Outline

Week 3: Introduction to Linear Systems

Course Notes: 2.6,3.1

Goals: Consider the solution to a system of linear equations as a geometric object; learn basic techniques (back substitution, row reduction) for solving systems of linear equations.

Intersections: $\mathbb{R}^2$

Which of the following could be the intersection of lines $a_1 x_1 + a_2 x_2 = a_3$ and $b_1 x_1 + b_2 x_2 = b_3$?

A. nothing
B. point
C. line
D. plane
E. two points
F. two lines

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The intersection of the two lines is the set of points (x_1, x_2) that are solutions to this system of linear equations:

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If the intersection is a **point**, what can we say about $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$?

A. zero B. nonzero C. positive D. negative

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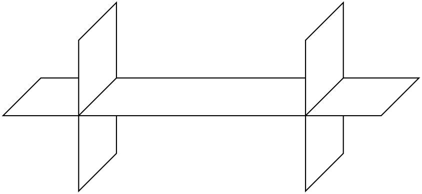

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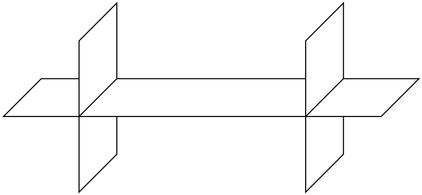

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$$\begin{array}{ccccccc} a_1x_1 & + & a_2x_2 & + & a_3x_3 & = & a_4 \\ b_1x_1 & + & b_2x_2 & + & b_3x_3 & = & b_4 \\ c_1x_1 & + & c_2x_2 & + & c_3x_3 & = & c_4 \end{array}$$

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\end{array}
$$

Possible solutions:

$\emptyset$

$\mathbf{x} = \mathbf{q}$

$\mathbf{x} = \mathbf{q} + s\mathbf{a}$

$\mathbf{x} = \mathbf{q} + s\mathbf{a} + t\mathbf{b}$

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Suppose $(1, 3, 5)$ and $(2, 6, 10)$ are solutions to the system of equations.

How many solutions total are there?

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Suppose $(1, 3, 5)$ and $(2, 6, 10)$ are solutions to the system of equations.

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Give another solution.

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Suppose the intersection of the three planes is a point. Then

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \neq 0$$

Think about this at home :)

Definition: Linear Combination

If $\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$ are a collection of vectors, and $s_1, s_2, \ldots, s_n$ are scalars, then

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n$$

is a *linear combination* of $\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$.

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Test for colinearity or planarity using determinant.

Linear (In)dependence

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So, the vectors are *linearly dependent* if there exist scalars $s_1, s_2, \ldots, s_n$, at least one of which is nonzero, such that $s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdot + s_n\mathbf{a}_n = \mathbf{0}$.

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Definition: Basis

In $\mathbb{R}^n$, a collection of n linearly independent vectors is called a *basis*.

Any $\mathbf{x}$ in $\mathbf{R}^n$ can be written as a linear combination of basis vectors.

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$$\begin{bmatrix} 7 \\ -2 \end{bmatrix} = -11 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Bases

In $\mathbb{R}^3$, what is the *easiest* basis to work with?
That is: find **a** , **b** , and **c** so that it is extremely easy to solve the system

$$s_1\mathbf{a} + s_2\mathbf{b} + s_3\mathbf{c} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Give one vector that can never be in a basis of $\mathbb{R}^2$.

(Remember: a basis in $\mathbb{R}^2$ is a collection of two vectors $\mathbf{a}$ and $\mathbf{b}$ so that the only solution to the equation $s\mathbf{a} + t\mathbf{b} = \mathbf{0}$ is $s = t = 0$.)

Suppose:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and also

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = p \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $s \neq p$.

Is $\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^2$?

Recall: a basis in $\mathbb{R}^2$ is two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $s_1\mathbf{a} + s_2\mathbf{b} = \mathbf{0}$ ONLY when $s_1 = s_2 = 0$.

