Math 361 Winter 2001/2002
Assignment 2 - Solutions

1. Solution for $A = \begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix}$

In general, $\lambda_{1,2} = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4\det(A)}}{2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$. Here: $\lambda_{1,2} = \frac{11 \pm \sqrt{11^2 - 4(24-14)}}{2} = \frac{11 \pm \sqrt{11^2 - 40}}{2} = \frac{11 \pm \sqrt{81}}{2} = \frac{11 \pm 9}{2} = 10, 1.$

Eigenvector w_1 for $\lambda_1 = 10$:

Let $w_1 = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix} w_1 = 10 w_1$, hence $\begin{pmatrix} 3x+7y \\ 2x+8y \end{pmatrix} = \begin{pmatrix} 10x \\ 10y \end{pmatrix}$. Multiplying the first equation by 2 and the second by 3, we get $\begin{pmatrix} 6x+14y \\ 6x+24y \end{pmatrix} = \begin{pmatrix} 20x \\ 30y \end{pmatrix}$, hence $-10y = 20x - 30y$, and finally $20y = 20x$, hence $x = y$. Thus, $w_1 = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, i.e. the vector corresponding to the eigenvalue $\lambda_1 = 10$, is some multiple of a 2x1 vector in which the upper component is equal to the bottom component

Eigenvector w_2 for $\lambda_2 = 1$:

Let $w_2 = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix} w_2 = w_2$, hence $\begin{pmatrix} 3x+7y \\ 2x+8y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. We get $\begin{pmatrix} 6x+14y \\ 6x+24y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$, hence $-10y = 2x - 3y$, and finally $y = -\frac{2}{7}x$. Thus $w_2 = c\begin{pmatrix} -3.5 \\ 1 \end{pmatrix}$ for some c.

Solution for $A = \begin{pmatrix} a & 1-b \\ 1-b & a \end{pmatrix}$

By the same procedure as in 1a:

$\lambda_1 = a + b - 1$; $w_1 = c\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = a - b + 1$; $w_2 = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2. $\begin{pmatrix} 6 & 10 \\ 3 & 5 \end{pmatrix}$

$\lambda_{1,2} = \frac{(6+5) \pm \sqrt{(6+5)^2 - 4(30-30)}}{2} = \frac{11 \pm \sqrt{11^2}}{2} = \frac{11 \pm 11}{2} = 11, 0$

$\begin{pmatrix} a & b \\ 3a & 3b \end{pmatrix}$

$\lambda_{1,2} = \frac{(a+3b) \pm \sqrt{(a+3b)^2 - 4(a3b - b3a)}}{2} = \frac{(a+3b) \pm \sqrt{(a+3b)^2}}{2} = \frac{(a+3b) \pm (a+3b)}{2} = a + 3b, 0.$

3. (a) and (c) are known from the class. For (b), we let λ be an eigenvalue of $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{\lambda - a_{11}}{a_{12}} \end{pmatrix} &= \begin{pmatrix} a_{11} + a_{12}\frac{\lambda - a_{11}}{a_{12}} \\ a_{21} + a_{22}\frac{\lambda - a_{11}}{a_{12}} \end{pmatrix} \\ &= \begin{pmatrix} \lambda \\ \frac{a_{21}a_{12} - a_{22}a_{11}}{a_{12}} + \frac{a_{22}\lambda}{a_{12}} \end{pmatrix} \\ &= \begin{pmatrix} \lambda \\ \frac{-\det(M) + a_{22}\lambda}{a_{12}} \end{pmatrix}. \end{aligned}$$

Since λ is an eigenvalue of M, it satisfies the equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + \det(M) = 0,$$

hence

$$-\det(M) + a_{22}\lambda = \lambda^2 - a_{11}\lambda.$$

Thus

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{\lambda - a_{11}}{a_{12}} \end{pmatrix} &= \begin{pmatrix} \lambda \\ \frac{\lambda^2 - a_{11}\lambda}{a_{12}} \end{pmatrix} \\ &= \lambda \cdot \begin{pmatrix} 1 \\ \frac{\lambda - a_{11}}{a_{12}} \end{pmatrix}. \end{aligned}$$

