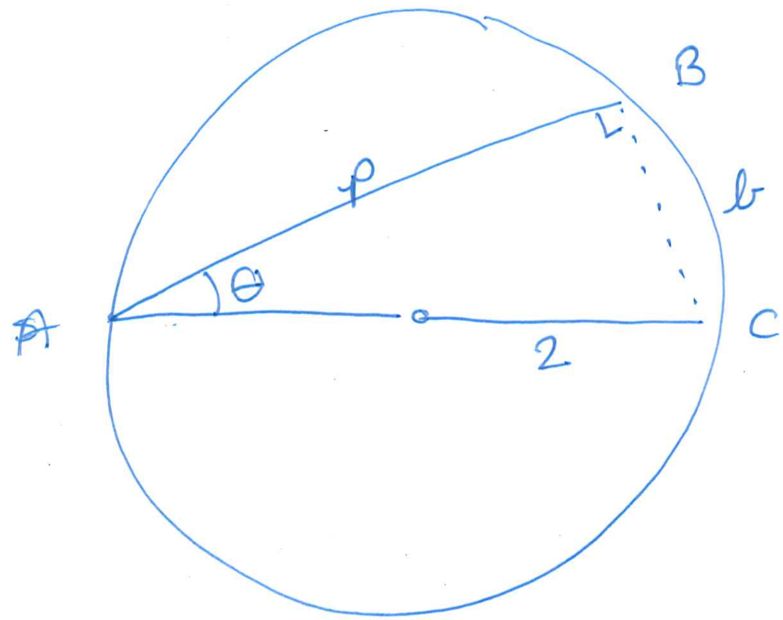
Webwork problem about rowing across a lake



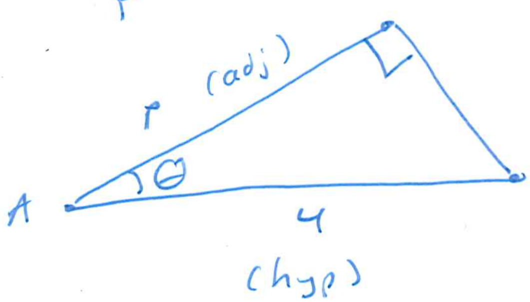
paddle : 2.5 kph

bike : 5 kph

MINIMIZE TIME

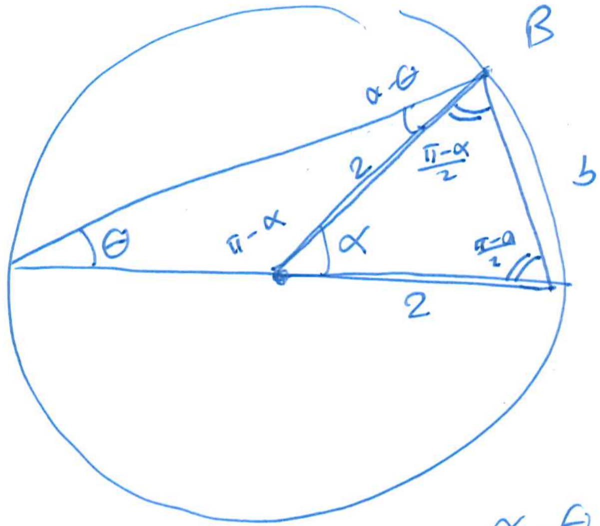
$$t = \frac{p}{2.5} + \frac{b}{5}$$

p :



$$\cos\theta = \frac{adj}{hyp} = \frac{p}{4}$$

$$\boxed{p = 4\cos\theta}$$



arc length:

$b = \alpha(2) = 2\alpha = \boxed{4\theta}$

↑ radius

$$\alpha - \theta + \frac{\pi - \alpha}{2} = \pi/2$$

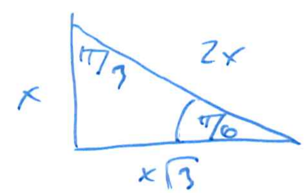
$$2\alpha - 2\theta + \cancel{\pi} - \alpha = \cancel{\pi}$$

