Math 361 Winter 2001/2002
Assignment 7 - Solutions

1. (a) At equilibrium, $\frac{dp}{dt} = 0 = mp^*(1-p^*) - ep^*$.
$-m(p^*)^2 + mp^* - ep^* = 0$
$p^*(-mp^* + m - e) = 0$
$p^* = 0$
$p^* = 1 - \frac{e}{m}$

(b) The parameter e must be less than the parameter m for p^* to take on a positive value. This makes sense because if the extinction rate (e) is greater than the colonization rate (m) then there will be insufficient migration to balance extinction.

Let $\frac{dp}{dt} = mp(1-p) - ep = F(p)$
$\frac{dF}{dp}(p) = m(1-p) - mp - e$
$= m - e - 2mp$

(c) Stability of the equilibrium $p^* = 0$:
$\frac{dF}{dp}(0) = m - e$. Which is positive (i.e., unstable) for $m > e$ and is negative (stable) for $m < e$. These dynamics make sense because if the migration rate is greater than the extinction rate then new subpopulations will be colonized at a greater rate than that at which local extinction occurs, therefore the metapopulation will persist. On the contrary, if the extinction rate is greater than the migration rate, the migration rate is insufficient to counteract extinction and the metapopulation will go extinct.

Stability for the equilibrium $p^* = 1 - \frac{e}{m}$:
$\frac{dF}{dp}(1 - \frac{e}{m}) = m(1 - (1 - \frac{e}{m})) - m(1 - \frac{e}{m}) - e$
$= e - (m - e) - e = e - m$. Thus, the equilibrium In contrast with the equilibrium $p^* = 1 - \frac{e}{m}$ is stable if it exists (i.e., if it positive, such that $e < m$). This makes sense because for a nonzero equilibrium to persist the migration rate must be at least as great as the extinction rate.

(d) In the logistic model there was a globally stable population size, namely K. Likewise in the metapopulation model, there is a globally stable metapopulation size, namely $1 - \frac{e}{m}$. The metapopulation model is related to the harvesting model in problem in assignment 6, in that the term ep is similar to the harvesting term HN.

2. (a) The set of equations describing a 2-species model of competition for space is
$\frac{dp_1}{dt} = m_1 p_1(1 - p_1) - ep_1$
$\frac{dp_2}{dt} = m_2 p_2(1 - p_1 - p_2) - m_1 p_1 p_2 - ep_2$.

The equilibrium fraction of habitat occupied by the dominant competitor ,p_1^*, is found by solving $\frac{dp_1}{dt} = 0$:

$p_1^*(m_1(1-p_1^*) - e) = 0$

$p_1^* = 0$

$p_1^* = \frac{m_1 - e}{m_1}$

The nonzero solution is $p_1^* = 1 - \frac{e}{m_1}$. For the equilibrium value of p_1 to be positive we need $m_1 > e$, i.e. the colonization rate must be bigger then the local extinction rate (which makes sense!)

(b) To determine the equilibrium fraction of habitat the subdominant competitor occupies when the dominant competitor is at its nonzero equilibrium proportion, substitute the equilibrium proportion for the dominant competitor in place of p_1 in the equation $\frac{dp_2}{dt} = m_2p_2(1-p_1-p_2) - m_1p_1p_2 - ep_2$ which describes the change in fraction of habitat occupied by the subdominant competitor and set $\frac{dp_2}{dt} = 0$.

$\frac{dp_2}{dt} = m_2p_2^*(1 - \frac{m_1-e}{m_1} - p_2^*) - m_1\frac{m_1-e}{m_1}p_2^* - ep_2^* = 0$

$p_2^*(m_2(1 - \frac{m_1-e}{m_1} - p_2^*) - m_1\frac{m_1-e}{m_1} - e) = 0$

$p_2^* = 0$

So one solution is $p_2^* = 0$, but this is rather uninteresting. Let's see if there is a nonzero equilibrium by finding the other root of the equation, $p_2^*(m_2(1 - \frac{m_1-e}{m_1} - p_2^*) - m_1\frac{m_1-e}{m_1} - e) = 0:$

$m_2(1 - \frac{m_1-e}{m_1} - p_2^*) - m_1 = 0$

$-m_2p_2^* = m_1 - m_2 + m_2\frac{m_1-e}{m_1}$

$p_2^* = 1 - \frac{m_1}{m_2} - \frac{m_1-e}{m_1}$

$p_2^* = \frac{e}{m_1} - \frac{m_1}{m_2}$

Provided $\frac{e}{m_1} > \frac{m_1}{m_2}$, the other equilibrium, $p_2^* = \frac{e}{m_1} - \frac{m_1}{m_2}$ is a positive nonzero equilibrium for the subdominant competitor. Since e is less than m_1 for the superior competitor to have a nonzero equilibrium, this means that for the subdominant competitor to persist, $m_2 > m_1$, which makes sense because the subdominant competitor must make up for its competitive disadvantage by having a higher colonization rate.