4. This material should be straightforward based on what was discussed in class.

5. If $b = 0$ we have a linear difference equation leading to simple exponential growth at rate a. For $b \neq 0$ introduce a second variable $y(t+1) = x(t)$. Then the given equation translates into the following system of equations:

$$\begin{aligned} x(t+1) &= ax(t) + by(t) \\ y(t+1) &= x(t) \end{aligned}$$

hence

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

The eigenvalues of the matrix $A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ are

$$\lambda_{1,2} = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

and the corresponding eigenvectors are (see problem 3):

$$w_{1/2} = \begin{pmatrix} 1 \\ \frac{\lambda_{1/2}-a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-a\pm\sqrt{a^2+4b}}{2b} \end{pmatrix}.$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r\lambda_1^t \begin{pmatrix} 1 \\ \frac{-a+\sqrt{a^2+4b}}{2b} \end{pmatrix} + s\lambda_2^t \begin{pmatrix} 1 \\ \frac{-a-\sqrt{a^2+4b}}{2b} \end{pmatrix},$$

For the first component of this general solution we get the general solution for $x(t)$ as

$$x(t) = r\lambda_1^t + s\lambda_2^t.$$

The coefficients r and s have to be determined from the initial conditions

$$\begin{aligned} x(0) &= r + s = c \\ x(1) &= r\lambda_1 + s\lambda_2 = d \end{aligned}$$

which yields

$$\begin{aligned} r &= \frac{d-\lambda_2 c}{\lambda_1-\lambda_2} \\ s &= \frac{-d+\lambda_1 c}{\lambda_1-\lambda_2} \end{aligned}$$

hence

$$x(t) = \frac{d-\lambda_2 c}{\lambda_1-\lambda_2}\lambda_1^t + \frac{-d+\lambda_1 c}{\lambda_1-\lambda_2}\lambda_2^t$$

6.

$$x_{n+2} - 3x_{n+1} + 2x_n = 0$$

Let $y_{n+1} = x_n$. The system reduces to

$$\begin{aligned} x_{n+1} &= 3x_n - 2y_n \\ y_{n+1} &= x_n \end{aligned}$$

We need to determine the eigenvalues of the matrix

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned} (3-\lambda)(-\lambda) + 2 &= 0 \\ \lambda^2 - 3\lambda + 2 &= 0 \\ \lambda &= 1, 2 \end{aligned}$$

The eigenvectors are given by

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}.$$

Thus the general solution is given by (see previous problem):

$$x_n = r + s \cdot 2^n$$

Given $x_0 = 10; x_1 = 20$

$$\begin{aligned} 10 &= r + s \\ 20 &= r + 2s \end{aligned}$$

and so $r = 0$ and $s = 10$. Thus $x_n = 10 \cdot 2^n$.

7. In each case the matrix for of the two dimensional system is given and then solved.

(a)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (5-\lambda)(-\lambda) + 6 &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0 \\ \lambda &= 3, 2 \end{aligned}$$

Eigenvectors:

$$\begin{aligned} v_2 &= \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} \\ v_3 &= \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 2^n a \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + 3^n b \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$$

Thus $x_n = 2^n a + 3^n b$.

Given initial conditions $x_0 = 2; x_1 = 5,\ a = 1, b = 1$ so that

$$x_n = 2^n + 3^n$$

(b)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned}(5-\lambda)(-\lambda)+4 &= 0\\ \lambda^2-5\lambda+4 &= 0\\ \lambda &= 1,4\end{aligned}$$

Eigenvectors:

$$\begin{aligned}v_1 &= \begin{pmatrix}1\\1\end{pmatrix}\\ v_4 &= \begin{pmatrix}1\\\frac{1}{4}\end{pmatrix}\end{aligned}$$

Solution:

$$\begin{pmatrix}x_n\\y_n\end{pmatrix} = 4^n a\begin{pmatrix}1\\\frac{1}{4}\end{pmatrix} + b\begin{pmatrix}1\\1\end{pmatrix}$$

Thus $x_n = 4^n a + b$.
Given initial conditions $x_1 = 9; x_2 = 33,\ a = 2, b = 1$ so that

$$x_n = 2\cdot\dot{4}^n + 1$$

(c)

$$\begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x_n\\y_n\end{pmatrix}$$

Eigenvalues:

$$\begin{aligned}(-\lambda)(-\lambda)-1 &= 0\\ \lambda^2-1 &= 0\\ \lambda &= 1,-1\end{aligned}$$

