INTEGRAL POINTS FOR DRINFELD MODULES

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Abstract. We prove that in the backward orbit of a nonpreperiodic (nontorsion) point under the action of a Drinfeld module of generic characteristic there exist at most finitely many points S-integral with respect to another nonpreperiodic point. This provides the answer (in positive characteristic) to a question raised by Sookdeo in [26]. We also prove that for each nontorsion point z there exist at most finitely many torsion (preperiodic) points which are S-integral with respect to z. This proves a question raised by Tucker and the author in [13], and it gives the analogue of Ih's conjecture [3] for Drinfeld modules.

1. Introduction

Let k be a number field, let S be a finite set of places of k containing all its archimedean places, and let $\mathfrak{o}_{k,S}$ be the subring of S-integers contained in k. The following two conjectures were made by Ih (and refined by Silverman and Zhang) as an analogue of the classical diophantine problems of Mordell-Lang, Manin-Mumford, and Lang; for more details, see [3].

Conjecture 1.1. *Let A be an abelian variety defined over k, and let $A_S/Spec(\mathfrak{o}_{k,S})$ be a model of A. Let D be an effective divisor on A, defined over k, at least one of whose irreducible components is not the translate of an abelian subvariety by a torsion point, and let D_S be its Zariski closure in A_S. Then the set of all torsion points of $A(\overline{k})$ whose closure in A_S is disjoint from D_S, is not Zariski dense in A.*

The next conjecture is for algebraic dynamical systems, and it is modeled by Conjecture 1.1 for elliptic curves where torsion points are seen as preperiodic points for the multiplication-by-2-map. In general, for any rational map f, we say that α is a preperiodic point for f if its orbit under f is finite. As always in arithmetic dynamics, we denote by f^n the n-th iterate of f. So, α is preperiodic if and only if there exist nonnegative integers $m \neq n$ such that $f^m(\alpha) = f^n(\alpha)$ (for more details on the theory of arithmetic dynamics we refer the reader to Silverman's book [24]).

Conjecture 1.2. *Let f be a rational function of degree at least 2 defined over k, and let $\alpha \in \mathbb{P}^1(k)$ be nonpreperiodic for f. Then there are only finitely many preperiodic points which are S-integral with respect to α, i.e. whose Zariski closures in $\mathbb{P}^1/Spec(\mathfrak{o}_{k,S})$ do not meet the Zariski closure of α.*

In [3], Baker, Ih and Rumely prove the first cases of the above conjectures. They prove Conjecture 1.1 for elliptic curves A, which in particular provides a proof of Conjecture 1.2 for Lattès maps f. Also in [3], the authors prove Conjecture 1.2 when f is a powering map (same proof works when f is a Chebyshev polynomial).

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The ingredients for proving their results are a strong form of equidistribution for torsion points of 1-dimensional algebraic groups (such as elliptic curves or $\mathbb{G}_m$), lower bounds for linear forms in (elliptic) logarithms, and a good understanding of the local heights associated to a dynamical system coming from an algebraic group endomorphism. At this moment it seems difficult to extend the above results beyond the case of 1-dimensional abelian varieties, or the case of rational maps associated to endomorphisms of 1-dimensional algebraic groups.

It is natural to consider the above conjectures in characteristic p, but one has to be careful in their reformulation. Indeed, if A is defined over $\overline{\mathbb{F}_p}$ then all its torsion points are also defined over $\overline{\mathbb{F}_p}$ and thus one can find infinitely many torsion points which are S-integral with respect to a divisor D of A. Similarly, if f is a rational map defined over $\overline{\mathbb{F}_p}$, all its preperiodic points are also in $\overline{\mathbb{F}_p}$ and thus again one can find infinitely many points which are S-integral with respect to a nonpreperiodic point α. On the other hand, the Drinfeld modules have always proven to be the right analogue in characteristic p of abelian varieties. Therefore we propose to study in this paper analogues of the above conjectures for Drinfeld modules. One could consider an analogue of Conjecture 1.1 for T-modules acting on $\mathbb{G}_a^n$, but similar to the case over number fields, proving any result towards Conjecture 1.1 (or its analogues) for groups varieties of dimension larger than 1 would be difficult.

We start by defining Drinfeld modules (for more details, see Section 2). Let p be a prime number, let q be a power of p, and let K be a finite extension of $\mathbb{F}_q(t)$. A Drinfeld module Φ (of generic characteristic) is a ring homomorphism from $\mathbb{F}_q[t]$ to $\mathrm{End}_K(\mathbb{G}_a)$. We fix an algebraic closure $\overline{K}$ of K, and we let K^{sep} be the separable closure of K inside $\overline{K}$.

Since each Drinfeld module is isomorphic (over K^{sep}) to a Drinfeld module for which $\Phi_t := \Phi(t)$ is a monic polynomial, we assume from now on that Φ is indeed in *normal form* i.e., Φ_t is monic. Note that a Drinfeld module Ψ is isomorphic to Φ (over K^{sep}) if there exists $\gamma \in K^{\mathrm{sep}}$ such that $\Psi_t(x) = \gamma^{-1}\Phi_t(\gamma x)$. So, our results about S-integral points are not affected by replacing Φ with an isomorphic Drinfeld module since conjugating by γ only affects the finitely many places where γ is not an S-unit (for a precise definition for S-integrality we refer the reader to Subsection 2.5).

The points of K^{sep} which have finite orbit under the action of Φ are called torsion; we denote by Φ_{tor} the set of all torsion points for Φ.

A Drinfeld module may have complex multiplication (similar to the case of abelian varieties), i.e. there exist endomorphisms g of $\mathbb{G}_a$ defined over K^{sep} such that $g\circ\Phi_t = \Phi_t\circ g$. In our results we assume Φ does not have complex multiplication since we employ a strong equidistribution theorem from [11] for torsion points of Drinfeld modules which uses the assumption that Φ has no complex multiplication.

The places of K split in two categories: *infinite places* and *finite places* depending on whether they lie (or not) over the place v_∞ of $\mathbb{F}_q(t)$, for which $v_\infty(f/g) = \deg(g) - \deg(f)$, for all nonzero $f, g \in \mathbb{F}_q[t]$. We assume Φ has good reduction at all its finite places, i.e., for all finite places v of K, the coefficients of Φ_t are v-integral (recall that we already assumed that Φ_t is monic). Also, for each place v of K we fix an extension of it to K^{sep}. Then we can prove the following result.

Theorem 1.3. *Assume Φ is in normal form and it has good reduction at all finite places of K, and also assume that Φ has no complex multiplication. Let $\beta \in K$ be* *a nontorsion point for Φ, and let S be a finite set of places of K. Then there exist at most finitely many $\gamma \in \Phi_{\mathrm{tor}}$ such that γ is S-integral with respect to β.*

In particular, our Theorem 1.3 applies to the Carlitz module, i.e., the Drinfeld module given by $\Phi_t(x) = tx + x^q$.

The proof of Theorem 1.3 goes through an intermediate result which offers an alternative way of computing the canonical height $\widehat{\mathrm{h}}_{\Phi}(x)$ of any point $x \in K$ (for more details, see Section 2).

Theorem 1.4. *Let Φ be a Drinfeld module as in Theorem 1.3. Let $\beta \in K$, and let $\{\gamma_n\} \subset \Phi_{\mathrm{tor}}$ be an infinite sequence. Then*

$$\sum_{v\in\Omega_K} \lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \sum_{\sigma\in\mathrm{Gal}(K^{\mathrm{sep}}/K)} \log|\beta - \gamma_n^{\sigma}|_v = \widehat{\mathrm{h}}_{\Phi}(\beta).$$

In the above result, and also later in the paper, a sum involving δ^{σ} over all $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ is simply a sum over all the Galois conjugates of δ.

Theorem 1.4 answers a conjecture of Tucker and the author ([13, Conjecture 5.2]). It is worth pointing out that there is no *Bogomolov-type* statement of Ih's Conjecture for Drinfeld modules, i.e., it is possible to find infinitely many points x of small canonical height which are S-integral with respect to a given nontorsion point β (see Example 2.4).

Theorem 1.3 may be interpreted as follows: for a nontorsion point β there exist at most finitely many γ such that $\Phi_Q(\gamma) = 0$ for some nonzero polynomial $Q \in \mathbb{F}_q[t]$, and γ is S-integral with respect to β. In other words, there exist at most finitely many points in the backward orbit of 0 (under Φ) which are S-integral with respect to β. In general, for each $\alpha \in K$ we let $\mathcal{O}_{\Phi}^{-}(\alpha)$ be the *backward orbit* of α under Φ, i.e. the set of all $\beta \in K^{\mathrm{sep}}$ such that there exists some nonzero $Q \in \mathbb{F}_q[t]$ such that $\Phi_Q(\beta) = \alpha$. So, Theorem 1.3 yields that for each torsion point α of Φ, there exist at most finitely many $\gamma \in \mathcal{O}_{\Phi}^{-}(\alpha)$ which are S-integral with respect to the nontorsion point β. It is natural to ask whether Theorem 1.3 holds when the point α is nontorsion.

Theorem 1.5. *Let Φ be a Drinfeld module without complex multiplication, and let $\alpha, \beta \in K$ be nontorsion points. Then there exist at most finitely many $\gamma \in \mathcal{O}_{\Phi}^{-}(\alpha)$ such that γ is S-integral with respect to β.*

It is expected that Theorem 1.5 holds also for Drinfeld modules with complex multiplication; however our proof of Theorem 1.5 relies heavily on the non-CM assumption about Φ since we use a result of Häberli [16] which is a Kummer-type result for non-CM Drinfeld modules (see Theorem 2.2). Theorem 1.5 is an analogue for Drinfeld modules of a result conjectured by Sookdeo in [26]. Motivated by a question of Silverman for S-integral points in the forward orbit of a rational map defined over $\overline{\mathbb{Q}}$, Sookdeo [26, Conjecture 1.2] asks whether for a rational map f defined over $\overline{\mathbb{Q}}$, and for a given nonpreperiodic point α of Φ, there exist at most finitely many points γ in the backward orbit of α (i.e., $f^n(\gamma) = \alpha$ for some nonnegative integer n) such that γ is S-integral with a given nonpreperiodic point β. In [26], Sookdeo proves that his conjecture holds when f is either a powering map, or a Chebyshev polynomial. For an arbitrary rational map f, the conjecture appears to be difficult. Furthermore, Sookdeo presents a general strategy of attacking his conjecture by reducing it to proving that the number of Galois orbits for $f^n(z) - \beta$ is bounded above independently of n.

We observe that Theorems 1.5 and 1.3 combine to yield the following result.

