Back to Limits!

$\lim_{x\to\infty} \frac{x^2}{5}$ $\lim_{x\to\infty} \frac{5}{x^2}$ $\lim_{x\to 0} \frac{x^2}{5}$ $\lim_{x\to 0} \frac{5}{x^2}$

Back to Limits!

$$\lim_{x\to\infty} \frac{x^2}{5} = \infty \qquad \lim_{x\to\infty} \frac{5}{x^2} \qquad \lim_{x\to 0} \frac{x^2}{5} \qquad \lim_{x\to 0} \frac{5}{x^2}$$

Back to Limits!

$$\lim_{x\to\infty} \frac{x^2}{5} = \infty \qquad \lim_{x\to\infty} \frac{5}{x^2} = 0 \qquad \lim_{x\to 0} \frac{x^2}{5} \qquad \lim_{x\to 0} \frac{5}{x^2}$$

Back to Limits!

$$\lim_{x\to\infty} \frac{x^2}{5} = \infty \qquad \lim_{x\to\infty} \frac{5}{x^2} = 0 \qquad \lim_{x\to 0} \frac{x^2}{5} = 0 \qquad \lim_{x\to 0} \frac{5}{x^2}$$

Back to Limits!

$$\lim_{x\to\infty} \frac{x^2}{5} = \infty \qquad \lim_{x\to\infty} \frac{5}{x^2} = 0 \qquad \lim_{x\to 0} \frac{x^2}{5} = 0 \qquad \lim_{x\to 0} \frac{5}{x^2} = \infty$$

Back to Limits!

$$\lim_{x\to\infty} \frac{x^2}{5} = \infty \qquad \lim_{x\to\infty} \frac{5}{x^2} = 0 \qquad \lim_{x\to 0} \frac{x^2}{5} = 0 \qquad \lim_{x\to 0} \frac{5}{x^2} = \infty$$

Indeterminate Forms

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is an indeterminate form of the type $\frac{0}{0}$.

Suppose $\lim_{x\to a} F(x) = \lim_{x\to a} G(x) = \infty$ (or $-\infty$). Then the limit

$$\lim_{x\to a} \frac{F(x)}{G(x)}$$

is an indeterminate form of the type $\frac{\infty}{\infty}$.

Back to Limits!

$$\lim_{x\to\infty} \frac{x^2}{5} = \infty \qquad \lim_{x\to\infty} \frac{5}{x^2} = 0 \qquad \lim_{x\to 0} \frac{x^2}{5} = 0 \qquad \lim_{x\to 0} \frac{5}{x^2} = \infty$$

Indeterminate Forms

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

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Suppose $\lim_{x\to a} F(x) = \lim_{x\to a} G(x) = \infty$ (or $-\infty$). Then the limit

$$\lim_{x\to a} \frac{F(x)}{G(x)}$$

is an indeterminate form of the type $\frac{\infty}{\infty}$.

When you see an indeterminate form, you need to do more work.

Indeterminate Forms

Example: L'Hôpital1

$$\lim_{x\to 5} \frac{x^2 - 3x - 10}{x - 5}$$

indeterminate form of the type $\frac{0}{0}$

Indeterminate Forms

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indeterminate form of the type $\frac{0}{0}$

To evaluate, factor the top:

$$\lim_{x\to 5} \frac{x^2 - 3x - 10}{x - 5} = \lim_{x\to 5} \frac{(x-5)(x+2)}{x-5} = \lim_{x\to 5} x + 2 = \boxed{7}$$