Eigenvectors:

$$\begin{aligned}v_1 &= \begin{pmatrix}1\\1\end{pmatrix}\\ v_3 &= \begin{pmatrix}1\\-1\end{pmatrix}\end{aligned}$$

Solution:

$$\begin{pmatrix}x_n\\y_n\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + (-1)^n b\begin{pmatrix}1\\-1\end{pmatrix}$$

Thus $x_n = a + (-1)^n b$.
Given initial conditions $x_1 = 3; x_2 = 5,\ a = 4, b = 1$ so that

$$x_n = 4 + (-1)^n$$

(d)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (2-\lambda)(-\lambda) &= 0 \\ \lambda^2 - 2\lambda &= 0 \\ \lambda &= 2, 0 \end{aligned}$$

Eigenvectors (NOTE: the standard formula cannot be used since $b = 0$ in the matrix):

$$\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This leads to

$$a - 2b = 0$$

Let $a = 1$ which implies $b = \frac{1}{2}$

An eigenvector corresponding to $\lambda = 0$ can be computed in a similar way and the result is

$$v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 2^n a \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

Thus $x_n = 2^n a$.

Given initial condition $x_0 = 10$, $a = 10$ so that

$$x_n = 10 \cdot 2^n$$

(e)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (-1-\lambda)(-\lambda) + 2 &= 0 \\ \lambda^2 + \lambda + 2 &= 0 \\ \lambda &= -2, 1 \end{aligned}$$

Eigenvectors:

$$\begin{aligned} v_{-2} &= \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \\ v_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (-2)^n a \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus $x_n = (-2)^n a + b$.
Given initial conditions $x_0 = 6; x_1 = 3,\ a = 1, b = 5$ so that

$$x_n = (-2)^n + 5$$

(f)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (3-\lambda)(-\lambda) &= 0 \\ \lambda &= 0, 3 \end{aligned}$$

Eigenvectors (again the eigenvectors have to be derived):

$$\begin{aligned} v_0 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ v_3 &= \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix} \end{aligned}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 3^n a \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$$

Thus $x_n = 3^n a$.
Given initial conditions $x_1 = 12$, $a = 4$ so that

$$x_n = 4 \cdot 3^n$$

8.
(a)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (3-\lambda)(4-\lambda) - 2 &= 0 \\ \lambda^2 - 7\lambda + 10 &= 0 \\ \lambda &= 5, 2 \end{aligned}$$

Eigenvectors:

$$\begin{aligned} v_5 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \end{aligned}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 5^n a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2^n b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

(b)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 1 \\ \frac{3}{16} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} \left(\frac{1}{4} - \lambda\right)\left(-\frac{1}{4} - \lambda\right) - \frac{3}{16} &= 0 \\ \lambda^2 - \frac{1}{4} &= 0 \\ \lambda &= \frac{1}{2}, -\frac{1}{2} \end{aligned}$$

Eigenvectors:

$$\begin{aligned} v_{\frac{1}{2}} &= \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix} \\ v_{-\frac{1}{2}} &= \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} \end{aligned}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \left(\frac{1}{2}\right)^n a \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix} + \left(-\frac{1}{2}\right)^n b \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix}$$

(c)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (\sigma_1 - \lambda)(-\lambda) - \beta\sigma_2 &= 0 \\ \lambda^2 - \sigma_1\lambda - \beta\sigma_2 &= 0 \\ \lambda &= \frac{\sigma_1 \pm \sqrt{\sigma_1^2 + 4\beta\sigma_2}}{2} \end{aligned}$$

Eigenvectors:

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 \\ \frac{-\sigma_1+\sqrt{\sigma_2+4\beta\sigma_2}}{2\sigma_2} \end{pmatrix} \\ v_2 &= \begin{pmatrix} 1 \\ \frac{-\sigma_1-\sqrt{\sigma_2+4\beta\sigma_2}}{2\sigma_2} \end{pmatrix} \end{aligned}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (\frac{\sigma_1 + \sqrt{\sigma_1^2 + 4\beta\sigma_2}}{2})^n a \begin{pmatrix} 1 \\ \frac{-\sigma_1+\sqrt{\sigma_2+4\beta\sigma_2}}{2\sigma_2} \end{pmatrix} + (\frac{\sigma_1 - \sqrt{\sigma_1^2 + 4\beta\sigma_2}}{2})^n b \begin{pmatrix} 1 \\ \frac{-\sigma_1-\sqrt{\sigma_2+4\beta\sigma_2}}{2\sigma_2} \end{pmatrix}$$