Theorem 1.6. *Let Φ and K be as in Theorem 1.3, and let $\alpha, \beta \in K$ such that β is not a torsion point for Φ. Then there exist at most finitely many $\gamma \in \mathcal{O}_{\Phi}^{-}(\alpha)$ such that γ is S-integral with respect to β.*

It is essential to ask that β is not a torsion point for Φ in Theorem 1.6 since otherwise the result is false. Indeed, if β were equal to 0, say, then one can find infinitely many points $\gamma \in \mathcal{O}_{\Phi}^{-}(\alpha)$ which are S-integral with respect to 0, for any set of places S which contains the infinite places and also all the places where α is not a unit. Indeed, if $\Phi_Q(\gamma) = \alpha$ for some nonzero $Q \in \mathbb{F}_q[t]$ then for each $v \notin S$, we have that if $|\gamma|_v > 1$ then $|\alpha|_v > 1$, while if $|\gamma|_v < 1$ then $|\alpha|_v < 1$ (because Φ has good reduction at v). Hence $|\gamma|_v = 1$ for all $v \notin S$, and thus each $\gamma \in \mathcal{O}_{\Phi}^{-}(\alpha)$ is S-integral with respect to 0.

Instead of considering S-integral points in the backward orbit of a point α under a rational map $f \in \overline{\mathbb{Q}}(z)$, one could consider S-integral points in its forward orbit. This question was settled by Silverman [23] who showed (as an application of Siegel's classical theorem [22] on S-integral points on curves) that if there exist infinitely many S-integers in the forward orbit of α, then $f \circ f$ is totally ramified at infinity, i.e., $f \circ f$ is a polynomial. In our setting for Drinfeld modules, studying S-integral points of the form $\Phi_Q(\alpha)$ (for arbitrary $Q \in \mathbb{F}_q[t]$) relative to a given point β was done by Tucker and the author in [11] and [14] (where a Siegel-type theorem for Drinfeld modules was proven). The result from [14] will be used in the proof of Theorem 1.5. The results of our present paper complete the picture for Drinfeld modules by studying S-integral points in the backward orbit of a given point.

Here is the plan of our paper. In Section 2 we introduce the notation and also state a crucial result (of Häberli [16] and Pink [19]) on Galois orbits of points in the backward orbit of a given nontorsion point α (see Theorem 2.2). This last result allows us to prove Theorem 1.5. In Section 3 we derive some results regarding torsion of Drinfeld modules which have everywhere good reduction away from the places at infinity. Finally, we conclude our proof of Theorems 1.3 and 1.4 in Section 4.

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2. Generalities

2.1. **Drinfeld modules.** Let p be a prime number, and let q be a power of p. We let K be a finite extension of $\mathbb{F}_q(t)$, let $\overline{K}$ be a fixed algebraic closure of K and K^{sep} be the separable closure of K inside K^{sep}.

Let $\Phi : \mathbb{F}_q[t] \longrightarrow \text{End}_K(\mathbb{G}_a)$ be a Drinfeld module of generic characteristic, i.e., Φ is a ring homomorphism with the property that

$$\Phi(t)(x) := \Phi_t(x) = tx + \sum_{i=1}^{r} a_i x^{q^i}$$

where $r \geq 1$ and each $a_i \in K$. We call r the rank of the Drinfeld module Φ.

We note that usually a Drinfeld module is defined on a ring A of functions defined on a projective irreducible curve C defined over $\mathbb{F}_q$, which are regular away from a given point η on C. In our definition, $C = \mathbb{P}^1$ and η is the point at infinity. There is no loss of generality for using our definition since $\mathbb{F}_q[t]$ embeds into each such ring A of functions.

At the expense of replacing K with a finite extension, and replacing Φ with an isomorphic Drinfeld module $\Psi = \gamma^{-1} \circ \Phi \circ \gamma$, for a suitable $\gamma \in K^{\mathrm{sep}}$, we may assume Φ_t is monic. We say in this case that Φ is in normal form.

2.2. **Endomorphisms of Φ.** We say that $f \in K^{\mathrm{sep}}[x]$ is an endomorphism of Φ if $\Phi_t \circ f = f \circ \Phi_t$, or equivalently if $\Phi_a \circ f = f \circ \Phi_a$ for all $a \in \mathbb{F}_q[t]$. The set of all endomorphisms of Φ is $\mathrm{End}_{K^{\mathrm{sep}}}(\Phi)$. Generically, Φ has no endomorphism other than Φ_a for $a \in \mathbb{F}_q[t]$; in this case $\mathrm{End}_{K^{\mathrm{sep}}}(\Phi) \xrightarrow{\sim} \mathbb{F}_q[t]$.

2.3. **Torsion points.** For each nonzero $Q \in \mathbb{F}_q[t]$, the set of all $x \in K^{\mathrm{sep}}$ such that $\Phi_Q(x) := \Phi(Q)(x) = 0$ is defined to be $\Phi[Q]$; each such point x is called a torsion point for Φ. The set of all torsion points for Φ is denoted by

$$\Phi_{\mathrm{tor}} := \bigcup_{Q \in \mathbb{F}_q[t] \setminus \{0\}} \Phi[Q].$$

It is immediate to see that the torsion points are precisely the points preperiodic under the action of the Drinfeld module. One can show that for each nonzero $Q \in \mathbb{F}_q[t]$, we have $\Phi[Q] \xrightarrow{\sim} (\mathbb{F}_q[t]/(Q))^r$. Moreover, since each Φ_Q is a separable polynomial, we obtain that $\Phi_{\mathrm{tor}} \subset K^{\mathrm{sep}}$. Furthermore, since any polynomial $f \in \overline{K}[z]$ satisfying $f \circ \Phi_t = \Phi_t \circ f$ has the property that $f(\Phi_{\mathrm{tor}}) \subset \Phi_{\mathrm{tor}}$ we conclude that $f \in K^{\mathrm{sep}}[z]$; i.e., all endomorphisms of Φ are indeed defined over K^{sep}. For more details on Drinfeld modules, see [15].

2.4. **Places of K.** Let Ω_K be the set of all places of K. The places from Ω_K lie above the places of $\mathbb{F}_q(t)$. We normalize each corresponding absolute value $|\cdot|_v$ so that we have a well-defined product formula for each nonzero $x \in K$:

$$\prod_{v \in M_K} |x|_v = 1.$$

Furthermore, we may assume $\log |t|_\infty = 1$, where $\infty := v_\infty$ is the place of $\mathbb{F}_q(t)$ corresponding to the degree of rational functions, i.e. (in exponential form), $v_\infty(f/g) = \deg(g) - \deg(f)$ for any nonzero $f, g \in \mathbb{F}_q[t]$. We let S_∞ be the set of (infinite) places in Ω_K which lie above v_∞. If $v \in \Omega_K \setminus S_\infty$, then we say that v is a finite place for Φ. For each $v \in \Omega_K$, we fix a completion K_v of K with respect to v, and also we fix an embedding of K^{sep} into $\overline{K_v}$. Finally, we let $\mathbb{C}_v$ be the completion of $\overline{K_v}$ with respect to $|\cdot|_v$.

2.5. **S-integrality.** For a set of places $S \subset M_K$ and $\alpha \in K$, we say that $\beta \in K^{\mathrm{sep}}$ is S-integral with respect to α if for every place $v \notin S$, and for every morphism $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ the following are true:

- if $|\alpha|_v \le 1$, then $|\alpha - \beta^\sigma|_v \ge 1$.
- if $|\alpha|_v > 1$, then $|\beta^\sigma|_v \le 1$.

For the results of our paper we could extend the definition of S-integrality to $\overline{K}$ (instead of K^{sep}), but this will not be necessary since we will mainly be concerned with preimages of a point in K under some Φ_Q and this preimage lives in K^{sep} because Φ_Q is a separable polynomial. For more details about the definition of S-integrality, we refer the reader to [3].

We note that if Ψ is an arbitrary Drinfeld module and Φ is a normalized Drinfeld module isomorphic to Ψ through the conjugation by $\gamma \in K^{\mathrm{sep}}$ (i.e., $\Phi_t(x) = \gamma^{-1}\Psi_t(\gamma x)$), then all torsion points of Φ are of the form $\gamma^{-1}\cdot z$, where $z \in \Psi_{\mathrm{tor}}$. So, if we let K be a finite extension of $\mathbb{F}_q(t)$ containing γ, and let S be a finite set of places of K containing all places where γ is not a unit, then $\alpha \in \Psi_{\mathrm{tor}}$ is S-integral with respect to $\beta \in K$ if and only if $\gamma^{-1}\alpha \in \Phi_{\mathrm{tor}}$ is S-integral with respect to $\gamma^{-1}\beta$. Therefore, our hypothesis from Theorems 1.3 and 1.5 (since a similar statement holds for the backward orbit of a nontorsion point) that Φ is in normal form is not essential. However, we prefer to work with Drinfeld modules in normal form since this makes easier to define the notion of good reduction for Φ, and also simplifies some of the technical points in our proof.

2.6. **Good reduction for Φ.** We say that the Drinfeld module Φ defined over K has good reduction at a place v of K, if each coefficient of Φ_t is integral at v, and moreover, the leading coefficient of Φ_t is a v-adic unit. It is immediate to see that if Φ has good reduction at v, then for each nonzero $Q \in \mathbb{F}_q[t]$ we have that Φ_Q has all its coefficients v-adic integers, while its leading coefficient is a v-adic unit. Clearly, a Drinfeld module cannot have good reduction at a place in S_∞. Since we assumed that Φ is in normal form, the condition of having good reduction at v reduces to the fact that Φ_t has all its coefficients v-adic integers.

2.7. **The action of the Galois group.** Since Φ is defined over K, we obtain that $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts continuously (where $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is endowed with the Krull topology) on $\Phi[Q^n]$ for each monic irreducible polynomial Q, and for each positive integer n. Because $\Phi[Q^n]$ is naturally ismorphic to $(\mathbb{F}_q[t]/Q^n)^r$, then taking inverse limits we obtain that $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts continuously on the associated Tate module $\mathbb{T}_Q(\Phi)$ for Φ (corresponding to irreducible, monic $Q \in \mathbb{F}_q[t]$); the Tate module is isomorphic to $\mathbb{F}_q[t]_{(Q)}^r$ (where $\mathbb{F}_q[t]_{(Q)}$ is the (Q)-adic completion of $\mathbb{F}_q[t]$ at the prime ideal (Q)). We let $\mathbb{T} := \prod_Q \mathbb{T}_Q$ (where the product is over all the monic irreducible polynomials $Q \in \mathbb{F}_q[t]$) be the profinite completion of $\mathbb{F}_q[t]$. Thus we obtain a continuous representation

$$\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathrm{GL}_r(\mathbb{T}), \tag{2.1}$$

which is analogous to the classical construction of the Galois representation on the Tate module corresponding to an elliptic curve (for the latter topic, see Silverman's classical book [25]).

