Discrete time dynamical systems

(Review of first part of Math 361, Winter 2001)

Basic problem:

$x_1(t), ..., x_n(t)$ dynamic variables (e.g. population size of age class i at time t); dynamics given by a set of n equations

$$\begin{aligned} x_1(t+1) &= F_1\left(x_1(t), ..., x_n(t)\right) \\ &... \\ x_n(t+1) &= F_n\left(x_1(t), ..., x_n(t)\right) \end{aligned} \tag{1}$$

where $F_1, ..., F_n$ are functions of n variables. Starting with some initial condition $x_1(0), ..., x_n(0)$, the time series of $x_1(t), ..., x_n(t)$ (i.e. the dynamics) are obtained by successively plugging in the present values of these variables into the functions$F_1, ..., F_n$ to obtain the new values in the next time step, etc.

Example:

$$x_1(t+1) = \lambda x_1(t).$$

Here $n = 1$, and $F_1(x_1) = \lambda x_1$ is linear. Starting with $x_1(0)$, we get

$$x_1(t) = \lambda^t x_1(0).$$

First case: linear difference equations

If the F_i are linear functions

$$F_i(x_1, ..., x_n) = a_{i1}x_1 + ... + a_{in}x_n,\ i = 1, ..., n,$$

(where we assume the a_{ij} to be real numbers), then the dynamics are given by the matrix multiplication

$$\begin{pmatrix} x_1(t+1) \\ ... \\ x_n(t+1) \end{pmatrix} = A \cdot \begin{pmatrix} x_1(t) \\ ... \\ x_n(t) \end{pmatrix}, \tag{2}$$

where A is the matrix

$$A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & .. & ... \\ a_{n1} & .. & a_{nn} \end{pmatrix}.$$

In 'most' cases, the matrix A has n eigenvalues $\lambda_1, ..., \lambda_n$ with corresponding eigenvectors $w_1, ..., w_n$:

$$A \cdot w_i = \lambda_i w_i, \ i = 1, ..., n.$$

The general solution for the dynamical system is:

$$\begin{pmatrix} x_1(t) \\ ... \\ x_n(t) \end{pmatrix} = b_1 \lambda_1^t w_1 + ... + b_n \lambda_n^t w_n,$$

where the initial condition is

$$\begin{pmatrix} x_1(0) \\ ... \\ x_n(0) \end{pmatrix} = b_1 w_1 + ... + b_n w_n.$$

In the long term, i.e. for $t \longrightarrow \infty$, there are two qualitatively distinct cases:

Let λ be that eigenvalue of A with the largest absolute value, i.e. the *dominant eigenvalue*, and let w be the corresponding eigenvector. Then, if λ is a real number, the long term behaviour is given by

$$\begin{pmatrix} x_1(t) \\ ... \\ x_n(t) \end{pmatrix} = b\lambda^t w \text{ for } t \to \infty$$

(where bw is the component of the initial condition in the direction of the eigenvector w). Thus, if λ is real, the dynamic variables will eventually grow at a rate λ, and the vector of variables will be a multiple of w.

If λ is not real, $\lambda = r + is = c \cdot (\cos\varphi + i \sin\varphi)$ with $\underline{\varphi} \neq 0$ (hence $s \neq 0$), then $w = u + iv$ with $v \neq 0$, and the complex conjugate $\overline{\lambda} = r - is$ is also an eigenvalue of A with corresponding eigenvector $w = u - iv$. The real valued solution will eventually converge to oscillating behaviour with exponentially growing (or decreasing) amplitude:

$$\begin{pmatrix} x_1(t) \\ ... \\ x_n(t) \end{pmatrix} = c^t \cdot [b_1 \cdot (\cos(t\varphi) \cdot u - \sin(t\varphi) \cdot v) + b_2 \cdot (\sin(t\varphi) \cdot u + \cos(t\varphi) \cdot v)] \quad \text{for } t \to \infty$$

where $c = \sqrt{r^2 + s^2}$ is the absolute value of λ, and where b_1 and b_2 are determined from the initial condition.

Special cases:

1. In the example given above, $x_1(t+1) = \lambda x_1(t)$, the matrix A simply consists of the number λ, which is its sole eigenvalue (with eigenvector 1).

