Math 361 Winter 2001/2002
Assignment 5 (Quiz on Wednesday, October 17)

1. Consider the difference equation

$$\begin{aligned} x(t+1) &= F(x(t)) \\ &= r \cdot x(t) \cdot (1 - x(t)), \end{aligned} \tag{1}$$

where $r > 0$ (see pages 45-49 in the textbook).

(a) Show that for $r > 1$ this dynamical system has a unique equilibrium $x^* > 0$. Show that this equilibrium is locally stable if and only if $r < 3$.

(b) By considering the twice iterated map

$$\begin{aligned} x(t+2) &= G(x(t)) \\ &= F(F(x(t))) \\ &= r \cdot [r \cdot x(t) \cdot (1 - x(t))] \cdot [1 - r \cdot x(t) \cdot (1 - x(t))], \end{aligned}$$

show that as r is increased above 3, a (non-degenerate) 2-cycle of the dynamical system (1) is born.

Hint: To find the solutions of the equation $x^* = G(x^*)$ you will have to solve a polynomial of degree 3. To do this, note that the equilibrium found in part (a) is necessarily a solution to $x^* = G(x^*)$, because it represents a degenerate 2-cycle, i.e. a 2-cycle on which the two values attained are equal.

(c) Show that the 2-cycle found in part (b) is locally stable if r is sufficiently close to 3.

2. Consider the population genetic model for the frequency dynamics of two genetic types A and B in a clonally (i.e. asexually) reproducing population (cf. class notes):

$$p(t+1) = \frac{p(t) \cdot w_A(t)}{p(t) \cdot w_A(t) + (1 - p(t)) \cdot w_B(t)} = p(t) \cdot \frac{w_A(t)}{\overline{w}(t)},$$

where $p(t)$ is the frequency of type A at time t, $1-p(t)$ is the frequency of type B at time t, w_A and w_B are the fitnesses of the two types, and $\overline{w}(t) = p(t) \cdot w_A(t) + (1-p(t)) \cdot w_B(t)$ is the mean fitness at time t. Here we assume that the fitnesses $w_A(t)$ and $w_B(t)$ are functions of the frequency $p(t)$ such that

$$\begin{aligned} w_A(t) &= a - p(t) \\ w_B(t) &= b + p(t) \end{aligned}$$

with $a > 1$ and $b = a - 1$. Thus, the fitnesses of the two types are assumed to be frequency-dependent, such that each type has an advantage when rare, i.e. such that $w_A(t) > w_B(t)$ when $p(t)$ is close to 0, and $w_A(t) < w_B(t)$ when $p(t)$ is close to 1.

(a) Show that there are 3 equilibrium states p^* for the frequency $p(t)$ of type A.

(b) Determine the stability of these equilibrium states and the long term dynamical behaviour of the model. Explain your results in biological terms based on the assumptions about the fitness functions $w_A(t)$ and $w_B(t)$.

3. Calculate all partial derivatives for the following functions:

(a)

$$f(x,y) = \sin(x^2 + y^{2)}$$

(b)

$$g(x,y) = \cos[x^3] \cdot \exp[-xy]$$

(c)

$$f(x,y) = x^n + x^{n-1}y + x^{n-2}y^2 + ... + x^2y^{n-2} + xy^{n-1} + y^n$$

(d)

$$h(x,y) = \frac{xy^3 + 4y^2}{x^4 + 7y}$$

(e)

$$g(x,y) = \ln\left[\cos\left[-xy\right]\right]$$

(f)

$$f(x,y,z) = xyz + (xyz)^2$$

(g)

$$f(x,y,z) = \exp[-(x^2 + y^2 + z^2)]$$

4. Consider the Nicholson-Bailey model with density dependence in the prey given by the Ricker model:

$$\begin{aligned} x(t+1) &= x(t) \cdot \exp[r(1 - x(t)/K] \cdot \exp\left[-ay(t)\right] = F\left(x(t), y(t)\right) \\ y(t+1) &= c \cdot x(t) \cdot (1 - \exp\left[-ay(t)\right]) = G\left(x(t), y(t)\right) \end{aligned}$$

Assume $r = 1$, $K = 1$ and $c = 1$, and show that the resulting model has a locally stable equilibrium (x^*, y^*) with $x^* > 0$ and $y^* > 0$ if the parameter a is sufficiently small.

5. Solve problem 11 on p. 63 in the textbook.

6. Solve problem 6 on p. 103 in the textbook.

7. Solve problem 7 on p. 103 in the textbook.

8. Solve problem 10 on p. 104 in the textbook.

9. Solve problems 11(a)-11(d) on p. 105 in the textbook.