Outline

Week 10: Eigenvalues and eigenvectors

Course Notes: 6.1

Goals: Understand how to find eigenvector/eigenvalue pairs, and use them to simplify calculations involving matrix powers.

When Matrix Multiplication looks like Scalar Multiplication

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

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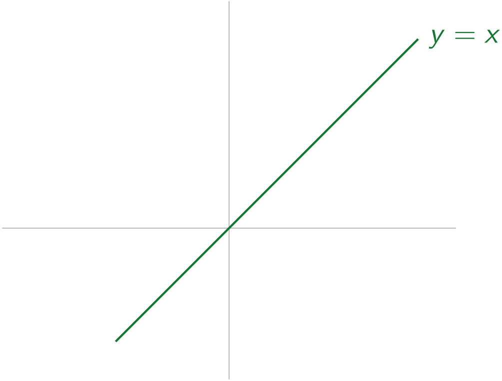
Recall: two vectors that are scalar multiples of one another are parallel.

Reflections, Revisited



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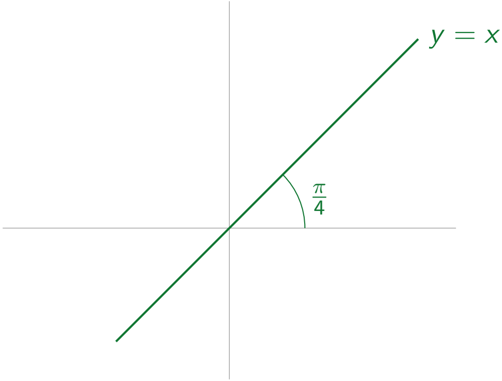

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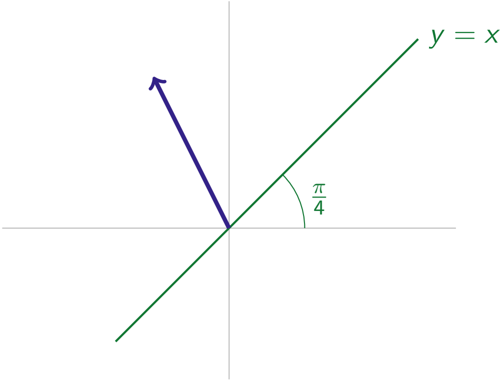

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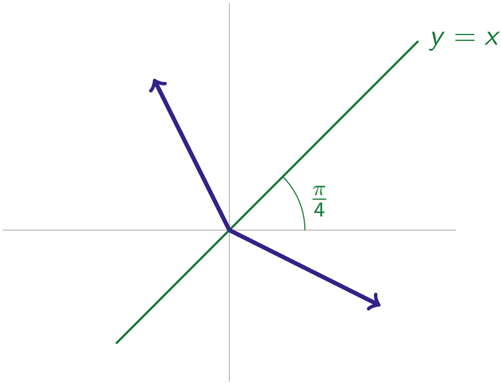

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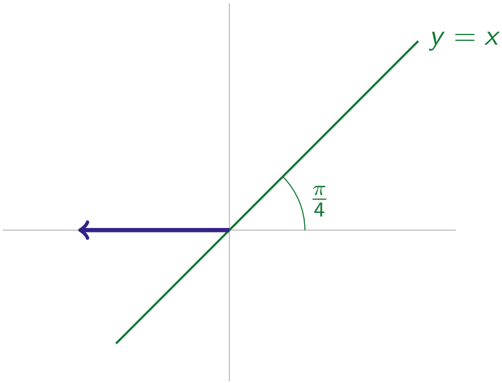

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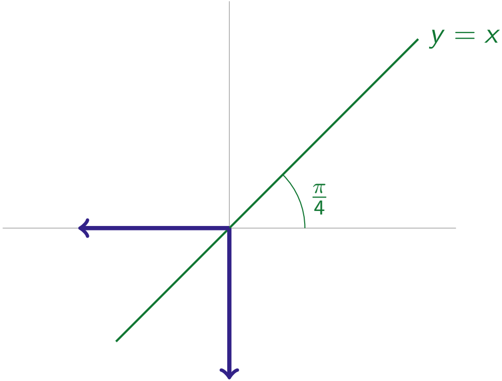

