Math 361 Winter 2001/2002
Assignment 3 (Note: Quiz on Monday, October 1)

1. In a population of an imaginary organism that dies at the end of the third year of its life the average number of offspring of 0-year-olds in their first year of life is 1/3, the average number of offspring of 1-year-olds in their second year is 4, and the average number of offspring of 2-year-olds in their last year is 2. The probability that 0-year-old survives its first year to become a 1-year-old is 2/3, and the probability that a 1-year-old survives the second year to become a 2-year-old is 1/2. Death is certain after 3 years. Set up a matrix model for this population. How would you go about determining the long-term growth rate and the stable age distribution for this population? (Just explain; no calculations needed.)

2. Using the methods of problem 5, Assignment 2, determine the eigenvalues corresponding to the (linear) dynamical systems of problem 9 on p. 31 in the textbook. Based on these eigenvalues, what can you say about the dynamics of x_n in each of these cases?

3. Consider the matrix
$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$
Calculate its eigenvalues and eigenvectors.

4. Solve problem 1 on p. 61 in the textbook.

5. Let A be a 2x2-matrix with a pair of complex conjugate eigenvalues
$$\lambda_{1/2} = r \pm i \cdot s = c(\cos\theta \pm i \cdot \sin\theta)$$
with corresponding eigenvectors
$$w_{1/2} = u \pm i \cdot v,$$
where $c > 0$, u and v are real 2-vectors, and where $\sin\theta \neq 0$.

   (a) Using the two eigenvector identities, show that
$$\begin{aligned} A \cdot u &= c(\cos\theta \cdot u - \sin\theta \cdot v) \\ A \cdot v &= c(\sin\theta \cdot u + \cos\theta \cdot v) \end{aligned}$$

   (b) Challenge: Show that
$$\begin{aligned} A^t \cdot u &= c^t(\cos(t\theta) \cdot u - \sin(t\theta) \cdot v) \\ A^t \cdot v &= c^t(\sin(t\theta) \cdot u + \cos(t\theta) \cdot v). \end{aligned}$$

6. Consider a population in which the average number of offspring per individual varies with population size, e.g. due to competition for resources. Specifically, assume that if x is the population size in a given year, then the average number of offspring per individual is given by the function

$$f(x) = \lambda \cdot \exp\left(1 - x/K\right),$$

where λ and K are (demographic) parameters.

   (a) Plot the function $f(x)$ for various values of K.

   (b) Think of possible biological interpretations of the parameters λ and K.

   (c) Formulate a population dynamical model for the population size in the next year, $x(t+1)$, as a function of the population size in the present year, $x(t)$:

$$x(t+1) = F(x(t)).$$

   Plot the function $F(x)$.

   (d) Determine the steady states and their stability of the model found in (c).

   Note: this problem is related to problems 3 and 4 on p. 62 in the textbook.

7. Solve problem 2 on p. 61 in the textbook.

8. Consider the non-linear population model

$$x(t+1) = F(x(t)) = \frac{\lambda \cdot x(t)}{1 + x(t)},$$

where $x(t)$ and $x(t+1)$ are population sizes in subsequent year, and $\lambda > 0$ is the average number of offspring per individual under ideal circumstances, i.e. if there is no competition for resources.

   (a) Plot $x(t+1)$ a function of $x(t)$ (consider both $\lambda < 1$ and $\lambda > 1$).

   (b) Determine the equilibria of this model and their stability. How many equilibria are there, depending on the value of λ? Use the method of cobwebbing to illustrate the dynamics of this population..

9. Consider the same population as in problem 6., but now assume that in addition to competition for resources there is also predation, which occurs in such a way that after reproduction has taken place, a constant number of individuals h are removed from the population in each year. Therefore, the population dynamical model changes to

$$x(t+1) = \frac{\lambda \cdot x(t)}{1 + x(t)} - h.$$

   (a) Use the method of cobwebbing to determine the dynamics of this population for various values of λ and h.

(b) How do the dynamics change compared to the population in problem 8? How does the equilibrium population size change?

(c) What is wrong with this model?

(d) Can you suggest a better way to incorporate constant predation pressure?

10. Solve problem 5 on p. 62 in the textbook.