Math 361 Winter 2001/2002
Assignment 6 - Solutions

1. (a) The equilibria are found graphically as the intersections of the graph of the function $rN[1-(N/K)^\theta]$ with the N-axis. The equilibrium at which the function changes sign form positive to negative as N is increased is stable, the other one is unstable.

(b) At equilibrium N^* the population size is constant, hence we have to solve $\frac{dN}{dt}(N^*) = rN^*[1-(N^*/K)^\theta] = 0$. Clearly, the two solutions are $N^* = K$ and $N^* = 0$. To evaluate the stability of these equilibria, let $\frac{dN}{dt} = rN[1 - (N/K)^\theta] = F(N)$, and determine whether $\frac{dF}{dN}(N^*)$ is positive or negative.

Stability analysis of the equilibrium $N^* = K$:

$\frac{dF}{dN}(N) = r - r\left(\frac{N}{K}\right)^\theta - r\theta\left(\frac{N}{K}\right)^\theta$

$\frac{dF}{dN}(N^*) = r - r - r\theta = -r\theta$

Since both r and θ are positive $\frac{dF}{dN}(N^*)$ is negative, and since $\frac{dF}{dN}(N^*)$ is negative the equilibrium, $N^* = K$, is stable.

Stability analysis of the equilibrium $N^* = 0$:

$\frac{dF}{dN}(N^*) = r$

Since r is positive, the equilibrium, $N^* = 0$, is unstable.

2. (a) A plot of $\frac{dN}{dt}$ versus N for $N > 0$ shows that $\frac{dN}{dt} < 0$ for $0 < N < a$, $\frac{dN}{dt} > 0$ for $a < N < K$ and $\frac{dN}{dt} < 0$ for $N > K$. There are three intersection points of the graph of dN/dt with the N-axis, hence there are three equilibrium points. The intermediate equilibrium is unstable because the graph changes from negative to positive as N is increased from below to above this equilibrium, while the opposite is true for the other two equilibria.

(b) At equilibrium, $\frac{dN}{dt} = 0 = rN^*(N^* - a)(1 - N^*/K)$.

The equation is already factored for us leaving $rN^* = 0 \Longrightarrow N^* = 0$, $N^* - a = 0 \Longrightarrow N^* = a$ or $(1 - N^*/K) = 0 \Longrightarrow N^* = K$. Thus we have three equilibria points $N^* = \{0, a, K\}$.

To determine stability set $\frac{dN}{dt} = rN(N-a)(1-N/K) = F(N)$, take the derivative of $F(N)$ with respect to N, and evaluate the derivative at the equilibrium points:

$\frac{dF(N)}{dN} = r(N-a)(1-N/K) + rN(1-N/K) - \frac{rN(N-a)}{K}$

At the equilibrium $N^* = 0$:

$\frac{dF}{dN}(0) = r(-a)$. Thus $\frac{dF(0)}{dN}$ is always negative which means the equilibrium, $N^* = 0$, is stable.

At the equilibrium $N^* = a$:

$\frac{dF}{dN}(a) = r(a-a)(1-a/K) + ra(1-a/K) - \frac{rN(a-a)}{K}\frac{dF(a)}{dN} = ra(1-a/K)$.

Because $a < K$ we have $\frac{dF}{dN}(a)$, which means that $N^* = a$ is unstable.

At the equilibrium $N^* = K$:

$\frac{dF(K)}{dN} = r(K-a)(1-K/K) + rK(1-K/K) - \frac{rK(K-a)}{K}$

$= -r(K-a)$. Again since $a < K$ this expression is negative for reasonable values of r, K, and a. Therefore this equilibrium is stable.

(c) The Allee effect implies that in contrast to the simple logistic equation the equilibrium $N^* = 0$ is locally stable, so that small populations go extinct, e.g. due to problems in finding suitable mates. To persist, i.e. to converge to the carrying capacity $N^* = K$, initial population sizes must be above the threshold given by the intermediate equilibrium $N^* = a$.

3. (a) At equilibrium $\frac{dN}{dt} = 0 = rN^*(1 - N^*/K) - HN^*$, hence $rN^*(1 - N^*/K) = HN^*$, which has the solutions $N^* = 0$ and $N^* = K(1 - \frac{H}{r})$

For the latter equilibrium to be positive we must have $0 < H < r$. For the stability analysis we calculate the derivative of $F(N) = rN(1 - N/K) - HN$:

$$\frac{dF}{dN} = r(1 - N/K) - rN/K - H.$$

Thus $\frac{dF}{dN}(0) = r - H$, and $\frac{dF}{dN}\left(K(1 - \frac{H}{r})\right) = H - r$. Thus, if $H > r$ there is only one feasible equilibrium, which is stable, and if $H < r$ there are two equilibria, with 0 being unstable and $K(1 - \frac{H}{r})$ being stable.