Pink and Rütsche [20] proved that the image of ρ is open, assuming that Φ has no complex multiplication; their result is a Drinfeld module analogue of the classical Serre Openess Conjecture for (non-CM) abelian varieties. Initially, Breuer and Pink [7] proved that the image of ρ is open assuming K is a transcendental extension of $\mathbb{F}_q(t)$, which can be viewed as the function field version of the result from [20]. In particular, Pink-Rütsche result yields that for any torsion point γ of order Q, where $Q(t) \in \mathbb{F}_q[t]$ is a polynomial of degree d, there exists a positive constant c_Φ such that

$$\frac{[K(\gamma) : K]}{\#\,\mathrm{GL}_r(\mathbb{F}_q[t]/(Q))} \geq c_\Phi .$$

Hence, $[K(\gamma) : K] \gg q^{rd}$, as $d \to \infty$. (As a matter of notation, we write $f(x) \gg g(x)$ whenever $|g(x)/f(x)|$ is bounded above as $x \to \infty$.)

Now, let $x \in K$ be a nontorsion point for Φ. For each monic irreducible $Q \in \mathbb{F}_q[t]$, for each $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, and for each sequence $\{x_i\}_{i\in\mathbb{N}} \subset K^{\mathrm{sep}}$ such that $\Phi_Q(x_{i+1}) = x_i$ for each $i \in \mathbb{N}$ while $\Phi_Q(x_1) = x$, we consider the map

$$\sigma \mapsto \{\sigma(x_i) - x_i\}_{i\in\mathbb{N}}.$$

This yields another continuous representations

$$\Psi_Q : \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathbb{T}_Q$$

and more generally

$$\Psi : \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathbb{T}.$$

Furthermore, we have the following Galois action on the entire backward orbit of x under the action of Φ

$$\tilde{\Psi} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathbb{T} \rtimes \mathrm{GL}_r(\mathbb{T}).$$

Häberli [16] proved that the image of $\tilde{\Psi}$ is open, again assuming that Φ has no complex multiplication. The assumption regarding the endomorphism ring for Φ is crucial; however Pink [19] proved an appropriately modified statement when Φ has complex multiplication. In particular, Häberli's result [16] yields that there exists a number $d := d(x) \in \mathbb{N}$ (bounded above by the index of the image of $\tilde{\Psi}$ in $\mathbb{T} \rtimes \mathrm{GL}_r(\mathbb{T})$) such that for each nonzero $Q \in \mathbb{F}_q[t]$, there are at most d distinct Galois orbits containing all the preimages of x under Φ_Q. Hence, we have the following theorem.

Theorem 2.2. *Let $\Phi : \mathbb{F}_q[t] \longrightarrow \mathrm{End}_K(\mathbb{G}_a)$ be a Drinfeld module such that $\mathrm{End}_{K^{\mathrm{sep}}}(\Phi) \xrightarrow{\sim} \mathbb{F}_q[t]$. Then for each $\alpha \in K$ which is not torsion, there exists a number $d(\alpha)$ such that for each nonzero $Q \in \mathbb{F}_q[t]$ there exist at most $d(\alpha)$ distinct Galois orbits of points $y \in K^{\mathrm{sep}}$ satisfying $\Phi_Q(y) = x$.*

Theorem 2.2 allows us to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. Our proof follows the strategy from [26, Theorems 2.5 and 2.6]. For each nonzero $Q \in \mathbb{F}_q[t]$ we let $d_Q(\alpha)$ be the number of Galois orbits contained in $\Phi_Q^{-1}(\alpha)$. Then $d_{Q_1} \le d_{Q_2}$ whenever $Q_1 \mid Q_2$, and $d_{Q_1Q_2} = d_{Q_1}d_{Q_2}$ whenever $\gcd(Q_1, Q_2) = 1$. Indeed, since $\gcd(Q_1, Q_2) = 1$ there exist $R_1, R_2 \in \mathbb{F}_q[t]$ such that $R_1Q_1 + R_2Q_2 = 1$. Then for each pair $(\delta_1, \delta_2) \in \Phi_{Q_1}^{-1}(\alpha) \times \Phi_{Q_2}^{-1}(\alpha)$, we let $\delta_{1,2} := \Phi_{R_2}(\delta_1) + \Phi_{R_1}(\delta_2) \in \Phi_{Q_1Q_2}^{-1}(\alpha)$. Furthermore, if (δ_1', δ_2') is another such pair such that δ_i' is not Galois conjugate with δ_i for some $i \in \{1, 2\}$, then $\delta_{1,2}'$ is not Galois conjugate with $\delta_{1,2}$. Indeed, if there is some $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ such that $\delta_{1,2}' = \delta_{1,2}^\sigma$, then also $\Phi_{Q_2}(\delta_{1,2}') = \Phi_{Q_2}(\delta_{1,2}^\sigma)$ and thus

$$\Phi_{Q_2R_2}(\delta_1') + \Phi_{R_1}(\alpha) = \Phi_{Q_2R_2}(\delta_1^\sigma) + \Phi_{R_1}(\alpha),$$

which yields that $\delta_1' - \delta_1^\sigma \in \Phi[Q_2R_2]$. On the other hand, $\delta_1' - \delta_1^\sigma \in \Phi[Q_1]$ (since both are in $\Phi_{Q_1}^{-1}(\alpha)$). Because $\gcd(Q_1, Q_2R_2) = 1$ we conclude that $\delta_1' = \delta_1^\sigma$; similarly we get that $\delta_2' = \delta_2^\sigma$ which yields that the pairs of points (δ_1', δ_2') and (δ_1, δ_2) are Galois conjugate, contrary to our assumption.

So indeed, $d_{Q_1Q_2} = d_{Q_1}d_{Q_2}$ if $\gcd(Q_1, Q_2) = 1$. By Theorem 2.2, d_Q is bounded above independently of Q. So, using that the function $Q \mapsto d_Q$ is multiplicative on the set of all (monic) polynomials, we conclude that for all but finitely many irreducible polynomials P, and for each positive integer n we have $d_{P^n} = 1$. Furthermore, for each polynomial P there exists a positive integer $n := n(P)$ such that

$d_{P^m} = d_{P^n}$ for all $m \geq n$. Hence, using that $Q \mapsto d_Q$ is multiplicative, and also using that $d_Q \leq d_P$ if $Q \mid P$, we obtain that there exists a nonzero $P := P_\alpha(t) \in \mathbb{F}_q[t]$ such that for all $Q \in \mathbb{F}_q[t]$ we have $d_Q(\alpha) = d_R(\alpha)$, where $R = \gcd(P, Q)$. In particular, with the above notation, if $\delta \in \Phi_R^{-1}(\alpha)$ then all points in $\Phi_{Q/R}^{-1}(\delta)$ are Galois conjugates.

At the expense of replacing S by a larger set we may assume it contains all places where either β or α is not a unit (note that neither α nor β are equal to 0 since they are nontorsion). Thus for each $v \notin S$ we have that each point in $\mathcal{O}_\Phi^-(\alpha)$ is also a v-adic unit. Therefore a point $\gamma \in \mathcal{O}_\Phi^-(\alpha)$ is S-integral with respect to β if and only if for all $v \notin S$ we have $|\gamma - \beta|_v = 1$.

Now, assume $\gamma \in \Phi_Q^{-1}(\alpha)$ is S-integral with respect to β for some $Q \in \mathbb{F}_q[t]$. Let $R = \gcd(P, Q)$ (where $P := P_\alpha(t) \in \mathbb{F}_q[t]$ is defined as above for α), and let $\delta \in \Phi_R^{-1}(\alpha)$ such that $\gamma \in \Phi_{Q/R}^{-1}(\delta)$. Because γ is S-integral with respect to β, then for each conjugate γ^σ, and for each place $v \notin S$, we have $|\gamma^\sigma - \beta|_v = 1$. Hence taking the product over all conjugates γ^σ satisfying $\Phi_{Q/R}(\gamma^\sigma) = \delta$ we obtain

$$|\Phi_{Q/R}(\beta) - \delta|_v = |\Phi_{Q/R}(\beta - \gamma)|_v = \prod_\sigma |\beta - \gamma^\sigma|_v = 1.$$

In the above computation we used that Φ_t is monic and therefore the leading coefficient of $\Phi_{Q/R}$ is a v-adic unit since it is in $\overline{\mathbb{F}_p}$. Also, we used that

$$|\Phi_{Q/R}(\beta - \gamma)|_v = \prod_{z \in \Phi[Q/R]} |\beta - \gamma - z|_v = \prod_\sigma |\beta - \gamma^\sigma|_v,$$

since all points in $\Phi_{Q/R}^{-1}(\delta)$ are Galois conjugates, and therefore for each $z \in \Phi[Q/R]$ we have that $\gamma + z = \gamma^\sigma$ for some $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

So, letting $S(\delta)$ be the places of $K(\delta)$ which lie above the places from S, we conclude that $\Phi_{Q/R}(\beta)$ is $S(\delta)$-integral with respect to δ (where the ground field is now $K(\delta)$). On the other hand, since $\beta \notin \Phi_{\mathrm{tor}}$, [14, Theorem 2.5] yields that there exist at most finitely $Q/R \in \mathbb{F}_q[t]$ such that $\Phi_{Q/R}(\beta)$ is $S(\delta)$-integral with respect to δ. Finally, noting that there are only finitely many

$$\delta \in \bigcup_{R|P} \Phi_R^{-1}(\alpha),$$

we conclude our proof. ☐

2.8. **Canonical height associated to Φ.** For a point x in K^{sep} its usual Weil height is defined as

$$h(x) = \sum_{v \in M_K} \frac{1}{[K(x) : K]} \sum_{\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \log^+ |x^\sigma|_v .$$

where by $\log^+ z$ we always denote $\log \max\{z, 1\}$ (for any real number z).

The *global canonical height* $\widehat{\mathrm{h}}_\Phi(x)$ associated to the Drinfeld module Φ was first introduced by Denis [9] (Denis defined the global canonical heights for general T-modules which are higher dimensional analogue of Drinfeld modules). For each $x \in K^{\mathrm{sep}}$, the global canonical height is defined as

$$\widehat{\mathrm{h}}_\Phi(x) = \lim_{n \to \infty} \frac{h\left(\Phi_{t^n}(x)\right)}{q^{rn}},$$

where h is the usual (logarithmic) Weil height on K^{sep}. Denis [9] showed that $\widehat{\mathrm{h}}$ differs from the usual Weil height h by a bounded amount, and also showed that $x \in \Phi_{\mathrm{tor}}$ if and only if $\widehat{\mathrm{h}}_\Phi(x) = 0$.

