Math 361 Winter 2001/2002
Assignment 3 - Solutions

1.

$$L = \begin{pmatrix} 1/3 & 4 & 2 \\ 2/3 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

To determine the long term growth rate and stable age distribution, one would have to find the dominant eigenvalue of L and the corresponding eigenvector.

2. a.

$$x_{n+2} + x_n = 0$$

Let $y_{n+1} = x_n$. The system reduces to:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (-\lambda)(-\lambda) + 1 &= 0 \\ \Longrightarrow \lambda_{1/2} &= \pm i = \cos(\pi/2) \pm i \sin(\pi/2) \end{aligned}$$

This system exhibits sustained oscillations with constant amplitude = 1 and frequency $(\pi/2)/(2\pi) = 1/4$.

b.

$$x_{n+2} - x_{n+1} + x_n = 0$$

Let $y_{n+1} = x_n$. The system reduces to:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (1-\lambda)(-\lambda) + 1 &= 0 \\ \Longrightarrow \lambda_{1/2} &= \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \cos(\pi/3) \pm i \sin(\pi/3) \end{aligned}$$

The has oscillatory solutions with constant amplitude = 1 and frequency 1/6.

c.

$$x_{n+2} - 2x_{n+1} + 2x_n = 0$$

Let $y_{n+1} = x_n$. The system reduces to:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (2-\lambda)(-\lambda) + 2 &= 0 \\ \Longrightarrow \lambda_{1/2} &= 1 \pm i = \sqrt{2}(\cos(\pi/4) \pm i \sin(\pi/4)) \end{aligned}$$

This has oscillatory solutions whose amplitude increases with time as $|\lambda_1|^t = |\lambda_2|^t = \sqrt{2}^t$ and with frequency 1/8.

d.

$$x_{n+2} + 2x_{n+1} + 3x_n = 0$$

Let $y_{n+1} = x_n$. The system reduces to:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (-2-\lambda)(-\lambda) + 3 &= 0 \\ \Longrightarrow \lambda &= -1 \pm i\sqrt{2} = \sqrt{3}\left(\cos(\pi - \tan^{-1}(\sqrt{2})) \pm i \sin(\pi - \tan^{-1}(\sqrt{2}))\right) \end{aligned}$$

This has oscillatory solutions whose amplitude increases with time as $|\lambda_1|^t = |\lambda_2|^t = \sqrt{3}^t$ and with frequency $\left(\pi - \tan^{-1}(\sqrt{2})\right)/(2\pi)$.

3. To calculate the eigenvalues we solve the equation

$$\mu^2 - tr(A)\mu + \det(A) = \mu^2 - 2\lambda\mu + \lambda^2 = (\mu - \lambda)^2 = 0$$

for μ, which yields

$$\mu_{1/2} = \lambda$$

i.e., both eigenvalues are equal to λ. To find the eigenvectors, we solve

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

hence

$$\begin{aligned}\lambda x + y &= \lambda x \\ \lambda y &= \lambda y\end{aligned}$$

Clearly, $y = 0$ because of the first of these equation, and we conclude that the only eigenvectors are multiples of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, there is only one eigenvector.

(Note: this is not a direct consequence of the fact that both eigenvalues are equal, for if you consider the matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then both eigenvalues of A are equal to λ, but clearly A has two different eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.)

4. a.

$$x_n = (1 - \alpha)x_{n-1} + \beta x_n$$

This equation is linear:

$$x_n = \left(\frac{1-\alpha}{1-\beta}\right) x_{n-1}$$

Solution:

$$x_n = \left(\frac{1-\alpha}{1-\beta}\right)^n x_0$$

b.

$$x_{n+1} = \frac{x_n}{1 + x_n}$$

This equation is non-linear.
Steady state:

$$\begin{aligned}x^* &= \frac{x^*}{1 + x^*} \\ x^{*2} &= 0\end{aligned}$$

Therefore $x^* = 0$ is the only steady state.
c.

$$x_{n+1} = x_n e^{-a x_n}$$

This equation is non-linear.

Steady states:

$$\begin{aligned} x^* &= x^* e^{-ax^*} \\ x^* &= 0 \end{aligned}$$

d.

$$(x_{n+1} - \alpha)^2 = \alpha^2(x_n^2 - 2x_n + 1)$$

This equation is non-linear.
Steady states:

$$\begin{aligned} x^*(x^*(1-\alpha^2) + 2\alpha(\alpha - 1)) &= 0 \\ x^* &= 0, \frac{2\alpha(1-\alpha)}{(1-\alpha^2)} \end{aligned}$$

e.

$$x_{n+1} = \frac{K}{k_1 + k_2/x_n}$$

This equation is non-linear.
Steady states:

$$\begin{aligned} k_1 x^* + k_2 &= K \\ x^* &= \frac{K - k_2}{k_1} \end{aligned}$$

5. a.