2. Leslie matrices: these are matrices describing the dynamics of age-structured populations. The dynamic variables $x_i(t)$ are the population sizes of age class i at time t. Leslie matrices L are of the form

$$L = \begin{pmatrix} f_1 & f_2 & ... & \cdots & f_n \\ s_1 & 0 & & & \\ & s_2 & 0 & & \\ & & ... & 0 & \\ & & & s_{n-1} & 0 \end{pmatrix}$$

where $f_1, ..., f_n$ are fecundities (and hence ≥ 0) and $s_1, ..., s_n$ are survival probabilities (and hence also ≥ 0). The dominant eigenvalue of such matrices is, in general, a real number, and hence the first of the above scenarios applies. In particular, whether a population described by a Leslie matrix goes extinct depends on whether the dominant eigenvalue has absolute value < 1. The eigenvector w corresponding to the dominant eigenvalue is called the *stable age distribution*.

3. Linear second order difference equations of the form

$$x(t+1) = a \cdot x(t) + b \cdot x(t-1)$$

can always be translated into a system of two linear first-order equations by defining $y(t) = x(t-1)$:

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = L \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

with

$$L = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

Example: Rabbit population dynamics and Fibonacci series; dynamics of blood cells.

Second case: non-linear difference equations.

If the functions F_i in (1) are non-linear, then the general approach is to first look for equilibrium states and then perform a linear stability analysis.

A point $(x_1^*, ..., x_n^*)$ is called an equilibrium for the dynamical system (1) if

$$F_i(x_1^*, ..., x_n^*) = x_i^* \text{ for } i = 1, ..., n.$$

Given an equilibrium state $(x_1^*, ..., x_n^*)$, the dynamics of a vector of small deviations $d_i(t) = x_i(t) - x_i^*$ from this equilibrium is given by

$$\begin{pmatrix} d_1(t+1) \\ ... \\ d_n(t+1) \end{pmatrix} = J \cdot \begin{pmatrix} d_1(t) \\ ... \\ d_n(t) \end{pmatrix},$$

where J is the *Jacobian matrix* of partial derivatives evaluated at the equilibrium

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x_1^*, ..., x_n^*) & ... & \frac{\partial F_1}{\partial x_n}(x_1^*, ..., x_n^*) \\ ... & & ... \\ \frac{\partial F_n}{\partial x_1}(x_1^*, ..., x_n^*) & ... & \frac{\partial F_n}{\partial x_n}(x_1^*, ..., x_n^*) \end{pmatrix}.$$

This dynamical equation holds approximately as long as $|d_i(t)| \ll 1$ are small.

The equilibrium is called *locally stable* if and only if

$$\begin{pmatrix} d_1(t) \\ ... \\ d_n(t) \end{pmatrix} \to \begin{pmatrix} 0 \\ ... \\ 0 \end{pmatrix} \text{ for } t \to \infty$$

for any initial condition $(d_1(0), ..., d_n(0))$ with $|d_i(0)| \ll 1$ for all i. The linear theory summarized in the first case above gives information about the dynamics of the vector of deviations $(d_1(t), ..., d_n(t))$. In particular, the equilibrium is locally stable if and only if the dominant eigenvalue of the Jacobian matrix J at this equilibrium has absolute value < 1.

Examples:

1. For linear systems of the form (2) the only equilibrium is $(x_1^*, ..., x_n^*) = (0, ..., 0)$, and the Jacobian matrix is the original matrix A itself, Thus, the linearized dynamical system is equal to the given system (as it should be, because the given system is linear), and the stability analysis simply tells us whether the system will converge to the 0 vector or not.

2. A single non-linear difference equation:

$$x(t+1) = F(x(t)).$$

The equilibria are given as solutions of

$$x^* = F(x^*).$$

The Jacobian matrix is simply

$$J = \frac{dF}{dx}(x^*),$$

and hence the equilibrium is locally stable if and only if

$$\left| \frac{dF}{dx}(x^*) \right| < 1.$$

For a single non-linear difference equation this stability condition can also be found graphically using the method of *cobwebbing*.