Following Poonen [21] and Wang [28], for each $x \in \mathbb{C}_v$, the *local canonical height* of x is defined as follows

$$\widehat{\mathrm{h}}_{\Phi,v}(x) := \lim_{n\to\infty} \frac{\log^+ |\Phi_{t^n}(x)|_v}{q^{rn}}.$$

It is immediate that $\widehat{\mathrm{h}}_{\Phi,v}(\Phi_{t^i}(x)) = q^{ir}\,\widehat{\mathrm{h}}_{\Phi,v}(x)$ and thus $\widehat{\mathrm{h}}_{\Phi,v}(x) = 0$ whenever $x \in \Phi_{\mathrm{tor}}$.

Now, if $f(x) = \sum_{i=0}^d a_i x^i$ is any polynomial defined over K, then $|f(x)|_v = |a_d x^d|_v > |x|_v$ when $|x|_v > M_v$, where

$$M_v = M_v(f) := \max\left\{ \left(\frac{1}{|a_d|}\right)^{\frac{1}{d-1}}, \max\left\{ \left|\frac{a_i}{a_d}\right|^{\frac{1}{d-i}} \right\}_{0\le i<d} \right\}. \tag{2.3}$$

Moreover, for a Drinfeld module Φ, if $|x|_v > M_v(\Phi_t)$ then $\widehat{\mathrm{h}}_{\Phi,v}(x) = \log|x|_v + \frac{\log|a_d|_v}{d-1} > 0$. In the special case that Φ has good reduction at v, then $M_v(\Phi_t) = 1$ and so, if $|x|_v > 1$ then $\widehat{\mathrm{h}}_{\Phi,v}(x) = \log|x|_v$, while if $|x|_v \le 1$ then $\widehat{\mathrm{h}}_{\Phi,v}(x) = 0$.

We define the *v-adic filled Julia set* $\mathcal{J}_v$ be the set of all $x \in \mathbb{C}_v$ such that $\widehat{\mathrm{h}}_{\Phi,v}(x) = 0$. So, we know that if $x \in \mathcal{J}_v$, then $|x|_v \le M_v$. In particular, if v is a place of good reduction for Φ, then $\mathcal{J}_v$ is the unit disk.

As shown in [21] and [28], the global canonical height decomposes into a sum of the corresponding local canonical heights, as follows

$$\widehat{\mathrm{h}}_\Phi(x) = \sum_{v\in M_K} \frac{1}{[K(x):K]} \sum_{\sigma\in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \widehat{\mathrm{h}}_{\Phi,v}(x^\sigma).$$

We note that the theory of canonical height associated to a Drinfeld module is a special case of the canonical heights associated to morphisms on algebraic varieties developed by Call and Silverman (see [8] for details). The definition for the canonical height functions given above seems to depend on the particular choice of the map Φ_t. On the other hand, one can define the canonical heights $\widehat{\mathrm{h}}_\Phi$ as in [9] by letting

$$\widehat{\mathrm{h}}_\Phi(x) = \lim_{\deg(R)\to\infty} \frac{h(\Phi_R(x))}{q^{r\deg(R)}},$$

and similar formula for canonical local heights $\widehat{\mathrm{h}}_{\Phi,v}(x)$ where R runs through all non-constant polynomials in $\mathbb{F}_q[t]$. In [21] and [28] it is proven that both definitions yield the same height function.

Finally, we observe that Ingram [18] defined the local canonical height in a slightly different way, i.e., Ingram's local canonical height $\lambda_v(x)$ for a point $x \in \mathbb{C}_v$ is defined as follows:

$$\lambda_v(x) := \widehat{\mathrm{h}}_{\Phi,v}(x) - \log|x|_v + c_v,$$

where $c_v := -\log|a_r|_v/(q^r - 1)$, where a_r is the leading coefficient of Φ_t. Using the product formula, we conclude that if $x \in K^{\mathrm{sep}}$ then

$$\widehat{\mathrm{h}}_\Phi(x) = \sum_{v\in \Omega_K} \frac{1}{[K(x):K]} \sum_{\sigma\in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \lambda_v(x^\sigma).$$

Furthermore, for normalized Drinfeld modules, one has

$$\lambda_v(x) = \widehat{\mathrm{h}}_{\Phi,v}(x) - \log|x|_v.$$

The advantage in Ingram's definition [18] is that $\lambda_v(x) := g_{\mu_v}(x,0)$, where $g_{\mu_v}(x,y)$ is the Arakelov-Green's function on the Berkovich space $\mathbb{P}^1 \times \mathbb{P}^1$ for the invariant measure μ_v associated to Φ_t (see [1, page 300]).

2.9. **Points of small canonical height.** Next we give a construction showing that there is no Bogomolov-type statement of Ih's Conjecture for Drinfeld modules, i.e., there exist infinitely many points of canonical height arbitrarily small which are S-integral with respect to a given nontorsion point β.

Example 2.4. *Indeed, let Φ be the Carlitz module corresponding to $\Phi_t(x) = tx+x^p$, where p is an odd prime number, and let $\beta = 1$. Then β is nontorsion for Φ because $\deg_t(\Phi_{t^n}(1)) = p^{n-1}$ for all $n \geq 1$. For each positive integer n, consider $x_n \in \mathbb{F}_p(t)^{\mathrm{sep}}$ which is a root of the equation*

$$\Phi_{t^n}(z) \cdot (z-1) = 1.$$

We let $S = \{v_\infty\} \subset \Omega_{\mathbb{F}_p(t)}$. Then it is immediate to see that x_n must be v-integral for each $v \notin S$ (otherwise $|\Phi_{t^n}(x_n)|_v > 1$ and also $|x_n - 1|_v > 1$, which is a contradiction). Furthermore, if $|x_n - 1|_v < 1$ for $v \notin S$, then $|\Phi_{t^n}(x_n)|_v > 1$, which is again a contradiction since it would imply that $|x_n|_v > 1$. Similarly, for each conjugate x_n^σ we have $|x_n^\sigma - 1|_v = 1$ for each $v \notin S$. On the other hand, x_n has height tending to 0. Indeed, as shown by Denis [9] there exists a positive constant C such that $|h(z) - \widehat{\mathrm{h}}_\Phi(z)| \leq C$ for all algebraic points z. So,

$$p^n\, \widehat{\mathrm{h}}_\Phi(x_n) = \widehat{\mathrm{h}}_\Phi(\Phi_{t^n}(x_n)) = \widehat{\mathrm{h}}_\Phi\left(\frac{1}{x_n-1}\right) \leq h\left(\frac{1}{x_n-1}\right) + C = h(x_n) + C \leq \widehat{\mathrm{h}}_\Phi(x_n) + 2C.$$

Hence, $\widehat{\mathrm{h}}_\Phi(x_n) \to 0$ as $n \to \infty$, and x_n is S-integral with respect to $\beta = 1$.

3. Preliminary results for torsion points

In this Section we assume that Φ is in normal form and that it has good reduction at all finite places of K.

Lemma 3.1. *Let s be a real number in $(0,1)$. If $v \in \Omega_K \setminus S_\infty$, then there exist at most finitely many $x \in \Phi_{\mathrm{tor}}$ such that $|x|_v < s$.*

Proof. Let $P \in \mathbb{F}_q[t]$ be the unique irreducible monic polynomial such that $|P|_v < 1$, i.e. the place v lies above the place coresponding to the polynomial P in $\mathbb{F}_q(t)$.

We first observe that if $|x|_v < 1$, then for each $a \in \mathbb{F}_q(t)$ we have $|\Phi_a(x)|_v \leq |x|_v < 1$ because each coefficient of Φ_a is integral at v.

Secondly, we claim that if $x \in \Phi_{\mathrm{tor}}$ such that $|x|_v < 1$, then there exists $n \in \mathbb{N}$ such that $\Phi_{P(t)^n}(x) = 0$. Indeed, using our first observation it suffices to prove that if $|x|_v < 1$ and $\Phi_{Q(t)}(x) = 0$, where $Q \in \mathbb{F}_q[t]$ is relatively prime with $P(t)$, then $x = 0$. Because $|x|_v < 1$ and each coefficient of $\Phi_{Q(t)}$ is integral at v while $|Q(t)|_v = 1$, we conclude that $|\Phi_{Q(t)}(x)|_v = |Q(t)x|_v = |x|_v$. Hence, indeed $x = 0$ as claimed.

So, if $0 \neq x \in \Phi_{\mathrm{tor}}$ satisfies $|x|_v < s < 1$ then there exists some $n \in \mathbb{N}$ such that $\Phi_{P(t)^n}(x) = 0$. Assume n is the smallest such positive integer, and let $y = \Phi_{P(t)^{n-1}}(x)$. Then $0 \neq y$ and $\Phi_{P(t)}(y) = 0$. Let $s_0 := |P(t)|_v^{1/(q-1)} < 1$; so, if

$|y|_v < s_0$, then $|\Phi_{P(t)}(y)|_v = |P(t)y|_v \neq 0$ which yields a contradiction. Therefore, $|y|_v \geq s_0$. On the other hand, if $0 < |z|_v < 1$ then

$$|\Phi_{P(t)}(z)|_v \leq \max\{|P(t)z|_v, |z|_v^q\} < |z|_v \tag{3.2}$$

since each coefficient of $\Phi_{P(t)}$ is integral at v. Because $|P(t)|_v < 1$ and also $|z|_v < 1$, then $|P(t)z|_v < s_0$. Thus, if $n > n_0 := 1+\log_q(\log_s(s_0))$ inequality (3.2) yields that $|\Phi_{P(t)^{n-1}}(x)|_v = |y|_v < s_0$. This yields a contradiction with the fact that $y \neq 0$ but $\Phi_{P(t)}(y) = 0$. So, in conclusion, if $x \in \Phi_{\rm tor}$ such that $|x|_v < s$ then $\Phi_{P(t)^{n_0}}(x) = 0$, where n_0 is a positive integer depending only on s and on v (note that s_0 depends only on v). Thus there exist at most finitely many torsion points x satisfying the inequality $|x|_v < s$. □

Lemma 3.1 is a special case of [12, Theorem 2.10]; however, our result is more precise since we assume each coefficient of Φ_t is integral at v. In particular, the following result is an immediate corollary.