(b) If $H > r$ then $dN/dt < 0$ for all $N > 0$, and hence the population is always decreasing. In other words, if harvesting given by H is too large the harvested population will go extinct.

4. (a) the per capita harvesting rate is given by $H/(0.1 + N)$ (the total harvesting is $H/(0.1 + N)$). Thus, the per capita harvesting rate goes to zero as the size of the harvested population becomes large.

Equilibria occur when $\frac{dN}{dt} = 0$.

So

$$\begin{aligned} N^*(1 - N^*) - \frac{HN^*}{0.1 + N^*} &= 0 \\ N^*\left(1 - N^* - \frac{H}{0.1 + N^*}\right) &= 0 \end{aligned}$$

Clearly, there is one equilibrium at $N^* = 0$.

Also

$$\left(1 - N^* - \frac{H}{0.1 + N^*}\right) = 0$$

implies that N^* is a solution of the quadratic equation

$$N^2 - 0.9N + (H - 0.1) = 0$$

This leads to two further equilibria at $N^* = \frac{0.9 \pm \sqrt{0.81 - 4(H-0.1)}}{2}$. The behavior of the system thus depends on the parameter H.

If $H < 0.1$, then one of the equilibria above will be > 0 and the other < 0 (which is biologically not feasible). Stability analysis shows that the positive equilibrium will be stable, while the equilibrium $N^* = 0$ is unstable in that case.

If $H > 0.3025$, then none of the values N^* given in the formula above is a real number, so that $N^* = 0$ is the only equilibrium, and it is stable (in fact, in this case $dN/dt < 0$ for all $N > 0$).

If $0.1 < H < 0.3025$, then both values of N^* given above are positive, so that there are three equilibrium points. Here the intermediate equilibrium is unstable, and the other two equilibrium (i.e. $N^* = 0$ and $N^* = \frac{0.9 + \sqrt{0.81 - 4(H-0.1)}}{2}$) are stable.

In sum, the parameter H acts as a bifurcation parameter, changing system behavior from one stable equilibrium and one unstable equilibrium to two stable ones and one unstable one as H increases above 0.1, and changing the behavior again to a single stable equilibrium as H increases above the second bifurcation point at 0.3025.

(c) In the model from the previous question per capita harvesting rate was equal to H independent of the size N of the harvested population. Clearly, the present model with decreasing per capita harvesting rate as a function of the size of the harvested population is more realistic (e.g. when harvester population size is constant or changes only slowly).

5. A model for E. coli growth is

$$N(t+1) = 2N(t)$$

which has the solution

$$N(t) = 2^t N_0$$

if $(t+1) - t$ is equivalent to 20 mins.

$$\begin{aligned} \text{mass of earth} &= 5.9763x10^{24}kg \\ \text{mass of E.coli} &= 10^{-12}g = 10^{-15}kg \end{aligned}$$

number of E. coli equal to the mass of the earth:

$$= \frac{5.9763x10^{24}}{10^{-15}} = 5.9763x10^{39}$$

Assuming that we start out with 1 E. coli, to calculate the length of time we must solve

$$\begin{aligned} 5.9763x10^{39} &= 2^t \\ t &= \frac{\ln 5.9763x10^{39}}{\ln 2} \\ &= 132.1 \end{aligned}$$

So it takes 133 time steps for the bacteria to surpass the weight of the earth. This is equivalent to 44.33 hours. The statement is a little exaggerated.

6. The solution to this differential equation is

$$N(t) = N_0 \cdot \exp[-Kt].$$

We are interested in the time t_0 at which $N(t_0) = N_0/2$, so we have to solve

$$N_0/2 = N_0 \cdot \exp[-Kt_0].$$

Note that N_0 cancels form this equation, so that the solution does not depend on N_0. The solution is given by

$$t_0 = \frac{\ln[1/2]}{-K} = \frac{\ln[2]}{K}.$$

7.

$$\frac{dN}{dt} = K(t)N$$

Using separation of variables and with the initial condition $N(0) = N_0$ we get:

$$\begin{aligned} \int_{N_0}^{N(t)} \frac{dN}{N} &= \int_0^t K(t)dt \\ \ln N(t) - \ln N_0 &= \ln \frac{N(t)}{N_0} = \int_0^t K(s)ds \\ N(t) &= N_0 \cdot \exp\left(\int_0^t K(s)ds\right). \end{aligned}$$

8. This problem deals with the logistic equation, which was explained in class. (For the last part of (d) note that if N_0 is very small, then $N(t)$ is approximately equal to $N_0 \exp[rt]$.)

