SIMULTANEOUSLY PREPERIODIC POINTS FOR FAMILIES OF POLYNOMIALS IN CHARACTERISTIC p

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ABSTRACT. The first author proved in [Ghi] that given a field K of characteristic $p > 0$, given an integer $d \geq 2$, and also given $\alpha, \beta \in K$, then for the family of polynomials $f_\lambda(x) := x^d + \lambda$ (parameterized by $\lambda \in \overline{K}$), there exist infinitely many $\lambda \in \overline{K}$ such that both α and β are preperiodic under the action of f_λ, if and only if at least one of the following three conditions holds: (i) $\alpha^d = \beta^d$; (ii) $\alpha, \beta \in \overline{\mathbb{F}}_p$; and (iii) $d = p^\ell$ for some $\ell \in \mathbb{N}$ and $\alpha - \beta \in \overline{\mathbb{F}}_p$. In the present paper, we generalize the results from [Ghi] by proving that for any polynomial $f \in \overline{\mathbb{F}}_p[x]$ of degree $d \geq 2$ satisfying a mild condition (which is already satisfied when $p \nmid d$), and for any starting points $\alpha, \beta \in K$, there exist infinitely many $\lambda \in \overline{K}$ such that both α and β are preperiodic for the polynomials $f_\lambda(x) := f(x) + \lambda$ if and only if at least one of the following two conditions hold: (i) $f(\alpha) = f(\beta)$ and (ii) $\alpha, \beta \in \overline{\mathbb{F}}_p$.

1. Introduction

In this paper, we continue the investigation and extend the results proved in [Ghi] on the unlikely intersection problem for families of dynamical systems in positive characteristic. Working over fields of characteristic 0, Baker and DeMarco [BD11] proved that for any integer $d \geq 2$ and given complex numbers a and b, if there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic under the action of $f_\lambda(x) = x^d + \lambda$, then $a^d = b^d$. The result of [BD11] was itself inspired by the groundbreaking work of Masser-Zannier [MZ10, MZ12] regarding simultaneous torsion sections for families of elliptic curves. Further extensions of the main result of [BD11] have been achieved for arbitrary families of polynomials (see [GHT13, BD13, GY18, FG18]). Similar results have also been established for certain families of rational maps (see [DWY15, DM20, GHT15]), including the case of maps parameterized by points in a higher dimensional space (see [GHT15, GHT16, GHN18]).

Each time, the proofs of these results involve two distinct components:

(I) First, one proves that a certain equidistribution theorem for points of small height holds for the given dynamical system. This leads to the conclusion that the canonical heights (suitably normalized) of the two starting points, computed with respect to the family of maps, are equal.

(II) Then using the equality of the above canonical heights, one derives the precise relation between the two starting points.

Part (I) above is obtained as a consequence from any of the equidistribution theorems established by Baker-Rumely [BR06], Chambert-Loir [CL06], Favre-Rivera-Letelier [FRL06] or Yuan [Yua08]. Verifying the hypotheses of the aforementioned equidistribution theorems is often the most challenging aspect, as it requires a detailed analysis of the arithmetical properties of the given dynamical system. Typically, completing step (II) is more straightforward and relies on a complex analytic argument.

The primary challenge in extending these results to fields of positive characteristic lies in the absence of an analogue to the complex analytic argument used in part (II). This presents a significant obstacle to obtaining similar results in the new setting of positive characteristic fields.

1.1. **Our results.** The family $f_\lambda(x) = x^d + \lambda$ of polynomials, as considered in [BD11], is the primary family studied in [Ghi] in the case of positive characteristic. In this paper, we extend the scope by considering more general families of polynomials and prove the following result.

Theorem 1.1. *Let $d \geq 3$ be an integer, let L be a field of characteristic p, and let $\alpha, \beta \in L$. We let $\overline{L}$ be a fixed algebraic closure of L, we let $\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ inside $\overline{L}$, and we let $f \in \overline{\mathbb{F}}_p[x]$ be a polynomial of degree d. We assume $f(x)$ has the following form:*

$$f(x) = \sum_{i=1}^{r} c_i x^{d_i}, \tag{1.1.1}$$

where each $c_i \in \overline{\mathbb{F}}_p^{\,*}$, while

$$1 \leq d_1 < d_2 < \cdots < d_r = d$$

and for each $i = 1, \dots, r$, we write $d_i = p^{\ell_i} \cdot s_i$ with $\ell_i \geq 0$, while $p \nmid s_i$ such that the following inequality holds:

$$p^{\ell_r}(s_r - 1) > \max\left\{1, p^{\ell_1}(s_1 - 1), p^{\ell_2}(s_2 - 1), \cdots, p^{\ell_{r-1}}(s_{r-1} - 1)\right\}. \tag{1.1.2}$$

We consider the family of polynomials

$$f_\lambda(x) := f(x) + \lambda \text{ parameterized by } \lambda \in \overline{L}.$$

Then there exist infinitely many $\lambda \in \overline{L}$ such that both α and β are preperiodic under the action of f_λ if and only if at least one of the following statements holds:

1. *$f(\alpha) = f(\beta)$.*
2. *$\alpha, \beta \in \overline{\mathbb{F}}_p$.*

Moreover, if either one of the conditions (1)-(2) holds, then for each $\lambda \in \overline{L}$, we have that α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

As a special case to Theorem 1.1, we consider the situation where the degree of $f(x)$ is coprime to the characteristic of the field in question.

Theorem 1.2. *Let $d \geq 2$ be an integer not divisible by a prime p, let L be a field of characteristic p, and let $\alpha, \beta \in L$. We let $\overline{L}$ be a fixed algebraic closure of L, we let $\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ inside $\overline{L}$, and we let $f \in \overline{\mathbb{F}}_p[x]$ be a polynomial of degree d. We consider the family of polynomials*

$$f_\lambda(x) := f(x) + \lambda \text{ parameterized by } \lambda \in \overline{L}.$$

Then there exist infinitely many $\lambda \in \overline{L}$ such that both α and β are preperiodic under the action of f_λ if and only if at least one of the following statements holds:

1. *$f(\alpha) = f(\beta)$.*
2. *$\alpha, \beta \in \overline{\mathbb{F}}_p$.*

Moreover, if either one of the conditions (1)-(2) holds, then for each $\lambda \in \overline{L}$, we have that α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

Remark 1.3. We observe that given any polynomial $f(x)$, when considering the family of polynomials $f(x)+\lambda$, automatically we could have assumed that $f(0)=0$ since any constant term in $f(x)$ can then be absorbed together with the parameter λ. Furthermore, we note that if $p \nmid d$ (as in Theorem 1.2), then the polynomial $f(x)$ has the form (1.1.1) satisfying the inequality (1.1.2) from Theorem 1.1, as long as $d \geq 3$. On the other hand, the case when $d=2$ and $p>2$ is already covered by [Ghi, Theorem 1.1] because then our family of quadratic polynomials $f_\lambda(x)$ can be conjugated (using a suitable linear polynomial $x \mapsto ax+b$) to the family of polynomials $x \mapsto x^2+\lambda$. So, indeed, Theorem 1.2 is a special case of Theorem 1.1.

Remark 1.4. It is tempting to think that our arguments might extend to more general classes of dynamical systems. For instance, using the notation as in Theorem 1.1, if each $s_i=1$ for $i=1,\dots,r$, meaning the polynomials $f(x)$ is additive, we can apply the same reasoning as in [Ghi, Section 6]. In this case, we obtain that there exist infinitely many $\lambda \in \overline{L}$ such that both α and β are preperiodic under the action of $f(x)+\lambda$ if and only if $\alpha-\beta \in \overline{\mathbb{F}}_p$. Actually, as shown in [Ghi, Section 6], the existence of a *single* parameter $\lambda \in \overline{L}$ such that both α and β are preperiodic under the action of $f(x)+\lambda$ yields the fact that $\alpha-\beta \in \overline{\mathbb{F}}_p$.

However, if we consider $s_r=1$, but not all $s_i=1$ for $i=1,\dots,r-1$ in Theorem 1.1 (i.e., when the degree of $f(x)$ is a power of the characteristic of the field in question but $f(x)$ is not additive), then the situation becomes very challenging. A specific example illustrating this difficulty arises when the characteristic of the field is 3, and we consider the family of polynomials $f_\lambda(x)=x^3+x^2+\lambda$. This situation introduces many technical difficulties, similar to those encountered by the authors in [GH13] when dealing with the unlikely intersection principle in families of Drinfeld modules.

Remark 1.5. If one were to pose the same question as in Theorem 1.2 for polynomials $f(x)$ whose coefficients are no longer in $\overline{\mathbb{F}}_p$, then the problem becomes significantly more complex.

The technical challenges encountered are similar to those arising in the study of the unlikely intersection problem in the context of Drinfeld modules, as discussed in [GH13]. In that case, the maps were additive, making it somewhat easier to manage their compositional behavior; yet, the full generality of the problem in [GH13] remains unsolved. The core difficulty lies in selecting sufficiently well-chosen parameters λ (and places v), as outlined in Section 3, to derive a statement similar to Proposition 3.3 for more general dynamical systems.

Additionally, any attempt to extend Theorem 1.2 to dynamical systems involving starting points α and β that are no longer constant, but instead vary polynomially with the parameter λ leads to deeper questions. To derive a precise relationship between such general starting points $\alpha(\lambda)$ and $\beta(\lambda)$, an analogue of the powerful result from [MS14] for fields of positive characteristic would be required. Indeed, the result of [MS14] was a key ingredient in the final step for the result of [BD13] when studying general dynamical systems $(f_\lambda, \alpha(\lambda))$ and $(f_\lambda, \beta(\lambda))$ over $\mathbb{C}$.

Despite these challenges, we still believe the following general result holds (see Conjecture 1.7). Before stating our conjecture, we first introduce the notion of *normalized polynomials* (or alternatively, *polynomials in normal form*).

