Outline

Week 11: Eigenvalues and eigenvectors:
complex numbers and random walks

Course Notes: 6.1, 6.2

Goals: More practice finding eigenvalues and eigenvectors; expanding these to the complex numbers; using them in the context of random walks.

Random Walks and Eigenvalues

to \ *from*	*pass*	*fail*
pass	$\frac{1}{2}$	$\frac{1}{3}$
fail	$\frac{1}{2}$	$\frac{2}{3}$

Random Walks and Eigenvalues

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$\lambda_1 = \frac{1}{6}$, $\mathbf{k}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$;

$\lambda_2 = 1$, $\mathbf{k}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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Suppose your initial state is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What happens after n tests?

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$\mathbf{x}_0 = \frac{3}{5}\mathbf{k}_1 + \frac{1}{5}\mathbf{k}_2$

Random Walks and Eigenvalues

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$P^n\mathbf{x}_0 = \frac{3}{5}P^n\mathbf{k}_1 + \frac{1}{5}P^n\mathbf{k}_2 = \frac{3}{5}\left(\frac{1}{6}\right)^n\mathbf{k}_1 + \frac{1}{5}\mathbf{k}_2$

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$\lim\limits_{n\to\infty} P^n\mathbf{x}_0 = \dfrac{1}{5}\mathbf{k}_2 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$

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$\lim\limits_{n\to\infty} P^n\mathbf{x}_0 = \dfrac{1}{5}\mathbf{k}_2 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$

What if you had failed the first test?

Eigenvalues of Probability Transition Matrices

Theorem

If P is a transition matrix (non-negative entries with all columns summing to one) that in addition has all positive entries then P has an eigenvalue 1 with a single eigenvector k_1 that can chosen to be a probability vector. All other eigenvalues satisfy $|\lambda| < 1$ with eigenvectors with components that sum to zero. Thus,

$$\lim_{n \to \infty} x_n = k_1$$

for any x_0. That is, k_1 is an equilibrium probability.

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[Proof, of sorts]

In short: that last example was typical. As long as a probability matrix has no zeroes:

- Probability matrices have eigenvalues
- There will be some equilibrium that the system will reach in the long run, regardless of initial state.

Equilibrium Probability

Which of these random walk models seems likely to have an equilibrium probability? Is it clear what would it be?

In any given day, your odds of dying are 1 in 1,000.

	alive	dead
alive	0.999	0
dead	0.001	1

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Equilibrium probability: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; everybody dies.

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Region of residence for a rat.

	N/S Am.	Other
N/S Am.	.99	.01
Other	.01	.99

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N/S Am.	.99	.01
Other	.01	.99

Equilibrium probability: $\begin{bmatrix} .5 \\ .5 \end{bmatrix}$; long-term average.

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Other	.01	.99

In a relationship or not, by year.

	single	partnered
single	.4	.25
partnered	.6	.75

Equilibrium Probability

Find the equilibrium probability of the system.

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Eigenvalues and eigenvectors:

$\lambda_1 = 1$	$\mathbf{k_1} = \begin{bmatrix} 1 \\ \frac{12}{5} \end{bmatrix}$
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For any $\mathbf{x}_0 = a_1\mathbf{k_1} + a_2\mathbf{k_2}$:

$$P^n\mathbf{x}_0 = a_2\mathbf{k_1} + a_2\left(\frac{3}{20}\right)^n\mathbf{k_2} \xrightarrow{n\to\infty} a_2\mathbf{k_1}$$

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Find the equilibrium probability of the system.

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$$P^n\mathbf{x}_0 = a_2\mathbf{k_1} + a_2\left(\frac{3}{20}\right)^n\mathbf{k_2} \xrightarrow{n\to\infty} a_2\mathbf{k_1}$$

$$\lim_{n\to\infty} P^n\mathbf{x}_0 = \begin{bmatrix} 5/17 \\ 12/17 \end{bmatrix} \text{ regardless of } \mathbf{x}_0$$

Computation Practice

$$P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} \qquad \mathbf{x}_0 = \begin{bmatrix} a \\ 1-a \end{bmatrix},\ a \in [0,1]$$

1. Find all eigenvalues of P, and an associated eigenvector to each.
2. Write $\mathbf{x}_0$ as a linear combination of eigenvectors of P.
3. Calculate $\mathbf{x}_n$, where n is some positive integer.
4. Find the equilibrium probability of P.

1. Find Eigenvalues and Eigenvectors

By our theorem, we know that 1 will be an eigenvalue. However, for the sake of practice, let's find them the old-fashioned way.

Eigenvalues of P are precisely those scalars λ such that $\det(P - \lambda I) = 0$. So we set the determinant equal to zero:

$$\det \begin{bmatrix} 1/3 - \lambda & 1/2 \\ 2/3 & 1/2 - \lambda \end{bmatrix} = \left(\frac{1}{3} - \lambda\right)\left(\frac{1}{2} - \lambda\right) - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \lambda^2 - \frac{5}{6}\lambda - \frac{1}{6}$$

And find $\lambda_1 = 1$ (as expected) and $\lambda_2 = -\frac{1}{6}$.

To find the associated eigenvectors, we set $P\mathbf{x} = \lambda\mathbf{x}$. (Next Slide)

1. Find Eigenvalues and Eigenvectors

$\boxed{\lambda_1 = 1}$

$$\begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \quad \to \quad \begin{bmatrix} -2/3 & 1/2 \\ 2/3 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions to this system are of the form $s \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ for some scalar s. Any vector of this form will do.

$\boxed{\lambda_2 = -\frac{1}{6}}$

$$\begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} x \\ y \end{bmatrix} \quad \to \quad \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions to this system are of the form $s \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for some scalar s. Any vector of this form will do.