(c) By looking at the equilibrium equations for species 1 and 2 it is clear that given a starting point such that both species are coexisting, if you increase e species 1 will go extinct. The equilibrium proportion for species 1 is $1 - \frac{e}{m_1}$, whereas the equilibrium for species 2 is $\frac{e}{m_1} - \frac{m_1}{m_2}$, it is clear from these equations that given m_1 and m_2 are fixed as you increase e species 1 will decline to 0 and species 2 will increase.

(d) The equilibrium of species 2 increases (see 5c). This makes sense because species 2 can only colonize uninhabited subpopulations, and for species 2 to coexist at all - its migration rate must be greater than that of species 1. Therefore, as the extinction rate of both species 1 and 2 increases, more habitat will be available for species 2 to colonize.

(e) Evaluating the Jacobian matrix J at the equilibrium $(p_1^*, p_2^*) = (1 - e/m_1, e/m_1 - m_1/m_2)$ yields

$$J = \begin{pmatrix} e - m_1 & 0 \\ -(m_1 + m_2)(e/m_1 - m_1/m_2) & m_1 - em_2/m_1 \end{pmatrix}.$$

The two diagonal entries $e - m_1$ and $m_1 - em_2/m_1$ are negative, in the first case because p_1^* is assumed to be positive, and in the second case because p_2^* is assumed to be positive. Therefore, $tr(J) < 0$ and $\det(J) > 0$, hence the equilibrium is a stable node.

3. With added terms $-mN_i$ because the competition equations are:

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N_1 + \alpha_{12} N_2}{K_1}) - mN_1 \quad \text{(eq. 6a)}$$

$$\frac{dN_2}{dt} = r_2 N_2 (1 - \frac{N_2 + \alpha_{21} N_1}{K_2}) - mN_2 \quad \text{(eq. 6b)}$$

A graphical analysis of the dynamics of two species involves drawing isoclines. The simplest way to draw isoclines is to determine at what points on the N_1 and N_2 axes the lines cross. These points will also provide information regarding parameter values that allow coexistence. The point at which an isocline crosses the N_1 axes is when $N_2 = 0$; likewise, the point at which an isocline crosses the N_2 axes is when $N_1 = 0$.

A 0-isocline is a solution to $\frac{dN}{dt} = 0$. For the first equation above, the 0-isocline is defined by the relation $r_1 N_1 (1 - \frac{N_1 + \alpha_{12} N_2}{K_1}) - mN_1 = 0$. We seek the points where the line described by this relation crosses the N_1 and N_2 axes. To find the point at which it crosses the N_2 axis, set $N_1 = 0$ and solve for N_2:

$1 - \frac{\alpha_{12} N_2}{K_1} - \frac{m}{r_1} = 0$ by dividing through by $r_1 N_1$ and then substituting $N_1 = 0$ we get $N_2 = \frac{K_1 (1 - \frac{m}{r_1})}{\alpha_{12}}$.

To determine at which point the relation $r_1 N_1 (1 - \frac{N_1 + \alpha_{12} N_2}{K_1}) - mN_1 = 0$ crosses the N_1 axis, set $N_2 = 0$ and solve for N_1:

$1 - \frac{N_1}{K_1} - \frac{m}{r_1} = 0$ by dividing through by $r_1 N_1$ and then substituting $N_2 = 0$ we get $N_1 = K_1 (1 - \frac{m}{r_1})$.

For the second equation above, the points at which the 0-isocline crosses the N_1 and N_2 axes are found in a similar way as before.