Corollary 3.3. *Let $v \in \Omega_K \setminus S_\infty$, let $z \in \mathbb{C}_v$, and let s be a real number such that $0 < s < 1$. Then there exist at most finitely many $x \in \Phi_{\rm tor}$ such that $|z - x|_v < s$.*

4. Ih's Conjecture for Drinfeld modules

Assume Φ is in normal form, that it has everywhere good reduction away from S_∞, and also that Φ has no complex multiplication. Also, using the notation from (2.3), for each place $v \notin S_\infty$, if $|x|_v > 1$ then $\widehat{\rm h}_{\Phi,v}(x) = \log|x|_v > 0$ (since the leading coefficient of Φ_t is a v-adic unit). Let $\beta \in K$ be a nontorsion point. We prove Theorem 1.3 as a consequence of Theorem 1.4.

Proof of Theorem 1.3. First, we enlarge S so that it contains S_∞; clearly enlarging S can only increase the number of torsion points which are S-integral with respect to β. Then for all $v \notin S$ we know that for a torsion point γ we have $|\gamma|_v \leq 1$ since Φ has good reduction at v. Hence for each $v \notin S$, if $\gamma \in \Phi_{\rm tor}$ is S-integral with respect to β, then

$$|\beta - \gamma^\sigma|_v = \max\{|\beta|_v, 1\},$$

for each $\sigma \in {\rm Gal}(K^{\rm sep}/K)$.

Assume there exist infinitely many torsion points γ_n which are S-integral with respect to β. By Theorem 1.4, we know that

$$\widehat{\rm h}_\Phi(\beta) = \sum_{v\in\Omega_K} \lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \sum_{\sigma\in{\rm Gal}(K^{\rm sep}/K)} \log|\beta - \gamma_n^\sigma|_v.$$

On the other hand, since γ_n is S-integral with respect to β, we know that

$$\log|\beta - \gamma_n^\sigma|_v = \log^+|\beta|_v.$$

Hence the above outer sum consists of only finitely many nonzero terms and therefore we may reverse the order of the summation with the limit, and conclude that

$$\widehat{\rm h}_\Phi(\beta) = \lim_{n\to\infty} \sum_{v\in\Omega_K} \frac{1}{[K(\gamma_n):K]} \sum_{\sigma\in{\rm Gal}(K^{\rm sep}/K)} \log|\beta - \gamma_n^\sigma|_v = 0,$$

by the product formula applied to each $\beta - \gamma_n$ (note that this element is nonzero since $\beta \notin \Phi_{\rm tor}$). Thus we obtain that $\widehat{\rm h}_\Phi(\beta) = 0$, which contradicts the fact that $\beta \notin \Phi_{\rm tor}$. So, indeed there are at most finitely many torsion points which are S-integral with respect to β. □

We are left to proving Theorem 1.4. This will follow from the following result.

Theorem 4.1. *Let $\beta \in K$, let $v \in M_K$, and let $\{\gamma_n\} \subset K^{\mathrm{sep}}$ be an infinite sequence of torsion points for the Drinfeld module Φ. Then*

$$\widehat{\mathrm{h}}_{\Phi,v}(\beta) = \lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)} |\beta - \gamma_n^{\sigma}|_v \, .$$

We prove Theorem 4.1 by analyzing two cases depending on whether $v \in S_\infty$ or not.

Proposition 4.2. *Theorem 4.1 holds if $v \notin S_\infty$.*

Proof. Firstly, since $v \notin S_\infty$, then v is a place of good reduction for Φ and then the v-adic filled Julia set $\mathcal{J}_v$ is the unit disk. In particular, if $\gamma \in \Phi_{\mathrm{tor}}$, then $|\gamma|_v \leq 1$.

There are two cases: either $|\beta|_v \leq 1$ or $|\beta|_v > 1$.

Case 1. Assume $|\beta|_v \leq 1$.

Then $\widehat{\mathrm{h}}_{\Phi,v}(\beta) = 0$, since in particular $\beta \in \mathcal{J}_v$. Also, for each $\gamma \in \Phi_{\mathrm{tor}}$ we have $|\gamma - \beta|_v \leq 1$. If for each torsion point γ we have that $|\beta - \gamma|_v = 1$, then clearly

$$\lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)} |\beta - \gamma_n^{\sigma}|_v = 0 = \widehat{\mathrm{h}}_{\Phi,v}(\beta).$$

Now, if there exists some torsion point γ such that $|\beta - \gamma|_v < 1$, let s be any real number satisfying

$$|\beta - \gamma|_v < s < 1.$$

By Lemma 3.3 we conclude that there exist finitely many torsion points γ' such that $|\beta - \gamma'|_v < s$. In particular, for all n sufficiently large, and for all $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, we have $|\beta - \gamma_n^{\sigma}|_v \geq s$ and thus

$$\frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma} \log |\beta - \gamma_n^{\sigma}|_v \geq \log(s)$$

and so, letting $s \to 1$ we obtain

$$\lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma} \log |\beta - \gamma_n^{\sigma}|_v \geq 0.$$

On the other hand, as explained above, $|\beta - \gamma_n^{\sigma}|_v \leq 1$ for all $\gamma_n \in \Phi_{\mathrm{tor}}$ and all $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$; so

$$\lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \log |\beta - \gamma_n^{\sigma}|_v \leq 0$$

and therefore, in conclusion

$$\lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \log |\beta - \gamma_n^{\sigma}|_v = 0 = \widehat{\mathrm{h}}_{\Phi,v}(\beta).$$

Case 2. Assume $|\beta|_v > 1$.

In this case, $|\gamma - \beta|_v = |\beta|_v > 1$ for each $\gamma \in \Phi_{\mathrm{tor}}$. So, if $|\beta|_v > 1$, then the above limit equals $\log |\beta|_v = \widehat{\mathrm{h}}_{\Phi,v}(\beta)$. ∎

Remark 4.3. The method of proof for Proposition 4.2 reveals the necessity for our hypothesis from Theorem 1.3 that Φ has good reduction at all finite places. Indeed, this allows us to conclude that $\mathcal{J}_v$ is the unit disk in $\mathbb{C}_v$ and moreover that for each $z \in \mathbb{C}_v$ we have $\widehat{\mathrm{h}}_{\Phi,v}(z) = \log^+ |z|_v$. It is likely that Proposition 4.2 holds without the above hypothesis but one would need a different approach.

Assume now that $v \in S_\infty$. We use the results from our paper [11]. As shown by Theorem 4.6.9 of [15], there exists an $\mathbb{F}_q[t]$-lattice $\Lambda_v \subset \mathbb{C}_v$ associated to the generic characteristic Drinfeld module Φ; $\mathbb{C}_v$ is the completion of $\overline{K_v}$ which is a complete, algebraically closed field. Let $\mathrm{E_v}$ be the exponential function defined in [15, Section 4.2] which gives a continuous (in the v-adic topology) isomorphism

$$\mathrm{E_v} : \mathbb{C}_v/\Lambda \to \mathbb{C}_v.$$

The torsion submodule of Φ in $\mathbb{C}_v$ is isomorphic naturally through ${\mathrm{E_v}}^{-1}$ to

$$\left(\mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} \Lambda_v\right)/\Lambda_v.$$

We show below that the filled Julia set $\mathcal{J}_v$ for the Drinfeld module Φ is the closure of the set of all torsion points. Indeed, we show first that the v-adic filled Julia set is compact. We note that the derivative of $\Phi_t(x)$ is constant equal to t. Hence, with respect to the absolute v-adic norm on $\mathbb{C}_v$,

$$|\Phi_t'(x)|_v = |t|_v > 1. \tag{4.4}$$

Therefore, Φ_t is uniformly expansive on $\mathcal{J}_v$ (according to Définition 3 in [5]) and so, by [5, Proposition 16], $\mathcal{J}_v$ is compact. We note that the results in [5] are stated in the case of an algebraically closed complete field of characteristic 0; however, Bézivin's argument from [5] goes through verbatim for an algebraically closed complete field in any characteristic (see also [4] for a treatment of the rational dynamics for algebraically closed non-archimedean fields of arbitrary characteristic).

Moreover, in the above case, because Φ_t is uniformly expansive on $\mathcal{J}_v$, then $\mathcal{J}_v$ equals its boundary, which is the (non-filled) Julia set. Even more it is true in this case. As shown in [17, Theorem 3.1], the Julia set is contained in the topological closure of the periodic points for P (which are torsion points for Φ). On the other hand, by its definition, the Julia set always contains the topological closure of the repelling periodic points for Φ_t. Because of (4.4), all periodic points for Φ_t are repelling. Hence the Julia set and the v_∞-adic filled Julia set are both equal to the topological closure of the torsion points of Φ, as claimed above.

We also note that the completion of $\mathbb{F}_q(t)$ with respect to the restriction of v on $\mathbb{F}_q(t)$ is $\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$. Then the restriction of $\mathrm{E_v}$ on $\left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right) \otimes_{\mathbb{F}_q[t]} \Lambda_v\right)/\Lambda_v$ gives an isomorphism between $\left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right) \otimes_{\mathbb{F}_q[t]} \Lambda_v\right)/\Lambda_v$ and $\mathcal{J}_v$.

Let r be the rank of Λ_v which is the same as the rank of the Drinfeld module Φ. Then $\left(\mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} \Lambda_v\right)/\Lambda_v \xrightarrow{\sim} (\mathbb{F}_q(t)/\mathbb{F}_q[t])^r$. Let $\omega_1, \dots, \omega_r$ be a fixed $\mathbb{F}_q[t]$-basis of Λ_v. Furthermore, we may assume the ω_i's form a basis of *successive minima* as defined by Taguchi [27], i.e., for each $P_1, \dots, P_r \in \mathbb{F}_q[t]$ we have

$$|P_1(t)\omega_1 + \cdots + P_r(t)\omega_r|_v = \max_{i=1}^{r} |P_i(t)\omega_i|_v. \tag{4.5}$$

Then the function E_v is defined as

$$E_v(u) := u \cdot \prod_{\omega \in \Lambda_v \setminus \{0\}} \left(1 - \frac{u}{\omega}\right).$$

So, if $|u|_v < |\omega_i|_v$ for each $i = 1, \dots, r$, then $|u|_v < |\omega|_v$ for each $\omega \in \Lambda_v \setminus \{0\}$ (see also (4.5)). Hence, there exists a sufficiently small positive real number r_v^0 such that (in a sufficiently small ball of radius r_v^0)

$$|E_v(u)|_v = |u|_v \text{ for each } u \in B(0, r_v^0). \tag{4.6}$$

Using [15, Proposition 4.6.3], $\left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right) \otimes_{\mathbb{F}_q[t]} \Lambda_v\right)/\Lambda_v$ is isomorphic to $\left(\mathbb{F}_q((\frac{1}{t}))/\mathbb{F}_q[t]\right)^r$. Then we have the isomorphism

$$\mathbb{E} : \left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t]\right)^r \to \mathcal{J}_v \text{ given by}$$

$$\mathbb{E}(\gamma_1, \dots, \gamma_r) := \mathrm{E_v}(\gamma_1\omega_1 + \cdots + \gamma_r\omega_r), \text{ for each } \gamma_1, \dots, \gamma_r \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t].$$