$$\alpha = 2\theta$$

$$t = \frac{4\cos\theta}{2.5} + \frac{4\theta}{5}$$

FIND MIN, $0 \le \theta \le \pi/2$

$$\frac{dt}{d\theta} = \frac{-4}{2.5}\sin\theta + \frac{4}{5} = 0$$



$$\frac{\cancel{4}}{2.5}\sin\theta = \frac{\cancel{4}}{5}$$

$\sin\theta = 1/2$, so $\theta = \pi/6$ ← CP

$$t(\pi/6) = \frac{4\cos(\pi/6)}{2.5} + \frac{4(\pi/6)}{5}$$

$$= \frac{4(\sqrt{3}/2)}{2.5} + \frac{4\pi}{30}$$

$$= 2\sqrt{3}\left(\frac{2}{5}\right) + \frac{4\pi}{30}$$

$$= \underbrace{\frac{4}{5}\sqrt{3} + \frac{4\pi}{30}}_{\approx}$$

$$\frac{4(1.5)}{5} + \frac{12}{30}$$

$$= \frac{6}{5} + \frac{12}{30}$$

$$t(0) = \frac{4}{2.5} + 0 = \frac{4\cdot 2}{5} = \frac{8}{5}$$ not minimum

$$t(\pi/2) = \frac{4(0)}{2.5} + \frac{4(\pi/2)}{5}$$

$$= \underbrace{\frac{2\pi}{5}}_{\approx \frac{6}{5}} < \frac{2.4}{5} = t(0)$$

min time

Webwork limit problem

$$\lim_{x \to \pi/4^-} 8(\tan x)^{\tan(2x)}$$

FORM: 1^∞

INDETERMINATE

$$8 \lim_{x \to \pi/4^-} e^{\ln\left((\tan x)^{\tan 2x}\right)}$$

$$\underbrace{\tan 2x}_{\to \infty} \cdot \underbrace{\ln(\underbrace{\tan x}_{1})}_{0}$$

$$= 8 \lim_{x \to \pi/4^-} e^{\boxed{\tan(2x) \ln(\tan x)}}$$

$$A \cdot B = \frac{B}{\left(\frac{1}{A}\right)} = B\left(\frac{A}{1}\right) = B \cdot A$$

$$= 8 e^{\lim_{x \to \pi/4^-} \tan(2x) \ln(\tan x)}$$

$$\tan 2x \cdot \ln(\tan x) = \frac{\ln(\tan x)}{\frac{1}{\tan 2x}} = \frac{\ln(\tan x)}{\cot(2x)}$$

$$= 8 e^{\underbrace{\lim_{x \to \pi/4^-} \frac{\ln(\tan x)}{\cot(2x)}}}$$

NUM $\to 0$

DEN $\to$ $1/\tan(\pi/2) \to 1/\infty \to 0$

$$= 8e^{\lim_{x \to \pi/4^-} \left(\frac{\frac{\sec^2 x}{\tan x}}{-\csc^2(2x) \cdot 2} \right)}$$

$$= 8e^{\lim_{x \to \pi/4^-} \left(\frac{\frac{\left(\frac{1}{\cos x}\right)^2}{\left(\frac{\sin x}{\cos x}\right)}}{-2 \cdot \left(\frac{1}{\sin 2x}\right)^2} \right)}$$

$$= 8e^{\lim_{x \to \pi/4^-} \left(-\frac{1}{2}\right)\left(\frac{\cos x \cdot (\sin 2x)^2}{\cos^2 x \cdot \sin x} \right)}$$

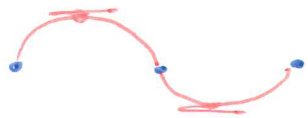
$$= 8e^{\lim_{x \to \pi/4^-} \left(-\frac{1}{2}\right)\left(\frac{(\sin 2x)^2}{\cos x \cdot \sin x} \right)}$$

$$= 8e^{\left(-\frac{1}{2}\right)\left(\frac{1}{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}} \right)} = 8e^{-\frac{1}{2}\left(1/\frac{1}{2} \right)} = 8e^{-\frac{1}{2} \cdot 2}$$

$$= 8e^{-1} = \boxed{\frac{8}{e}}$$

4 marks 12. (a) Let $g(x)$ be a continuous function for which $g'(x)$ and $g''(x)$ exist. If $g(x)$ has at least three zeros, then how many zeros must $g'(x)$ and $g''(x)$ have? Explain your answer carefully.

$g(x)$:



$g(x)$ is diff'able and there exist a, b, c (all different) such that $g(a) = g(b) = g(c)$. ($a < b < c$)

By Rolle's Thm there are d_1 and d_2 (d_1 in (a,b), d_2 in (b,c)) such that $g'(d_1) = g'(d_2) = 0$.

So g' has at least 2 zeros.

Continuied on next page

4 marks (b) Consider the equation $2x^2 - 3 + \sin(x) + \cos(x) = 0$. Show that this has at least two solutions.

Using the IVT:

Let $f(x) = 2x^2 - 3 + \sin x + \cos x$
f is continuous everywhere. IVT

$f(10) = 200 - 3 + \sin 10 + \cos 10 > 0$

$f(0) = -3 + 0 + 1 < 0$

$f(-10) = 200 - 3 - \sin 10 + \cos 10 > 0$

Since 0 is btw $f(10)$ + $f(0)$, $f(x)$ has a root in $(0, 10)$.

Since 0 is btw $f(-10)$ + $f(0)$, $f(x)$ has a second root in $(-10, 0)$

2 marks (c) Show that the same equation cannot have more than two solutions. Part (a) will help you here.

$f'(x) = 4x + \cos x - \sin x$

$f''(x) = 4 - \sin x - \cos x > 0$.

So $f''(x)$ has no roots.

So by part (a), f does not have 3 roots.

So f has at most 2 roots.

This part is continuing part (a) from the previous page (Question 12)

since $\boxed{g'' \text{ exists}}$, by Rolle:

there is e in (d_1, d_2)

s.t. $g''(e) = 0$

So g'' has at least one zero.

This was a reminder of the logic behind our conclusion in part (c) Question 12

$\boxed{(a)}$

Roots of $f(x)$

$\cancel{\exists^{+}}$

$\cancel{f''(x)=0}$

$\boxed{\text{not ?}}$

anything could happen