Math 361   Winter 2001/2001
Assignment 2
(Quiz Friday, September 21)

1. Find the eigenvalues and the corresponding eigenvectors for the following matrices:

$$\begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix} \quad \begin{pmatrix} a & 1-b \\ 1-b & a \end{pmatrix}$$

2. Show that the following matrices have an eigenvalue that is 0:

$$\begin{pmatrix} 6 & 10 \\ 3 & 5 \end{pmatrix} \quad \begin{pmatrix} a & b \\ 3a & 3b \end{pmatrix}$$

   Matrices for which one or more eigenvalues are zero are known as *singular* matrices. Such matrices have rows that are functions of the other rows in the matrix (in the first example the second row is half the first row; in the second example the second row is three times the first row). Singular matrices are special and generally rare.

3. Solve problem 4, p. 30 in the textbook.

4. Let A be a 2x2 matrix that has the two eigenvalues r and s with corresponding eigenvectors x and y (so that $A \cdot x = rx$ and $A \cdot y = sy$). Consider vectors w of the form

$$w = ax + by,$$

   where a and b are arbitrary real numbers (cf. Problem 1 above).

   (a) Show that $A \cdot w = arx + bsy$. (Hint: Use Problem 2 and the fact that x and y are eigenvectors of A).

   (b) Since the $A \cdot w = arx + bsy$ is again a vector, we can apply the matrix A to this vector, i.e. we can calculate $A \cdot (A \cdot w) = A \cdot (arx + bsy)$, which we denote by $A^2 \cdot w$. (Thus, $A^2 \cdot w$is simply the vector that is obtained by applying the matrix A twice.) Show that

$$A^2 \cdot w = ar^2x + bs^2y.$$

   (c) Similarly, we can reapply the matrix A to the vector $A^2 \cdot w$ to get a vector $A^3 \cdot w$, to which we can apply A again to get a vector $A^4 \cdot w$, and so on. In this way, we can calculate the vector $A^t \cdot w$ by applying the matrix A t times. Show that

$$A^t \cdot w = ar^tx + bs^ty.$$

   (d) What happens with $A^t \cdot w$ for large t when both r and s have an absolute value that is smaller than 1?

5. Consider the (linear) second-order difference equation

$$x(t+2) = ax(t+1) + bx(t) \tag{1}$$

with initial conditions $x(0) = c$ and $x(1) = d$, and where we assume $b \neq 0$ (why?). Solve this equation by converting it into a system of two (linear) first-order difference equations using the new variable $y(t+1) = x(t)$, and by using the techniques of linear algebra (see class notes). Write down the solution for $x(t)$ satisfying the given initial conditions.

6. Solve problem 1, p. 29 in the textbook, using the methods of problem 5.

7. Solve problem 2, p. 29 in the textbook, using the methods of problem 5.

8. Solve problem 7, p. 31 in the textbook.

9. In a population of an imaginary organism that lives 2 years the average number of births for 1-year-olds is 1/3, the average number of births for 2-year-olds is 4, and the survival probability from 1 to 2 is 2/3. Death is certain after 3 years.

   (a) Set up a matrix model for the dynamics of this population.

   (b) Find the long term growth rate (largest eigenvalue) and the stable age distribution (corresponding eigenvector; the ratio of 1-year-olds to 2-year-olds) for this population.

10. Solve problem 16, p. 33 in the textbook.

11. Solve problem 20, p. 35 in the textbook.