9. (a) Original model:

$$\frac{dN}{dt} = \square(C - \alpha N)\, N$$

Let

$$\begin{aligned} N &= N^* \hat{N} \\ t &= t^* \tau \end{aligned}$$

with two parameters $\hat{N}, \tau$ that have to be determined, and two new variables N^*, t^*. Substituting this into the original equation gives:

$$\begin{aligned} \frac{\hat{N}}{\tau}\frac{dN^*}{dt^*} &= \square\left(C_0 - \alpha N^* \hat{N}\right) N^* \hat{N} \\ \frac{dN^*}{dt^*} &= \square\tau \left(C_0 - \alpha N^* \hat{N}\right) N^* \end{aligned}$$

For $\tau = \frac{1}{\square}$, $\hat{N} = \frac{1}{\alpha}$ we get

$$\frac{dN^*}{dt^*} = (C_0 - N^*)\, N^*,$$

Dropping the stars for convenience, the dimensionless equation becomes

$$\frac{dN}{dt} = (C_0 - N)\, N$$

(b) Steady states

$$\begin{aligned} \frac{dN}{dt} &= 0 = (C_0 - N)\, N \\ \Longrightarrow N &= 0, C_0 \end{aligned}$$

(c) To determine the stability, linearize the model:

$$f(N) = (C_0 - N)\, N$$

$$f'(N) = C_0 - 2N$$

Thus $f'(0) = C_0$ and $f'(C_0) = -C_0$. Since $C_0 > 0$, we see that $N = 0$ is unstable, while $N = C_0$ is stable.

(d) For the dimensionless model $N^* = 0, C_0$. Recall from (a) that $N = N^*\hat{N}$ and $\hat{N} = \frac{1}{\alpha}$. Therefore, in the original model the equilibria are $N = 0, \frac{C_0}{\alpha}$.

The exact solution is given by

$$N(t) = \frac{N_0 B}{N_0 + (B - N_0)\, e^{-rt}}.$$

The equilibrium points corresponding to this exact solution are 0 and $\frac{C_0}{\alpha}$(i.e. if $N(0) = 0$ or $\frac{C_0}{\alpha}$, then $N(t) = N_0$ for all t.) Moreover, all trajectories of the exact solution converge to the stable equilibrium $\frac{C_0}{\alpha}$ (except if $N(0) = 0$). Thus the dimensionless model agrees with the exact solution.

10. (a) $c_1 = 3$, $c_2 = -2$

(b)

$$\begin{aligned} y' &= 10y \\ y(0) &= 0.001 \end{aligned}$$

The general solution of this equation is given by

$$y(t) = c \cdot e^{10t}$$

Now

$$y(0) = c_1 = 0.001$$

So the solution is given by $y(t) = 0.001e^{10t}$.

(c)

$$\begin{aligned} y'' - 3y' - 4y &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

The characteristic equation is given by $r^2 - 3r - 4 = 0$ which has solutions $r = 1, 4$. Thus (see p. 132-133 in the textbook) the general solution is given by

$$y(t) = c_1 e^{-t} + c_2 e^{4t}$$

Using the initial conditions we have to solve the simultaneous equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ 4c_2 - c_1 &= 1 \end{aligned}$$

which yield $c_1 = -\frac{1}{5}$, $c_2 = \frac{1}{5}$, hence the solution is

$$y(t) = \frac{1}{5}\left(e^{4t} - e^{-t}\right)$$

(d)

$$\begin{aligned} y'' - 9y &= 0 \\ y(0) &= 5 \\ y'(0) &= 0 \end{aligned}$$

The characteristic equation is given by $r^2 - 9 = 0$ which has solutions $r = -3, 3$. Thus the general solution is given by

$$y(t) = c_1 e^{-3t} + c_2 e^{3t}$$

Using the initial conditions we have to solve the simultaneous equations

$$\begin{aligned} c_1 + c_2 &= 5 \\ 3c_2 - 3c_1 &= 0 \end{aligned}$$

which yield $c_1 = -\frac{5}{2}$, $c_2 = \frac{5}{2}$, hence the solution is

$$y(t) = \frac{5}{2}\left(e^{3t} - e^{-3t}\right)$$

(e)

$$\begin{aligned} y'' - 5y' &= 0 \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

The characteristic equation is given by $r^2 - 5r = 0$ which has solutions $r = 0, 5$. Thus the general solution is given by

$$y(t) = c_1 e^{5t} + c_2$$

Using the initial conditions we have to solve the simultaneous equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ 5c_1 &= 2 \end{aligned}$$

which yields $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, hence the solution is

$$y(t) = \frac{1}{5}\left(2e^{5t} + 3\right).$$