Word problems with max/min

Example: Optimization 1

A rancher wants to build a rectangular pen, using one side of her barn for one side of the pen, and using 100m of fencing for the other three sides. What are the dimensions of the pen built this way that has the largest area?

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Let the wide of the fence parallel to the barn be w meters long, and the two sides of the fence touching the barn be l meters long. Then $2l + w = 100$, and $A = lw$, where A is the area of the pen.

We want to find where A is maximum, given $2l + w = 100$. So, we can turn A into a function of just one variable by substituting $w = 100 - 2l$. Then

$$A(l) = l(100 - 2l) = 100l - 2l^2$$

so A is a parabola pointing down; its maximum occurs at its only critical point. Now to find its critical point, we differentiate:

$$A'(l) = 100 - 4l$$

so $l = 25$ and $w = 50$ give the pen with the biggest area.

General Idea

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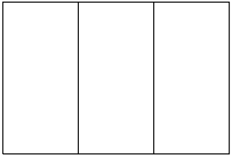
Find all the variables in terms of ONE variable, so we can find extrema.

Example: Optimization 2

You want to build a pen, as shown below, in the shape of a rectangle with two interior divisions. If you have 1000m of fencing, what is the greatest area you can enclose?



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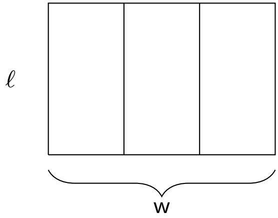

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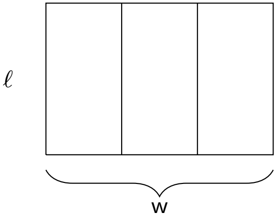

The area is $A = \ell w$, and the amount of fencing gives us $4\ell + 2w = 1000$. Then $w = 500 - 2\ell$, so

$$A = (500 - 2\ell)\ell = 500\ell - 2\ell^2$$

We differentiate to find CPs, and the maximum of A over the domain $0 < \ell < 250$ is at $\ell = 125$, which gives $w = 250$ and $A = (125)(250)$ sq metres.

Example: Optimization 3

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A 100-by-100 square.

Example: Optimization 4

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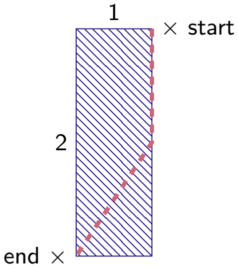

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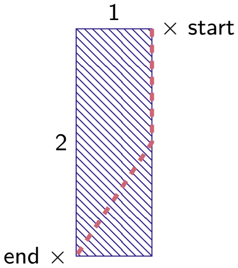

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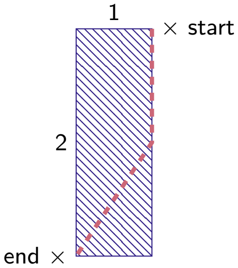

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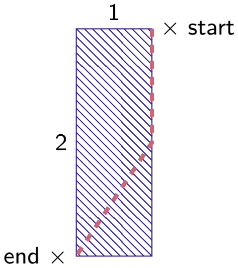
Let $2-x$ be the distance you travel portaging the boat, and then $\sqrt{x^2+1}$ is the distance you row. Time it takes is given by $T(x)=\frac{1}{6}(2-x)+\frac{1}{3}\sqrt{x^2+1}$. By differentiating, we see the only critical point for positive x is $x=\frac{1}{\sqrt{3}}$. This is the global minimum, so the fastest route is to portage for $2-\frac{1}{\sqrt{3}}$ kilometers, then row the rest of the way.

Example: Optimization 5

You are standing on the bank of a river that is 1km wide, and you want to reach the opposite side, two miles down the river. You can paddle 6 kilometers per hour, and walk 3 kph while carrying your boat. What route takes you to your desired destination in the least amount of time?



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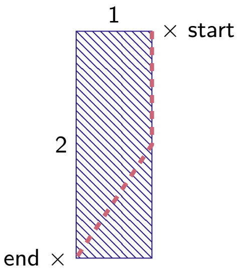
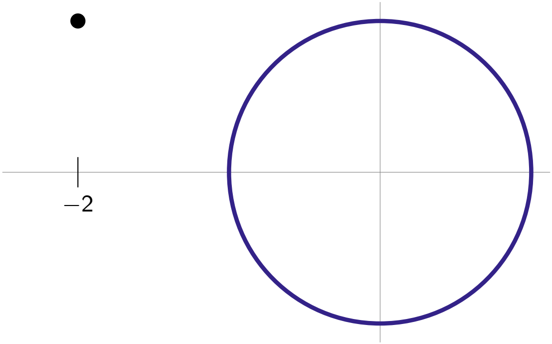


Paddle the whole way: it's faster, and more direct.

