we make a series of specializations of the a_i's which we call in turn $a_{i,0}, a_{i,1}, \cdots$, until we specialize all the variables to values in k.

First, we let $k_0 := \overline{k(a_1, a_2, \ldots, a_{d-2})}$, and we apply Theorem 1.3 (see Corollary 1.6) to the polynomial

$$f_0(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_1 z + a_0$$

defined over the function field $K_0 := k_0(a_0)$ with the starting point c_0. We note that by Lemma 5.3, we know that $\widehat{h}_{f_0}(c_0) \geq \frac{1}{d}$. Take the exceptional set of of places S_0 to be the set containing only the place "at infinity" of $K_0 = k_0(a_0)$ which is the only pole of a_0. Hence we obtain the existence of a co-finite set $Z^{(0)} \subset \mathbb{Z}_{\geq 0} \times \mathbb{N}$ of portraits such that for all $(m_0, n_0) \in Z^{(0)}$, there exists $a_{0,0} \in k_0$ such that c_0 has portrait (m_0, n_0) with respect to

$$f_1(z) := z^d + a_{d-2}z^{d-2} + \cdots + a_2 z^2 + a_1 z + a_{0,0}.$$

This is the polynomial f_0 obtained after specializing a_0 to $a_{0,0} \in k_0$. This specialization is equivalent with reducing f_0 modulo a place of K_0. Fix $(m_0, n_0) \in Z^{(0)}$ and a corresponding $a_{0,0}$.

Next we let $k_1 := \overline{k(a_2, \ldots, a_{d-2})}$ and we regard $f_1(z)$ above as a polynomial defined over the function field $K_1 := k_1(a_1, a_{0,0})$ (note that $\operatorname{trdeg}_{k_1} K_1 = 1$ because $a_{0,0} \in \overline{k_1(a_1)}$). Applying Lemma 5.3 to $f_1(z)$ we conclude that there exists at most one constant point $c \in k_1$ such that $\widehat{h}_{f_1}(c) < \frac{1}{d}$. Since, by construction, $c_0 \in k \subset k_1$ is preperiodic of portrait (m_0, n_0) for f_1, we must have $\widehat{h}_{f_1}(c_1) \geq \frac{1}{d}$.

Let the exceptional set of places S_1 consist of places $\mathfrak{p}$ of K_1 where a_1 or $a_{0,0}$ has a pole, or when c_0 does *not* have portrait (m_0, n_0) modulo $\mathfrak{p}$. The set S_1 is finite by Lemma 5.1. Therefore we can apply Theorem 1.3 (see Corollary 1.6) and obtain a co-finite set $Z^{(1)} \subset \mathbb{Z}_{\geq 0} \times \mathbb{N}$ of portraits such that for each $(m_1, n_1) \in Z^{(1)}$, there exists $a_{1,1} \in k_1$ (and in turn $a_{0,1} \in k_1$) such that c_0 has portrait (m_0, n_0) and c_1

has portrait (m_1, n_1) under the action of

$$f_2(z) := z^d + a_{d-2}z^{d-2} + \cdots + a_2 z^2 + a_{1,1}z + a_{0,1}.$$

This comes from reducing a_1 and $a_{0,0}$ modulo a place in K_1 outside S_1. Fix $(m_1, n_1) \in Z^{(1)}$ and also fix corresponding $a_{1,1}$ and $a_{0,1}$.

The above process could be done inductively as follows. Let $i \in \{1, \ldots, d-2\}$, and assume we previously found $Z^{(0)}$, $Z^{(1)}$,..., $Z^{(i-1)}$ and fixed $(m_j, n_j) \in Z^{(j)}$ for $0 \le j \le i-1$ together with $a_{0,i-1}, a_{1,i-1}, \ldots, a_{i-1,i-1} \in k_{i-1}$ where $k_{i-1} := \overline{k(a_i, \ldots, a_{d-2})}$ such that the following hold. Let $k_i := \overline{k(a_{i+1}, \ldots, a_{d-2})}$ with the understanding that $k_i = k$ when $i = d-2$. Let

$$f_i(z) := z^d + a_{d-2}z^{d-2} + \ldots + a_i z^i + a_{i-1,i-1}z^{i-1} + \ldots + a_{1,i-1}z + a_{0,i-1},$$

which is a polynomial defined over the function field $K_i := k_i(a_i, a_{i-1,i-1}, \ldots, a_{0,i-1})$. We now have that for $0 \le j \le i-1$, the point c_j has portrait (m_j, n_j) under the action of f_i.

