Properties of a Taylor Polynomial

Constant: $f(x) \approx f(a)$

Linear: $f(x) \approx f(a) + f'(a)(x-a)$

Quadratic: $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$

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$T_n(a) = f(a)$
$T_n'(a) = f'(a)$
$T_n''(a) = f''(a)$
$\vdots$
$T_n^{(n)}(a) = f^{(n)}(a)$
$T_n^{(n+1)}(a) = 0$

Taylor Polynomials

Definition

For a natural number n, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

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Given a function $f(x)$ that is differentiable n times at a point a, the n-th degree **Taylor polynomial** for $f(x)$ about a is

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If $a = 0$, we call the function a **Maclaurin polynomial**.

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Example: Taylor 1

Find the 7th degree Maclaurin[1] polynomial for e^x.

[1] Remember: a Maclaurin polynomial is just a Taylor polynomial centered about $a = 0$

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Find the 7th degree Maclaurin[1] polynomial for e^x.

Let $f(x) = e^x$. Then every derivative of e^x is just e^x, and $e^0 = 1$. So:

$$\begin{aligned} T_7(x) &= f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 + \cdots + \frac{1}{7!}f^{(7)}(0)(x-0)^7 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \\ &= \sum_{k=0}^{7} \frac{x^k}{k!} \end{aligned}$$

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e^x approximations - link

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Example: Taylor 2

Find the 8th degree Maclaurin[2] polynomial for $f(x) = \sin x$.

[2] Remember: a Maclaurin polynomial is just a Taylor polynomial centered about $a = 0$

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Example: Taylor 2

Find the 8th degree Maclaurin[2] polynomial for $f(x) = \sin x$.

$$\begin{array}{llll} f(x) = \sin x & f(0) = 0 & f^{(4)}(0) = 0 & f^{(8)}(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 & f^{(5)}(0) = 1 & \\ f''(x) = -\sin x & f''(0) = 0 & f^{(6)}(0) = 0 & \\ f'''(x) = -\cos x & f'''(0) = -1 & f^{(7)}(0) = -1 & \end{array}$$

$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 + \cdots + \frac{1}{8!}f^{(8)}(0)(x-0)^8 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\ &= \sum_{k=0}^{3} \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

Link: sine approximations

[2] Remember: a Maclaurin polynomial is just a Taylor polynomial centered about $a = 0$

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Example: Taylor 3

Find the 7th degree Taylor polynomial for $f(x) = \ln x$, centered at $a = 1$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Example: Taylor 3

Find the 7th degree Taylor polynomial for $f(x) = \ln x$, centered at $a = 1$.

$$\begin{array}{llll}
f(x) = \ln x & f(1) = 0 & f^{(4)}(x) = -3!x^{-4} & f^{(4)}(1) = -3! \\
f'(x) = x^{-1} & f'(1) = 1 & f^{(5)}(x) = 4!x^{-5} & f^{(5)}(1) = 4! \\
f''(x) = -x^{-2} & f''(1) - 1 & f^{(6)}(x) = -5!x^{-6} & f^{(6)}(1) = -5! \\
f'''(x) = 2x^{-3} & f'''(1) = 2 & f^{(7)}(x) = 6!x^{-7} & f^{(7)}(1) = 6!
\end{array}$$

$$\begin{aligned}
T_8(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \cdots + \frac{1}{7!}f^{(7)}(1)(x-1)^7 \\
&= 0 + (1)(x-1) + (-1)\frac{1}{2}(x-1)^2 + (2)\frac{1}{3!}(x-1)^3 - 3!\frac{1}{4!}(x-1)^4 \\
&\quad + 4!\frac{1}{5!}(x-1)^5 - 5!\frac{1}{6!}(x-1)^6 + 6!\frac{1}{7!}(x-1)^7 \\
&= (x-1) - \frac{x-1}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4} + \frac{(x-1)^4}{5} - \frac{(x-1)^5}{6} + \frac{(x-1)^6}{7!} \\
&= \sum_{k=0}^{7} (-1)^{k+1} \frac{(x-1)^k}{k}
\end{aligned}$$

log approximations

Error: What Makes an Approximation Accurate?

degree	conditions	formula
0	$f(a) = T_0(a)$	$T_0(x) = f(a)$
1	$f(a) = T_1(a)$ $f'(a) = T_1'(a)$	$T_1(x) = f(a) + f'(a)(x-a)$
2	$f(a) = T_2(a)$ $f'(a) = T_2'(a)$ $f''(a) = T_2''(a)$	$T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$
n	$f(a) = T_n(a)$ $f'(a) = T_n'(a)$ $f''(a) = T_n''(a)$ $\vdots$ $f^{(n)}(a) = T_n^{(n)}(a)$	$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$

Error Term: Taylor's Theorem

Error

The error in an estimation $f(x) \approx T_n(x)$ is $f(x) - T_n(x)$. We often use $|f(x) - T_n(x)|$ if we don't care whether the approximation is too big or too little, but only that it is not too egregious.