Solution for the point at which the relation $r_2N_2(1-\frac{N_2+\alpha_{21}N_1}{K_2})-mN_2$ $=0$ crosses the N_1 axis:

$1-\frac{\alpha_{21}N_1}{K_2}-\frac{m}{r_2}=0$ by dividing through by r_2N_2 and then substituting $N_2=0$ we get $N_2=\frac{K_2(1-\frac{m}{r_2})}{\alpha_{21}}$

Solution for the point at which the relation $r_2N_2(1-\frac{N_2+\alpha_{21}N_1}{K_2})-mN_2$ $=0$ crosses the N_2 axis:

$1-\frac{N_2}{K_2}-\frac{m}{r_2}=0$ by dividing through by r_2N_2 and then substituting $N_1=0$ we get $N_2=K_2(1-\frac{m}{r_2})$.

Now, note that when we include removal, K_1 is replaced by $K_1(1-\frac{m}{r_1})$ and K_2 is replaced by $K_2(1-\frac{m}{r_2})$ in comparison with the points at which the 0-isoclines cross the N_1 and N_2 axes in the standard Lotka-Volterra models. Provided $m=0$ (i.e. no removal), cases where coexistence is impossible occur e.g. when $K_1>\frac{K_2}{\alpha_{21}}$ and $\frac{K_1}{\alpha_{12}}>K_2$. Assuming that parameter values are such that these inequalities are satisfied, we are looking for values of m such that $K_1(1-\frac{m}{r_1})>\frac{K_2(1-\frac{m}{r_2})}{\alpha_{21}}$ and $\frac{K_1(1-\frac{m}{r_1})}{\alpha_{12}}>K_2(1-\frac{m}{r_2})$, because in that case coexistence is possible with removal. (Note that coexistence is stable in that case according to the standard analysis of the Lotka-Volterra system.). The simple but important point is to note that as m is increased from 0, the intersection points of the 0-isoclines with the N_1-axis switch arrangements, while the intersection points of the 0-isoclines with the N_2-axis keep their order. This results in the 0-isoclines now having an intersection point in the first quadrant, leading to coexistence at a stable equilibrium.

5. (a)

$$K(C)=\frac{K_{max}C}{K_n+C}$$

If $C=K_n$ then

$$K(K_n)=\frac{K_{max}K_n}{2K_n}=\frac{K_{max}}{2}$$

(b) If $K(C)=K_mC$ then the model becomes

$$\begin{aligned}\frac{dN}{dt} &= K_mCN-\frac{FN}{V}\\ \frac{dC}{dt} &= -\alpha K_mCN-\frac{FC}{V}+\frac{FC_0}{V}\end{aligned}$$

Steady states:

$$\begin{aligned} N\left(K_m C - \frac{F}{V}\right) &= 0 \\ \alpha K_m CN - \frac{FC}{V} + \frac{FC_0}{V} &= 0 \end{aligned}$$

Form the first equation we see that either $N = 0$ or $C = \frac{F}{K_m V}$. From the second equation, we see that $N = 0$ implies

$$\begin{aligned} \frac{F}{V}(C_0 - C) &= 0 \\ \Longrightarrow C &= C_0 \end{aligned}$$

On the other hand, $C = \frac{F}{K_m V}$ implies

$$N = \frac{1}{\alpha}\left(C_0 - \frac{F}{K_m V}\right),$$

which is obtained by rearranging the second equation.

Thus, the new steady state with positive population density N is $(N_1, C_1) = \left(\frac{1}{\alpha}\left(C_0 - \frac{F}{K_m V}\right), \frac{F}{K_m V}\right)$

(c) Jacobian at (N_1, C_1):

$$J(N, C) = \begin{pmatrix} K_m C - \frac{F}{V} & K_m N \\ -\alpha K_m C & -\alpha K_m N - \frac{F}{V} \end{pmatrix}$$

So

$$J(N_1, C_1) = \begin{pmatrix} 0 & \frac{K_m}{\alpha}\left(C_0 - \frac{F}{K_m V}\right) \\ -\frac{\alpha F}{V} & -K_m C_0 \end{pmatrix}$$

So

$$\begin{aligned} \beta &= -K_m C_0 < 0 \\ \gamma &= -\alpha K_m^2 C_1 N_1 < 0 \end{aligned}$$

Since both β and γ are less than zero, this steady state is not stable (see textbook, p. 142).