Lemma 5.3 now asserts that there are at most i constants $c \in k_i$ such that $\widehat{h}_{f_i}(c) < \frac{1}{d}$. Since $c_0, \ldots, c_{i-1}$ are such constants, we must have $\widehat{h}_{f_i}(c_i) \ge \frac{1}{d}$. Let S_i be the set of places $\mathfrak{p}$ of K_i such that a_i has a pole, or for some $0 \le j \le i-1$ the element $a_{j,i-1}$ has a pole, or for some $0 \le j \le i-1$ the point c_j modulo $\mathfrak{p}$ does not have portrait (m_j, n_j). This set S_i is finite by Lemma 5.1. By Theorem 1.3 (see also Corollary 1.6), there exists a co-finite set $Z^{(i)} \subset \mathbb{Z}_{\ge 0} \times \mathbb{N}$ of portraits such that for every $(m_i, n_i) \in Z^{(i)}$ there exist $a_{0,i}, \ldots, a_{i,i} \in k_i$ satisfying the following. For $0 \le j \le i$, the point c_j has portrait (m_j, n_j) under:

$$f_{i+1}(z) := z^d + a_{d-2}z^{d-2} + \ldots + a_{i+1}z^{i+1} + a_{i,i}z^i + \ldots + a_{1,i}z + a_{0,i}.$$

We continue the above process until $i = d-2$, which finishes the proof of Theorem 2.2. □

5.2. **Almost any portrait is realized by almost any starting point.** Let $\varphi(z) \in K(z)$ having degree $d \ge 2$. In the proof of Theorem 2.5 we will use the following easy fact.

Lemma 5.5. *Let $\varphi(z) \in K(z)$ be non-isotrivial. Then the set $\{\alpha \in \mathbb{P}^1(K) : Y(\varphi, \alpha) \ne \emptyset\}$ is finite.*

Proof. We use the following two properties following from the fact that φ is non-isotrivial (see [1]):

(i) The set $\mathrm{Prep}_\varphi(K)$ of preperiodic points in $\mathbb{P}^1(K)$ is finite.
(ii) There is a positive lower bound τ for the canonical height of points in $\mathbb{P}^1(K) \setminus \mathrm{Prep}_\varphi(K)$.

Let R be the finite set (possibly empty) of points in $\mathbb{P}^1(\overline{K})$ where φ is totally ramified at. There exists M such that for every $m > M$ and every $\alpha \in \mathbb{P}^1(K) \setminus \mathrm{Prep}_\varphi(K)$, φ is not totally ramified at $\varphi^m(\alpha)$. To see this, we simply require that $\widehat{h}_\varphi(\varphi^m(\alpha)) > d^M\tau$ is greater than the canonical height of any point in R. Then the given set in the lemma is contained in the finite set:

$$\mathrm{Prep}_\varphi(K) \cup R \cup \varphi^{-1}(R) \ldots \cup \left(\varphi^M\right)^{-1}(R).$$

□

Proof of Theorem 2.5. Let T_1 be the finite set of points in $\mathbb{P}^1(k)$ consisting of either preperiodic points or the points in the set in Lemma 5.5. We now have $Y(\varphi,\alpha)=\emptyset$ for every $\alpha\in\mathbb{P}^1(k)\setminus T_1$. Note that if $(m,n)\notin W(\varphi)$ then $n\notin X(\varphi)$ (see Remark 2.6).

There is a lower bound τ on the canonical heights of points in $\mathbb{P}^1(k)\setminus T_1$ (see [1]). By Theorem 1.3, there is a finite set $\mathcal{Z}(\tau,|S|)$ such that for every $(m,n)\in(\mathbb{Z}_{\geq 0}\times\mathbb{N})\setminus(\mathcal{Z}(\tau,|S|)\cup W(\varphi))$ and for every $\alpha\in\mathbb{P}^1(k)\setminus T_1$ there exists a place $\mathfrak{p}\notin S$ such that α has portrait (m,n) modulo $\mathfrak{p}$.

Hence it suffices to *fix* an $(m,n)\in\mathcal{Z}(\tau,|S|)\setminus W(\varphi)$ and prove that there exists a finite subset T_2 of $\mathbb{P}^1(k)$ (possibly depending on K, φ, (m,n), and S) such that the following holds. For every $\alpha\in\mathbb{P}^1(k)\setminus T_2$, there exists a place $\mathfrak{p}\notin S$ such that α has portrait (m,n) modulo $\mathfrak{p}$.

Now let C be a nonsingular projective curve over k whose function field is K. We identify places of K with points in $C(k)$. Choose a Zariski open subset V of C such that $V\subseteq C\setminus S$ and φ extends to a morphism from $\mathbb{P}^1_k\times_k V$ to itself. For every $(\mu,\eta)\in\mathbb{Z}_{\geq 0}\times\mathbb{N}$, the equation $\varphi^{\mu+\eta}(z)=\varphi^{\mu}(z)$ defines a Zariski closed subset $V_{\mu,\eta}$ of $\mathbb{P}^1_k\times_k V$ which is equidimensional of dimension 1. Define:

$$U=V_{m,n}\setminus\left(\bigcup_{\mu=0}^{m-1}V_{\mu,n}\cup\bigcup_{p|n}V_{m,n/p}\right)$$

where p ranges over all prime factors of n. We have that U is a Zariski open subset of $V_{m,n}$. Since φ has a point of portrait (m,n), the set U is non-empty.