$$A \cdot w_i = \lambda_i \cdot w_i,\ i = 1, 2$$

Therefore:

$$\begin{aligned} A(u + iv) &= c(\cos\theta + i\sin\theta)(u + iv) \\ A(u - iv) &= c(\cos\theta - i\sin\theta)(u - iv) \\ A \cdot u + iA \cdot v &= c(\cos\theta \cdot u - \sin\theta \cdot v + i(\cos\theta \cdot v + \sin\theta \cdot u)) \\ A \cdot u - iA \cdot v &= c(\cos\theta \cdot u - \sin\theta \cdot v - i(\cos\theta \cdot v + \sin\theta \cdot u)) \end{aligned}$$

and so

$$\begin{aligned} 2A \cdot u &= c(2\cos\theta \cdot u - 2\sin\theta \cdot v) \\ 2A \cdot v &= c(2\cos\theta \cdot v + 2\sin\theta \cdot u) \end{aligned}$$

b.

$$\begin{aligned} A \cdot (A \cdot w_i) &= A \cdot (\lambda_i w_i) \\ &= \lambda_i (A \cdot w_i) \\ &= \lambda_i (\lambda_i w_i) \\ &= \lambda_i^2 w_i \end{aligned}$$

This can be continued to show that

$$A^t \cdot w_i = \lambda_i^t w_i$$

So

$$\begin{aligned} A^t \cdot w_1 &= c^t(\cos\theta + i\sin\theta)^t \cdot w_1 \\ A^t \cdot w_2 &= c^t(\cos\theta - i\sin\theta)^t \cdot w_2 \end{aligned}$$

Using trigonometric identities one can show that

$$c^t(\cos\theta \pm i\sin\theta)^t = c^t(\cos(t\theta) \pm i\sin(t\theta))$$

(De Moivre's theorem). Thus:

$$\begin{aligned} A^t \cdot w_1 &= c^t(\cos(t\theta) + i\sin(t\theta)) \cdot w_1 \\ A^t \cdot w_2 &= c^t(\cos(t\theta) - i\sin(t\theta)) \cdot w_2 \end{aligned}$$

The result follows from these identities as in part (a).

6. a. The function $f(x)$ is a monotone decreasing function of x, which attains its maximal value at 0: $f(0) = \lambda$. In general, the function $f(x)$ takes on larger values for larger λ and K.

b. K determines how fast the per capita number of descendants decreases with population size, e.g. due to competition. λ determines the maximal number of per capita descendants, realized when the population size is 0, i.e. when there is no competition. (The maximal number of per capita descendants is $\lambda \cdot e$.)

c.

$$x(t+1) = f(x(t)) \cdot x(t) = F(x(t)).$$

The function $F(x)$ has the following general properties:

$F(0) = 0$;

$F(x) \to 0$ for $x \to \infty$;

$F(x)$ has a single maximum, which lies to the left of the single equilibrium $x^* > 0$ for which $F(x^*) = x^*$. (Such an equilibrium exists is $\lambda \cdot e > 1$.)

$F(x)$ has a single inflection point, which lies to the right of the equilibrium x^* for which $F(x^*) = x^*$.

d. Clearly, $x^* = 0$ i a steady state of the dynamical system, because $F(0) = 0$, and to find its stability we have to calculate

$$\frac{dF}{dx}(0) = \lambda \cdot e.$$

Thus, the steady state $x^* = 0$ is locally stable if and only if $\lambda \cdot e < 1$.

If $\lambda \cdot e > 1$, i.e. if $x^* = 0$ is unstable, then there is an additional steady state $x^* > 0$, which we can find by solving the equation $F(x^*) = x^*$, or, equivalently, the equation

$$f(x^*) = \lambda \exp(1 - x^*/K) = 1,$$

which yields

$$x^* = K(1 + \ln(\lambda)).$$

To determine the stability of x^*, we have to calculate

$$\frac{dF}{dx}(x^*) = -\ln(\lambda).$$

Thus, the second steady state is locally stable if and only if $\ln(\lambda) < 1$, i.e., if and only if $\lambda < e$.

7.a.

$$\begin{aligned} x_{n+1} &= rx_n(1 - x_n) \\ f(x) &= rx(1 - x) \\ \frac{df}{dx} &= r - 2rx \\ \frac{df}{dx}|_{x^*=0} &= r \end{aligned}$$

Stable when $r < 1$.

b.

$$\begin{aligned} x_{n+1} &= -x_n^2(1 - x_n) \\ f(x) &= -x^2(1 - x) \\ \frac{df}{dx} &= -2x + 3x^2 \\ \frac{df}{dx}\Big|_{x^*=\frac{1+\sqrt{5}}{2}} &= \frac{7 + \sqrt{5}}{2} > 1. \end{aligned}$$

This equilibrium is unstable.

c.

$$\begin{aligned} x_{n+1} &= \frac{1}{2 + x_n} \\ f(x) &= \frac{1}{2 + x} \\ \frac{df}{dx} &= -\frac{1}{(2 + x)^2} \\ \frac{df}{dx}\Big|_{x^*=\sqrt{2}-1} &= 3 + 2\sqrt{2} > 1 \end{aligned}$$