Definition 1.6. Let K be an arbitrary algebraically closed field and let $f \in K[x]$ be a polynomial of degree $d \geq 2$. We say that f is is normalized form if it has the following shape:

$$f(x)=x^d+c_{d-2}x^{d-2}+c_{d-3}x^{d-3}+\cdots+c_1x, \tag{1.1.3}$$

for some $c_1, \dots, c_{d-2} \in K$. In particular, if $d = 2$, then the only quadratic polynomial in normalized form is x^2.

Conjecture 1.7. *Let K be an algebraicaly closed field of characteristic p, let $\alpha(z), \beta(z) \in K[z]$ and let let $f_\lambda(x) \in K[x]$ be a family (parameterized by $\lambda \in K$) of polynomials of degree $d \geq 2$ in normalized form, i.e.*

$$f_\lambda(x) = x^d + \sum_{i=0}^{d-2} c_i(\lambda) \cdot x^i, \tag{1.1.4}$$

for some polynomials $c_i(z) \in K[z]$ (for $i = 0, \dots, d-2$). Then there exist infinitely many $\lambda \in K$ such that both $\alpha(\lambda)$ and $\beta(\lambda)$ are preperiodic under the action of f_λ if and only if at least one of the following conditions holds:

(1) *there exists a family of polynomials $g_\lambda(x)$ (similar to (1.1.4)) and there exist integers $k > 0$ and $m, n \geq 0$ such that*

$$f_\lambda^k \circ g_\lambda = g_\lambda \circ f_\lambda^k \text{ and } f_\lambda^m(\alpha(\lambda)) = g_\lambda\left(f_\lambda^n(\beta(\lambda))\right) \text{ (or } f_\lambda^m(\beta(\lambda)) = g_\lambda\left(f_\lambda^n(\alpha(\lambda))\right)\text{)}, \tag{1.1.5}$$

for all $\lambda \in K$.

(2) *$c_i(z) \in \overline{\mathbb{F}}_p[z]$ for $i = 0, \dots, d-2$ and also, $\alpha(z), \beta(z) \in \overline{\mathbb{F}}_p[z]$;*

(3) *for* each *$\lambda \in K$, the polynomial $\tilde{f}_\lambda(x) := f_\lambda(x) - c_0(\lambda)$ is additive (i.e., $\tilde{f}_\lambda(x+y) = \tilde{f}_\lambda(x) + \tilde{f}_\lambda(y)$ for all x, y) and $\gamma(\lambda) := \alpha(\lambda) - \beta(\lambda)$ is preperiodic under the action of $\tilde{f}_\lambda(x)$.*

(4) *there exists $n > m \geq 0$ such that either $f_\lambda^n(\alpha(\lambda)) = f_\lambda^m(\alpha(\lambda))$ for all λ, or $f_\lambda^n(\beta(\lambda)) = f_\lambda^m(\beta(\lambda))$ for all λ.*

Remark 1.8. One could formulate Conjecture 1.7 for an arbitrary (unnormalized) family of polynomials $f_\lambda \in K[\lambda][x]$ (of degree $d \geq 2$), but then the condition (2) would become more complicated; on the other hand, when $p \nmid d$ it suffices to deal with dynamical systems corresponding to polynomials in normalized form since modulo a linear conjugation, one can achieve the normalized form for our dynamical system.

Condition (4) does not appear for the dynamical systems from [Ghi] or from our Theorems 1.1 and 1.2, but clearly, for general dynamical systems, one needs to take into account the possibility that either $\alpha(\lambda)$ or $\beta(\lambda)$ are *persistent* preperiodic.

Condition (3) asks that the only monomials x^i in $f_\lambda(x)$ (for $i > 0$) appearing with a nonzero coefficient correspond to $i = p^j$ for some $j \geq 0$. The following Example shows some of the subtleties coming from condition (3).

Example 1.9. Let K be an algebraically closed field of characteristic p. For some given integers $\ell \geq 1$ and $0 \leq r_1 < r_2 < \cdots < r_\ell$, and for some given $C_1, \cdots, C_\ell \in \overline{\mathbb{F}_p}^*$, we consider the family of polynomials:

$$f_\lambda(x) := \sum_{i=1}^{\ell} C_i x^{p^{r_i}} + \lambda \text{ parameterized by } \lambda \in K.$$

Also, we consider $\alpha, \beta \in K$. Then arguing as in [Ghi, Section 6], we get that $\alpha - \beta \in \overline{\mathbb{F}}_p$ if and only if there exist infinitely many $\lambda \in K$ such that both α and β are preperiodic under the action of $f_\lambda(x)$.

On the other hand, considering $K = \overline{\mathbb{F}_p(t)}$ and the family of polynomials

$$f_\lambda(x) := x^{p^2} + tx^p + \lambda \text{ parameterized by } \lambda \in K$$

and *any distinct* starting points $\alpha, \beta \in K$, it is expected that there are only *finitely many* $\lambda \in K$ such that both α and β are preperiodic under the action of $f_\lambda(x)$. However, based on the difficulties we encountered when dealing with similar families of maps in [GH13], we also expect this to be difficult to prove.

Traditionally, the function field arithmetic in characteristic p presented additional subtleties when studying unlikely intersection questions (see [Bre05, Bos02, BM17, BM22, Sca02]). To our knowledge, there are only a handful of results concerning the unlikely intersection principle for dynamical systems over fields of characteristic p (see [GH13, Ghi24, Ghi]). Generally, the arithmetic properties for algebraic dynamical systems are more subtle over fields of positive characteristic (see [CHT23]).

In the next subsection, we outline the strategy for proving our main result, while also highlighting the challenging technical aspects that limit further extensions of Theorem 1.2.

1.2. **Proof strategy and further remarks.** Our overall strategy is similar to the one employed in [GH13, Ghi]. As mentioned earlier, the main ideas of the proof are divided into two parts. The first part involves establishing the equality of the canonical heights of two given points. As is typical in much of the work on the unlikely intersection problem in arithmetic dynamics, we follow the approach outlined in the proof of [BD11]. Specifically, for a product formula field L, let $\widehat{h}_{f_\lambda}(\gamma)$ denote the canonical height associated with the specialized polynomial f_λ at $\lambda \in \overline{L}$ for a point $\gamma \in L$. Assuming there exist infinitely many parameters λ_n, called preperiodic parameters, such that both the given points α and β are preperiodic under the action of f_{λ_n}, one can show that the sequence $\{\lambda_n\}_{n\geq 1}$ gives rise to a sequence of small points with respect to certain height functions induced by $\widehat{h}_{f_\lambda}(\alpha)$ (respectively, $\widehat{h}_{f_\lambda}(\beta)$) on the space $\overline{L}$ of parameters. We then apply the main equidistribution theorem from [BR10] to deduce that $\widehat{h}_{f_\lambda}(\alpha) = \widehat{h}_{f_\lambda}(\beta)$ for each parameter $\lambda \in \overline{L}$. To state the equidistribution theorem employed in our proof, we need a technical setup involving both the theory of Berkovich spaces and arithmetic dynamics. The hypothesis of having an infinite sequence of preperiodic parameters λ_n can be replaced by a weaker condition:

$$\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\beta) = 0; \tag{1.2.1}$$

which still establishes the equality of the canonical heights of α and β. As noted in Remark 2.5, once α (or β) is preperiodic under the action of f_λ, its global canonical height (with respect to f_λ) is 0; thus, the condition (1.2.1) is indeed weaker than the hypothesis of Theorem 1.1 and 1.2. Moreover, as a consequence of the equidistribution theorem, we establish the crucial Theorem 2.11 in subsection 2.4, which shows that the existence of an infinite sequence of parameters λ_n satisfying equation (1.2.1) implies that for *each* parameter λ and for each place v of L, we have

$$\widehat{h}_{v,\lambda}(\alpha) = \widehat{h}_{v,\lambda}(\beta); \tag{1.2.2}$$

for the precise definition of the local canonical heights $\widehat{h}_{v,\lambda}$, we refer the reader to Section 2.

As mentioned above, the main ingredient for proving Theorem 2.11 is the powerful equidistribution theorem which relies on potential theory on the Berkovich line and results from the theory of dynamical systems, as detailed in [BR10]. Since the setup and the proof are identical to that of [Ghi, Theorem 4.1], we will provide only a sketch of the proof in Subsection 2.4 and refer the reader to [Ghi] for the full details.

We proceed with the second part of the proof for Theorem 1.1 in Section 3 and complete the argument in Section 4. We first establish the result for the case when the field L is of transcendence degree 1 over $\overline{\mathbb{F}}_p$; this is done in Proposition 3.3. The aforementioned proposition states that assuming equation (1.2.2) holds, the desired conclusion in Theorem 1.1 must follow. Its proof involves a careful analysis of the valuations of $f(\alpha)-f(\beta)$, utilizing equation (1.2.2) for suitably chosen parameters λ.

In Section 4 we extend the result to the general case where L has arbitrary transcendence degree of $\overline{\mathbb{F}}_p$. The key observation here is that if α and β share a common preperiodic parameter $\lambda\in\overline{L}$ then $\operatorname{trdeg}_{\overline{\mathbb{F}}_p}\overline{\mathbb{F}}_p(\alpha,\beta)\le 1$. This is formalized in Proposition 4.2, which allows us to reduce the general case to the case of transcendence degree one, thereby completing the proof.