Taylor's Theorem

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

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The trick is bounding $f^{(n+1)}(c)$. It's usually OK to be sloppy here! Also, usually what we care about is the magnitude of the error: $|f(x) - T_n(x)|$.

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T_3(x) &= f(0) + f'(0)(x-0) + \frac{1}{2!}f''(0)(x-0)^2 + \frac{1}{3!}f'''(0)(x-0)^3 \\
&= e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}
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Example: Taylor 4

Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

Recall $f(x) - T_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$ for some c between x and a, and $2 < e < 3$.

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Recall $f(x) - T_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$ *for some* c *between* x *and* a*, and* $2 < e < 3$.

For some c strictly between 0 and .1: $\underbrace{f(.1) - T_3(.1)}_{\text{error}} = \frac{1}{4!}f^{(4)}(c)(.1-0)^4 = \frac{1}{4!10^4}e^c$

For c in $(0, .1)$, $1 \le e^c < e^1 < 3$, so
$\frac{1}{4!10^4} \cdot 1 \le \underbrace{f(.1) - T_3(.1)}_{\text{error}} \le \frac{1}{4!10^4} \cdot 3$

$0.0000041\overline{6} \le f(.1) - T_3(.1) \le 0.0000125$

Example: Taylor 5

Suppose we use the 5th degree Taylor polynomial centered at $a = \pi/2$ to approximate $f(x) = \cos x$. What could magnitude of the error be if we approximate $\cos(2)$?

Recall $f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$ *for some* c *between* x *and* a.

We don't actually have to compute $T_5(x)$, but if you want to as an exercise, click here to see it.

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We don't actually have to compute $T_5(x)$, but if you want to as an exercise, click here to see it.

For some c in $(\pi/2, 2)$:

$$|\underbrace{f(2) - T_5(2)}_{error}| = \left|\frac{1}{6!} f^{(6)}(c)(2-\pi/2)^6\right|$$

Note $f^{(6)}(x)$ is going to be positive or negative sine or cosine, so $|f^{(6)}(c)| \leq 1$. Also, $|2 - \pi/2| < \frac{1}{2}$. Now:

$$|f(2) - T_5(2)| < \frac{1}{6!}(1)\left(\frac{1}{2}\right)^6 = \frac{1}{6!2^6}$$

And $\frac{1}{6!2^6} \approx 0.0000217$.

Example: Taylor 6

Suppose we use a third degree Taylor polynomial centered at 4 to approximate $f(x) = \sqrt{x}$. If we use this Taylor polynomial to approximate $\sqrt{4.1}$, give a bound for the magnitude of our error.

Recall $f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$ *for some* c *between* x *and* a.

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Recall $f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$ *for some* c *between* x *and* a.

For some c in $(4, 4.1)$, $|f(4.1) - T_3(4.1)| = |\frac{1}{4!} f^{(4)}(c)(4.1-4)^4| = \frac{.1^4}{4!} f^{(4)}(c)$

So, let's investigate $f^{(4)}(c)$. First we find that the fourth derivative of $f(x) = x^{1/2}$ is $f^{(4)}(x) = \frac{-15}{16} x^{-7/2}$. So, for c in $(4, 4.1)$, we have
$|f^{(4)}(c)| = \left|\frac{-15}{16\sqrt{c^7}}\right| = \frac{15}{16\sqrt{c^7}} \leq \frac{15}{16\sqrt{4}^7} = \frac{15}{16 \cdot 2^7}$

So, the error is bounded by: $|f(4.1) - T_3(4.1)| \leq \frac{.1^4}{4!} \cdot \frac{15}{16 \cdot 2^7} \approx 0.00000003$

Example: Taylor 7

Suppose you want to approximate the value of e, knowing only that it is somewhere between 2 and 3. You use a 4th degree Maclaurin polynomial for $f(x) = e^x$ to approximate $f(1) = e^1 = e$. Bound the magnitude of your error.

Example: Taylor 7

Suppose you want to approximate the value of e, knowing only that it is somewhere between 2 and 3. You use a 4th degree Maclaurin polynomial for $f(x) = e^x$ to approximate $f(1) = e^1 = e$. Bound the magnitude of your error.

For some c in $(0, 1)$: $|f(1) - T_4(1)| = \left|\frac{1}{5!}f^{(5)}(c)(1-0)^5\right| = \frac{1}{5!}e^c \leq \frac{1}{5!}e^1 < \frac{3}{5!} = 0.025$

Which Degree?

Example: Taylor 8

Suppose you want to approximate $\sin 3$ using a Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

Which Degree?

Example: Taylor 8

Suppose you want to approximate $\sin 3$ using a Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

We want the magnitude of the error, so let's deal with absolute values. For some c in $(3, \pi)$:

$$|f(3) - T_n(3)| = \left|\frac{1}{(n+1)!} f^{(n+1)}(c)(3-\pi)^{n+1}\right| = \frac{|3-\pi|^{n+1}}{(n+1)!}\left|f^{(n+1)}(c)\right| \leq \frac{(.2)^{n+1}}{(n+1)!}(1) = \frac{.2^{n+1}}{(n+1)!}$$

If we plug in $n = 2$, we get $\frac{.2^{n+1}}{(n+1)!} = 0.00133...$, which is not SMALLER than 0.001. If we plug in $n = 3$, we get $\frac{.2^{n+1}}{(n+1)!} = 0.000066...$ which IS smaller than 0.001. So we have to use the degree 3 Taylor polynomial.