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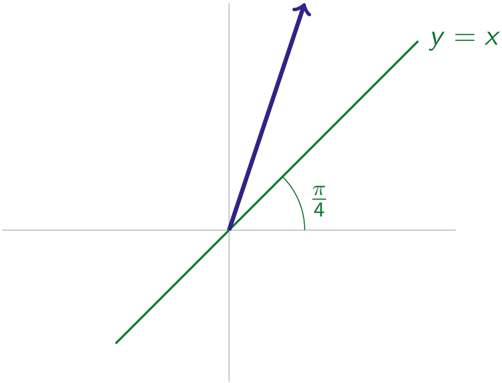

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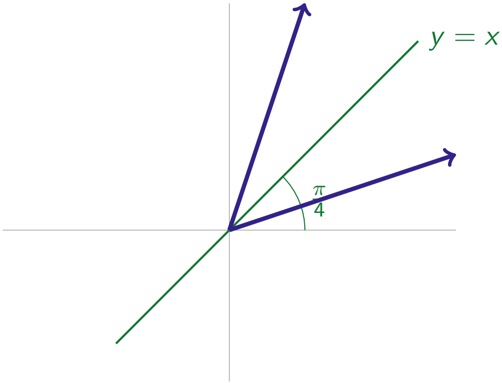

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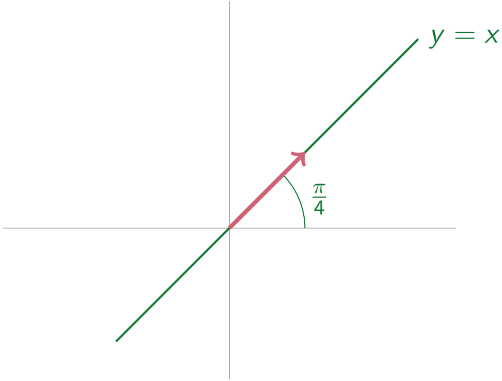

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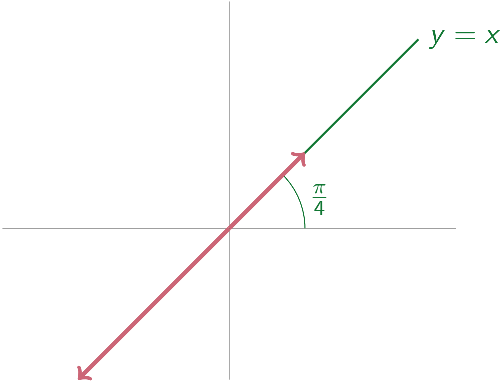

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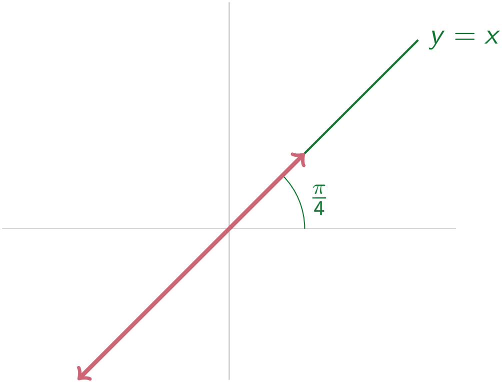

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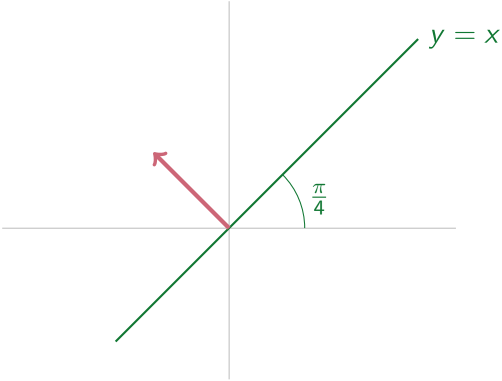

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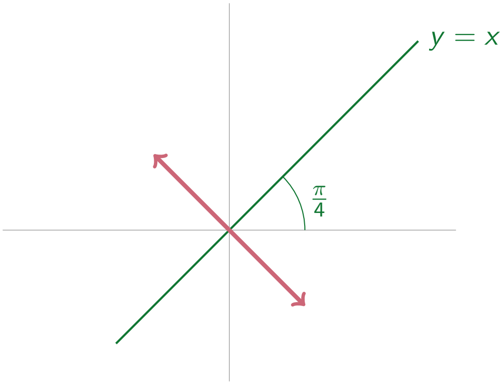