2. Dynamics and heights associated to our family of polynomials

Similar to the analysis from [Ghi, Sections 2-4], we let L_0 be the perfect closure of the rational function field $\overline{\mathbb{F}}_p(t)$, i.e.,

$$L_0 := \overline{\mathbb{F}}_p\left(t, t^{1/p}, t^{1/p^2}, \cdots, t^{1/p^n}, \cdots\right) \tag{2.0.1}$$

and then we let L be a given finite extension of L_0. Then each finite extension of L is separable, i.e., $L^{\mathrm{sep}}=\overline{L}$; we also fix an algebraic closure $\overline{L}$ of L. Note that each place of $\overline{\mathbb{F}}_p(t)$ (which, geometrically, corresponds to a point of $\mathbb{P}^1(\overline{\mathbb{F}}_p)$) extends uniquely to a place w of L_0, thus making L_0 a product formula field. Above each given place w of L_0 there exist finitely many places v of L; we denote by $\Omega:=\Omega_L$ the set of places of L. Then, the following conditions hold:

(i) for each nonzero $x\in L$, we have $|x|_v=1$ for all but finitely many $v\in\Omega_L$; and
(ii) for each nonzero $x\in L$, we have

$$\prod_{v\in\Omega_L}|x|_v=1 \tag{2.0.2}$$

where we denote by $|\cdot|_v$ the absolute value at the place $v\in\Omega$ such that the product formula (2.0.2) holds. Finally, we have the following fact: only the elements in $\overline{\mathbb{F}}_p$ are those $x\in L$ which are integral at each place in Ω, i.e.,

$$\text{if } |x|_v\le 1 \text{ for each } v\in\Omega, \text{ then } x\in\overline{\mathbb{F}}_p. \tag{2.0.3}$$

2.1. **Preperiodic parameters for a given starting point.** We let $f\in\overline{\mathbb{F}}_p[x]$ be a monic polynomial of degree $d\ge 2$ satisfying $f(0)=0$. As we will show in Proposition 4.1, both Theorems 1.1 and 1.2 reduce easily to the case the respective polynomial $f(x)$ is monic and it has no constant term (see also Remark 1.3). Furthermore, these two assumptions on $f(x)$, besides not affecting the generality of our main results, they also reduce a bit the technical details in some of our arguments.

As before, we let $f_\lambda(x)=f(x)+\lambda$ be the corresponding family of polynomials parameterized by $\lambda\in\overline{L}$. Given $\gamma\in L$, we define

$$P_{n,\gamma}(\lambda) := f_\lambda^n(\gamma) \text{ for each } n\in\mathbb{N}; \tag{2.1.1}$$

then $P_{n,\gamma}(\lambda)$ is a polynomial in λ. A simple induction on n yields the following result.

Lemma 2.1. *With the above hypothesis, for each $n \in \mathbb{N}$, the polynomial $P_{n,\gamma}(\lambda)$ is monic and has degree d^{n-1} in λ.*

In fact, an easy induction yields that each coefficient of λ^i in $P_{n,\gamma}(\lambda)$ for $i = 1, \dots, d^{n-1} - 1$ is itself a polynomial *in* γ, i.e.,

$$P_{n,\gamma}(\lambda) = \lambda^{d^{n-1}} + \sum_{i=1}^{d^{n-1}-1} c_{n,i}(\gamma) \cdot \lambda^i + \gamma^{d^n} + \sum_{j=1}^{d^n-1} b_{n,j} \cdot \gamma^j \tag{2.1.2}$$

with each $c_{n,i} \in \overline{\mathbb{F}}_p[x]$ being a polynomial of degree less than d^n, and also, each $b_{n,j} \in \overline{\mathbb{F}}_p$ for $j = 1, \dots, d^n - 1$.

Remark 2.2. We immediately obtain as a corollary of Lemma 2.1 the fact that if $\gamma \in \overline{L}$ is preperiodic for f_λ, then $\lambda \in \overline{L}$.

Next, we establish the fact that for any starting point γ, one can find infinitely many parameters λ such that γ is preperiodic under the action of f_λ; our result is valid for an arbitrary field L.

Proposition 2.3. *Let L be an arbitrary field of characteristic p, let $f \in L[x]$ be a monic polynomial of degree $d \geq 2$ for which $f(0) = 0$, and let $\gamma \in L$. Then there exist infinitely many $\lambda \in \overline{L}$ such that γ is preperiodic under the action of f_λ.*

Proof of Proposition 2.3. If $\gamma \in \overline{\mathbb{F}}_p$, then the statement is obvious because then γ is preperiodic under f_λ for each $\lambda \in \overline{\mathbb{F}}_p$. So, from now on, we assume $\gamma \in L \setminus \overline{\mathbb{F}}_p$. The desired conclusion in Proposition 2.3 follows from the next Lemma, which provides a more refined conclusion.

Lemma 2.4. *Assume $\gamma \notin \overline{\mathbb{F}}_p$. Then there exist infinitely many $\lambda \in \overline{L}$ with the property that there exists some prime number q such that $f_\lambda^q(\gamma) = \gamma$.*

Proof of Lemma 2.4. The argument is similar to the proof of [Ghi, Proposition 6.3].

We argue by contradiction and so, assume the set

$$\mathcal{P} := \left\{\lambda \in \overline{L}\colon \text{ there exists a prime } q \text{ such that } f_\lambda^q(\gamma) = \gamma\right\}$$

is finite. In particular, this means that there exists a positive integer $M \geq d$ with the property that for each prime $q > M$ and for each $\lambda \in \overline{L}$ such that

$$f_\lambda^q(\gamma) = \gamma, \tag{2.1.3}$$

there exists a prime $q_0 < M$ (with q_0 depending on λ, of course) such that

$$f_\lambda^{q_0}(\gamma) = \gamma. \tag{2.1.4}$$

However, since q and q_0 are distinct primes, then equations (2.1.3) and (2.1.4) yield that $f_\lambda(\gamma) = \gamma$, i.e., $\lambda = \gamma - f(\gamma)$. Hence, letting $P_{q,\gamma}(\lambda) := f_\lambda^q(\gamma)$ as before (see equation (2.1.1)), the only solution $\lambda \in \overline{L}$ to the equation $P_{q,\gamma}(\lambda) = \gamma$ is $\lambda_0 := \gamma - f(\gamma)$. Now, using the shape of the polynomial $P_{q,\gamma}(\lambda)$ (see equation (2.1.2)), we conclude that

$$P_{q,\gamma}(\lambda) = (\lambda - \gamma + f(\gamma))^{d^{q-1}}. \tag{2.1.5}$$

In particular, this means that the constant term in the polynomial $P_{q,\gamma}$ must be $(f(\gamma)-\gamma)^{d^{q-1}}$. On the other hand, we know that the constant term in the polynomial $P_{q,\gamma}$ is $f^q(\gamma)$; this leads to the equation:

$$(f(\gamma)-\gamma)^{d^{q-1}} = f^q(\gamma). \tag{2.1.6}$$

Now, if we do *not* have an identity:

$$(f(x)-x)^{d^{q-1}} = f^q(x), \tag{2.1.7}$$

then the only solutions γ to equation (2.1.6) must live in $\overline{\mathbb{F}}_p$ (since all coefficients of $f(x)$ are from $\overline{\mathbb{F}}_p$). However, this would contradict the hypotheses of Lemma 2.4. Thus, it means that the equation (2.1.7) *must be* an identity.

Now, equation (2.1.7) combined with the fact that $f(0)=0$ yields that the order of vanishing of $f^q(x)$ at $x=0$ must be at least d^{q-1}. Thus, the order of vanishing ℓ of $f(x)$ at $x=0$ must be strictly larger than 1. Then the order of vanishing at 0 in $f^q(x)$ must be ℓ^q; on the other hand, the function from the left hand side of equation (2.1.7) would have the order of vanishing at 0 equal to d^{q-1}. So, we must have $\ell^q = d^{q-1}$, which is impossible (note that q is a prime larger than d).

Hence, this contradiction yields that the conclusion in Lemma 2.4 must hold, as desired. □

Lemma 2.4 shows that also when $\gamma \notin \overline{\mathbb{F}}_p$, there exist infinitely many $\lambda \in \overline{L}$ such that γ is preperiodic under the action of f_λ. This concludes our proof for Proposition 2.3. □

2.2. **Canonical heights.** As usual, for each $x \in \overline{L}$, its Weil height is defined as

$$h(x) := \frac{1}{[L(x):L]} \cdot \sum_{v\in\Omega} \sum_{y\in \mathrm{Gal}(L^{\mathrm{sep}}/L)\cdot x} \log^+ |y|_v, \tag{2.2.1}$$

where $\log^+(z) = \log\max\{z,1\}$ for each real number z. Then, for each $\lambda \in \overline{L}$, the global canonical height of $x \in \overline{L}$ with respect to the polynomial f_λ is given by

$$\widehat{h}_{f_\lambda}(x) = \lim_{n\to\infty} \frac{h\left(f_\lambda^n(x)\right)}{d^n}. \tag{2.2.2}$$

Remark 2.5. If γ is preperiodic under the action of f_λ, then it is immediate to see (based on equation (2.2.2)) that $\widehat{h}_{f_\lambda}(\gamma)=0$ (since there are finitely many distinct points $f_\lambda^n(\gamma)$).

However, using [Ben05, Theorem B], one can also establish the converse statement as well, i.e., once $\widehat{h}_{f_\lambda}(\gamma)=0$, then γ must be preperiodic under the action of f_λ. Indeed, as long as $\lambda \notin \overline{\mathbb{F}}_p$, then f_λ is not isotrivial and therefore, [Ben05, Theorem B] shows that a point is preperiodic if and only if its canonical height equals 0. Finally, if $\lambda \in \overline{\mathbb{F}}_p$, then it is immediate to see that γ is preperiodic if and only if also $\gamma \in \overline{\mathbb{F}}_p$. Similarly, if $\widehat{h}_{f_\lambda}(\gamma)=0$ (and $\lambda \in \overline{\mathbb{F}}_p$), then we must have that $|\gamma|_v \le 1$ for each place $v \in \Omega$ (see also Lemma 2.6 (ii)) and therefore, we must also have that $\gamma \in \overline{\mathbb{F}}_p$ (see (2.0.3)).