6. (a) First equation: The change in the amount of organisms is dependent on a per capita growth rate that depends on the nutrient via Michaelis-Menten kinetics (s. textbook p. 125) and a per capita death rate μ. Second equation: The rate of change of the amount of nutrients present is dependent on a

diffusion term, $D(C_0 - C)$, where D is dependent on the properties of the membrane. That is, nutrient flow into the growth chamber at a rate that is proportional to the difference in the nutrient concentration between the reservoir and the growth chamber, with the constant of proportionality depending on the membrane between the two compartments. In addition, there is loss of nutrients due to consumption, i.e. at a rate that is proportional to the production rate of new cells, with α representing this proportionality.

(b) N - number of organisms

t - time

C, C_0 - concentration (mass/volume)

μ - death rate (1/time)

K_n - concentration (mass/volume)

K_{max} - 1/time

D - 1/time

α - concentration/number of organisms

(c) (cf. textbook, section 4.5) Let

$$\begin{aligned} N &= N^* \hat{N} \\ C &= C^* \hat{C} \\ t &= t^* \tau \end{aligned}$$

Substituting this into the model gives the equations:

$$\begin{aligned} \frac{dN^*}{dt^*} &= \tau N^* \left(\frac{K_{max} C^*}{\frac{K_n}{\hat{C}} + C^*} \right) - \tau \mu N^* \\ \frac{dC^*}{dt^*} &= \frac{\tau D C_0}{\hat{C}} - \tau D C^* - \frac{\tau \alpha \hat{N} K_{max}}{\hat{C}} \left(\frac{C^*}{\frac{K_n}{\hat{C}} + C^*} \right) N^* \end{aligned}$$

If we let $\hat{C} = K_n$, $\tau = \frac{1}{\mu}$ and $\hat{N} = \frac{K_n \mu}{\alpha K_{max}}$, then we get the following dimensionless equations (dropping
the star for convenience):

$$\begin{aligned} \frac{dN}{dt} &= \alpha_1 \left(\frac{C}{1+C} \right) N - N \\ \frac{dC}{dt} &= \alpha_2 - \alpha_3 C - \left(\frac{C}{1+C} \right) N \end{aligned}$$

where $\alpha_1 = \frac{K_{max}}{\mu}$, $\alpha_2 = \frac{DC_0}{\mu K_n}$ and $\alpha_3 = \frac{D}{\mu}$. (Of course there are other possible choices for the dimensionless variables!)

(d) Working with the first equation, we see that at an equilibrium (N^*, C^*) we must have

$$N^* \left(\alpha_1 \left(\frac{C^*}{1 + C^*} - 1 \right) - 1 \right) = 0.$$

Hence either $N^* = 0$, in which case we see from the second equation that $C^* = \alpha_2/\alpha_3$, or $C^* = 1/(\alpha_1 - 1)$, in which case we see from the second equation that $N^* = \alpha_1(\alpha_2 - \alpha_3 C^*) = \alpha_1\alpha_2 - \frac{\alpha_1\alpha_3}{\alpha_1 - 1}$. Thus, the steady states are at $\left(0, \frac{\alpha_2}{\alpha_3}\right)$ and $\left(\alpha_1\alpha_2 - \frac{\alpha_1\alpha_3}{\alpha_1-1}, \frac{1}{\alpha_1-1}\right)$. Note that we need $\alpha_1 > 1$ and $\alpha_2 - \frac{\alpha_3}{\alpha_1-1} > 0$ and for a non-trivial steady state to exist at which both variables have positive values.

(e) Carrying out stability analysis yields the following entries in the Jacobian:

$$\begin{aligned} a_{11} &= \alpha_1 \left(\frac{C}{1+C} \right) - 1 \\ a_{12} &= \frac{\alpha_1 N}{(1+C)^2} \\ a_{21} &= \frac{-C}{1+C} \\ a_{22} &= -\frac{N}{(1+C)^2} - \alpha_3 \end{aligned}$$