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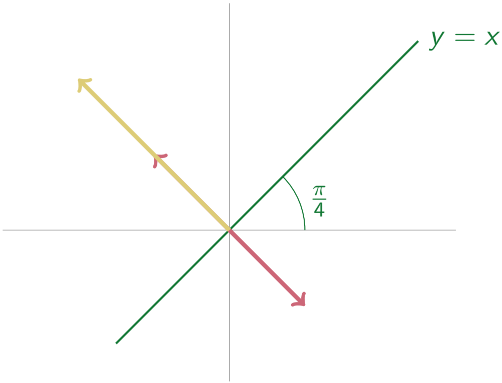

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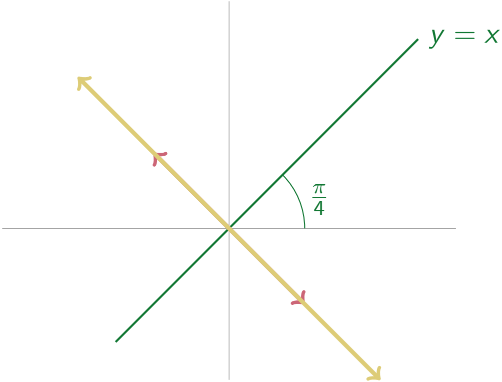

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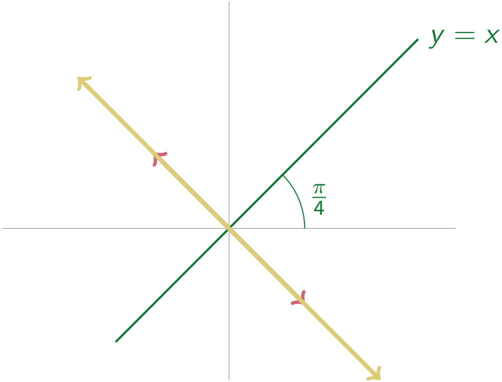

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$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} = -1 \cdot \begin{bmatrix} x \\ -x \end{bmatrix}$$

Eigenvectors and Eigenvalues

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ x \end{bmatrix}$$

The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has **eigenvector** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with **eigenvalue** 1.

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Tthe matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has **eigenvector** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with **eigenvalue** -1.

Given a matrix A, a scalar λ, and a NONZERO vector $\mathbf{x}$ with

$$A\mathbf{x} = \lambda\mathbf{x}$$

we say $\mathbf{x}$ is an *eigenvector* of A with *eigenvalue* λ.

Rotation Matrix Eigenvalues

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Search for eigenvectors:

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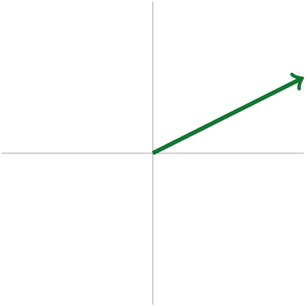

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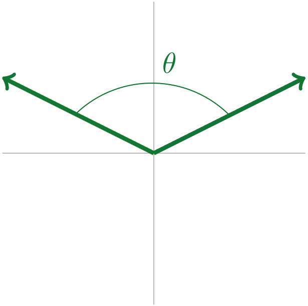

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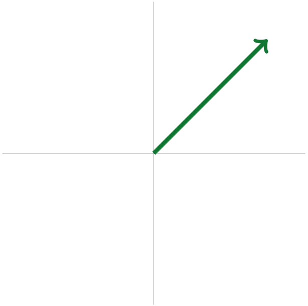

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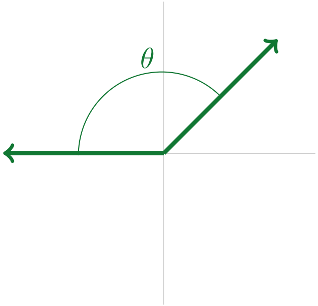

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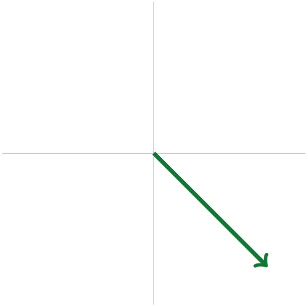

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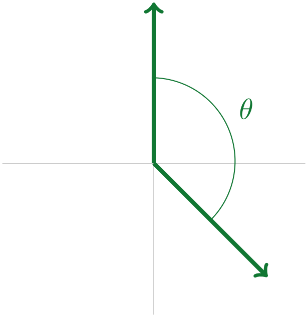

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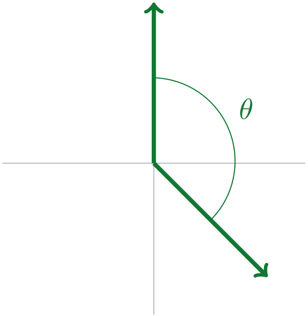

In general, no (real) eigenvalues of Rot_θ.