Let ρ denote the projection from $\mathbb{P}^1_k\times_k V$ to $\mathbb{P}^1_k$. We prove that the image $\rho(U)$ cannot be a finite subset of $\mathbb{P}^1_k$. Assume otherwise, say $\rho(U)=\{u_1,\dots,u_r\}$. Then this implies that all points of portrait (m,n) under φ are the constant points $u_1,\dots,u_r$ contradicting the assumption $(m,n)\notin W(\varphi)$. Hence $\rho(U)$ is infinite. Since $\rho(U)$ is constructible in $\mathbb{P}^1_k$ by Chevalley's theorem, we must have that $\rho(U)$ is co-finite in $\mathbb{P}^1_k$.

Now for every $\alpha\in\rho(U)$, pick any $P\in\rho^{-1}(\alpha)$, and let $\mathfrak{p}\in V$ be the image of P under the projection from $\mathbb{P}^1_k\times_k V$ to V. We have that α has portrait (m,n) under the action of φ modulo $\mathfrak{p}$. This finishes the proof of Theorem 2.5. ☐

We now prove of Theorem 2.3, which in turn relies on Theorem 2.5.

Proof of Theorem 2.3. The proof uses an inductive process which is "dual" to the proof of Theorem 2.2. First, we let $k_0:=\overline{k(a_1,a_2,\dots,a_{d-2})}$, and we apply Theorem 2.5 to the polynomial

$$f_0(z)=z^d+a_{d-2}z^{d-2}+\cdots+a_1z+a_0$$

defined over the function field $K_0:=k_0(a_0)$. Then Lemma 5.2 shows that f_0 is non-isotrivial. By Lemma 5.3, there does not exist $c\in k$ which is preperiodic under f_0. Therefore by Remark 2.6 and Kisaka's list [15], we have $W(f_0)=\emptyset$, hence $(m_0,n_0)\notin W(f_0)$. Take the exceptional set of of places S_0 to be the set containing only the place "at infinity" of $K_0=k_0(a_0)$ which is the only pole of a_0. Hence we obtain the existence of the co-finite set $T^{(0)}\subset\mathbb{P}^1(k)$ such that for all $c_0\in T^{(0)}$, there exists $a_{0,0}\in k_0$ such that c_0 has portrait (m_0,n_0) under the action of

$$f_1(z):=z^d+a_{d-2}z^{d-2}+\cdots+a_2z^2+a_1z+a_{0,0}.$$

This is the polynomial f_0 obtained after specializing a_0 to $a_{0,0} \in k_0$. This specialization is equivalent with reducing f_0 modulo a place of K_0. Fix $c_0 \in T^{(0)}$ and a corresponding $a_{0,0}$.

The above process could be done inductively as follows. Let $i \in \{1, \dots, d-2\}$, and assume we previously found co-finite sets $T^{(0)}$, $T^{(1)}$,..., $T^{(i-1)} \subset \mathbb{P}^1(k)$ and fixed $c_j \in T^{(j)}$ for $0 \le j \le i-1$ together with $a_{0,i-1}, a_{1,i-1}, \dots, a_{i-1,i-1} \in k_{i-1}$ where $k_{i-1} := \overline{k(a_i, \dots, a_{d-2})}$ such that the following hold. Let $k_i := \overline{k(a_{i+1}, \dots, a_{d-2})}$ with the understanding that $k_i = k$ when $i = d-2$, and let

$$f_i(z) := z^d + a_{d-2}z^{d-2} + \dots + a_i z^i + a_{i-1,i-1}z^{i-1} + \dots + a_{1,i-1}z + a_{0,i-1},$$

which is as a polynomial defined over the function field $K_i := k_i(a_i, a_{i-1,i-1}, \dots, a_{0,i-1})$. Also, for each $j = 0, \dots, i-1$, the point c_j has portrait (m_j, n_j) under the action of f_i.