Example: Optimization 6

Let C be the circle given by $x^2 + y^2 = 1$. What is the closest point on C to the point $(-2, 1)$?



We can reasonably see that the y-coordinate will be positive, and the x-coordinate negative. For any point (x, y) on C, the distance D to $(-2, 1)$ is given by:

$$\begin{aligned} D^2 &= (x - -2)^2 + (y-1)^2 \\ &= x^2 + 4x + 4 + \left(\sqrt{1-x^2} - 1\right)^2 \\ &= x^2 + 4x + 4 + (1-x^2) - 2\sqrt{1-x^2} + 1 \\ &= 4x + 6 - 2\sqrt{1-x^2} \end{aligned}$$

where this only makes sense for x between -1 and 1. Minimizing D is the same as minimizing D^2; so we take the derivative of the above function:

$$4 - 2\frac{-2x}{2\sqrt{1-x^2}} = 4 + \frac{2x}{\sqrt{1-x^2}}$$

There is a singular point at the endpoints $x = \pm 1$, and a critical point at $c = -\sqrt{4/5}$. We see the critical point is a local min, so the closest point is $\boxed{(-\sqrt{4/5}, \sqrt{1/5})}$.

Example: Optimization 7

Suppose you want to manufacture a closed cylindrical can on the cheap. If the can should have a volume of one litre (1000 cm^3), what is the smallest surface area it can have?

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Let the can have radius r and height h. Then its volume is $1000 = V = \pi r^2 h$, and its surface area is $2(\pi r^2) + (2\pi r)h$.

Using the volume equation, we know $h = \frac{1000}{\pi r^2}$, so for surface area, this gives us

$$S = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

S makes sense only for $r > 0$. We want the minimum value of S, so we find the critical and singular points.

$$S'(r) = 4\pi r - \frac{2000}{r^2}$$

The singular point is not in the range of S; the only critical point is $r = \sqrt[3]{\frac{500}{\pi}}$. We see that for r smaller than the CP, $S(r)$ is decreasing, then $S(r)$ is increasing for r larger than the CP. So the minimum of $S(r)$ occurs at $r = \sqrt[3]{\frac{500}{\pi}}$. This gives us a minimum surface area of:

$$S\left(\sqrt[3]{\frac{500}{\pi}}\right) = 2\pi\left(\frac{500}{\pi}\right)^{2/3} + 2000\left(\frac{\pi}{500}\right)^{1/3}$$

Example: Optimization 8

A cylindrical can is to hold 20π cubic metres. The material for the top and bottom costs \$10 per square metre, and material for the side costs \$8 per square metre. Find the radius r and height h of the most economical can.

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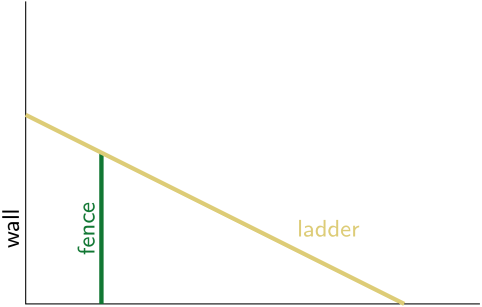
The cost is $C = 8(2\pi rh) + 2 \cdot 10 \cdot \pi r^2$. Since $20\pi = V = \pi r^2 h$, we see $rh = \frac{20}{r}$, so

$$C(r) = 16\pi \frac{20}{r} + 20\pi r^2$$

This is minimized at $r = 2$, hence $h = 5$.

Example: Optimization 9

Suppose a 2-metre high fence stands 1 metre away from a high wall. What is the shortest ladder that will reach over the fence to the wall?



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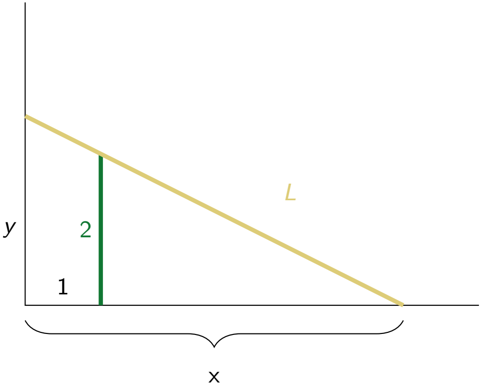