Indeterminate Forms

Example: L'Hôpital1

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Example: L'Hôpital2

$$\lim_{x \to \infty} \frac{3x^2 - 4x + 2}{8x^2 - 5}$$

indeterminate form of the type $\frac{\infty}{\infty}$

Indeterminate Forms

Example: L'Hôpital1

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Example: L'Hôpital2

$$\lim_{x\to \infty} \frac{3x^2-4x+2}{8x^2-5}$$

indeterminate form of the type $\frac{\infty}{\infty}$

To evaluate, pull out x^2:

$$\lim_{x\to \infty} \frac{3x^2-4x+2}{8x^2-5} = \lim_{x\to \infty} \frac{x^2(3-\frac{4}{x}+\frac{2}{x^2})}{x^2(8-\frac{5}{x^2})} = \lim_{x\to \infty} \frac{3-\frac{4}{x}+\frac{2}{x^2}}{8-\frac{5}{x^2}} = \lim_{x\to \infty} \frac{3-0+0}{8-0} = \boxed{\frac{3}{8}}$$

Harder Indeterminate Forms

Example: L'Hôpital3

$$\lim_{x\to 0} \frac{3\sin x - x^4}{x^2 + \cos x - e^x}$$

indeterminate form of the type $\frac{0}{0}$

Harder Indeterminate Forms

Example: L'Hôpital3

$$\lim_{x\to 0} \frac{3\sin x - x^4}{x^2 + \cos x - e^x}$$

indeterminate form of the type $\frac{0}{0}$

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Suppose also that f and g are continuous and differentiable at a, and $g'(a) \neq 0$. Then:

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - 0}{g(x) - 0} \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \frac{(x-a)^{-1}}{(x-a)^{-1}} \\
&= \lim_{x\to a} \left(\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \right) \\
&= \frac{f'(a)}{g'(a)}
\end{aligned}$$

Harder Indeterminate Forms

Example: L'Hôpital3

$$\lim_{x\to 0} \frac{3\sin x - x^4}{x^2 + \cos x - e^x}$$

indeterminate form of the type $\frac{0}{0}$

$$\lim_{x\to 0} \frac{3\sin x - x^4}{x^2 + \cos x - e^x} = \frac{\frac{d}{dx}[3\sin x - x^4]|_{x=0}}{\frac{d}{dx}[x^2 + \cos x - e^x]|_{x=0}} =$$

$$\frac{[3\cos x - 4x^3]|_{x=0}}{[2x - \sin x - e^x]|_{x=0}} = \frac{3-0}{0-0-1} = \boxed{-3}$$

L'Hôpital's Rule: First Part

Let f and g be functions such that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x).$$

If $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

If f and g are differentiable on an open interval containing a, and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This works even for $a = \pm\infty$.

Extremely Important Note:
L'Hôpital's Rule only works on indeterminate forms.

L'Hôpital's Rule: Second Part

Let f and g be functions such that

$$\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x).$$

If $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

If f and g are differentiable on an open interval containing a, and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

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This works even for $a = \pm\infty$.

Extremely Important Note:
L'Hôpital's Rule only works on indeterminate forms.

Example: L'Hôpital4

Evaluate:

$$\lim_{x \to 2} \frac{3x \tan(x-2)}{x-2}$$

Example: L'Hôpital4

Evaluate:

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$$\lim_{x \to 2} \frac{3x \tan(x-2)}{x-2} \qquad \text{form } \frac{0}{0}$$

$$= \frac{3\left[x \sec^2(x-2) + \tan(x-2)\right]_{x=2}}{1}$$

$$= 3\left[2\sec^2 0 + \tan 0\right] = \boxed{6}$$

Little Harder

Example: L'Hôpital5

$$\lim_{x \to 0} \frac{x^4}{e^x - \cos x - x}$$

indeterminate form of the type $\frac{0}{0}$

Little Harder

Example: L'Hôpital5

$$\lim_{x\to 0} \frac{x^4}{e^x - \cos x - x}$$ indeterminate form of the type $\frac{0}{0}$

$$\lim_{x\to 0} \frac{x^4}{e^x + \sin x - x} \overset{?}{\underset{?}{=}} \frac{4x^3}{e^x + \sin x - 1}\bigg|_{x=0} = \frac{0}{0}$$ oops