Lemma 5.3 now asserts that there are at most i constants $c \in k_i$ such that $\widehat{h}_{f_i}(c) < \frac{1}{d}$. Since $c_0, \dots, c_{i-1}$ are such constants and since (m_i, n_i) is distinct from $(m_0, n_0), \dots, (m_{i-1}, n_{i-1})$, there exists no $c \in k$ such that c has portrait (m_i, n_i) under f_i. Therefore we have $(m_i, n_i) \notin W(f_i)$. Let S_i be the set of places $\mathfrak{p}$ of K_i such that a_i has a pole, or for some $0 \le j \le i-1$ the element $a_{j,i-1}$ has a pole, or for some $0 \le j \le i-1$ the point c_j does not have portrait (m_j, n_j) under the action of f_i modulo $\mathfrak{p}$. This set S_i is finite by Lemma 5.1. By Theorem 2.5, there exists a co-finite set $T^{(i)} \subset \mathbb{P}^1(k)$ such that for every $c_i \in T^{(i)}$ there exist $a_{0,i}, \dots, a_{i,i} \in k_i$ satisfying the following. For $0 \le j \le i$, the point c_j has portrait (m_j, n_j) under the action of

$$f_{i+1}(z) := z^d + a_{d-2}z^{d-2} + \dots + a_{i+1}z^{i+1} + a_{i,i}z^i + \dots + a_{1,i}z + a_{0,i}.$$

Continuing the above process until $i = d-2$, we finish the proof of Theorem 2.3. □

We conclude this section by proving Corollary 2.7.

Proof of Corollary 2.7. As in the proof of Theorem 2.3, let $k_0 = \overline{k(a)}$, $K_0 = k_0(b)$. We apply Theorem 2.5 to obtain a co-finite subset $U^{(0)}$ of k such that for every $c_0 \in U^{(0)}$, the following holds. For every $(m_0, n_0) \in \mathbb{Z}_{\ge 0} \times \mathbb{N}$, there exists $\overline{b} \in \overline{k(a)}$ such that c_0 has portrait (m_0, n_0) under:

$$\varphi_{c_0,m_0,n_0}(z) := z^3 + az + \overline{b}$$

regarded as a polynomial in $K_1[z]$. Here $K_1 := k(a, \overline{b})$ is a function field over $k_1 := k$.

We claim that $W(\varphi_{c_0,m_0,n_0})$ is empty. By Lemma 5.3, c_0 is the only constant preperiodic point of φ_{c_0,m_0,n_0}. Hence for every portrait $(m_1, n_1) \neq (m_0, n_0)$, we have $(m_1, n_1) \notin W(\varphi_{c_0,m_0,n_0})$. It suffices to show $(m_0, n_0) \notin W(\varphi_{c_0,m_0,n_0})$ by proving that φ_{c_0,m_0,n_0} has a point $\gamma \neq c_0$ of portrait (m_0, n_0). The case $n_0 > 1$ is easy: if $m_0 = 0$ we pick $\gamma_0 = \varphi_{c_0,m_0,n_0}(c_0)$, while if $m_0 > 0$ we pick $\gamma_0 \neq c_0$ such that $\varphi_{c_0,m_0,n_0}(\gamma_0) = \varphi_{c_0,m_0,n_0}(c_0)$. Note that this is possible since φ_{c_0,m_0,n_0} has no totally ramified point other than infinity. We now consider the case $n_0 = 1$. Since the polynomial:

$$\varphi_{c_0,m_0,n_0}(z) - z = z^3 + (a-1)z + \overline{b}$$

is not the cube of a linear polynomial in $K_1[z]$, we have that φ_{c_0,m_0,n_0} has at least two distinct points α_0, α_1 having portrait $(0, 1)$. Using the fact that φ has no totally ramified point (other than infinity) and looking at appropriate backward orbits of α_0 and α_1, we get at least 2 points having portrait $(m_0, 1)$. Hence $W(\varphi_{c_0,m_0,n_0}) = \emptyset$.

Let $S(c_0, m_0, n_0)$ be the set of places $\mathfrak{p}$ of K_1 such that φ_{c_0,m_0,n_0} has bad reduction at $\mathfrak{p}$ or c_0 does not have portrait (m_0, n_0) modulo $\mathfrak{p}$. Then Applying Theorem 2.5, we see then that for each (m_1, n_1) there is a co-finite set $\mathcal{U}(c_0, m_0, n_0, m_1, n_1)$ such that for all $c_1 \in \mathcal{U}(c_0, m_0, n_0, m_x 1, n_1)$, there is a polynomial $f(z) = z^3 + \tilde{a}z + \tilde{b} \in k[z]$ such that, for $i = 1, 2$, c_i has portrait (m_i, n_i) under f. Define:

$$U^{(1)}(c_0) := \bigcap_{((m_0,n_0),(m_1,n_1))} \mathcal{U}(c_0, m_0, n_0, m_1, n_1)$$

which is a co-countable subset of k. From our construction, the sets $U^{(0)}$ and $U^{(1)}(c_0)$ for every $c_0 \in U^{(0)}$ satisfy the assertion in Corollary 2.7.

For the second assertion in the corollary, we simply pick the elements $(c_0, c_1) \in k^2$ satisfying $c_0 \in U^{(0)}$ and $c_1 \in U^{(1)}(c_0)$. $\square$

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