We construct the following group isomorphism

$$\tau : \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t] \to \frac{1}{t} \cdot \mathbb{F}_q\left[\left[\frac{1}{t}\right]\right], \text{ given by}$$

$$\tau\left(\sum_{i \geq -n} \alpha_i \left(\frac{1}{t}\right)^i\right) = \sum_{i \geq 1} \alpha_i \left(\frac{1}{t}\right)^i, \tag{4.7}$$

for every natural number n and for every $\sum_{i \geq -n} \alpha_i \left(\frac{1}{t}\right)^i \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ (obviously, τ vanishes on $\mathbb{F}_q[t]$). The group $\frac{1}{t} \cdot \mathbb{F}_q[[\frac{1}{t}]]$ is a topological group with respect to the restriction of v on $\frac{1}{t} \cdot \mathbb{F}_q[[\frac{1}{t}]]$. Hence, the isomorphism τ^{-1} induces a topological group structure on $\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t]$. Therefore, τ becomes a continuous isomorphism of topological groups. We endow $\left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t]\right)^r$ with the corresponding product topology. The isomorphism τ extends diagonally to another continuous isomorphism, which we also call

$$\tau : \left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)/\mathbb{F}_q[t]\right)^r \to \left(\frac{1}{t} \cdot \mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right)^r =: G.$$

Moreover, using that $\mathrm{E_v}$ is a continuous morphism, we conclude

$$\mathbb{E}\tau^{-1} : G \to \mathcal{J}_v \text{ is a continuous isomorphism.} \tag{4.8}$$

Since $\omega_1, \dots, \omega_r$ is a basis of Λ_v formed by successive minima, for each $a_1, \dots, a_r \in \frac{1}{t} \cdot \mathbb{F}_q[[1/t]]$ we have

$$\left|\sum_{i=1}^r a_i(1/t)\omega_i\right|_v = \max_{i=1}^r |a_i(1/t)\omega_i|_v.$$

Indeed, without loss of generality, we assume $a_i \neq 0$ for $i = 1, \dots, s$ and $a_i = 0$ for $i > s$ (for some $s \leq r$). Let $N \in \mathbb{N}$ such that $|1/t^N|_v = \min_{i=1}^s |a_i(1/t)|_v$. Then there exist nonzero $P_1, \dots, P_s \in \mathbb{F}_q[t]$ of degree less than N such that for each $i = 1, \dots, s$ we have

$$a_i(1/t) = P_i(t)/t^N + b_i(1/t),$$

where each $b_i(1/t) \in 1/t^{N+1} \cdot \mathbb{F}_q[[1/t]]$. Then

$$\left|\sum_{i=1}^s b_i(1/t)\omega_i\right|_v \leq |1/t^{N+1}|_v \cdot \max_{i=1}^s |\omega_i|_v < |1/t^N|_v \cdot \max_{i=1}^s |\omega_i|_v.$$

On the other hand,

$$\left|\frac{\sum_{i=1}^s P_i(t)\omega_i}{t^N}\right|_v = \frac{\max_{i=1}^s |P_i(t)\omega_i|_v}{|t^N|_v} \geq \frac{\max_{i=1}^s |\omega_i|_v}{|t^N|_v},$$

since $|P_i(t)|_v \geq 1$ because each P_i is nonzero. Therefore,

$$\left|\sum_{i=1}^r a_i(1/t)\omega_i\right|_v = \frac{\max_{i=1}^s |P_i(t)\omega_i|_v}{|t^N|_v} = \max_{i=1}^r |a_i(1/t)\omega_i|_v. \tag{4.9}$$

So, letting

$$r_v := \frac{r_v^0}{1+\max_{i=1}^r |\omega_i|} < r_v^0,$$

we obtain that for each $a_1(1/t), \dots, a_r(1/t) \in \frac{1}{t}\cdot \mathbb{F}_q[[1/t]]$ such that $|a_i(1/t)|_v < r_v$ we have that

$$\left|\sum_{i=1}^r a_i(1/t)\omega_i\right|_v = \max_{i=1}^r |a_i(1/t)\omega_i|_v < r_v^0.$$

Then using (4.6), we conclude that if $|a_i(1/t)|_v < r_v$, we have

(4.10)

$$\left|\mathbb{E}\tau^{-1}\left(a_1(1/t),\dots,a_r(1/t)\right)\right|_v = \left|E_v\left(\sum_{i=1}^r a_i(1/t)\omega_i\right)\right|_v = \left|\sum_{i=1}^r a_i(1/t)\omega_i\right|_v = \max_{i=1}^r |a_i(1/t)\omega_i|_v.$$

Since $\mathbb{E}\tau^{-1}$ induces an isomorphism between $\left(\frac{1}{t}\cdot\mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right)^r$ and $\mathcal{J}_v$, and using (4.10), we conclude that if $0 < r < r_v$, then

$$\tau\mathbb{E}^{-1}(B(0,r)\cap\mathcal{J}_v) = \prod_{i=1}^r \left(B\left(0, \frac{r}{|\omega_i|}\right)\cap\left(\frac{1}{t}\cdot\mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right)\right). \tag{4.11}$$

Furthermore, since E_v is an additive map, then for each $\beta\in\mathcal{J}_v$, letting $\tau\mathbb{E}^{-1}(\beta) = (b_1,\dots,b_r)\in\left(\frac{1}{t}\cdot\mathbb{F}_q[[1/t]]\right)^r$, we get

$$\tau\mathbb{E}^{-1}(B(\beta,r)\cap\mathcal{J}_v) = \prod_{i=1}^r \left(B\left(b_i, \frac{r}{|\omega_i|}\right)\cap\left(\frac{1}{t}\cdot\mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right)\right). \tag{4.12}$$

Notation. Let ν_v be the Haar measure on G, normalized so that its total mass is 1. Let $\mu_v := \left(\mathbb{E}\tau^{-1}\right)_*\nu_v$ be the induced measure on $\mathcal{J}_v$ (i.e. $\mu_v(V) := \nu_v\left(\tau\mathbb{E}^{-1}(V)\right)$ for every measurable $V\subset\mathcal{J}_v$).

Because ν_v is a probability measure, then μ_v is also a probability measure. Because ν_v is a Haar measure on G and $\mathbb{E}\tau^{-1}$ is a group ismorphism, then μ_v is a Haar measure on $\mathcal{J}_v$.

Definition 4.13. *Given $x\in K^{\mathrm{sep}}$, we define a probability measure $\overline{\delta}_x$ on $\mathbb{C}_v$ by*

$$\overline{\delta}_x = \frac{1}{[K(x):K]}\sum_{\sigma\in\mathrm{Gal}(K^{\mathrm{sep}}/K)}\delta_{x^\sigma},$$

where δ_y is the Dirac measure on $\mathbb{C}_v$ supported on $\{y\}$.

Before we can state the equidistribution result from [11, Theorem 2.7] (see our Theorem 4.15), we need to define the concept of weak convergence for a sequence of probability measures on a metric space.

Definition 4.14. *A sequence $\{\lambda_k\}$ of probability measures on a metric space S weakly converges to λ if for any bounded continuous function $f : S \to \mathbb{R}$, $\int_S f d\lambda_k \to \int_S f d\lambda$ as $k \to \infty$. In this case we use the notation $\lambda_k \xrightarrow{\text{w}} \lambda$.*

Theorem 4.15. *Let $\Phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic such that $\operatorname{End}_{K^{\text{sep}}}(\Phi) \xrightarrow{\sim} \mathbb{F}_q[t]$. Let $\{x_k\}$ be a sequence of distinct torsion points in Φ. Then $\overline{\delta}_{x_k} \xrightarrow{\text{w}} \mu_v$.*

Remark 4.16. Theorem 4.15 is stated in [11, Theorem 2.7] when K is a transcendental extension of $\mathbb{F}_q(t)$ since the author needed to use the fact that the image of the Galois group is open in the adèlic Tate module for a Drinfeld module, and at that moment the result was only known under the assumption that K is transdendental over $\mathbb{F}_q(t)$ (see [7]). However, since then Rütsche and Pink [20] removed the assumption on K, and thus Theorem 4.15 holds in the above generality (see also [11, Remarks 3.2]).

Furthermore the proof from [11] yields that the points in $\Phi[Q]$ are equidistributed in $\mathcal{J}_v$ with respect to $d\mu_v$ as $\deg(Q) \to \infty$. This follows as the main result of [11] using the fact that $\mathcal{J}_v$ is isomorphic (as a topological group) to G and the points in $\Phi[Q]$ correspond in G to all points of the form

$$\left(\frac{P_1}{Q}, \dots, \frac{P_r}{Q}\right)$$

where the monic polynomials P_i have degrees less than $\deg(Q)$. More precisely, for a generic open subset of $\left(\frac{1}{t} \cdot \mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right)^r$ which is of the form (see [11, page 847, equation (6)])

$$U := \left(a_1\left(\frac{1}{t}\right), \cdots, a_r\left(\frac{1}{t}\right)\right) + \left(\frac{1}{t^{n_1+1}} \cdot \mathbb{F}_q\left[\left[\frac{1}{t}\right]\right], \cdots, \frac{1}{t^{n_r+1}} \cdot \mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right)$$

for some polynomials $a_i(t) \in t \cdot \mathbb{F}_q[t]$ of degree at most n_i, we have to show (see [11, page 847, equation (7)]) that the number of tuples

$$\left(\frac{P_1}{Q}, \cdots, \frac{P_r}{Q}\right) \in U$$

where $\deg(P_i) < \deg(Q) = d$ is asymptotic to

$$q^{dr - \sum_{i=1}^r n_i} \text{ as } d \to \infty.$$

As argued in [11], it suffices to prove this claim when $r = 1$, in which case the above statement reduces to show that (as $d \to \infty$) there are q^{d-n_1} distinct polynomials P_1 of degree less than d satisfying

$$\frac{P_1}{Q} - a_1\left(\frac{1}{t}\right) \in \frac{1}{t^{n_1+1}} \cdot \mathbb{F}_q\left[\left[\frac{1}{t}\right]\right].$$

This last statement follows at once since this last condition induces n_1 conditions on the d coefficients of P_1.

Furthermore, a strong equidistribution result for torsion points is obtained in the proof of the main result from [11] (a similar result was proven in a more general context by Favre and Rivera-Letelier [10]).