Now, for each $v \in \Omega$, we let $\mathbb{C}_v$ be an algebraically closed field containing L, which is also complete with respect to a fixed extension of $|\cdot|_v$ to $\mathbb{C}_v$; more precisely, $\mathbb{C}_v$ is the completion of an algebraic closure of the completion of L at the place v. Then $\mathbb{C}_v$ is both complete and algebraically closed; furthermore, we let

$$\mathcal{O}_v := \{z \in \mathbb{C}_v \colon |z|_v \le 1\} \text{ and } \mathcal{M}_v := \{z \in \mathbb{C}_v \colon |z|_v < 1\}.$$

By our construction of $\mathbb{C}_v$, we have that for any given $z \in \mathcal{O}_v$, there exists a unique $\xi \in \overline{\mathbb{F}}_p$ such that $z - \xi \in \mathcal{M}_v$; then we write

$$z \equiv \xi \pmod{\mathcal{M}_v}.$$

Let $\lambda \in \mathbb{C}_v$ and define the local canonical height $\widehat{h}_{v,\lambda}(x)$ of $x \in \mathbb{C}_v$ with respect to the polynomial f_λ; more precisely, we have the formula

$$\widehat{h}_{v,\lambda}(x) := \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(x)|_v}{d^n}. \tag{2.2.3}$$

Then $\widehat{h}_{v,\lambda}(x)$ is a continuous function of both λ and x on $\mathbb{C}_v$; furthermore, the following holds:

$$\widehat{h}_{v,\lambda}(x) = \frac{\widehat{h}_{v,\lambda}(f_\lambda^m(x))}{d^m} \text{ for each } m \in \mathbb{N} \text{ and for each } x \in \mathbb{C}_v. \tag{2.2.4}$$

The following easy Lemma 2.6 is employed multiple times in our proof.

Lemma 2.6. *Let* $\gamma, \lambda \in \mathbb{C}_v$.

(i) If $\max\{|\lambda|_v, |\gamma|_v\} \le 1$, *then*

$$\widehat{h}_{v,\lambda}(\gamma) = 0.$$

(ii) If $|\gamma|_v^d > \max\{1, |\lambda|_v\}$, *then*

$$\widehat{h}_{v,\lambda}(\gamma) = \log|\gamma|_v > 0. \tag{2.2.5}$$

(iii) If $|\lambda|_v > \max\left\{1, |\gamma|_v^d\right\}$, *then*

$$\widehat{h}_{v,\lambda}(\gamma) = \frac{\log|\lambda|_v}{d} > 0.$$

Proof of Lemma 2.6. We first note that for any $\gamma \in \mathbb{C}_v$, we have the following:

$$\max\{1, |f(\gamma)|_v\} = \max\left\{1, |\gamma|_v^d\right\}, \tag{2.2.6}$$

since $f \in \overline{\mathbb{F}}_p[x]$ has degree d (and therefore, each nonzero coefficient of f is a v-adic unit). Then conclusion (i) is immediate since knowing that both λ and γ are integral at the place v yields that each $f_\lambda^n(\gamma)$ is integral at v, thus showing that $\widehat{h}_{v,\lambda}(\gamma) = 0$.

Next, we work under the hypotheses from part (ii). The fact that $|\gamma|_v^d > \max\{1, |\lambda|_v\}$ yields (see also equation (2.2.6)) that

$$|f_\lambda(\gamma)|_v = |f(\gamma) + \lambda|_v = |\gamma|_v^d > |\gamma|_v.$$

An easy induction on n shows that for each $n \ge 1$, we have that

$$|f_\lambda^n(\gamma)|_v = |\gamma|_v^{d^n};$$

then the desired conclusion in part (ii) follows.

Finally, part (iii) is a consequence of part (ii) because the inequality $|\lambda| > \max\left\{1, |\gamma|_v^d\right\}$ yields

$$|f_\lambda(\gamma)|_v = |f(\gamma) + \lambda|_v = |\lambda|_v > |\lambda|_v^{\frac{1}{d}}. \tag{2.2.7}$$

Equation (2.2.7) allows us to apply the conclusion from part (ii) to the point $f_\lambda(\gamma)$ and the parameter λ and thus, we get

$$\widehat{h}_{f_\lambda}(f_\lambda(\gamma)) = |f_\lambda(\gamma)|_v = |\lambda|_v.$$

Then equation (2.2.4) yields the desired conclusion in Lemma 2.6, part (iii). □

As a corollary to Lemma 2.6, we have the following.

Corollary 2.7. *If $\lambda \in \overline{\mathbb{F}}_p$ then for $v \in \Omega$, the local canonical height $\widehat{h}_{v,\lambda}(\gamma) = \log^+ |\gamma|_v$ for each $\gamma \in \mathbb{C}_v$.*

2.3. **The filled Julia set for polynomial dynamics.** For each $v \in \Omega$ and $\lambda \in \overline{L}$, recall that the v-adic filled Julia set for f_λ is defined to be the set

$$K_{v,\lambda} := \{z \in \mathbb{C}_v \mid \widehat{h}_{v,\lambda}(z) = 0\} = \{z \in \mathbb{C}_v \mid |f_\lambda^n(z)|_v \not\to \infty \text{ as } n \to \infty\}.$$

It follows from Lemma 2.6 that $K_{v,\lambda}$ is contained in the closed disk $D_{v,\lambda}$ (centered at 0) of radius $R_{v,\lambda} = \max\{1, |\lambda|_v^{1/d}\}$. It follows from Lemma 2.6 (ii) that for $z \in \mathbb{C}_v$ with $|z|_v > R_{v,\lambda}$ we have $\widehat{h}_{v,\lambda}(x) = |z|_v$.

In the following, we set $E_{1,\lambda} = D_{v,\lambda}$ and $E_{n,\lambda} = f_\lambda^{-1}(E_{n-1,\lambda})$ for $n > 1$.

Remark 2.8. Clearly, the sets $E_{n,\lambda}$ refer to a fixed place v, but we prefer (for the sake of simplifying our notation) that we do not add v in the subscript for these sets. The same observation applies next to the level sets $L_{n,\lambda}$.

Note that in the case where $|\lambda|_v \le 1$, our polynomial f_λ induces a surjective self-map on $D_{v,\lambda} = D_v(0,1)$ the closed unit disk in $\mathbb{C}_v$. It is not hard to see that $E_{n,\lambda} = D_v(0,1)$ for all n and $K_\lambda = D_v(0,1)$.

We will be mainly concerned with the case where $|\lambda|_v > 1$. In this case, we see that $D_{v,\lambda}$ is mapped by f_λ into a disk (again centered at 0) of larger radius which is equal to $|\lambda|_v$. We denote by $E_{0,\lambda} := f_\lambda(D_{v,\lambda}) \subseteq D_v(0, |\lambda|_v)$. It is also easy to check that

$$E_{n+1,\lambda} \subsetneq E_{n,\lambda} \text{ for all } n \ge 0 \text{ and moreover, } K_{v,\lambda} = \bigcap_{n \ge 0} E_{n,\lambda}.$$

For any integer $n \ge 0$, we set $L_{n,\lambda} := E_{n,\lambda} \setminus E_{n+1,\lambda}$, called the *level set* for $\widehat{h}_{v,\lambda}$ of level n. In particular, each point $z \in L_{0,\lambda} = E_{0,\lambda} \setminus E_{1,\lambda}$ has the property that $|\lambda|_v^{1/d} < |z|_v \le |\lambda|_v$. Furthermore, if $z \in L_{n,\lambda}$ then we have that

$$f_\lambda^n(z) \in L_{0,\lambda} \quad \text{and} \quad \widehat{h}_{v,\lambda}(z) = \frac{\widehat{h}_{v,\lambda}(f_\lambda^n(z))}{d^n} = \frac{\log |f_\lambda^n(z)|_v}{d^n} > 0, \tag{2.3.1}$$

according to Lemma 2.6, part (ii).

Remark 2.9. If $|\lambda|_v > 1$, the filled Julia set $K_{v,\lambda}$ is the v-adic Julia set for f_λ.

The introduction of the level sets $L_{n,\lambda}$ is a novelty compared to [Ghi] (in which the analysis was more ad-hoc); we believe that the key to settling Conjecture 1.7 lies in a thorough analysis of these level sets.

2.4. **Equality of the respective local canonical heights.** In this subsection, we outline the proof of Theorem 2.11 below. The structure and arguments are the same as those used in the proof of [Ghi], although the polynomial $f(x)$ considered here is more general than that in [Ghi]. To avoid repetition, we will provide only a sketch and refer the reader to [Ghi, Section 2 and Section 3] for the full details.

Let $v \in \Omega$ be a given place of L and let $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ denote the Berkovich affine line over $\mathbb{C}_v$ (see [BR10] or [BD11, Section 2] for more details). Let $\gamma \in L$. Then, the *generalized Mandelbrot set* $M_{\gamma,v} \subset \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ associated to γ at v is defined to be the closure in $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ of the subset of $\mathbb{C}_v$ consisting of all parameters $\lambda \in \mathbb{C}_v$ such that the orbit of γ is v-adically bounded under the action of f_λ. Note that for such a parameter λ, since the orbit of γ is bounded under f_λ we have that $\widehat{h}_{v,\lambda}(\gamma) = 0$. As $\mathbb{C}_v$ is a dense subspace of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, continuity in λ implies that the canonical local height function $\widehat{h}_{v,\lambda}(\gamma)$ has a natural extension on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. It follows that $\lambda \in M_{\gamma,v}$ if and only if $\widehat{h}_{v,\lambda}(\gamma) = 0$. Thus, $M_{\gamma,v}$ is a closed subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ and in fact, one can show that $M_{\gamma,v}$ is a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$.