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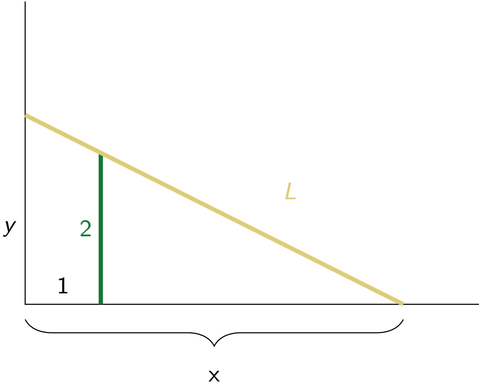
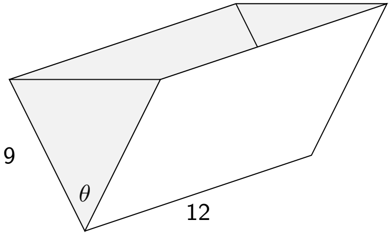


To get y in terms of x, we notice similar triangles tell us $\frac{y}{x} = \frac{2}{1-x}$, hence $y = \frac{2x}{1-x}$. The values of x that make sense are $(1, \infty)$.

Noting $L^2 = x^2 + y^2 = x^2 + \frac{4x^2}{(1-x)^2}$, we see L^2 gets very large as x gets close to 1 or as x gets very large. The only critical point is at $x = \sqrt[3]{4} + 1$, and this gives us our minimum $L = (\sqrt[3]{4} + 1)^2 + \dfrac{4(\sqrt[3]{4}+1)^2}{\sqrt[3]{4}^2}$.

Example: Optimization 10

Suppose a file folder is 12 inches long and 9 inches wide. You want to make a box by opening the folder and capping the ends. What angle makes should you open the folder to, to make the box with the most volume?



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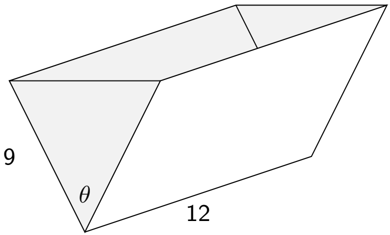
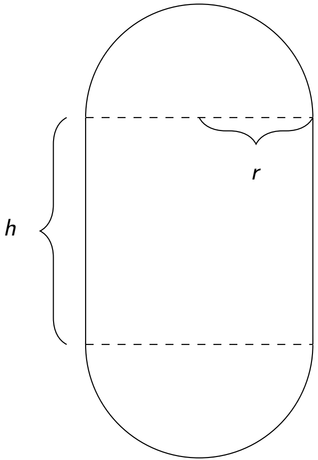


The volume of a shape like this is the area of the base, times the height, which is 12. So we maximize the volume by maximizing the area of the triangle with two sides of length 9, and angle between them θ. The area of this triangle is $\frac{9^2}{2}\sin\theta$, and θ only makes sense between 0 and π; the maximum occurs at $\pi/2$. So, you want the folder to make a right angle.

Example: Optimization 11

We want to bend a piece of wire into the perimeter of the shape shown below: a rectangle of height h and width $2r$, with a half circle of radius r on the top and bottom.

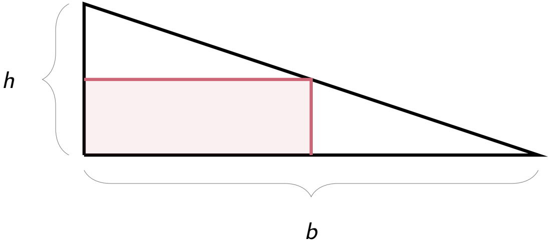


If you only have 100cm of wire, what values of r and h give the largest enclosed area?

The perimeter is $100 = 2\pi r + 2h$, so $h = 50 - \pi r$.
$A = \pi r^2 + 2rh = \pi r^2 + 2r(50 - \pi r) = 100r - \pi r^2$. The maximum of this function occurs at $r = \frac{50}{\pi}$. Then $h = 0$. That is: there's no rectangular part, only a circle.
Circles are somehow more efficient at storing space than rectangles. More precisely, we mean that the ratio of area to surface area is higher in circles than in rectangles. So the most efficient way to make the shape is to have the rectangular part have height 0.

Example: Optimization 12

Suppose we take a right triangle, with height h and base b. We inscribe a regrangle in it that shares a right angle, as shown below. What are the dimensions of the rectangle with the biggest area?



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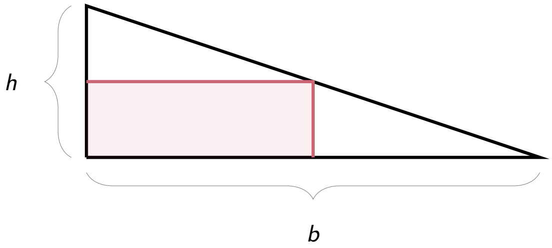

$b/2$ by $h/2$.

ACTIVITY

By cutting out squares from the corners, turn a piece of paper into an open-topped box that holds a lot of beans.