Theorem 4.17. *Given $\gamma \in \Phi_{\rm tor}$, and also given an open subset $\mathbb{E}\tau^{-1}(U)$ of $\mathcal{J}_v$, where $U \subset G$ is defined as above:*

$$U := \left(a_1\left(\frac{1}{t}\right), \cdots, a_r\left(\frac{1}{t}\right)\right) + \left(\frac{1}{t^{n_1+1}} \cdot \mathbb{F}_q\left[\left[\frac{1}{t}\right]\right], \cdots, \frac{1}{t^{n_r+1}} \cdot \mathbb{F}_q\left[\left[\frac{1}{t}\right]\right]\right),$$

we let

$$N(\gamma, U) := \{\sigma \in \mathrm{Gal}(K^{\rm sep}/K) \, : \, \gamma^\sigma \in \mathbb{E}\tau^{-1}(U)\}.$$

Let δ be a real number in the interval $(0,1)$. Then for all $\gamma \in \Phi_{\rm tor}$ and for all open subsets U as above,

$$\frac{N(\gamma, U)}{[K(\gamma) : K]} = \mu_v(U) + O_\delta\left([K(\gamma) : K]^{-\delta}\right).$$

Proof. This is the Drinfeld module analogue of the strong equidistribution result from [3, Proposition 2.4], and it is essentially proven in [11] when deriving formula (7), page 847. One simply needs to be more careful when estimating the error term in [11, (39), page 854] since this time we do not fix the open set U. The differences are as follows. In [11, (33), page 852], one estimates the number of all polynomials q_i' (for $i = 1, \ldots, r$) to be

$$q^{\sum_{i=1}^r \deg(b) - \deg(b') - \deg(d) - n_i} + O(1),$$

where $O(1)$ is independent of all previously defined quantities. Then the error term in [11, (39), page 854] is bounded above by the number of divisors of the polynomial $b \in \mathbb{F}_q[t]$ (which is the order of γ). Finally, noting that the number of divisors of b is bounded above by $q^{\epsilon \deg(b)}$, and that [11, (44), page 855] is bounded below (see also [11, Remarks 3.2, page 856]) by the number of polynomials relatively prime with b and of degree less than $\deg(b)$, and this number is larger than $q^{(1-\epsilon)\deg(b)}$, for any positive real number ϵ, we obtain the conclusion of Proposition 4.17. □

Using (4.12), Theorem 4.17 is equivalent with the following statement.

Corollary 4.18. *Let $\delta \in (0,1)$, let $\beta \in \mathcal{J}_v$ and let $U := B(\beta, r) \cap \mathcal{J}_v$ for some $r < r_v$. Then for all $\gamma \in \Phi_{\rm tor}$ we have*

$$\frac{\#\{\sigma \in \mathrm{Gal}(K^{\rm sep}/K) \, : \, \gamma^\sigma \in U\}}{[K(\gamma) : K]} = \mu_v(U) + O_\delta\left([K(\gamma) : K]^{-\delta}\right).$$

The following result follows from the powerful lower bound for linear forms in logarithms for Drinfeld modules established by Bosser [6].

Fact 4.19. *Assume $v \in \Omega_K$ is an infinite place. Let $\beta \in K$ be a nontorsion point and let $\gamma \in \Phi[Q]$ where $Q \in \mathbb{F}_q[t]$ is a monic polynomial of degree d. Then there exist (negative) constants C_0 and C_1 (depending only on Φ and β) such that*

$$\log|\gamma - \beta|_v \geq C_0 + C_1 d \log d.$$

Proof. In [14, Fact 3.1], Tucker and the author showed that Bosser's result yields the existence of some (negative) constants C_2 and C_3 such that for all polynomials $P \in \mathbb{F}_q[t]$ we have

$$\log|\Phi_P(\beta)|_v \geq C_2 + C_3 \deg(P) \log \deg(P). \tag{4.20}$$

On the other hand, if $|y|_v$ is sufficiently small but positive, then

$$\log|\Phi_t(y)|_v = \log|ty|_v = \log|y|_v + \log|t|_v.$$

Note that $\log|t|_v > 0$ since v is an infinite place. So assuming that d is sufficiently large, say $d \geq d_0 \geq 3$, if

$$\log|\beta - \gamma|_v < C_2 + (C_3 - 1)d \log d$$

then

$$\log|\Phi_Q(\beta - \gamma)|_v = \log|\Phi_Q(\beta)|_v = \log|Q\beta|_v = d\log|t|_v + \log|\beta|_v < C_2 + C_3 d \log d$$

contradicting thus (4.20). Therefore for all $d \geq d_0$ we have that

$$\log|\beta - \gamma|_v \geq C_2 + (C_3 - 1)d \log d.$$

Since $\beta \notin \Phi_{\mathrm{tor}}$ we conclude that there exists $C_4 < 0$ such that for all torsion points $\gamma \in \Phi[Q]$ for some monic polynomial Q of degree less than d_0 we have

$$\log|\beta - \gamma|_v \geq C_4.$$

In conclusion, Fact 4.19 holds with $C_0 := \min\{C_2, C_4\}$ and $C_1 := C_3 - 1$. □

Proposition 4.21. *Theorem 4.1 holds if $v \in S_\infty$.*

Proof. Again we split our analysis into two cases depending on whether β is in the (filled) Julia set $\mathcal{J}_v$ or not.

Case 1. Assume $\beta \notin \mathcal{J}_v$.

As previously discussed, if $\beta \notin \mathcal{J}_v$, then $f(z) = \log|z - \beta|_v$ is a continuous function on $\mathcal{J}_v$ and therefore using the result of [11, Theorem 2.7] (see our Theorem 4.15 above, or alternatively use [2, Corollary 4.6] which can be used since $\mathcal{J}_v$ is a compact set)

$$\lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma\in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \log|\beta - \gamma_n^\sigma|_v = \int_{\mathcal{J}_v} \log|\beta - z| d\mu_v(z).$$

Because $\deg(Q_n) \to \infty$ (since the torsion points γ_n are distinct), we know that $\{\Phi[Q_n]\}_n$ is equidistributed in $\mathcal{J}_v$ and thus

$$\int_{\mathcal{J}_v} \log|\beta - z| d\mu_v(z) = \lim_{\deg(Q)\to\infty} \frac{1}{q^{r\deg(Q)}} \sum_{\Phi_Q(z)=0} \log|\beta - z|_v = \lim_{\deg(Q)\to\infty} \frac{\log|\Phi_Q(\beta)|_v}{q^{r\deg(Q)}}.$$

By [13, Corollary 3.13] we conclude that

$$\lim_{\deg(Q)\to\infty} \frac{\log|\Phi_Q(\beta)|_v}{q^{r\deg(Q)}} = \widehat{\mathrm{h}}_{\Phi,v}(\beta)$$

which yields that indeed

$$\lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma\in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \log|\beta - \gamma_n^\sigma|_v = \widehat{\mathrm{h}}_{\Phi,v}(\beta).$$

Case 2. Assume $\beta \in \mathcal{J}_v$.

First we note that in this case $\widehat{\mathrm{h}}_{\Phi,v}(\beta) = 0$. We need to show that

$$\lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma\in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \log|\beta - \gamma_n^\sigma|_v = 0 = \widehat{\mathrm{h}}_{\Phi,v}(\beta).$$

For each $n \in \mathbb{N}$, let $Q_n \in \mathbb{F}_q[t]$ be the monic polynomial of minimal degree d_n such that $\Phi_{Q_n}(\gamma_n) = 0$. Then we know that $e_n = [K(\gamma_n):K] >> q^{rd_n}$ since $\frac{\#\,\mathrm{GL}_r(\mathbb{F}_q[t]/(Q_n))}{[K(\gamma_n):K]}$ is bounded above (by [20]).

We claim that $\int_{\mathcal{J}_v} \log|z|_v d\mu_v = 0$. Indeed, for each point z we have

$$\log|z|_v = \widehat{\mathrm{h}}_{\Phi,v}(z) - \lambda_v(z) + c_v(\Phi),$$

where λ_v is the local height as defined by Ingram [18] and

$$c_v(\Phi) := \frac{-\log|a_r|_v}{q^r - 1},$$

where a_r is the leading coefficient of $\Phi_t(x)$ (for more details see [18]). Since we assumed $a_r = 1$, then for each $z \in \mathcal{J}_v$ we have

$$\log|z|_v = -\lambda_v(z)$$

because $\widehat{\mathrm{h}}_{\Phi,v}(z) = 0$. However $\lambda_v(z) = g_{\mu_v}(z, 0)$ where $g_{\mu_v}(x, y)$ is the Arakelov-Green function as defined in [1, Section 10.2]. Therefore, by [1, Proposition 10.12] we have

$$\int_{\mathcal{J}_v} \lambda_v(z) d\mu_v = 0$$

since the invariant measure μ_v is supported on the Julia set $J_v = \mathcal{J}_v$. Thus indeed

$$\int_{\mathcal{J}_v} \log|z|_v d\mu_v = 0. \tag{4.22}$$

Next we employ the strategy of proof from [3]. Let $\epsilon := |t^m|_v^{-1} < r_v$ (where r_v is defined as above); in particular, E_v induces an isomorphism restricted on the closed ball $\overline{B}(0, \epsilon)$. Also, since $\epsilon < r_v$ then we may apply the conclusion of Corollary 4.18 with $r = \epsilon$.

We consider

$$J_{v,\beta,\epsilon} := \{z \in \mathcal{J}_v \,:\, |z - \beta|_v \leq \epsilon\}.$$

Also, we define $h_{\beta,\epsilon} : \mathcal{J}_v \longrightarrow \mathbb{R}$ as follows

$$h_{\beta,\epsilon}(z) := \min\left\{0, \log\left(\frac{|z-\beta|_v}{\epsilon}\right)\right\}.$$

Then $h_{\beta,\epsilon}$ is supported on $J_{v,\beta,\epsilon}$ and it has a logarithmic singularity at β. So, there exists a continuous function $g_{\beta,\epsilon} : \mathcal{J}_v \longrightarrow \mathbb{R}$ such that $\log|z-\beta|_v = g_{\beta,\epsilon}(z) + h_{\beta,\epsilon}(z)$.

