Math 361 Winter 2001/2002
Assignment 1 - Solutions

1. **(a)** If t is time in hours, and if each cell has 2 descendants per hour, then the population size doubles every hour:
$$x(t+1) = 2x(t).$$
The general solution for this dynamical system is
$$x(t) = 2^t x_0,$$
where $x_0 = x(0)$ is the initial population size at time 0.

   (b) From (a), with $x_0 = 1$:
$$x(t) = 2^t.$$

   (c) The question is: what is t such that $x(t) = 2^t > 10^8$? Solving for t yields $t\ln(2) > 8\ln(10)$, i.e. $t > 8\ln(10)/\ln(2) = 26.58...$. Therefore, the population size will be bigger than 10^8 after 27 hours.

2. The earth weighs ca. $5.9763 \cdot 10^{24}kg = 5.9763 \cdot 10^{27}g$. If E.coli divides once every 20 minutes, we get $3 \cdot 24 = 72$ generations per day. Starting with a single E.coli bacterium, the population size after 72 generations is, by problem 1, $x(t) = 2^{72} = 4.7224 \cdot 10^{21}$. With a single bacterium weighing $10^{-12}g$,this yields a total mass of $4.7224 \cdot 10^9 g$. Thus, Crichton was off by 18 orders of magnitude...

   (a) $x(t+1) = \frac{2}{3}x(t)$.

   (b) $x(t) = (\frac{2}{3})^t x_0$, with $x_0 = x(0)$, the initial condition.

   (c) Here $x(0) = 10^8$, and we are looking for t such that $x(t)$ □ 1, hence for t such that $(\frac{2}{3})^t 10^8$ □ 1,which upon taking logarithms yields $t \geq 45.43...$ time units. Since each time unit is 2 hours, we have to wait approximately 91 hours until the population goes extinct.

   (d) The model used here is deterministic and assumes that in each time interval exactly 1/3 of the population dies off, because for each individual bacterium the chance of dying is 1/3. In very large populations this per capita chance of dying indeed translates into a loss of 1/3 of the population in each time step, but as the population size becomes very small this may not be true anymore, because the process of dying is stochastic. For example, if there are only three bacteria left, we would predict that on average (that is, if we consider a large number of such populations consisting of three individuals) the population size in the next time unit would be 2. However, for any single population consisting of three individuals it is quite likely that all three individuals will die during the next time step (with a per capita probability of dying of 1/3, the chance that all three individuals die is 1/27), or that none of them dies (which happens with probability 8/27), so that the culling of 1/3 might not be an accurate prediction. Thus, because stochasticity becomes important when population sizes get small, predictions from deterministic models have to be treated with caution. Rather than speaking of a time of extinction of, say, 91 hours, it would be better to calculate a probability distribution of extinction times, with some times to extinctions being more likely than others, and with the mean time to extinction being the values calculated from the deterministic model.

(a) In order to just prevent extinction the number of per capita descendants over one time step must be ≥ 1. Thus, to prevent extinction we need

$$f \geq \frac{1}{r(1-m)}$$

Here $m = 0.8, r = 0.5$, so the condition is

$$f \geq \frac{1}{0.5 \cdot 0.2} = 10.$$

(b)

For population expansion $a_{n+1} > a_n$

$$\begin{aligned} a_{n+1} &= fr(1-m)a_n > a_n \\ f &> \frac{1}{r(1-m)} \end{aligned}$$

6. **(a)** Let $J(t)$ be the number of juveniles at time t, and let $A(t)$ be the number of adults at time t. Then we have:

$$\begin{aligned} J(t+1) &= 0.9A(t) \\ A(t+1) &= 0.5J(t) \end{aligned}$$

In matrix notation:

$$\begin{pmatrix} J(t+1) \\ A(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 0.9 \\ 0.5 & 0 \end{pmatrix} \cdot \begin{pmatrix} J(t) \\ A(t) \end{pmatrix}. \tag{1}$$

according to the assumptions made.

(b)

t	$J(t)$	$A(t)$
0	100	200
1	180	50
2	45	90
3	81	22
4	19	40
5	36	9

(c) From (b) it is evident that the population is decreasing in size towards 0 as t becomes large. The reason for this is that the average number of offspring per adult that survive to adulthood and reproduction is less than 1: (per capita offspring) x (survival to reproduction)$= 0.9 \cdot 0.5 = 0.45 < 1$. Formally:

$$A(t+1) = 0.5 \cdot 0.9 \cdot A(t-1),$$

hence

$$A(t) = (0.45)^{t/2} A(0),$$

where $A(0)$ is the initial adult population size. Since $(0.45)^{t/2} \to 0$ as $t \to \infty$, we also have $A(t) \to 0$ as $t \to \infty$.

(d) If c is the number of offspring per adult and d is the chance of survival of juveniles to adulthood, then we clearly need

$$cd \geq 1$$

in order to sustain the population.