(d)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (1-\lambda)(-\lambda) - 2 &= 0 \\ \lambda^2 - \lambda - 2 &= 0 \\ \lambda &= -1, 2 \end{aligned}$$

Eigenvectors:

$$\begin{aligned} v_{-1} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (-1)^n a \begin{pmatrix} 1 \\ -2 \end{pmatrix} + (2)^n b \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(e)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (1-\lambda)(\frac{1}{3} - \lambda) &= 0 \\ \lambda &= -1, \frac{1}{3} \end{aligned}$$

Eigenvectors:

$$\begin{aligned} v_{-1} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ v_{\frac{1}{3}} &= \begin{pmatrix} 1 \\ \frac{4}{9} \end{pmatrix} \end{aligned}$$

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (-1)^n a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{1}{3}\right)^n b \begin{pmatrix} 1 \\ \frac{4}{9} \end{pmatrix}$$

(f)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 3 \\ -\frac{1}{8} & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (\frac{1}{4} - \lambda)(1 - \lambda) + \frac{3}{8} &= 0 \\ \lambda^2 - \frac{5}{4} + \frac{5}{8} &= 0 \\ \lambda &= \frac{5}{8} \pm \frac{i\sqrt{15}}{8} \end{aligned}$$

Eigenvectors:

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 \\ \frac{1}{8} + \frac{i\sqrt{15}}{24} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{8} \end{pmatrix} + i \begin{pmatrix} 0 \\ \frac{\sqrt{15}}{24} \end{pmatrix} \\ v_2 &= \begin{pmatrix} 1 \\ \frac{1}{8} - \frac{i\sqrt{15}}{24} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{8} \end{pmatrix} - i \begin{pmatrix} 0 \\ \frac{\sqrt{15}}{24} \end{pmatrix} \end{aligned}$$

General real-valued solution (see class notes):

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= a \cdot \left[\mathrm{Re}\left\{ (\frac{5}{8} + \frac{i\sqrt{15}}{8})^n \right\} \cdot \begin{pmatrix} 1 \\ \frac{1}{8} \end{pmatrix} - \mathrm{Im}\left\{ (\frac{1}{8} + \frac{i\sqrt{15}}{24})^n \right\} \cdot \begin{pmatrix} 0 \\ \frac{\sqrt{15}}{24} \end{pmatrix} \right] \\ &\quad + b \cdot \left[\mathrm{Im}\left\{ (\frac{5}{8} + \frac{i\sqrt{15}}{8})^n \right\} \cdot \begin{pmatrix} 1 \\ \frac{1}{8} \end{pmatrix} + \mathrm{Re}\left\{ (\frac{1}{8} + \frac{i\sqrt{15}}{24})^n \cdot \begin{pmatrix} 0 \\ \frac{\sqrt{15}}{24} \end{pmatrix} \right\} \right] \end{aligned}$$

where a and $b$have to be determined from the initial conditions.

9. Let

$$\begin{aligned} x(t) &= \text{number of 1 year olds present in year n} \\ y(t) &= \text{number of 2 year olds present in year n} \end{aligned}$$

Then the Leslie matrix model will look as follows:

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 4 \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

The eigenvalues of this system are:

$$\lambda = \frac{1 \pm \sqrt{97}}{6}$$

and thus the growth rate of the population is $\frac{1+\sqrt{97}}{6} \approx 1.8081$.

The corresponding eigenvector is:

$$v = \begin{pmatrix} 1 \\ \frac{\sqrt{97}-1}{24} \end{pmatrix}$$

and so the proportion of 1 year olds to 2 year olds is $\approx 1 : 0.3687$.

10. The model can be reformulated as:

$$R_{n+1} = (1-f)R_n + \gamma f R_{n-1}$$

The eigenvalues of this system are given from the original model:

$$\lambda_{1/2} = \frac{(1-f) \pm \sqrt{(1-f)^2 + 4\gamma f}}{2}$$

We know already that $f > 0, \gamma > 0$ so $\lambda_1 = \frac{(1-f)+\sqrt{(1-f)^2+4\gamma f}}{2}$ is the dominant eigenvalue determining the growth rate of the active blood cells.