Little Harder

Example: L'Hôpital5

$$\lim_{x\to 0} \frac{x^4}{e^x - \cos x - x}$$

indeterminate form of the type $\frac{0}{0}$

$$\lim_{x\to 0} \frac{x^4}{e^x + \sin x - x} \ {}^?_? \! =^?_? \ \frac{4x^3}{e^x + \sin x - 1}\Bigg|_{x=0} = \frac{0}{0}$$

oops

Iterate!

$$\lim_{x\to 0} \frac{x^4}{e^x + \sin x - x} = \lim_{x\to 0} \frac{4x^3}{e^x + \sin x - 1} = \lim_{x\to 0} \frac{12x^2}{e^x + \cos x} = \frac{0}{2} = \boxed{0}$$

Example: L'Hôpital6

Evaluate:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

Example: L'Hôpital6

Evaluate:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} =^{L'H} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{2\sqrt{x}}{x}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \boxed{0}$$

Other Indeterminate Forms

Example: L'Hôpital7

$\lim_{x \to \infty} e^{-x} \ln x$ form $0 \cdot \infty$

Other Indeterminate Forms

Example: L'Hôpital7

$\lim_{x\to\infty} e^{-x}\ln x$ form $0\cdot\infty$

$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty}\frac{\ln x}{e^x} \qquad \text{form } \frac{\infty}{\infty}$$

$$=^{L'H} \lim_{x\to\infty}\frac{1/x}{e^x}$$

$$= \lim_{x\to\infty}\frac{1}{xe^x} = \boxed{0}$$

Vote Vote Vote

Which of the following can you *immediately* apply L'Hôpital's rule to?

A. $\dfrac{e^x}{2e^x + 1}$

B. $\displaystyle\lim_{x\to 0} \frac{e^x}{2e^x + 1}$

C. $\displaystyle\lim_{x\to \infty} \frac{e^x}{2e^x + 1}$

D. $\displaystyle\lim_{x\to \infty} e^{-x}(2e^x + 1)$

E. $\displaystyle\lim_{x\to 0} \frac{e^x}{x^2}$

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E. $\displaystyle\lim_{x\to 0} \frac{e^x}{x^2}$

Votey McVoteface

Suppose you want to use L'Hôpital's rule to evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$, which has the form $\frac{0}{0}$. How does the quotient rule fit into this problem?

A. You should use the quotient rule because the function you are differentiating is a quotient.

B. You will not use the quotient rule because you differentiate the numerator and the denominator separately

C. You may use the quotient rule because perhaps $f(x)$ or $g(x)$ is itself in the form of a quotient

D. You will not use L'Hôpital's rule because $\frac{0}{0}$ is not an appropriate indeterminate form

E. You will not use L'Hôpital's rule because, since the top has limit zero, the whole function has limit 0

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More Questions

Which of the following is NOT an indeterminate form?

A. $\frac{\infty}{\infty}$ for example, $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$

B. $\frac{0}{0}$ for example, $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$

C. $\frac{0}{\infty}$ for example, $\displaystyle\lim_{x\to 0^+} \frac{x}{\ln x}$

D. $0 \cdot \infty$ for example, $\displaystyle\lim_{x\to\infty} x(\arctan(x) - \pi/2)$

E. all of the above are indeterminate forms

More Questions

Which of the following is NOT an indeterminate form?

A. $\frac{\infty}{\infty}$ for example, $\lim_{x\to\infty} \frac{e^x}{x^2}$

B. $\frac{0}{0}$ for example, $\lim_{x\to 0} \frac{e^x - 1}{x}$

C. $\frac{0}{\infty}$ for example, $\lim_{x\to 0^+} \frac{x}{\ln x} = 0$

D. $0 \cdot \infty$ for example, $\lim_{x\to\infty} x(\arctan(x) - \pi/2)$

E. all of the above are indeterminate forms

I have so many questions

Which of the following is NOT an indeterminate form?