Associated with γ, we define

$$G_{\gamma,v}(\lambda) := \lim_{n\to\infty} \frac{\log^+ |f^n_\lambda(\gamma)|_v}{d^{n-1}} = d \cdot \widehat{h}_{v,\lambda}(\gamma). \tag{2.4.1}$$

Note that $G_{\gamma,v}(\lambda) \geq 0$ for all $\lambda \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$; also, $\lambda \in M_{\gamma,v}$ if and only if $G_{\gamma,v}(\lambda) = 0$. It turns out that $G_{\gamma,v}$ is the Green's function for $M_{\gamma,v}$ relative to ∞. We define the *generalized adèlic Mandelbrot set* $\mathbb{M}_\gamma = \prod_{v\in\Omega} M_{\gamma,v}$ associated with γ and for each $\lambda \in \overline{L}$, we set

$$h_{\mathbb{M}_\gamma}(\lambda) := h_{\mathbb{M}_\gamma}(S) = \sum_{v\in\Omega_L} \left(\frac{1}{|S|} \sum_{z\in S} G_{\gamma,v}(z) \right) \text{ where } S \text{ is the } \mathrm{Gal}(L^{\mathrm{sep}}/L)\text{-orbit of } \lambda. \tag{2.4.2}$$

It turns out that $\mathbb{M}_\gamma$ is a compact Berkovich adèlic set with the *logarithmic capacity* $\mathbf{c}(\mathbb{M}_\gamma) = 1$ and $h_{\mathbb{M}_\gamma}(\lambda)$ represents the height of λ relative to $\mathbb{M}_\gamma$. Consequently, the equidistribution result [BR10, Theorem 7.52] applies. The following is a special case we need for our application:

Theorem 2.10. *With the above notation, let $\mathbb{E} = \prod_{v\in\Omega} E_v$ be a compact Berkovich adèlic set with $\mathbf{c}(\mathbb{E}) = 1$. Suppose that S_n is a sequence of $\mathrm{Gal}(L^{\mathrm{sep}}/L)$-invariant finite subsets of L^{sep} with $|S_n| \to \infty$ and $h_{\mathbb{E}}(S_n) \to 0$ as $n \to \infty$. For each $v \in \Omega_L$ and for each n let δ_n be the discrete probability measure supported equally on the elements of S_n. Then the sequence of measures $\{\delta_n\}$ converges weakly to μ_v the equilibrium measure on E_v.*

Now, we can now apply Theorem 2.10 to prove the following result.

Theorem 2.11. *Let L, f_λ, $\widehat{h}_{f_\lambda}$, $\widehat{h}_{v,\lambda}$ be defined as in Section 2; also, let $\alpha, \beta \in L$. Assume there exists an infinite sequence $\{\lambda_n\}$ in $\overline{L}$ with the property that*

$$\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\beta) = 0. \tag{2.4.3}$$

Then for each each $v \in \Omega$ and for each $\lambda \in \mathbb{C}_v$, we have that $\widehat{h}_{v,\lambda}(\alpha) = \widehat{h}_{v,\lambda}(\beta)$.

Proof. Since the proof is identical with the one of [Ghi, Theorem 4.1], we omit it here. ☐

3. Proof of the precise relation between the starting points

We begin to establish the proof for our main result, which is Theorem 1.1 (as explained in Remark 1.3, Theorem 1.2 is an immediate consequence of Theorem 1.1). We will actually prove a more general result in Theorem 3.2 below.

Recall from Section 2 that the perfect closure of $\overline{\mathbb{F}}_p(t)$ is denoted by

$$L_0 := \overline{\mathbb{F}}_p\left(t, t^{1/p}, t^{1/p^2}, \cdots, t^{1/p^n}, \cdots\right).$$

Let L be a finite extension of L_0 and let $f \in \overline{\mathbb{F}}_p[x]$ be a monic polynomial which has the following form (for some $r \geq 1$):

$$f(x) = \sum_{i=1}^{r} c_i x^{d_i} \text{ with each } c_i \in \overline{\mathbb{F}}_p^{\,*} \text{ (and } c_r = 1), \tag{3.0.1}$$

where $1 \leq d_1 < d_2 < \cdots < d_r = d$; furthermore, writing (for each $i = 1, \ldots, r$) $d_i = p^{\ell_i} \cdot s_i$ where $\ell_i \geq 0$ and $p \nmid s_i$, then we assume the following inequality holds:

$$p^{\ell_r}(s_r - 1) > \max\left\{1, p^{\ell_1}(s_1 - 1), p^{\ell_2}(s_2 - 1), \cdots, p^{\ell_{r-1}}(s_{r-1} - 1)\right\}. \tag{3.0.2}$$

Remark 3.1. We observe that automatically inequality (3.0.2) yields that $d \geq 3$.

The following key result will be proven in Section 4.

Theorem 3.2. *Let $f \in \overline{\mathbb{F}}_p[x]$ be a polynomial of degree $d \geq 3$ of the form (3.0.1) satisfying inequality (3.0.2). We consider the family of polynomials $f_\lambda(x) := f(x) + \lambda$ parameterized by $\lambda \in \overline{L}$ and let $\alpha, \beta \in L$. Then there exists an infinite sequence $\{\lambda_n\}_{n\geq 1}$ in $\overline{L}$ with the property that*

$$\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\beta) = 0, \tag{3.0.3}$$

if and only if at least one of the following statements holds:

(1) $f(\alpha) = f(\beta)$; or
(2) $\alpha, \beta \in \overline{\mathbb{F}}_p$.

Moreover, if either one of the conditions (1)-(2) holds, then for each $\lambda \in \overline{L}$, we have that α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

It follows from Theorem 2.11 that Equation (3.0.3) leads to the equalities between the local canonical heights $\widehat{h}_{v,\lambda}(\alpha)$ and $\widehat{h}_{v,\lambda}(\beta)$ for each place $v \in \Omega$ and for each $\lambda \in \mathbb{C}_v$. We can then derive the precise relationship between α and β from the equalities of the local canonical heights as in the following result.

Proposition 3.3. *Let $f \in \overline{\mathbb{F}}_p[x]$ and let $f_\lambda(x) := f(x) + \lambda$ be as given in Theorem 3.2. Let $\alpha, \beta \in L$, not both of them contained in $\overline{\mathbb{F}}_p$. If for each $v \in \Omega$ and for each $\lambda \in \mathbb{C}_v$, we have that*

$$\widehat{h}_{v,\lambda}(\alpha) = \widehat{h}_{v,\lambda}(\beta), \tag{3.0.4}$$

then we must have that $f(\alpha) = f(\beta)$.

Therefore, Proposition 3.3 is the heart of the argument for all our results and we devote the remaining part of Section 3 for its proof.

3.1. **General strategy for proving Proposition 3.3.** From now on in this Section, we work under the hypotheses from Proposition 3.3. Our strategy follows the proof of [Ghi, Proposition 5.1]; however, there are some technical difficulties in our more general setting.

So, we let S be the (finite) set of places $v \in \Omega$ with the property that

$$\max\{|\alpha|_v, |\beta|_v\} > 1. \tag{3.1.1}$$

Note that our hypothesis from Proposition 3.3 that not both α and β live in $\overline{\mathbb{F}}_p$ yields that S is a *nonempty* set. We also let

$$\epsilon := f(\alpha) - f(\beta). \tag{3.1.2}$$

Our strategy will be to prove that

$$|\epsilon|_v < 1 \text{ for each } v \in S. \tag{3.1.3}$$

Now, since S consists of all the places v where α or β is not v-adic integral (see inequality (3.1.1)), then the only places of Ω for which ϵ may not be a v-adic integer are exactly the ones from the set S (see also equation (2.2.6)). So, inequality (3.1.3) would prove that ϵ is integral at each place $v \in \Omega$ and furthermore, there is at least one place v_0 in Ω such that $|\epsilon|_{v_0} < 1$. Due to the product formula (2.0.2) on L (see also (2.0.3)), this delivers the desired conclusion that $\epsilon = 0$ (i.e., $f(\alpha) = f(\beta)$).

3.2. $|\epsilon|_v$ **is strictly less than** $|\alpha|_v$**.** In Subsection 3.2, we fix some place $v \in S$.

We first observe that the canonical local height $\widehat{h}_{v,\lambda}$ is just the Weil local height $h_v(\cdot) = \log^+|\cdot|$ for any $\lambda \in \overline{\mathbb{F}}_p$, by Corollary 2.7. Thus, we have the following.

Lemma 3.4. *We have* $|\alpha|_v = |\beta|_v > 1$.

Proof. The conclusion follows by choosing any parameter $\lambda \in \overline{\mathbb{F}}_p$. Then, we have that $\widehat{h}_{v,\lambda}(\alpha) = \log^+|\alpha|_v$ and $\widehat{h}_{v,\lambda}(\beta) = \log^+|\beta|_v$ for each $v \in \Omega$ by Corollary 2.7. In particular, for $v \in S$, we have $\log^+|\alpha|_v = \log|\alpha|_v$ or $\log^+|\beta|_v = \log|\beta|_v$. The equality between $\widehat{h}_{v,\lambda}(\alpha)$ and $\widehat{h}_{v,\lambda}(\beta)$ implies that $|\alpha|_v = |\beta|_v > 1$ for $v \in S$, as desired. ☐

The next step is to choose appropriate parameters λ such that both α and β are in the same level sets $L_{n,\lambda}$ for some n. To do this, we take parameters λ satisfying $|\lambda|_v = |\alpha|_v^d$ for $v \in S$. Then, we have that $E_{1,\lambda} = D_{v,\lambda}$ is a disk of radius $|\alpha|_v$. Note that the disk $D_{v,\lambda}$ is partitioned into a disjoint union of smaller disks. Each disk contains a unique element $\gamma \cdot \alpha$ for some $\gamma \in \overline{\mathbb{F}}_p$ as its center. That is,

$$D_{v,\lambda} = \sqcup_{\gamma \in \overline{\mathbb{F}}_p} D(\gamma \cdot \alpha, |\alpha|_v)^-$$

where $D(\gamma \cdot \alpha, |\alpha|_v)^-$ denotes the open disk centered at $\gamma \cdot \alpha$ of radius $|\alpha|_v$. Also, note that for each $w \in D_{v,\lambda}$, there exists a unique $u_w \in \mathcal{O}_v$ such that $w = u_w \cdot \alpha$.