In order to keep approximately the same number of active blood cells we should have

$$\lambda_1 \approx 1$$

For this condition we get from the expression for λ_1 that:

$$\sqrt{(1-f)^2 + 4\gamma f} \approx 1 + f$$

hence

$$(1-f)^2 + 4\gamma f \approx (1+f)^2$$

for which we need γ to be approximately equal to 1.

11.

(a) In a given year n, members of age class $i-1$ survive with probability σ_{i-1} to constitute the members of age class i in year $n+1$. Translating this into equations yields:

$$p^i_{n+1} = \sigma_{i-1} p^{i-1}_n \quad i = 1 \ldots m.$$

For the first age class, we sum up the contributions from all age classes to get the equation

$$p^0_{n+1} = \Sigma^m_{i=1} \alpha_i p^i_n.$$

If the above equations are put into vector form, then we get the matrix equation $P_{n+1} = A \cdot P$,where A the required form.

(b) The characteristic equation for an $n \times n$- matrix A is the analogue of the equation $\lambda^2 - tr(A)\lambda + \det(A) = 0$ whose solutions are the eigenvalues of a $2 \times 2-$matrix A. In particular, the solutions of the characteristic equation of a matrix are the eigenvalues of that matrix. The characteristic equation is simply obtained by setting the determinant of the matrix $\lambda I - A$, which is obtained from a given matrix A by subtracting A from a matrix which has λ in the diagonal and 0's elsewhere, equal to 0. Thus, the characteristic equation is $\det(\lambda I - A) = 0$, where I is the identity matrix, having 1's in the diagonal and 0's elsewhere. For example, for a $2 \times 2-$matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $\lambda I - A = \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix}$, hence $\det(\lambda I - A) = (\lambda - a)(\lambda - d) - bc = \lambda^2 - tr(A)\lambda + \det(A)$.

The claim of this problem follows directly from the application of the rule for finding determinants of $n \times n$ matrices, which you can find in any textbook on linear algebra.

(c) With

$$p(\lambda) = \lambda^n - \alpha_1 \lambda^{n-1} - \alpha_2 \sigma_1 \lambda^{n-2} - ... - \alpha_n \sigma_1 \sigma_2 ... \sigma_{n-1}$$

we have

$$\frac{p(\lambda)}{\lambda^n} = 1 - \alpha_1 \lambda^{-1} - \alpha_2 \sigma_1 \lambda^{-2} - ... - \alpha_n \sigma_1 \sigma_2 ... \sigma_{n-1} \lambda^{-n},$$

hence

$$f(\lambda) = \alpha_1 \lambda^{-1} + \alpha_2 \sigma_1 \lambda^{-2} + ... + \alpha_n \sigma_1 \sigma_2 ... \sigma_{n-1} \lambda^{-n}$$

Thus clearly

$$\lim_{\lambda \to 0} f(\lambda) = \infty$$

and

$$\lim_{\lambda \to \infty} f(\lambda) = 0$$

Moreover, for the derivative of f with respect to λ we have:

$$f'(\lambda) = -\alpha_1 \lambda^{-2} - 2\alpha_2 \sigma_1 \lambda^{-3} - ... - n\alpha_n \sigma_1 \sigma_2 ... \sigma_{n-1} \lambda^{-n-1}.$$

Thus, $f'(\lambda) < 0$ for all $\lambda > 0$, which means that $f(\lambda)$ is a monotone decreasing function for $0 < \lambda < \infty$. Since $f(\lambda) > 0$ for small λ and $f(\lambda) < 0$ for large

λ, there must therefore be a unique value λ^* such that $f(\lambda^*) = 1$, hence, we conclude form the definition of $f(\lambda)$ that there is a unique value λ^* with $0 < \lambda^* < \infty$ such that $p(\lambda^*) = 0$.

(d) This was explained in class: the general solution is a linear combination of n components corresponding to the n eigenvalues. If λ^* is the dominant eigenvalue, then the population will eventually become a multiple of the corresponding eigenvector, because all other components become unimportant. Thus, there will be convergence to the age distribution given by this eigenvector.