Math 361 Winter 2001/2002
Assignment 1
(Quiz: Friday, September 14)

1. Consider a population of yeast cells growing on an agar plate. Suppose that each cell divides into two daughter cells once every hour.

   (a) Let t be time in hours, and let $x(t)$ be the size of the yeast population after t hours. Formulate an equation that describes the dynamics of the yeast population by giving $x(t+1)$ as a function of $x(t)$. What is the general solution to this equation?

   (b) Suppose we start out with a single yeast cell. Write down the number of yeast cells after t hours.

   (c) Suppose that there is enough food on the plate to support a population of 10^8 yeast cells. Starting with one cell, how many hours does it take until the yeast population reaches this size?

2. Consider the following statement in Michael Crichton's book *Andromeda Strain* (Dell, N.Y., 1969, p. 247):

   "The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth."

   Assume that Crichton's ideal circumstances hold and determine whether his statement is correct under the realistic assumption that the mass of an E. coli bacterium is roughly 10^{-12} grams and by taking into account that the mass of the earth is roughly $5.9763 \cdot 10^{24}$ kilograms.

3. Going back to the population of yeast cells in problem 1, assume that after the population has reached its limiting size of 10^8, individuals stop dividing and instead start to die off due to lack of food. Assume that for any individual cell that is alive the chance of dying during the next two hours is $1/3$.

   (a) Formulate a difference for the dynamics of the yeast population under this assumption.

   (b) What is the solution to the model found in (a)?

   (c) How long does it take for the population to die out? (*Hint*: how many time units does it take for $N(t)$ to be smaller than 1?)

   (d) Can you think of reasons for why your answer in b) might be wrong? (*Hint*: Keep in mind that the fate of individuals is given by the expected probability of surviving a certain time period. In reality, some might live longer and some might live shorter than expected. As long as the population size is large these differences will average out, but what happens when population sizes get very small?)

4. Solve problem 10, p. 31 in the textbook.

5. **Optional:** Consider the rabbit model of Fibonacci:

$$R(t+2) = R(t+1) + R(t) \tag{1}$$

with $R(0) = 0$ and $R(1) = 1$ (see class notes). Use a spreadsheet (e.g. Excel) to calculate the rabbit population $R(t)$ for $t = 1, ..., 50$.

6. Imagine a population with two age classes, so that each individual in the population is either a 'juvenile' or an 'adult'. Suppose that only adults reproduce, and that they do so by producing on average 0.9 juveniles per year. Assume also that adults die after reproduction. Suppose further that in each year 1/2 of all the juveniles survive to become adults, while the other half dies.

   (a) Formulate a model for the dynamics of this population with one year as the basic time unit. (Hint: you will end up with two equations, one describing the size of the juvenile population in the next year as a function of the adult population size in the present year, and one describing the adult population size next year as a function of the juvenile population size in the present year.)

   (b) Starting out with 100 juveniles and 200 adults, give the population sizes (juveniles, adults, and total) in the following 5 years.

   (c) What is the long term fate of the populations? What is the reason for this in terms of the demographic parameters, i.e. in terms of average reproductive output per adult individual and of juvenile survival probability? (Hint: no calculations needed.)

   (d) Suggest a way of 'salvaging' the population by changing the demographic parameters so that the population becomes viable (i.e. survives in the long run).

7. Consider the function $f : [0, 1) \longrightarrow [0, 1)$ which assigns to each number $x \in [0, 1)$, i.e. to each real number x satisfying $0 \,\square\, x < 1$ the fractional part of $10x$. That is:

$$f(x) = 10x - Int[10x], \tag{2}$$

where $Int[y]$ denotes the integral part of a real number y (see class notes for further details on this function). Consider the dynamical system given by the difference equation

$$x(t+1) = f(x(t)) \tag{3}$$

with initial condition $x(0) \in [0, 1)$.

   (a) Find all equilibrium states of this dynamical system, i.e. all solutions of the equation $x^* = f(x^*)$.

   (b) For an equilibrium x^* as found in (a) and for a given integer n, find an initial condition $x(0)$ such that a trajectory starting from $x(0)$ ends up at x^* after exactly n iterations (i.e. such that $x(n) = x^*$ and $x(t) \neq x^*$ for $t < n$).

(c) For each integer $n \geq 1$, find an initial condition $x(0)$ such that the trajectory starting from $x(0)$ moves on a cycle of period n.

(d) Amplification of small differences in initial conditions: Show that no matter how small $\varepsilon > 0$ is, there are always 2 initial conditions $x(0)$ and $x'(0)$ that satisfy $|x(0) - x'(0)| < \varepsilon$, but for which the trajectories $x(t)$ and $x'(t)$ starting from these initial conditions diverge in the sense that there is an integer t for which $|x(t) - x'(t)| > \delta$, where δ is any number between 0 and 1.

8. Consider two 2-vectors
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$
Then we can define the vector addition
$$w = x + y$$
by saying that the two components of the new vector w are the sum of the respective components of x and y:
$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}.$$
Thus the two components of $w = x + y$ are $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$.

Similarly, for any number r we can define a new vector rx whose components are the components of x multiplied by r: $rx = \begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix}$. The new vector rx is simply obtained by stretching the vector x by a factor r.

Now let
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
be a 2x2-matrix, and consider two vectors x and y, as well as their sum $w = x + y$ defined above, and the stretched vector rx. Then we can consider the new vectors $A \cdot w$, $A \cdot x$ and $A \cdot y$, and $A \cdot rx$which are obtained by applying the matrix A to the vectors w, x, y, and rx. Show that for these new vectors we have:
$$A \cdot w = A \cdot x + A \cdot y$$
and
$$A \cdot rx = r(A \cdot x).$$
Conclude that for any two vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and any two numbers r and x we have:
$$A \cdot (rx + sy) = r(A \cdot x) + s(A \cdot y).$$

9. For any two 2x2-matrices A, B, the multiplication rule (row) x (column) yields a new 2x2-matrix AB, whose entries in the first row are: (first row of A) x (first column of B), (first row of A) x (second column of B), and whose entries in the second row are

(second row of A) x (first column of B), (second row of A) x (second column of B). Using the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad C = \begin{pmatrix} i & j \\ k & l \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

show that:

(a) AB does, in general, not equal BA (matrix multiplication is not commutative, unlike multiplication of real numbers!).

(b) $A(B+C) = AB+AC$ (matrix multiplication is distributive, just like multiplication of real numbers).

(c) $(AB)C = A(BC)$ (matrix multiplication is associative, just like multiplication of real numbers).

(d) $AI = IA = A$ (multiplication of a matrix by the *identity matrix* I from either side leaves the matrix unchanged, just like multiplication by 1 leaves any real number unchanged).

10. Solve problem 11, p. 31 in the textbook.

11. Solve problem 12, p. 31 in the textbook.