Proposition 3.5. *If* $\lambda = -\kappa\alpha^d$ *for some* $\kappa \in \mathcal{O}_v^*$ *with the property that* $\kappa \equiv 1 \pmod{\mathcal{M}_v}$, *then the set*

$$E_{2,\lambda} = f_\lambda^{-1}(E_{1,\lambda}) \subset \sqcup_{\zeta \in \mu_d} D(\zeta \cdot \alpha, |\alpha|_v)^-,$$

where μ_d *denotes the set of all the* d*-th roots of unity. In particular, we have that*

$$L_{2,\lambda} \subset E_{2,\lambda} \subset \sqcup_{\zeta \in \mu_d} D(\zeta \cdot \alpha, |\alpha|_v)^-$$

Moreover, the set $E_{2,\lambda} \cap D(\zeta \cdot \alpha, |\alpha|_v)^-$ *is nonempty for every* $\zeta \in \mu_d$.

Proof. Suppose that $w = u_w\alpha \in E_{2,\lambda}$ for some $u_w \in \mathcal{O}_v$, then

$$\begin{aligned} f_\lambda(u_w\alpha) &= (u_w\alpha)^d + \sum_{i=1}^{r-1} c_i(u_w\alpha)^{d_i} - \kappa\alpha^d \\ &= \alpha^d \left[(u_w^d - \kappa) + \sum_{i=1}^{r-1} c_i u_w^{d_i}\alpha^{d_i-d} \right] \in E_{1,\lambda} (= D_{v,\lambda}). \end{aligned}$$

Since $f_\lambda(w) \in E_{1,\lambda}$, we must have $|f_\lambda(u_w\alpha)|_v \le |\alpha|_v < |\alpha|_v^d$. It follows that $|u_w^d - \kappa|_v < 1$ (note that $|\alpha|_v^{-1} < 1$, by Lemma 3.4 and the fact that $v \in S$) and thus $u_w^d \equiv \kappa \equiv 1 \pmod{\mathcal{M}_v}$. Equivalently, $u_w \equiv \zeta \pmod{\mathcal{M}_v}$ for some d-th root of unity ζ. Hence, $w \in D(\zeta\cdot\alpha, |\alpha|_v)^-$ as desired.

Now, let ζ be a d-th root of unity. In order to show that the intersection $E_{2,\lambda} \cap D(\zeta \cdot \alpha, |\alpha|_v)^-$ is nonempty, it suffices to show that there exists a $z \in \mathcal{O}_v$ with $|z|_v < 1$ such that $f_\lambda\left((\zeta + z)\cdot\alpha\right) = \xi\cdot\alpha$ for some $\xi \in \mathcal{O}_v$. This is equivalent with asking that

$$(\zeta + z)^d \cdot \alpha^d + \sum_{i=1}^{r-1} c_i(\zeta+z)^{d_i}\cdot\alpha^{d_i} - \kappa\alpha^d = \xi\alpha,$$

which leads to the following equation:

$$(z+\zeta)^d + \sum_{i=1}^{r-1} c_i (z+\zeta)^{d_i}\alpha^{d_i-d} - \kappa - \xi\cdot\alpha^{1-d} = 0. \tag{3.2.1}$$

We note that $\zeta^d = 1 \equiv \kappa \pmod{\mathcal{M}_v}$ (according to the hypotheses of Proposition 3.5) and also that $|\alpha|_v^{d_i-d} < 1$ for each $i = 1, \dots, r-1$ (since $d_i < d$ and also, because $v \in S$ and so, $|\alpha|_v > 1$). Hence, the equation (3.2.1) has a solution $z \in \mathbb{C}_v$ with $|z|_v < 1$, as claimed. Therefore, the set $E_{2,\lambda} \cap D(\zeta\cdot\alpha, |\alpha|_v)^-$ is nonempty as desired in the conclusion of Proposition 3.5. □

Remark 3.6. It is not necessary to choose $\kappa \equiv 1 \pmod{\mathcal{M}_v}$ in Proposition 3.5. The arguments also apply if $\kappa \equiv \rho \pmod{\mathcal{M}_v}$ for any $\rho \in \overline{\mathbb{F}}_p^{\,*}$. We still get that the subset $E_{2,\lambda}$ (and hence $K_{v,\lambda}$) is contained in *finitely many* open disks inside $D_{v,\lambda}$.

For any $\gamma \in \mathcal{O}_v$, we consider the parameter λ_γ such that $f_{\lambda_\gamma}(\alpha) = \gamma\cdot\alpha$; more precisely, $\lambda_\gamma := \gamma\cdot\alpha - f(\alpha)$. Note that if $\gamma^d \not\equiv 1 \pmod{\mathcal{M}_v}$ then $\gamma\alpha \in L_{1,\lambda_\gamma}$, since

$$\left|f_{\lambda_\gamma}(\gamma\alpha)\right|_v = |f(\lambda_\gamma\alpha) + \gamma\alpha - f(\alpha)|_v = \left|(\lambda_\gamma^d - 1)\cdot\alpha^d\right|_v = |\alpha|_v^d > |\alpha|_v. \tag{3.2.2}$$

Moreover, $f_{\lambda_\gamma}(\beta) = f(\beta) + \gamma\cdot\alpha - f(\alpha) = \epsilon + \gamma\cdot\alpha$.

Lemma 3.7. *We have that* $|\epsilon|_v < |\alpha|_v$.

Proof. Choose a $\gamma \in \mathcal{O}_v$ such that $\gamma^d \not\equiv 1 \pmod{\mathcal{M}_v}$ and a parameter λ_γ such that $f_{\lambda_\gamma}(\alpha) = \gamma\cdot\alpha \in L_{1,\lambda}$; i.e., $\lambda_\gamma = \gamma\alpha - f(\alpha)$ (see also equation (3.2.2)). The fact that $f_{\lambda_\gamma}(\alpha) \in L_{1,\lambda_\gamma}$ gives us that $f^2_{\lambda_\gamma}(\alpha) \in E_{0,\lambda_\gamma} \setminus E_{1,\lambda_\gamma}$ and so, $\left|f^2_{\lambda_\gamma}(\alpha)\right|_v > |\lambda_\gamma|_v^{1/d} = |\alpha|_v$. This yields that (see also equation (2.3.1))

$$\widehat{h}_{v,\lambda_\gamma}(\alpha) = \frac{\widehat{h}_{v,\lambda_\gamma}\left(f^2_{\lambda_\gamma}(\alpha)\right)}{d^2} = \frac{\log\left|f^2_{\lambda_\gamma}(\alpha)\right|_v}{d^2} > \frac{\log|\alpha|_v}{d^2}. \tag{3.2.3}$$

On the other hand, since $f_{\lambda_\gamma}^2(\alpha) \in E_{0,\lambda_\gamma}$, we have that $\left|f_{\lambda_\gamma}^2(\alpha)\right|_v \le |\lambda_\gamma|_v = |\alpha|_v^d$. So, we also get that

$$\widehat{h}_{v,\lambda_\gamma}(\alpha) = \frac{\widehat{h}_{v,\lambda_\gamma}\left(f_{\lambda_\gamma}^2(\alpha)\right)}{d^2} = \frac{\log\left|f_{\lambda_\gamma}^2(\alpha)\right|_v}{d^2} \le \frac{\log|\alpha|_v}{d}. \tag{3.2.4}$$

Now, since $\widehat{h}_{v,\lambda_\gamma}(\beta) = \widehat{h}_{v,\lambda_\gamma}(\alpha)$, we must have that $\beta \in E_{1,\lambda_\gamma}$ because otherwise, Lemma 2.6 (ii) yields that $\widehat{h}_{v,\lambda_\gamma}(\beta) = \log|\beta|_v = \log|\alpha|_v > \widehat{h}_{v,\lambda_\gamma}(\alpha)$ (see also inequality (3.2.4)). So, because $f_{\lambda_\gamma}(\beta) = \lambda_\gamma\alpha + \epsilon$ and $\beta \in E_{1,\lambda_\gamma} = D_v(0, |\alpha|_v)$, we must have that $|\epsilon|_v \le |\alpha|_v$.

Now, suppose that $|\epsilon|_v = |\alpha|_v$. Then, we may write $\epsilon = u\alpha$ for some $u \in \mathcal{O}_v^*$ and thus $f_{\lambda_\gamma}(\beta) = (u+\gamma)\cdot\alpha$. We prove the following useful claim.

Claim 3.8. There exists a $\gamma \in \mathcal{O}_v$ such that $\gamma^d \not\equiv 1 \pmod{\mathcal{M}_v}$ and $f_{\lambda_\gamma}(\beta) \in E_{2,\lambda_\gamma}$.

Proof of Claim 3.8. Let ξ be the unique element in $\overline{\mathbb{F}}_p^{\,*}$ such that $u \equiv \xi \pmod{\mathcal{M}_v}$. By [Ghi, Claim 5.7], there exists a $\nu \in \overline{\mathbb{F}}_p$ satisfying $\nu^d \ne 1$ and $(\xi+\nu)^d = 1$. We let $\zeta := \xi + \nu \in \mu_d$ and write $\gamma = \nu + z$ for some suitable $z \in \mathcal{M}_v$ which will be determined next.

By Proposition 3.5, the set $E_{2,\lambda} \cap D(\zeta\cdot\alpha, |\alpha|_v)^-$ is nonempty. Hence, there exists an element in E_{2,λ_γ} of the form $\kappa\cdot\alpha$ for some $\kappa \in \mathcal{O}_v$ such that $\kappa \equiv \zeta \pmod{\mathcal{M}_v}$. We let $z := \kappa - (u+\nu)$; clearly, $|z|_v < 1$. Then the point $\gamma = \nu + z \in \mathcal{O}_v$ satisfies the property that $\gamma^d \not\equiv 1 \pmod{\mathcal{M}_v}$ along with the fact that $f_{\lambda_\gamma}(\beta) = \kappa\cdot\alpha \in E_{2,\lambda_\gamma}$, as desired in the conclusion of Claim 3.8. □

Now, with γ satisfying the desired conditions from Claim 3.8, we have that $f_{\lambda_\gamma}(\beta) \in E_{2,\lambda_\gamma}$ and therefore, $\beta \in E_{3,\lambda_\gamma}$. Thus, $\left|f_{\lambda_\gamma}^3(\beta)\right|_v \le |\lambda_\gamma|_v = |\alpha|_v^d$ and so, we get

$$\widehat{h}_{v,\lambda_\gamma}(\beta) = \frac{\widehat{h}_{v,\lambda_\gamma}\left(f_{\lambda_\gamma}^3(\beta)\right)}{d^3} \le \frac{\log|\alpha|_v^d}{d^3} = \frac{\log|\alpha|_v}{d^2}. \tag{3.2.5}$$

Inequalities (3.2.5) and (3.2.3) contradict the fact that $\widehat{h}_{v,\lambda_\gamma}(\beta) = \widehat{h}_{v,\lambda_\gamma}(\alpha)$. Hence, $|\epsilon|_v < |\alpha|_v$ as desired in the conclusion of Lemma 3.7. □

3.3. Final step in the proof of Proposition 3.3.

We continue our analysis for $|\epsilon|_v = |f(\beta) - f(\alpha)|_v$ (for a given $v \in S$) with the goal of proving the inequality from (3.1.3).