Finding Eigenvectors, Given Eigenvalues

Given: 7 is an eigenvalue of the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$.

What is an eigenvector associated to that eigenvalue?

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In equation form:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 4x & = & 7x \\ 2x & + & 4y & + & z & = & 7y \\ 4x & + & y & + & 2z & = & 7z \end{array}$$

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In better equation form:

$$\begin{array}{rcrcrcr} -6x & + & 2y & + & 4x & = & 0 \\ 2x & - & 3y & + & z & = & 0 \\ 4x & + & y & - & 5z & = & 0 \end{array}$$

Finding Eigenvectors, Given Eigenvalues

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Gaussian elimination on augmented matrix:

$$\left[\begin{array}{ccc|c} -6 & 2 & 4 & 0 \\ 2 & -3 & 1 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right] \rightarrow \rightarrow \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Finding Eigenvectors, Given Eigenvalues

Gaussian elimination on augmented matrix:

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Solutions:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Finding Eigenvectors, Given Eigenvalues

Gaussian elimination on augmented matrix:

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Solutions:

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So these are the solutions to:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 7 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Finding Eigenvectors, Given Eigenvalues

The solutions to:

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are precisely the vectors:

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Finding Eigenvectors, Given Eigenvalues

The solutions to:

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are precisely the vectors:

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So, we can choose $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as an example of an eigenvector with eigenvalue 7.

Generalized Eigenvector Finding

Suppose a matrix A has eigenvalue λ. Find an associated eigenvector $\mathbf{x}$.

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$$
\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} \\
Ax - \lambda\mathbf{x} &= \mathbf{0} \\
(A - \lambda I)\mathbf{x} &= \mathbf{0}
\end{aligned}
$$

Finding Eigenvectors from Eigenvalues

The matrix $\begin{bmatrix} 3 & 6 \\ 6 & -2 \end{bmatrix}$ has eigenvalues -6 and 7.

Find an eigenvector associated to each eigenvalue.

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$\lambda = -6$, $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ $\qquad$ $\lambda = 7$, $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

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Basis of $\mathbb{R}^2$: $\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

Finding Eigenvectors from Eigenvalues

The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has eigenvalues -2 and 5.

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$\lambda = -2$, $\mathbf{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ $\qquad$ $\lambda = 5$, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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Basis of $\mathbb{R}^2$: $\left\{ \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

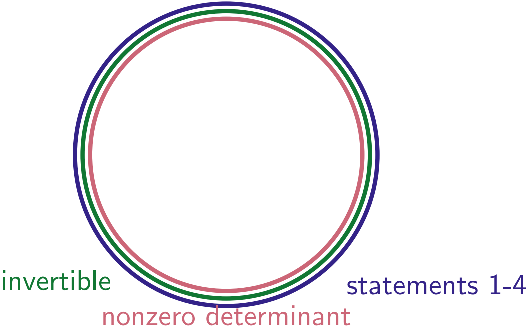
How do we Find Eigenvalues, Though?

First, a reminder....

Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
2) $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
3) The rank of A is n.
4) The reduced form of A has no zeroes along the main diagonal.
5) A is invertible
6) $\det(A) \neq 0$



How do we Find Eigenvalues, Though?

A matrix; λ eigenvalue, $\mathbf{x}$ eigenvector (so $\mathbf{x} \neq \mathbf{0}$)

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A matrix; λ eigenvalue, $\mathbf{x}$ eigenvector (so $\mathbf{x} \neq \mathbf{0}$)

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \text{ has a nonzero solution} \\ \det(A - \lambda I) &= 0 \end{aligned}$$

Find Eigenvalues and Associated Eigenvectors

$$\lambda \text{ eigenvalue } \Leftrightarrow \det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Find Eigenvalues and Associated Eigenvectors

$$\lambda \text{ eigenvalue } \Leftrightarrow \det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$\lambda = 1, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\lambda = -1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$

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Find Eigenvalues and Associated Eigenvectors

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$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$\lambda = 1,\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ NOT a basis of $\mathbb{R}^2$!