A. 1^{∞} for example, $\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x}$

B. 0^{∞} for example, $\lim_{x\to\infty}\left(\frac{1}{x}\right)^{x}$

C. ∞^{0} for example, $\lim_{x\to\infty} x^{\frac{1}{x}}$

D. 0^{0} for example, $\lim_{x\to 0^+} x^{x}$

E. all of the above are indeterminate forms

F. none of the above are indeterminate forms

I have so many questions

Which of the following is NOT an indeterminate form?

A. 1^{∞} for example, $\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x$

B. 0^{∞} for example, $\lim_{x\to\infty}\left(\frac{1}{x}\right)^x = 0$

C. ∞^0 for example, $\lim_{x\to\infty} x^{\frac{1}{x}}$

D. 0^0 for example, $\lim_{x\to 0^+} x^x$

E. all of the above are indeterminate forms

F. none of the above are indeterminate forms

Exponential Indeterminate Forms

Example: L'Hôpital8

$$\lim_{x\to\infty} x^{1/x}$$

Exponential Indeterminate Forms

Example: L'Hôpital8

$$\lim_{x\to\infty} x^{1/x}$$

$$\begin{aligned}
\lim_{x\to\infty} x^{1/x} &= \lim_{x\to\infty} e^{\left(\ln\left(x^{1/x}\right)\right)} \\
&= \lim_{x\to\infty} e^{\left(\frac{\ln x}{x}\right)} \\
&= e^{\left(\lim_{x\to\infty} \frac{\ln x}{x}\right)} \\
&\overset{L'H}{=} e^{\left(\lim_{x\to\infty} \frac{1}{x}\right)} \\
&= e^0 = \boxed{1}
\end{aligned}$$

Exponential Indeterminate Forms

Example: L'Hôpital9

$$\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x}$$

Exponential Indeterminate Forms

Example: L'Hôpital9

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$$

First we calculate:

$$\begin{aligned}\lim_{x\to\infty}\ln\left(\left(1+\frac{2}{x}\right)^{3x}\right) &= \lim_{x\to\infty}3x\ln\left(1+\frac{2}{x}\right)\\ &= \lim_{x\to\infty}\frac{3\ln\left(1+\frac{2}{x}\right)}{x^{-1}}\\ &\overset{L'H}{=} \lim_{x\to\infty}\frac{3\left(\frac{-2x^{-2}}{1+2/x}\right)}{-x^{-2}}\\ &= \lim_{x\to\infty}\frac{6}{1+2/x} = 6\end{aligned}$$

So, now:

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x} = \boxed{e^6}$$

Evaluate: Example: L'Hôpital10

$$\lim_{x \to \infty} \frac{\ln x}{\ln \sqrt{x}}$$

Example: L'Hôpital11

$$\lim_{x \to \infty} (\ln x)^{\sqrt{x}}$$

Example: L'Hôpital12

$$\lim_{x \to 0} \frac{\arcsin x}{x}$$

Evaluate: Example: L'Hôpital10

$$\lim_{x\to\infty} \frac{\ln x}{\ln \sqrt{x}}$$

Easier to simplify first.

Example: L'Hôpital11

$$\lim_{x\to\infty} (\ln x)^{\sqrt{x}}$$

Not an indeterminate form: huge number to a huge power. Limit is infinity.

Example: L'Hôpital12

$$\lim_{x\to 0} \frac{\arcsin x}{x}$$

L'Hôpital: $\lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$

More Examples

Example: L'Hôpital12.25 (CLP #14, 3.7)

$$\lim_{x\to\infty} \sqrt{2x^2+1} - \sqrt{x^2+x}$$

Example: L'Hôpital12.5 (CLP #19, 3.7)

$$\lim_{x\to 0} \sqrt[x^2]{\sin^2 x}$$

Example: L'Hôpital12.5 (CLP #20, 3.7)

$$\lim_{x\to 0} \sqrt[x^2]{\cos x}$$

Example: L'Hôpital13

Sketch the graph of $f(x) = x \ln x$.

Note: when you want to know $\lim_{x\to 0} f(x)$, you'll need to use L'Hôpital.

Example: L'Hôpital14

Evaluate $\lim_{x\to 0^+} (\csc x)^x$