Evaluating these at the non-trivial steady state $(N^*, C^*) = \left(\alpha_1\alpha_2 - \frac{\alpha_1\alpha_3}{\alpha_1-1}, \frac{1}{\alpha_1-1}\right)$ gives:

$$\begin{aligned} a_{11} &= 0 \\ a_{12} &= \alpha_1 A \\ a_{21} &= \frac{-1}{\alpha_1} \\ a_{22} &= -(A + \alpha_3) \end{aligned}$$

where $A = \frac{N^*}{(1+C^*)^2}$. The trace of this Jacobian is $\beta = -(A + \alpha_3)$, the determinant is $\gamma = A > 0$. So this steady state is

always stable as long as it exists (see textbook, p. 142). As mentioned, conditions for existence of a non-trivial steady state are $\alpha_1 > 1$, which implies that $K_{max} > \mu$ according to the definition of α_1, and

$$\alpha_2 > \frac{\alpha_3}{\alpha_1 - 1}$$

Substituting in the original variables this says that

$$\frac{C_0}{\mu K_n} > \frac{1}{K_{max} - \mu}.$$

7. (a) linear, second order, non-homogenous, constant coefficients
(b) non-linear, second order, homogenous, constant coefficients
(c) non-linear, third order, homogenous, constant coefficients
(d) non-linear, first order, homogenous, constant coefficients
(e) linear, second order, non-homogenous, non-constant coefficients
(f) non-linear, first order, homogeneous, constant coefficients.
(g) linear, first order, non-homogeneous, non-constant coefficients
(h) linear, fifth order, non-homogeneous, non-constant coefficients
(i) linear, first order, non-homogenous, non-constant coefficients

8. (a)

$$\begin{aligned} \frac{dx}{dt} &= x^2 - y^2 \\ \frac{dy}{dt} &= x(1-y) \end{aligned}$$

Steady states:

$$\begin{aligned} x^2 &= y^2 \\ x(1-y) &= 0 \end{aligned}$$

$x = 0 \implies y = 0$
$y = 1 \implies x = \pm 1$. Steady states: $(0,0), (-1,1), (1,1)$.
Jacobian:

$$J(x,y) = \begin{pmatrix} 2x & -2y \\ 1-y & -x \end{pmatrix}$$

So

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J(-1,1) = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$$

$$J(1,1) = \begin{pmatrix} 2 & -2 \\ 0 & -1 \end{pmatrix}$$

(b)

$$\begin{aligned} \frac{dx}{dt} &= y - xy \\ \frac{dy}{dt} &= xy \end{aligned}$$

Steady states:

$$\begin{aligned} y(1-x) &= 0 \\ xy &= 0 \end{aligned}$$

$y = 0 \implies x = c$, where c is any real number. $x = 1 \implies y = 0$. Thus, the steady states are of the form $(c, 0)$, where c is any real number.

Jacobian:

$$J(x,y) = \begin{pmatrix} -y & 1-x \\ y & x \end{pmatrix}$$

So

$$J(c,0) = \begin{pmatrix} 0 & 1-c \\ 0 & 0 \end{pmatrix}$$

(c)

$$\begin{aligned} \frac{dx}{dt} &= x - x^2 - xy \\ \frac{dy}{dt} &= y(1-y) \end{aligned}$$

Steady states:

$$\begin{aligned} x(1-x-y) &= 0 \\ y(1-y) &= 0 \end{aligned}$$

$y = 0 \implies x = 0, 1$, $y = 1 \implies x = 0$. Steady states: $(0,0), (0,1), (1,0)$.

Jacobian:

$$J(x,y) = \begin{pmatrix} 1-2x-y & -x \\ 0 & 1-2y \end{pmatrix}$$

So

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$J(0,1) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

(d)

$$\begin{aligned} \frac{dx}{dt} &= x(1-y) \\ \frac{dy}{dt} &= y(x-1) \end{aligned}$$

Steady states:

$$\begin{aligned} x(1-y) &= 0 \\ y(x-1) &= 0 \end{aligned}$$

$x = 0 \implies y = 0$, $y = 1 \implies x = 1$. Steady states: $(0,0), (1,1)$.
Jacobian:

$$J(x,y) = \begin{pmatrix} 1-y & -x \\ y & x-1 \end{pmatrix}$$

So

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J(1,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

9. Given solutions $x_1(t)$ and $x_2(t)$ and the identity it can be seen that

$$\begin{aligned} u(t) = \frac{x_1(t) + x_2(t)}{2} &= e^{at}\cos(bt) \\ w(t) = \frac{x_1(t) - x_2(t)}{2i} &= e^{at}\sin(bt) \end{aligned}$$