2. Basis Vectors

To find $\mathbf{x}_0$ as a combination of eigenvectors, we have to CHOOSE our eigenvectors. I like integers, so I'll use $\mathbf{k_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{k_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Your vectors may be scalar multiples of these.

The equation we have to solve is:

$$\begin{bmatrix} a \\ 1-a \end{bmatrix} = x \begin{bmatrix} 3 \\ 4 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 1-a \end{bmatrix}$$

Since a is a constant, we can solve this using an augmented matrix and row reduction.

$$\left[\begin{array}{cc|c} 3 & 1 & a \\ 4 & -1 & 1-a \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 3 & 1 & a \\ 7 & 0 & 1 \end{array}\right]$$

So $x = \frac{1}{7}$ and $y = a - \frac{3}{7}$. That is, $\mathbf{x}_0 = \frac{1}{7}\mathbf{k_1} + (a - \frac{3}{7})\mathbf{k_2}$.

3. Find $\mathbf{x}_n$

Recall $\mathbf{x}_n = P^n \mathbf{x}_0$. With our previous work, the answer is an easy calculation.

$$
\begin{aligned}
\mathbf{x}_n = P^n \mathbf{x}_0 = P^n \left(\frac{1}{7} \mathbf{k_1} + \left(\frac{3}{7} - a \right) \mathbf{k_2} \right) \\
&= \frac{1}{7} P^n \mathbf{k_1} + \left(\frac{3}{7} - a \right) P^n \mathbf{k_2} \\
&= \frac{1}{7} \mathbf{k_1} + \left(\frac{3}{7} - a \right) \left(-\frac{1}{6} \right)^n \mathbf{k_2} \\
&= \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} + \begin{bmatrix} \left(\frac{3}{7} - a\right)(-1/6)^n \\ -\left(\frac{3}{7} - a\right)(-1/6)^n \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{7} + \left(\frac{3}{7} - a\right)(-1/6)^n \\ \frac{4}{7} - \left(\frac{3}{7} - a\right)(-1/6)^n \end{bmatrix}
\end{aligned}
$$

4. Find the Equilibrium Probability

Recall the equilibrium probability is $\lim_{n\to\infty} \mathbf{x}_n = \lim_{n\to\infty} P^n \mathbf{x}_0$. With our previous work, the answer is an easy calculation. Using an intermediate result from the last slide:

$$\begin{aligned}\lim_{n\to\infty} \mathbf{x}_n &= \lim_{n\to\infty} \left[\frac{1}{7}\mathbf{k_1} + \left(\frac{3}{7} - a\right)\left(-\frac{1}{6}\right)^n \mathbf{k_2}\right] \\ &= \frac{1}{7}\mathbf{k_1} \\ &= \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}\end{aligned}$$

ALTERNATELY, our theorem tells us that the equilibrium probability will always be an eigenvector associated with the eigenvalue $\lambda = 1$. Since our eigenvectors were of the form $s[3, 4]$, we can find the equilibrium probability by figuring out which value of s gives us a vector whose entries sum to one; $s = 7$ is that scalar.

Complex Eigenvalues:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Calculate $A^{90}\mathbf{x}$.

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Calculate $A^{90}\mathbf{x}$.

$$\lambda_1 = i, \mathbf{k}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \qquad \lambda_2 = -i, \mathbf{k}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

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$$\mathbf{x} = (1.5 + i)\mathbf{k}_1 + (1.5 - i)\mathbf{k}_2$$

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$$\begin{aligned} A^{90}\mathbf{x} &= (1.5 + i)A^{90}\mathbf{k}_1 + (1.5 - i)A^{90}\mathbf{k}_2 \\ &= (1.5 + i)(i)^{90}\mathbf{k}_1 + (1.5 - i)(-i)^{90}\mathbf{k}_2 \\ &= (1.5 + i)(-1)\mathbf{k}_1 + (1.5 - i)(-1)\mathbf{k}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \end{aligned}$$

Complex Eigenvalues: A Neat Trick

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

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The eigenvalues and eigenvectors are complex conjugates of one another.

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The eigenvalues and eigenvectors are complex conjugates of one another.

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow \quad \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} \quad \Leftrightarrow \quad \overline{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}} \quad \Leftrightarrow \quad A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$$

Complex Eigenvalues: A Neat Trick

Find all eigenvalues and eigenvectors of :

$$A = \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix}$$

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Find all eigenvalues and eigenvectors of :

$$A = \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} -3-\lambda & 5 \\ -2 & -1-\lambda \end{bmatrix} = \lambda^2 + 4\lambda + 13$$

$$\text{Roots:} \quad \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3i$$

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$$\lambda_1 = -2 - 3i,$$

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$$\lambda_1 = -2-3i, \qquad \mathbf{x}_1 = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$$

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$$\lambda_1 = -2 - 3i, \qquad \mathbf{x}_1 = \begin{bmatrix} 1 + 3i \\ 2 \end{bmatrix}$$

$$\lambda_2 = -2 + 3i, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix}$$

Eigenfact or Eigenfiction?

Let A be a matrix with eigenvalue λ and associated eigenvector $\mathbf{k}$.
True or False:

1. $2\mathbf{k}$ is an eigenvector of A, associated with λ.
2. It is possible that $\mathbf{k}$ is an eigenvalue of A associated with a *different* eigenvalue (that is, other than λ).
3. All eigenvectors of A associate with λ are scalar multiples of $\mathbf{k}$.
4. $\mathbf{k}$ might be the zero vector.
5. λ might be zero.
6. If A has only real entries, and $\mathbf{k}$ has only real entries, then λ is real.
7. If A has only real entries, then λ is real.