Lemma 3.9. *For each $v \in S$, we must have that $|\epsilon|_v < 1$.*

Proof of Lemma 3.9. First, we recall that from Lemma 3.7, we already know

$$|\epsilon|_v < |\alpha|_v \text{ for each } v \in S. \tag{3.3.1}$$

We let $\lambda_1 := \alpha - f(\alpha)$. Then clearly, $f_{\lambda_1}(\alpha) = \alpha$, which means that for any place v (not just the ones from the set S), we have that

$$\widehat{h}_{v,\lambda_1}(\alpha) = 0. \tag{3.3.2}$$

Thus, by hypothesis (3.0.4) of Proposition 3.3, we must also have that $\widehat{h}_{v,\lambda_1}(\beta) = 0$ for each $v \in S$. We compute

$$f_{\lambda_1}(\beta) = f(\beta) + \alpha - f(\alpha) = \epsilon + \alpha$$

(note that $\epsilon = f(\beta) - f(\alpha)$) and

$$f_{\lambda_1}^2(\beta) = f(\epsilon+\alpha) - f(\alpha) + \alpha. \tag{3.3.3}$$

We assume that $|\epsilon|_v \geq 1$ for some $v \in S$ and we will derive a contradiction. More precisely, using also inequality (3.3.1), we assume that for some $v \in S$, we have the inequalities:

$$1 \leq |\epsilon|_v < |\alpha|_v. \tag{3.3.4}$$

Observe that for any integers $\ell \geq 0$ and $s > 1$ with $p \nmid s$, we have

(3.3.5)
$$(x+y)^{p^\ell s} = \left(x^{p^\ell} + y^{p^\ell}\right)^s = x^{p^\ell s} + y^{p^\ell s} + s x^{p^\ell (s-1)} y^{p^\ell} + \binom{s}{2} x^{p^\ell (s-2)} y^{2p^\ell} + \cdots + s x^{p^\ell} y^{p^\ell (s-1)}.$$

From inequalities (3.3.4), we have that

$$\left|(\epsilon+\alpha)^{p^\ell s} - \epsilon^{p^\ell s} - \alpha^{p^\ell s}\right|_v = |\epsilon|_v^{p^\ell} \cdot |\alpha|_v^{p^\ell (s-1)}. \tag{3.3.6}$$

Indeed, since $1 \leq |\epsilon|_v < |\alpha|_v$, it follows that $\left|\binom{s}{j} \epsilon^{p^\ell (s-j)} \alpha^{p^\ell j}\right|_v < \left|s \epsilon^{p^\ell} \alpha^{p^\ell (s-1)}\right|_v$ for $1 \leq j < s-1$ and hence equality (3.3.6) holds. Furthermore, we claim that inequalities (3.0.2) and (3.3.4) yield the following inequalities:

(3.3.7)
$$\left|(\epsilon+\alpha)^{d_i} - \epsilon^{d_i} - \alpha^{d_i}\right|_v \leq |\epsilon|_v^{p^{\ell_i}} \cdot |\alpha|_v^{p^{\ell_i}(s_i-1)} < |\epsilon|_v^{p^{\ell_r}} \cdot |\alpha|_v^{p^{\ell_r}(s_r-1)} = \left|(\epsilon+\alpha)^{d_r} - \epsilon^{d_r} - \alpha^{d_r}\right|_v.$$

for all $1 \leq i < r$. In order to justify the first inequality in (3.3.7), note that if $d_i = p^{\ell_i}$ (i.e., $s_i = 1$), then simply $(\epsilon+\alpha)^{d_i} = \epsilon^{d_i} + \alpha^{d_i}$; on the other hand, if $s_i > 1$, the first inequality in (3.3.7) follows from (3.3.6) (in which case, it is an equality).

Now, in order to prove the second inequality from (3.3.7), suppose to the contrary that

$$|\epsilon|_v^{p^{\ell_i}} |\alpha|_v^{p^{\ell_i}(s_i-1)} \geq |\epsilon|_v^{p^{\ell_r}} |\alpha|_v^{p^{\ell_r}(s_r-1)} \quad \text{for some } i < r. \tag{3.3.8}$$

Then,

$$\left|\frac{\epsilon}{\alpha}\right|_v^{p^{\ell_i}} |\alpha|_v^{p^{\ell_i} s_i} \geq \left|\frac{\epsilon}{\alpha}\right|_v^{p^{\ell_r}} |\alpha|_v^{p^{\ell_r} s_r}$$

and thus

$$\left|\frac{\epsilon}{\alpha}\right|_v^{p^{\ell_i} - p^{\ell_r}} \geq |\alpha|_v^{d-d_i} \geq |\alpha|_v > 1.$$

Since $|\epsilon|_v < |\alpha|_v$, we must have $p^{\ell_i} - p^{\ell_r} < 0$. On the other hand, $|\epsilon|_v \geq 1$ by our assumption (3.3.4), we obtain that $|\epsilon|_v^{p^{\ell_i}} \leq |\epsilon|_v^{p^{\ell_r}}$. Inequality (3.0.2) gives that $|\alpha|_v^{p^{\ell_i}(s_i-1)} < |\alpha|_v^{p^{\ell_r}(s_r-1)}$. Therefore,

$$|\epsilon|_v^{p^{\ell_i}} |\alpha|_v^{p^{\ell_i}(s_i-1)} < |\epsilon|_v^{p^{\ell_r}} |\alpha|_v^{p^{\ell_r}(s_r-1)}$$

which contradicts (3.3.8) and hence the inequality (3.3.7) must hold for all i with $1 \leq i < r$.

Clearly,

$$f(\epsilon+\alpha) = f(\epsilon) + f(\alpha) + \sum_{i=1}^{r} c_i \left[(\epsilon+\alpha)^{d_i} - \epsilon^{d_i} - \alpha^{d_i}\right]. \tag{3.3.9}$$

We write $g(\epsilon, \alpha) = f(\epsilon+\alpha) - f(\epsilon) - f(\alpha)$ so that

$$f_{\lambda_1}^2(\beta) = f_{\lambda_1}(\alpha+\epsilon) = g(\epsilon, \alpha) + \alpha + f(\epsilon). \tag{3.3.10}$$

We have from (3.3.7) that

$$|g(\epsilon, \alpha)|_v = |\epsilon|_v^{p^{\ell_r}} |\alpha|_v^{p^{\ell_r}(s_r-1)}. \tag{3.3.11}$$

Furthermore, another application of the left inequality from (3.3.4) along with inequality (3.0.2) yields that

$$|\epsilon|_v^{p^{\ell_r}} \cdot |\alpha|_v^{p^{\ell_r}(s_r-1)} \geq |\alpha|_v^{p^{\ell_r}(s_r-1)} > |\alpha|_v. \tag{3.3.12}$$

Also, the right inequality from (3.3.4) yields (see also equation (2.2.6))

$$|\epsilon|_v^{p^{\ell_r}} \cdot |\alpha|_v^{d-p^{\ell_r}} > |\epsilon|_v^d \geq |f(\epsilon)|_v. \tag{3.3.13}$$

Equations (3.3.10), (3.3.11), (3.3.12) and (3.3.13) deliver the fact that

$$\left|f_{\lambda_1}^2(\beta)\right|_v = |g(\epsilon,\alpha)|_v = \left|\epsilon^{p^{\ell_r}} \cdot \alpha^{d-p^{\ell_r}}\right|_v \geq |\alpha|_v^{d-p^{\ell_r}} > |\alpha|_v \tag{3.3.14}$$

where in the last two inequalities we also used equations (3.3.4) and the inequality $p^{\ell_r}(s_r-1) > 1$ from (3.0.2).

It follows that $f_{\lambda_1}^2(\beta) \notin E_{1,\lambda_1} = D_{v,\lambda_1}$ since D_{v,λ_1} is the closed disk of radius $|\alpha|_v$. Thus, the v-local canonical height is computed as follows:

$$\widehat{h}_{v,\lambda_1}\left(f_{\lambda_1}^2(\beta)\right) = \log\left|f_{\lambda_1}^2(\beta)\right|_v = \log\left|\epsilon^{\ell_r} \cdot \alpha^{d-p^{\ell_r}}\right|_v > 0. \tag{3.3.15}$$

Finally, using equations (3.3.15) and (2.2.4), we conclude that $\widehat{h}_{v,\lambda_1}(\beta) > 0$. Coupled with equation (3.3.2), this contradicts the main hypothesis (3.0.4) of Proposition 3.3 that $\widehat{h}_{v,\lambda_1}(\beta) = \widehat{h}_{v,\lambda_1}(\alpha)$. This concludes our proof of Lemma 3.9. □

Now we can finish our proof of Proposition 3.3.