Since the differential equation is linear (see p. 132 in the textbook), and since $u(t)$ and $w(t)$ are linear combinations of $x_1(t)$ and $x_2(t)$, $u(t)$ and $w(t)$ are also solutions of the original equation. Moreover, any solution can be obtained as a linear combination of $x_1(t)$ and $x_2(t)$, and therefore any real-valued solution can be obtained as a linear combination of $u(t)$ and $w(t)$. In other words, the general real values solution is of the form

$$x(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt).$$

10. (a)

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues: $\lambda = \pm 1$
To find the eigenvectors, we solve

$$A \cdot v_i = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \lambda_i \cdot \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}$$

for $i = 1, 2$. This amounts to solving the matrix equation

$$\begin{pmatrix} a_{11} - \lambda_i & a_{12} \\ a_{21} & a_{22} - \lambda_i \end{pmatrix} \cdot \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for the components v_{i1} and v_{i2} of the eigenvectors. Thus, for the eigenvector v_1 with eigenvalue $+1$ we have to solve

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

which has e.g. the solution $v_{11} = 0, v_{12} = 1$ (all other solutions are multiples of this one).

So the first eigenvector is

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For the eigenvector v_2 with eigenvalue -1 we have to solve:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix},$$

which has e.g. the solutions $v_{21} = 1, v_{22} = 0$ (again, all other solutions are multiples of this one).

So

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore, the general solution of the dynamical system

$$\begin{matrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{matrix} = A \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$$

(b)

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} (3-\lambda)^2 - 1 &= 0 \\ \lambda^2 - 6\lambda + 8 &= 0 \\ \lambda_1 &= 2 \\ \lambda_2 &= 4 \end{aligned}$$

Eigenvectors:
v_1:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

Solution:

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

v_2:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

Solution:

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

(c)

$$A = \begin{pmatrix} -2 & 7 \\ 2 & 3 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -4$, $\lambda_2 = 5$.
Eigenvectors:
v_1:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

Solution:

$$v_1 = \begin{pmatrix} 1 \\ -\frac{2}{7} \end{pmatrix}$$

v_2:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -7 & 7 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

Solution

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -\frac{2}{7} \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$$

(d)

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 3$.
Eigenvectors:
v_1:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

Solution:

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

v_2:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

Solution:

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

(e)

$$A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$
Eigenvectors:
v_1:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

Solution:

$$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

v_2:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

Solution:

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t$$

(f)

$$A = \begin{pmatrix} -4 & 1 \\ 3 & 0 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -2 + \sqrt{7}$, $\lambda_1 = -2 - \sqrt{7}$
Eigenvectors:
v_1:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2-\sqrt{7} & 1 \\ 3 & 2-\sqrt{7} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

Solution:

$$v_1 = \begin{pmatrix} 1 \\ 2+\sqrt{7} \end{pmatrix}$$

v_1:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+\sqrt{7} & 0 \\ 0 & 2+\sqrt{7} \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

Solution:

$$v_2 = \begin{pmatrix} 1 \\ 2-\sqrt{7} \end{pmatrix}$$

Thus the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2+\sqrt{7} \end{pmatrix} e^{(-2+\sqrt{7})t} + c_2 \begin{pmatrix} 1 \\ 2-\sqrt{7} \end{pmatrix} e^{-(2+\sqrt{7})t}$$

11. (a) We have

$$\frac{dx_1}{dt} = D - ux_1 - k_{12}x_1 + k_{21}x_2$$

Using a similar approach as to x_1, we can write the following equations:

$$\begin{aligned} \frac{dx_2}{dt} &= k_{12}x_1 - (k_{21} + s + k_{23})\,x_2 + k_{32}x_3 \\ \frac{dx_3}{dt} &= k_{23}x_2 - k_{32}x_3 \end{aligned}$$

(b) Steady states: Set each equation equal to zero. Use the third and first equations to note that at steady state

$$\begin{aligned} x_1^* &= \frac{D + k_{21}x_2^*}{u + k_{12}} \\ x_3^* &= \frac{k_{23}x_2^*}{k_{32}} \end{aligned}$$

Substituting these into equation 2 yields

$$x_2^* = \frac{k_{12}D}{(u + k_{12})(k_{21} + s - k_{12}k_{21})}.$$

Substituting this expression back into the expression for x_1^* and x_3^* gives all equilibrium values as functions of the parameters in the system.

(c) Calculate the Jacobian at the equilibrium determined in (b) and check whether all eigenvalues have negative real parts.