For each $n \in \mathbb{N}$ we define $\mu_n := \overline{\delta}_{\gamma_n}$ be the probability measure on $\mathcal{J}_v$ supported on the Galois orbit of γ_n (see Definition 4.13). Then for each continuous function $f : \mathcal{J}_v \longrightarrow \mathbb{R}$ we have

$$\int_{\mathcal{J}_v} f d\mu_n := \frac{1}{[K(\gamma_n) : K]} \cdot \sum_{\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)} f\left(\gamma_n^\sigma\right).$$

Lemma 4.23.

$$\int_{\mathcal{J}_v} h_{\beta,\epsilon} d\mu_v = -\frac{\epsilon^r}{q^r - 1}.$$

Proof. Since we know that $h_{\beta,\epsilon}$ is supported on $J_{v,\beta,\epsilon}$ and moreover we know that $\mathcal{J}_v$ is closed under translations (and μ_v is translation-invariant) it suffices to prove

$$\int_{J_{v,0,\epsilon}} h_{0,\epsilon} d\mu_v = -\frac{\epsilon^r}{q^r - 1},$$

i.e., we may assume $\beta = 0$. So, we have to prove that

$$\int_{J_{v,0,\epsilon}} \log\left(\frac{|z|_v}{\epsilon}\right) d\mu_v = -\frac{\epsilon^r}{q^r - 1}.$$

By our assumption that $\epsilon < r_v$, we know that $\mathrm{E_v}$ induces an isometric analytic automorphism of $\overline{B}(0,\epsilon)$; so using the change of variables $z = \mathrm{E_v}(u)$ we compute

$$\int_{J_{v,0,\epsilon}} \log\left(\frac{|z|_v}{\epsilon}\right) d\mu_v$$

$$= \int_{\mathrm{E_v}^{-1}(J_{v,0,\epsilon})} \log\left(\frac{|\mathrm{E_v}(u)|_v}{\epsilon}\right) d\nu_v$$

$$= \int_{\mathrm{E_v}^{-1}(J_{v,0,\epsilon})} \log\left(\frac{|u|_v}{\epsilon}\right) d\nu_v,$$

where ν_v is the measure on $\mathrm{E_v}^{-1}(J_{v,0,\epsilon})$ which is isomorphic to

$$\left(\mathbb{F}_q((1/t)) \otimes_{\mathbb{F}_q[t]} \Lambda\right) \cap B(0,\epsilon) = \oplus_{i=1}^r B\left(0, \epsilon|\omega_i|_v^{-1}\right) \cdot \omega_i.$$

Furthermore, we recall that for $u_1, \dots, u_r \in \mathbb{F}_q[[1/t]]$ we have that

$$\log|u_1\omega_1 + \cdots + u_r\omega_r|_v = \max_{i=1}^r |u_i\omega_i|.$$

Making another change of variables $u_i = \frac{x_i}{\omega_i t^m}$ and noting that $\epsilon = |1/t^m|_v$ and that the measure ν has mass equal to 1, we are left to show that

$$I := \int_{(\mathbb{F}_q[[1/t]])^r} \log \max_{i=1}^r |x_i|_v d\nu_0(x_1,\dots,x_r) = -\frac{1}{q^r-1},$$

where ν_0 is the (probability) measure on $(\mathbb{F}_q[[1/t]])^r$. We let

$$S_1 := (\mathbb{F}_q[[1/t]])^r \setminus (1/t \cdot \mathbb{F}_q[[1/t]])^r,$$

we note that $\max_{i=1}^r \log|x_i|_v$ restricted on S_1 equals 0, and so we obtain

$$I = \int_{(1/t\mathbb{F}_q[[1/t]])^r} \max_{i=1}^r \log|x_i|_v d\nu_0$$

$$= \frac{1}{q^r} \cdot \int_{(\mathbb{F}_q[[1/t]])^r} \left(-1 + \max_{i=1}^r \log|y_i|_v\right) d\nu_0 \text{ (by the change of variables } x_i = y_i/t)$$

$$= -\frac{1}{q^r} + \frac{I}{q^r}$$

Hence $I = -\frac{1}{q^r-1}$. □

So, using (4.22) coupled with Lemma 4.23 we obtain that

$$\int_{\mathcal{J}_v} g_{\beta,\epsilon}(z)d\mu_v = -\int_{\mathcal{J}_v} h_{\beta,\epsilon}(z)d\mu_v = \frac{\epsilon^r}{q^r-1}.$$

Using the fact that the measures μ_n converge (weakly) to μ_v (according to [11]; see also Theorem 4.15) we conclude that for n sufficiently large we have

$$\left|\int_{\mathcal{J}_v} g_{\beta,\epsilon}(z)d\mu_n\right| < 2\epsilon^r, \tag{4.24}$$

because $g_{\beta,\epsilon}$ is a continuous function. It remains to bound $\left|\int_{\mathcal{J}_v} h_{\beta,\epsilon}(z)d\mu_n\right|$.

Since $h_{\beta,\epsilon}$ is supported on $J_{v,\beta,\epsilon}$ it suffices to analyze the conjugates of γ_n which land in $J_{v,\beta,\epsilon}$. For this we let D_n be the smallest integer larger than $[K(\gamma_n):K]^{1/2r}$ and we split $J_{v,\beta,\epsilon}$ into D_n subsets as follows. For each interval $[c,d] \subset [0,1]$ let $J_v([c,d])$ be the subset of $J_{v,\beta,\epsilon}$ containing all $z \in \mathcal{J}_v$ such that $c\epsilon \le |z-\beta|_v \le d\epsilon$. Then

$$J_{v,\beta,\epsilon} = \bigcup_{i=1}^{D_n} J_v\left(\left[\frac{i-1}{D_n}, \frac{i}{D_n}\right]\right).$$

We note that $\mu_v(J_v([c,d])) = (d\epsilon)^r - (c\epsilon)^r$ for each $0 \le c < d \le 1$ since $J_{v,\beta,\epsilon}$ is isomorphic to $(1/t^m \cdot \mathbb{F}_q[[1/t]])^r$ and $|1/t^m|_v = \epsilon$. So, for large n, Corollary 4.18 (applied with $\delta := 2/3$ to the annular region $J_v([c,d]) = \mathcal{J}_v \cap (B(\beta, d\epsilon) \setminus B(\beta, c\epsilon)))$ yields each such subset contains at most

$$2\mu_v(J_v([c,d])) \cdot [K(\gamma_n) : K] \le 2\epsilon^r(d^r - c^r)D_n^{2r}.$$

conjugates of γ_n. Note that we can apply Corollary 4.18 because we chose $\epsilon < r_v$. Also, the reason for which $\delta = 2/3$ works in Corollary 4.18 is that in this case

$$\mu_v(J_v([c,d])) \cdot [K(\gamma_n) : K] \gg_\epsilon D_n^r \gg D_n^{2r(1-\delta)} \gg [K(\gamma_n) : K]^{1-\delta},$$

and thus Corollary 4.18 yields that $2\mu_v(J_v([c,d])) \cdot [K(\gamma_n) : K]$ is the main term for computing the number of conjugates of γ_n landing in $J_v([c,d])$ (note that ϵ is fixed for this computation).

We analyze the first interval: $J_v([0, 1/D_n])$. We recall that d_n is the degree of the minimal monic polynomial Q_n such that $\Phi_{Q_n(t)}(\gamma_n) = 0$. Without loss of generality we assume $\gamma_n \in J_v([0, 1/D_n])$ is the conjugate of γ_n closest to β. By Bosser's theorem (see [6] and also our Fact 4.19), we have

$$\log |\gamma_n - \beta|_v \ge C_0 + C_1 d_n \log d_n.$$

On the other hand, there are at most

$$2\mu_v(J_v([0, 1/D_n]))[K(\gamma_n) : K] = \frac{2\epsilon^r[K(\gamma_n) : K]}{D_n^r} \le 2\epsilon^r[K(\gamma_n) : K]^{1/2}$$

conjugates of γ_n in $J_v([0, 1/D_n])$. We denote by I_0 the set of all $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ such that $\gamma_n^\sigma \in J_v([0, 1/D_n])$. Using also that $\epsilon < 1$, we conclude that

(4.25)
$$0 \ge \int_{J_v([0,1/D_n])} h_{\beta,\epsilon}(z) d\mu_n = \frac{\sum_{\sigma \in I_0} \log \frac{|\beta - \gamma_n^\sigma|_v}{\epsilon}}{[K(\gamma_n) : K]} > \frac{2\epsilon^r \cdot (C_0 + C_1 d_n \log d_n)}{[K(\gamma_n) : K]^{1/2}} > -\epsilon^r,$$

for large n since $D_n = [K(\gamma_n) : K]^{1/2} \gg q^{d_n/2}$ (by [20]).

Finally, consider the remaining subsets of $J_{v,\beta,\epsilon}$. For each $i = 2, \dots, D_n$ there are at most $2\mu_v(J_v([(i-1)/D_n, i/D_n])) \cdot [K(\gamma_n) : K]$ conjugates of γ_n in $J_v([(i-1)/D_n, i/D_n])$ and for each such conjugate γ_n^σ we have

$$h_{\beta,\epsilon}(\gamma_n^\sigma) = \log \frac{|\beta - \gamma_n^\sigma|_v}{\epsilon} \ge \log((i-1)/D_n).$$

Using that $J_{v,\beta,\epsilon}$ is isomorphic to $(1/t^m \cdot \mathbb{F}_q[[1/t]])^r$ we conclude

$$0 \ge \int_{\mathcal{J}_v \setminus J_v([0,1/D_n])} h_{\beta,\epsilon}(z) d\mu_n$$

$$\ge \sum_{i=1}^{D_n - 1} \log\left(\frac{i}{D_n}\right) \cdot 2\mu_v(J_v[(i-1)/D_n, i/D_n]) > 2\int_{J_{v,\beta,\epsilon}} \log\left(\frac{|z|_v}{\epsilon}\right) = -\frac{2\epsilon^r}{q^r - 1},$$

by Lemma 4.23. Hence, using also inequality (4.25) we obtain that

$$\left| \int_{\mathcal{J}_v} h_{\beta,\epsilon}(z) d\mu_n \right| < 3\epsilon^r. \tag{4.26}$$

Combining inequalities (4.24) with (4.26) we conclude that

$$\left| \int_{\mathcal{J}_v} \log |z - \beta|_v d\mu_n \right| < 5\epsilon^r,$$

for all n sufficiently large, and so, letting $\epsilon \to 0$ we obtain

$$\lim_{n\to\infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma\in \mathrm{Gal}(K^{\mathrm{sep}}/K)} \log|\gamma_n^\sigma - \beta| = 0 = \widehat{\mathrm{h}}_{\Phi,v}(\beta),$$

if $\beta \in \mathcal{J}_v$. This concludes our proof. □

Propositions 4.2 and 4.21 finish the proof of Theorem 4.1.

Remark 4.27. In the proof of Proposition 4.21 we use in an essential way the hypothesis that Φ has no complex multiplication because we employ the strong equidistribution result (from [11]) for torsion points of a Drinfeld module. However, we expect that Theorem 1.3 holds without this hypothesis on Φ, only that one would need a different approach.

References

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