7. **(a)** A number x^* is an equilibrium for this system if and only if cutting the first digit after the comma in the decimal representation of x^* does not change x^*. Obviously, this happens if and only if $x^*$is of the form

$$x^* = 0.kkkk\ldots,$$

where $k = 1, 2, ...9$.

(b) Let $x(0) = 0.a_1a_2...a_nkkk..$, where $k = 1, 2, ...9$, and where $a_n \neq k$. Clearly, $x(n)$ is one of the equilibria found in (a), and because $a_n \neq k$, none of the $x(t)$ with $t < n$ are equal to such an equilibrium.

(c) Let $x(0) = 0.a_1a_2...a_k....$ Then the trajectory starting with $x(0)$ will be periodic with period n if for every $k \geq 1$, $a_{k+n} = a_k$.

(d) Let n be an integer such that $\varepsilon > 10^{-n}$, and let $u = 0.a_1a_2...a_k...$ and $v = 0.a'_1a'_2...a'_k...$ be two numbers such that $|u - v| > \delta$. Consider the two initial conditions $x(0) = 0.11....1a_1a_2...a_k...$ and $x'(0) = 0.11....1a'_1a'_2...a'_k...$, in which n 1's after the comma are followed by the a_k's and the a'_k's from the decimal representations of u and v, respectively. Then, by the choice of n, $|x(0) - x'(0)| < \varepsilon$, and for $t = n + 1$ we have $x(t) = u$ and $x'(t) = v$, hence the claim.

8.

$$A \cdot w = A \cdot x + A \cdot y$$

LHS:

$$\begin{aligned} A \cdot w &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\ &= \begin{pmatrix} a(x_1 + y_1) + b(x_2 + y_2) \\ c(x_1 + y_1) + d(x_2 + y_2) \end{pmatrix} \end{aligned}$$

RHS:

$$\begin{aligned} A\cdot x + A\cdot y &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} + \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} \\ &= \begin{pmatrix} a(x_1 + y_1) + b(x_2 + y_2) \\ c(x_1 + x_2) + d(x_2 + y_2) \end{pmatrix} \\ &= A\cdot w \end{aligned}$$

$$A\cdot rx = r(A\cdot x)$$

Proof:

$$\begin{aligned} A\cdot rx &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix} \\ &= \begin{pmatrix} arx_1 + brx_2 \\ crx_1 + drx_2 \end{pmatrix} \\ &= r\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} \\ &= r\,(A\cdot x) \end{aligned}$$

By what we did above, we have:

$$\begin{aligned} A\cdot(rx + sy) &= A\cdot(rx) + A\cdot(sy) \\ &= r(A\cdot x) + s(A\cdot y) \end{aligned}$$

9. These calculations are straightforward.

10. **(a)** With the initial condition $x(0) = x_0$, we have

$$\begin{aligned} x(1) &= ax_0 + b \\ x(2) &= ax(1) + b = a^2x_0 + ab + b \\ x(3) &= ax(2) + b = a^3x_0 + a^2b + ab + b \\ &\ldots \\ x(t) &= ax(t-1) + b = a^tx_0 + a^{t-1}b + \ldots + ab + b \end{aligned}$$

(b) With $K = \frac{d}{a+b+c}$ we see that $x_n = K$ for all n satisfies the dynamic equation, hence $K = \frac{d}{a+b+c}$ is a solution.

(c) We substitute the proposed solution into the dynamic equation:

$$\begin{aligned} & a(K + c_1\lambda_1^{n+2} + c_2\lambda_2^{n+2}) + b(K + c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1}) + c(K + c_1\lambda_1^n + c_2\lambda_2^n) \\ =\ & (aK + bK + cK) + a(c_1\lambda_1^{n+2} + c_2\lambda_2^{n+2}) + b(c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1}) + c(c_1\lambda_1^n + c_2\lambda_2^n) \\ =\ & d + 0 \\ =\ & d \end{aligned}$$

by the choice of K and because $c_1\lambda_1^n + c_2\lambda_2^n$ is a homogenous solution by assumption. Therefore, $x_n = K + c_1\lambda_1^n + c_2\lambda_2^n$ is a solution.

11. **(a)** Dividing $n_{k+2} = n_{k+1} + n_k$ by n_{k+1} yields

$$\frac{n_{k+2}}{n_{k+1}} = 1 + \frac{n_k}{n_{k+1}}.$$

As $k \longrightarrow \infty$ we have $\frac{n_{k+2}}{n_{k+1}} \longrightarrow \tau$ and $\frac{n_k}{n_{k+1}} \longrightarrow 1/\tau$ by definition, hence we get

$$\tau = 1 + 1/\tau$$

as claimed.

(b) This simply results from repeatedly inserting the value $1 + 1/\tau$ for τ on the right hand side of the above equation.