Example: Taylor 9

Suppose you want to approximate e^5 using a Maclaurin polynomial of $f(x) = e^x$. If the magnitude of your error must be less than 0.001, what degree Maclaurin polynomial should you use?

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The magnitude of the error means its absolute value. Our error is, for some c in $(0, 5)$:
$f(5) - T_n(5) = \frac{1}{(n+1)!} f^{(n+1)}(c)(5-0)^{n+1} = \frac{1}{(n+1)!} e^c 5^{n+1}$.
We can bound e^c for c in $(0, 5)$ by $1 = e^0 < e^c < e^5 < 3^5$. So now:
$f(5) - T_n(5) \leq \frac{1}{(n+1)!} \cdot 3^5 \cdot 5^{n+1}$
We set $\frac{1}{(n+1)!} \cdot 3^5 \cdot 5^{n+1} > 0.001$, and by plugging in different values of n, we find the smallest n that makes the inequality true is $n = 21$. So we can use the 21st-degree Maclaurin polynomial and get our desired error.

Example: Taylor 10

Suppose you want to approximate $\ln 3$ using a Taylor polynomial of $f(x) = \ln x$ centered at $a = 1$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

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Suppose you want to approximate $\ln 3$ using a Taylor polynomial of $f(x) = \ln x$ centered at $a = 1$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

Your error will be: $f(3) - T_n(3) = \frac{1}{(n+1)!}f^{(n)}(c)(3-1)^{n+1}$ for some c in $(0, 3)$. So, we need to bound $f^{(n)}(c)$. by writing out a number of derivatives of natural log, we notice that for $n \geq 1$, $f^{(n)}(c) = (-1)^{n-1}(n-1)!c^{-n}$. So, $f^{(n+1)}(c) = (-1)^n n!c^{-n}$. For c in $(0, 3)$:

$$\frac{n!}{1^n} = n! \leq |f^{(n+1)}(c)| \leq \frac{n!}{3^n}$$

Now for the error:

$$|f(3) - T_n(3)| \leq \frac{1}{(n+1)!} \cdot \frac{n!}{3^n} \cdot 2^{n+1} = \frac{2^{n+1}}{(n+1)3^n}$$

Setting this < 0.001, we find by plugging in values of n that $n = 5$ is the smallest n that makes the inequality true. So , using $T_5(x)$ will give us our desired error.

Taylor Error

Example: Taylor 11

Let $f(x) = \sqrt[4]{x}$. Suppose you use a second-degree Taylor polynomial of $f(x)$ centered at $a = 81$ to approximate $\sqrt[4]{81.2}$. Bound your error, and tell whether $T_2(10)$ is an overestimate or underestimate.

Taylor's formula tells us that, for some c in $(81, 81.2)$:

$$f(10) - T_2(10) = \frac{1}{3!}f^{(3)}(c)(81.2-81)^3 = \frac{1}{6}\cdot(\frac{1}{5})^3 f^{(3)}(c) = \frac{1}{6\cdot 5^3}f^{(3)}(c)$$

So, we should probably find out what $f^{(3)}(x)$ is. Since $f(x) = x^{1/4}$, it's not too hard to figure out $f'''(x) = \frac{21}{4^3}x^{-11/4}$. So, $f'''(c) = \frac{21}{4^3c^{11/4}}$. Plugging in:

$$f(10) - T_2(10) = \frac{1}{6\cdot 5^3}\cdot\frac{21}{4^3c^{11/4}} = \frac{7}{2\cdot 4^3\cdot 5^3\cdot c^{11/4}}$$

Now our job is to bound this, and we should use reasonable numbers.

$$\begin{aligned}
81 &\le c \le 81.2\\
81^{11/4} &\le c^{11/4} \le 81.2^{11/4}\\
(\sqrt[4]{81})^{11} &\le c^{11/4} \le \sqrt[4]{81.2}^{11}\\
3^{11} &\le c^{11/4} \le 4^{11}\\
\frac{1}{4^{11}} &\le \frac{1}{c^{11/4}} \le \frac{1}{3^{11}}\\
\frac{7}{2\cdot 4^3\cdot 5^3\cdot 4^{11}} &\le \frac{7}{2\cdot 4^3\cdot 5^3c^{11/4}} \le \frac{7}{2\cdot 4^3\cdot 5^3\cdot 3^{11}}
\end{aligned}$$

So, $\frac{7}{2\cdot 4^3\cdot 5^3\cdot 4^{11}} \le f(x) - T_2(x) \le \frac{7}{2\cdot 4^3\cdot 5^3\cdot 3^{11}}$ And, since $f(x) - T_2(x)$ is positive, $T_2(x)$ is an underestimate.