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Find a scalar constant c so that $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = c \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

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$$c = \frac{q-t}{s-p}$$

Substitution

For the vectors below, either show that is is not a basis of $\mathbb{R}^3$, or give $\begin{bmatrix} 2 \\ 24 \\ 49 \end{bmatrix}$ as a linear combination.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 2 \\ 24 \\ 49 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix} - 8 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Substitution

Suppose a thrown ball at time t has height $h = At^2 + Bt + C$.
At $t = 1$, the ball is at height 0;
at $t = 2$, the ball is at height 1; and
at $t = 3$, the ball is at height 6.
Find A, B, and C.

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Find A, B, and C.

$A = 2$, $B = -5$, $C = 3$

General Form

$$
\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & c_1 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & c_2 \\
 & \vdots & & & \vdots & & & \vdots & \\
a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & c_m
\end{array}
$$

Where $a_{i,j}$ and c_i are known and fixed.

Goal: easily-solvable system

$$
\begin{array}{rcrcrcrcr}
x_1 & + & 3x_2 & + & 17x_3 & + & 9x_4 & = & 10 \\
-3x_1 & + & -6x_2 & + & 8x_3 & + & 5x_4 & = & 17 \\
\pi x_1 & + & -8x_2 & + & 3x_3 & + & x_4 & = & -2 \\
8x_1 & + & -8x_2 & + & 5x_3 & + & 2x_4 & = & 2
\end{array}
$$

Goal: easily-solvable system

$$\begin{aligned} x_1 + 3x_2 + 17x_3 + 9x_4 &= 10 \\ -6x_2 + 8x_3 + 5x_4 &= 17 \\ 3x_3 + x_4 &= -2 \\ 2x_4 &= 2 \end{aligned}$$

Upper Triangular

Goal: easily-solvable system

$$\begin{array}{rcrcrcrcr}
x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & = & 10 \\
 & & -6x_2 & + & 0x_3 & + & 0x_4 & = & 17 \\
 & & & & 3x_3 & + & 0x_4 & = & -2 \\
 & & & & & & 2x_4 & = & 2
\end{array}$$

Diagonal

Equivalent Equations

Notice:

$$3x_1 + 5x_2 + 7x_3 = 10$$

and

$$6x_1 + 10x_2 + 14x_3 = 20$$

have the same solutions.

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Caution:

$$0x_1 + 0x_2 + 0x_3 = 0$$

has more solutions.

Elementary Row Operations: Multiplication of a row by a non-zero number

$$\begin{aligned} 3x_1 &- 9x_2 + 6x_3 = 30 \\ -x_1 &+ 3x_2 + 5x_3 = 4 \\ x_1 &+ x_2 + x_3 = -6 \end{aligned}$$

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Same solutions as:

$$\begin{array}{rcrcrcr} 1x_1 & - & 3x_2 & + & 2x_3 & = & 10 \\ -x_1 & + & 3x_2 & + & 5x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & -6 \end{array}$$

Elementary Row Operations: Adding a Multiple of a Row to Another Row

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Same solutions as:

$$\begin{array}{rcrcrcr} 0x_1 & - & 0x_2 & + & 7x_3 & = & 14 \\ -x_1 & + & 3x_2 & + & 5x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & -6 \end{array}$$

Elementary Row Operations: Interchanging Rows

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Same solutions as:

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Streamlined Notation: Augmented Matrices

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 2x_3 & = & 10 \\ -x_1 & + & 3x_2 & + & 5x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & -6 \end{array}$$

We'll write this as:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 10 \\ -1 & 3 & 5 & 4 \\ 1 & 1 & 1 & -6 \end{array}\right]$$

Augmented Matrices

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution:

Augmented Matrices

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution:

Using Elementary Row Operations

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right]$$

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$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] (1,:) = \frac{1}{3}(1,:)$$

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Using Elementary Row Operations

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$$x_1 = 11$$
$$x_2 = -4$$
$$x_3 = 0$$

Row Operation Calculator (link)