Example:

$$x(t+1) = \frac{\lambda x(t)}{1 + x(t)^b} \tag{3}$$

with $\lambda, b > 0$. Then the equilibria are

$$\begin{aligned} x^* &= 0 \\ x^* &= (\lambda - 1)^{1/b} \quad (\text{for } \lambda > 1). \end{aligned}$$

The first of these is stable if and only if $\lambda < 1$, and the second is stable if and only if

$$1 - b\frac{\lambda - 1}{\lambda} > -1.$$

3. The Nicholson-Bailey predator prey model:

$$\begin{aligned} N(t+1) &= \lambda N(t) \cdot \exp\left[-aP(t)\right] \\ P(t+1) &= cN(t)\left(1 - \exp\left[-aP(t)\right]\right) \end{aligned}$$

where $N(t)$ = prey population size at time t, $P(t)$ = predator population size at time t, and where $\lambda, a, c > 0$ are population parameters (λ is the maximal per capita number of offspring in the prey, a is the searching efficiency of the predator, and c is the conversion rate, i.e. the number of predators produced, on average, from a single attacked prey). The equilibria of this model are given by

$$\begin{aligned} N^* &= 0,\ P^* = 0 \\ N^* &= \frac{\lambda \ln \lambda}{ac(\lambda - 1)},\ P^* = \frac{\ln \lambda}{a} \quad (\text{for } \lambda > 1) \end{aligned}$$

The Jacobian matrix at the first of these equilibrium states is

$$J = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix},$$

hence this equilibrium is stable if and only if $\lambda < 1$. The Jacobian matrix at the second equilibrium is

$$J = \begin{pmatrix} 1 & -\frac{\lambda \ln \lambda}{c(\lambda-1)} \\ c\frac{\lambda-1}{\lambda} & \frac{\ln \lambda}{\lambda-1} \end{pmatrix}.$$

It can be shown that for $\lambda > 1$ this matrix has a pair of complex conjugate eigenvalues with absolute value >1. Thus, the second equilibrium is always unstable when it exists, with a vector of small difference spiraling away from the equilibrium and exhibiting oscillations with increasing amplitude.

What happens when equilibria are unstable? In the case of a single non-linear difference equation such as (3), we observe the *period-doubling route to chaos*. First of all, as the equilibrium $x^* > 0$ becomes unstable, a new 2-cycle appears, i.e. an equilibrium of the twice iterated map

$$\begin{aligned} x(t+2) &= F\left(F\left(x(t)\right)\right) \\ &= G\left(x(t)\right). \end{aligned} \tag{4}$$

This 2-cycle is initially stable, but becomes unstable itself as $|dF/dx(x^*)|$ increases further. These assertions can be proved by doing a stability analysis (either graphically or analytically) of the dynamical system (4). Due to similarity of the functions F and G near the respective equilibria considered, the process repeats itself when applied repeatedly. In particular, when the 2-cycle becomes unstable, a 4-cycle appears, which is initially stable, but then becomes itself unstable and gives way to a stable 8-cycle, etc. Hence the period-doubling route to chaos. The values of $|dF/dx(x^*)|$ for which the transitions from a stable 2^n-cycle to a stable 2^{n+1}-cycle occur are called *bifurcation points*. These bifurcation points have a limit point. For values of $|dF/dx(x^*)|$ larger than this limit value the dynamical system can exhibit *chaos*. This type of dynamics is characterized by irregular time series and by *sensitive dependence on initial conditions*, which means that trajectories (time series) starting at arbitrarily close initial conditions will eventually diverge. In other words, in chaotic systems small initial differences will become large over time.

Such complicated dynamics can also be observed in higher dimensional systems of non-linear difference equations, e.g. in neural networks, or in predator-prey systems with density dependence in the prey. One of the important questions in this context is how one can simplify such complexity. For example, in predator-prey systems simplification is achieved if the driving forces for destabilization are kept in check, e.g. in predator-prey systems by including strongly stabilizing density dependence in the prey (prevents overshooting in the prey) or by including predator interference or prey refuges (prevents overexploitation of the prey). Simplification can also be achieved by applying external perturbations to the dynamical system at regular intervals. Such methods of chaos control can for example induce periodic dynamics in chaotic neural networks.