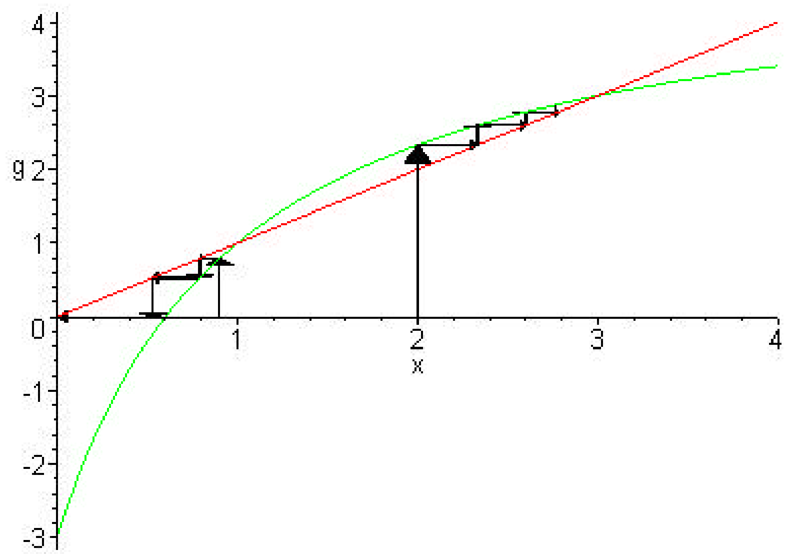

Figure 1: Cobwebbing in a single species model with predation

This equilibrium is unstable.

d.

$$\begin{aligned} x_{n+1} &= x_n \ln x_n^2 \\ f(x) &= x \ln x^2 \\ \frac{df}{dx} &= 2 + \ln x^2 \\ \frac{df}{dx}|_{x=e^{1/2}} &= 3 > 1 \end{aligned}$$

This equilibrium is unstable.

8. a. The function $F(x)$ has the following properties:

$F(0) = 0$;

$F(x) \to \lambda$ for $x \to \infty$;

$F'(x) > 0$ for all x, i.e.$F(x)$is monotonically increasing.

b. If $\lambda < 1$, then 0 is the only equilibrium of the dynamical system defined by $F(x)$.

If $\lambda > 1$, there is an additional steady state $x^* > 0$ for which $F(x^*) = x^*$.

The steady state $x^* = 0$ is locally stable if and only if $\lambda < 1$. The steady state $x^* > 0$ is always locally stable.

9.

a. Figure 1 shows typical cobwebbing behavior for this model, and we see some interesting behaviour, in that there are 2 non-zero equilibrium points, i.e.

two intersection points between the graph of $F(x)$ and the diagonal (assuming that $\lambda > 1$, and that h is not too large). In this example, if the population starts at a density greater than 1, the population will converge to the upper equilibrium. On the other hand, if the population starts at a density less than one it will go to negative values, i.e. it will go extinct due to predation. In particular, the lower equilibrium is unstable.

b. For $\lambda < 1$, the curve of $x(t+1)$ *vs.* $x(t)$ is always lower than the $y = x$ line regardless of the value of h, which tells us that the population will eventually go extinct (if we assume negative population sizes is extinction). For $\lambda > 1$, changing h will cause the curve $x(t+1)$ *vs.* $x(t)$ to move up or down. As h approaches 0, the dynamics are the same as in problem 8. As the predation pressure h increases from 0, the range of initial population sizes increases for which the population will go to negative values. For sufficiently large h, the curve $x(t+1)$ *vs.* $x(t)$ will again always be lower than the $y = x$ which means that, as with the case when $\lambda < 1$, the population will always go to negative values (extinction).

c. The reason why this model is problematic is because it allows for negative population sizes!

d. A better way to model predation is to multiply the per capita growth rate in the prey, $f(x(t))$, by some factor h with $0 < h < 1$, resulting in a model of the form

$$x(t+1) = h \cdot f(x(t)) \cdot x(t) = h \cdot F(x(t)).$$

10. We are given that

$$\left| \left(\frac{df}{dx} \left|_{x_1^{**}} \right) \left(\frac{df}{dx} \left|_{x_2^{**}} \right) \right| < 1$$

where $x_i^{**}, i = 1, 2$ are steady states of $f(f(x)) = g(x)$ (i.e. $g(x_i^{**}) = x_i^{**}$ for $i = 1, 2$; note that $x_1^{**} = f(x_2^{**})$ and $x_2^{**} = f(\ x_1^{**})$.)

Taking the derivative of $g(x) = f(f(x))$ at either of the stable states gives:

$$\frac{d}{dx} g(x) \left|_{x_i^{**}} \right. = \frac{d}{dx} f(f(x)) \left|_{x_i^{**}} \right. = \frac{df}{dx} \left|_{x_i^{**}} \right. \frac{df}{dx} \left|_{x_j^{**}} \right.$$

Thus,$\left|\frac{d}{dx} g(x) \left|_{x_i^{**}} \right. \right| < 1$ by assumption, hence both steady states of g are stable, and hence these two steady states give rise to a stable 2-point cycle of the dynamical system defined by the function f.