Find Eigenvalues and Associated Eigenvectors

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Find Eigenvalues and Associated Eigenvectors

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$\lambda = 1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\lambda = 2, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $\lambda = 3, \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$

Find Eigenvalues and Associated Eigenvectors

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

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$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix} \right\} \text{ Basis of } \mathbb{R}^3$$

Using Eigenvalues to Compute Matrix Powers

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

$\lambda = 1$, $\mathbf{k}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\lambda = 2$, $\mathbf{k}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $\lambda = 3$, $\mathbf{k}_3 = \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$

$$\mathbf{x} = \begin{bmatrix} 47 \\ 16 \\ 2 \end{bmatrix}$$

Using Eigenvalues to Compute Matrix Powers

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

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$$\mathbf{x} = \begin{bmatrix} 47 \\ 16 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix} = 2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3$$

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$A\mathbf{x} =$

Using Eigenvalues to Compute Matrix Powers

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$$A\mathbf{x} = A(2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3)$$

Using Eigenvalues to Compute Matrix Powers

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$$A\mathbf{x} = A(2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3) = 2A\mathbf{k}_1 + 4A\mathbf{k}_2 + A\mathbf{k}_3$$

Using Eigenvalues to Compute Matrix Powers

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Using Eigenvalues to Compute Matrix Powers

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$$\mathbf{x} = 2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3$$

$A^{10}\mathbf{x} =$

Using Eigenvalues to Compute Matrix Powers

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$$\mathbf{x} = 2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3$$

$$A^{10}\mathbf{x} = A^{10}(2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3)$$

Using Eigenvalues to Compute Matrix Powers

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$2\mathbf{k}_1 + 4(2^{10})\mathbf{k}_2 + 3^{10}\mathbf{k}_3$

Using Eigenvalues to Compute Matrix Powers

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$$2\mathbf{k}_1 + 4(2^{10})\mathbf{k}_2 + 3^{10}\mathbf{k}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2^{14} \\ 2^{12} \\ 0 \end{bmatrix} + \begin{bmatrix} 29 \cdot 3^{10} \\ 12 \cdot 3^{10} \\ 2 \cdot 3^{10} \end{bmatrix}$$

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Random Walks and Eigenvalues

to \ *from*	*pass*	*fail*
pass	$\frac{1}{2}$	$\frac{1}{3}$
fail	$\frac{1}{2}$	$\frac{2}{3}$

Random Walks and Eigenvalues

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$\lambda_1 = \frac{1}{6}$, $\mathbf{k}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$;

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Random Walks and Eigenvalues

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Suppose your initial state is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What happens after n tests?

Random Walks and Eigenvalues

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Random Walks and Eigenvalues

to \ *from*	*pass*	*fail*
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$P^n\mathbf{x}_0 = \frac{3}{5}P^n\mathbf{k}_1 + \frac{1}{5}P^n\mathbf{k}_2 = \frac{3}{5}\left(\frac{1}{6}\right)^n\mathbf{k}_1 + \frac{1}{5}\mathbf{k}_2$

Random Walks and Eigenvalues

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$$\lim_{n\to\infty} P^n\mathbf{x}_0 = \frac{1}{5}\mathbf{k}_2 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

Complex Eigenvalues

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Calculate $A^{90}\mathbf{x}$.

Complex Eigenvalues

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Calculate $A^{90}\mathbf{x}$.

$$\lambda_1 = i, \mathbf{k}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \qquad \lambda_2 = -i, \mathbf{k}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

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$$\mathbf{x} = (1.5 + i)\mathbf{k}_1 + (1.5 - i)\mathbf{k}_2$$

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$$\begin{aligned} A^{90}\mathbf{x} &= (1.5 + i)A^{90}\mathbf{k}_1 + (1.5 - i)A^{90}\mathbf{k}_2 \\ &= (1.5 + i)(i)^{90}\mathbf{k}_1 + (1.5 - i)(-i)^{90}\mathbf{k}_2 \\ &= (1.5 + i)(-1)\mathbf{k}_1 + (1.5 - i)(-1)\mathbf{k}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \end{aligned}$$

Complex Eigenvalues: A Neat Trick

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The eigenvalues and eigenvectors are complex conjugates of one another.

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Find all eigenvalues and eigenvectors of :

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$$\text{Roots:} \quad \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3i$$

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