Proof of Proposition 3.3. As before, we recall the notation $\epsilon = f(\beta) - f(\alpha)$. By definition of the set S (see (3.1.1)), we have that $|\epsilon|_v \leq 1$ for every $v \notin S$. On the other hand, Lemma 3.9 yields that $|\epsilon|_v < 1$ if $v \in S$. Hence, ϵ is integral at all places and furthermore for the places $v \in S$ (note that S is nonempty due to our assumption that not both α and β are in $\overline{\mathbb{F}}_p$), we have that $|\epsilon|_v < 1$; this contradicts the product formula (2.0.2), unless $\epsilon = 0$, which is precisely the desired conclusion from Proposition 3.3. □

4. Proof of our main results

In this section, we prove our main results, Theorem 1.1, Theorem 1.2 and Theorem 3.2. We show in Subsection 4.1 that Theorem 1.1 follows from Theorem 3.2 and that Theorem 1.2 follows from Theorem 1.1. Then we finish the proof of Theorem 3.2 in Subsection 4.2.

4.1. **Proof of Theorems 1.1 and 1.2 assuming Theorem 3.2 holds.** First, we obtain a simple reduction in both Theorems 1.1 and 1.2.

Proposition 4.1. *We may assume in both Theorems 1.1 and 1.2 that the corresponding polynomial $f(x)$ is monic and that $f(0) = 0$.*

Proof. First of all, as already observed in Remark 1.3, we may assume that $f(0) = 0$ (in Theorem 1.2) by absorbing the constant term of $f(x)$ with the parameter λ in the definition of the polynomial family $f_\lambda(x)$.

Second, for any polynomial $f \in \overline{\mathbb{F}}_p[x]$ for which we also assume that $f(0) = 0$, there exists some $c \in \overline{\mathbb{F}_p}^*$ with the property that the polynomial $g := \tilde{\mu}_c \circ f \circ \tilde{\mu}_c^{-1}$ is monic (and again $g(0) = 0$), where $\tilde{\mu}_c(x) := cx$. We consider then two families of polynomials:

$$f_\lambda(x) := f(x) + \lambda \text{ and } g_\lambda(x) := g(x) + \lambda,$$

both parameterized by λ. By construction, we have that

$$g_{c\lambda} = \tilde{\mu}_c \circ f_\lambda \circ \tilde{\mu}_c^{-1}. \tag{4.1.1}$$

Equation (4.1.1) shows that for any given starting point γ, we have that γ is preperiodic under the action of $f_\lambda(x)$ if and only if $c\gamma$ is preperiodic under the action of $g_{c\lambda}(x)$. This reduces the proof of both Theorems 1.1 and 1.2 for the corresponding triples $(f_\lambda, \alpha, \beta)$ to the proof of the aforementioned theorems for the triples $(g_\lambda, c\alpha, c\beta)$, as claimed in the conclusion of Proposition 4.1. □

Next, we note that in Theorem 1.1 we do not assume the field L in question is a function field of transcendence degree one over $\overline{\mathbb{F}}_p$. However, as in [GH13, Ghi], we show in Proposition 4.2 below that we may assume $\text{trdeg}_{\overline{\mathbb{F}}_p} L \le 1$, essentially making the reduction of Theorem 1.1 to Theorem 3.2.

Proposition 4.2. *Let $f \in \overline{\mathbb{F}}_p[x]$ be a monic polynomial of degree $d \ge 2$ satisfying $f(0) = 0$, let L be a field of characteristic $p > 0$, and let $\alpha, \beta \in L$. If there exists $\lambda_1 \in \overline{L}$ such that both α and β are preperiodic for the polynomial $f_{\lambda_1}(x) = f(x) + \lambda_1$, then* $\text{trdeg}_{\mathbb{F}_p}\mathbb{F}_p(\alpha, \beta) \le 1$.

Proof. We recall the notation from Subsection 2.1 (see Lemma 2.1) that for each $\gamma \in L$, we have

$$P_{n,\gamma}(\lambda) := f_\lambda^n(\gamma), \tag{4.1.2}$$

which is a monic polynomial of degree d^{n-1} (in λ). Furthermore, the constant term is $P_{n,\gamma}(0) = f^n(\gamma)$, which is a monic polynomial of degree d^n in γ (note that f is a monic polynomial of degree d). Moreover, since $f(0) = 0$, then $P_{n,\gamma}(0)$ has no constant term (as a polynomial in γ). Also, from (2.1.2) we know that

$$P_{n,\gamma}(\lambda) = \lambda^{d^{n-1}} + \sum_{i=1}^{d^{n-1}-1} c_{n,i}(\gamma) \cdot \lambda^i + \gamma^{d^n} + \sum_{j=1}^{d^n-1} b_{n,j} \cdot \gamma^j$$

with each $c_{n,i} \in \overline{\mathbb{F}}_p[x]$ being a polynomial of degree less than d^n, and also, each $b_{n,j} \in \overline{\mathbb{F}}_p$ for $j = 1, \dots, d^n - 1$. Therefore, imposing the condition that α is a preperiodic point under the action of some f_{λ_1} yields an equation of the form:

$$P_{n,\alpha}(\lambda_1) = P_{m,\alpha}(\lambda_1) \text{ for some } 0 \le m < n.$$

Using equation (2.1.2) (along with the information about the degrees of each corresponding polynomials $c_{m,i}$ and $c_{n,j}$), we obtain that $\alpha \in \overline{\mathbb{F}_p(\lambda_1)}$. A similar reasoning, using this time that β is preperiodic under the action of f_{λ_1}, yields that also $\beta \in \overline{\mathbb{F}_p(\lambda_1)}$. Hence, we conclude that

$$\text{trdeg}_{\mathbb{F}_p}\mathbb{F}_p(\alpha, \beta) \le \text{trdeg}_{\mathbb{F}_p}\overline{\mathbb{F}_p(\lambda_1)} \le 1,$$

as desired for the conclusion of Proposition 4.2. □

Proof of Theorem 1.1. Using Propositions 4.1, 4.2 and 2.3 and arguing identically as in [Ghi, Section 6.1] proves that the conclusion in Theorem 1.1 is a consequence of Theorem 3.2. □

Next, we show that Theorem 1.2 follows from Theorem 1.1.

Proof of Theorem 1.2. First, we note that according to Proposition 4.1, we may assume from now on that $f(x)$ is monic and $f(0) = 0$.

Next, we consider the case where $d \geq 3$. Since $p \nmid d$, we obtain that the polynomial $f(x)$ has the shape (1.1.1) satisfying the inequality (1.1.2), which means that the hypothesis of Theorem 1.1 are met. Hence, we see that Theorem 1.2 is a consequence of Theorem 1.1 in this case.

The remaining case is when $d = 2$ and p is odd. As noted in Remark 1.3, our family f_λ can be then conjugated to the family of (normalized) polynomials $x \mapsto x^2 + \lambda$, in which case, the desired conclusion follows from [Ghi, Theorem 1.1]. This concludes our proof of Theorem 1.2. □

4.2. **Proof of Theorem 3.2.** We work with the notation and the assumptions from Theorem 3.2. We first prove the direct implication in Theorem 3.2.

Proposition 4.3. *With the notation as in Theorem 3.2 for L, α, β, $f(x)$ and $f_\lambda(x)$, assume there exists an infinite sequence $\{\lambda_n\}_n$ in $\overline{L}$ such that the condition (3.0.3) holds, i.e., we have*

$$\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\beta) = 0.$$

Then at least one of the following two conditions must hold:

(1) $f(\alpha) = f(\beta)$;
(2) $\alpha, \beta \in \overline{\mathbb{F}}_p$.

Proof. Theorem 2.11 shows that condition (3.0.3) yields that for each place $v \in \Omega = \Omega_L$ and for each $\lambda \in \mathbb{C}_v$, we have

$$\widehat{h}_{v,\lambda}(\alpha) = \widehat{h}_{v,\lambda}(\beta), \tag{4.2.1}$$

i.e., hypothesis (3.0.4) from Proposition 3.3 is met. Then Proposition 3.3 yields that

(1) either $f(\alpha) = f(\beta)$;
(2) or both α and β live in $\overline{\mathbb{F}}_p$.

This concludes our proof of Proposition 4.3. □

Next we prove that the converse and also, the "moreover" statement in Theorem 3.2 hold as well.

Proposition 4.4. *With the notation from Theorem 3.2 for L, α, β, $f(x)$ and $f_\lambda(x)$, assume in addition that at least one of the following two conditions are met:*

(1) $f(\alpha) = f(\beta)$;
(2) $\alpha, \beta \in \overline{\mathbb{F}}_p$.

Then there must exist an infinite sequence $\{\lambda_n\}_n$ in $\overline{L}$ such that the condition (3.0.3) holds, i.e., we have

$$\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\beta) = 0.$$

Moreover, for each $\lambda \in \overline{L}$, we have that α is preperiodic udner the action of $f_\lambda(x)$ if and only if β is preperiodic under the action of $f_\lambda(x)$.

Proof. We show first that if either one of conditions (1)-(2) holds, then for each $\lambda \in \overline{L}$, we have that α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

We argue case by case, as follows.

(1) If $f(\alpha) = f(\beta)$, then for each $\lambda \in \overline{L}$, we have that $f_\lambda(\alpha) = f_\lambda(\beta)$ and therefore, α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

(2) For any $\gamma \in \overline{\mathbb{F}}_p$, using equations (4.1.2) and (2.1.2), we get that for each $\lambda \in \overline{L}$, we have that γ is preperiodic under the action of f_λ if and only if $\lambda \in \overline{\mathbb{F}}_p$. Therefore, if both α and β live in $\overline{\mathbb{F}}_p$, we have that for each $\lambda \in \overline{L}$, α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

Next, we note that according to Proposition 2.3, we know there exist infinitely many $\lambda \in \overline{L}$ such that α (and therefore, also β) is preperiodic under the action of f_λ. Therefore, either one of the conditions (1)-(2) yields the existence of infinitely many $\lambda_n \in \overline{L}$ such that both α and β are preperiodic under the action of f_{λ_n}. Finally (see also Remark 2.5), for each such preperiodic parameter $\lambda_n \in \overline{L}$, we have

$$\widehat{h}_{f_{\lambda_n}}(\alpha) = \widehat{h}_{f_{\lambda_n}}(\beta) = 0.$$

This concludes our proof of Proposition 4.4. □

Combining Propositions 4.3 and 4.4 finishes our proof of Theorem 3.2.

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