Outline

Week 8: Inverses and determinants

Course Notes: 4.5, 4.6

Goals: Be able to calculate a matrix's inverse;
understand the relationship between the invertibility of a matrix and the solutions of associated linear systems;
calculate the determinant of a square matrix of any size, and learn some tricks to make the computation more efficient.

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix, I, is a square matrix with 1s along its diagonal, and 0s everywhere else.

For any matrix A that can be multiplied with I, $AI = IA = A$.

Matrix Inverses: The Closest we can Get to Division

Linear System Setup:

$$\begin{cases} x + 2y + 3z &= 10 \\ 4x + 5y + 6z &= 20 \\ 7x + 8y + 9z &= 30 \end{cases}$$

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

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Solve for $\mathbf{x}$.

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Can't we just divide by A?
We want: a matrix A^{-1} with the property $A^{-1}A = I$, the identity matrix.

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A matrix A^{-1} is the **inverse** of a square matrix A if $A^{-1}A = I$, where I is the identity matrix.

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If $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x} = A^{-1}\mathbf{b}$

If an Inverse Exists....

Theorem

If an n-by-n matrix A has an inverse A^{-1}, then for any $\mathbf{b}$ in $\mathbb{R}^n$,

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has precisely one solution, and that solution is

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Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
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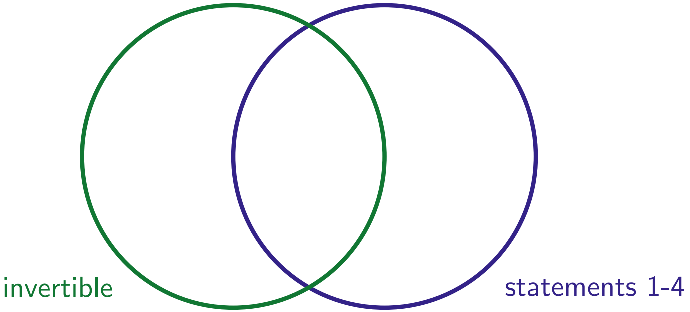

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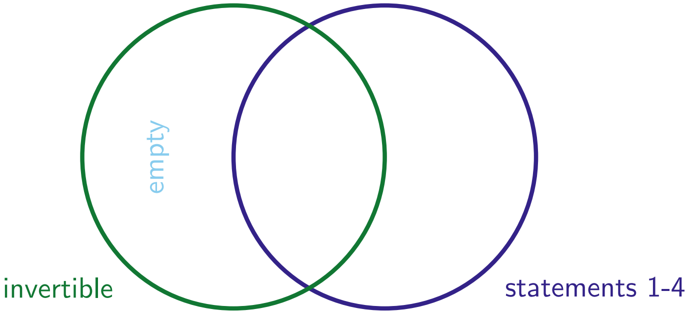

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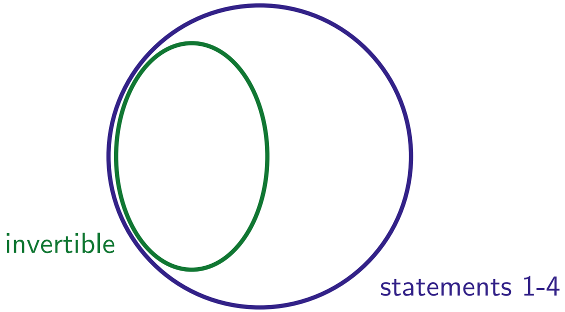

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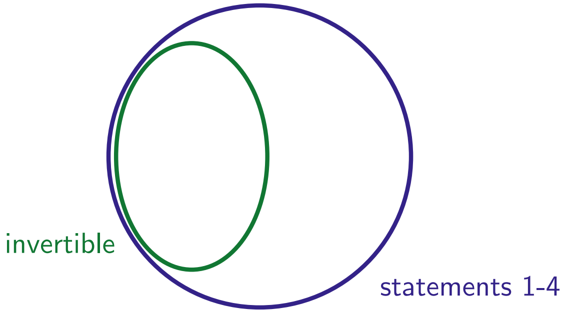

Are the statements **equivalent** to A being invertible?

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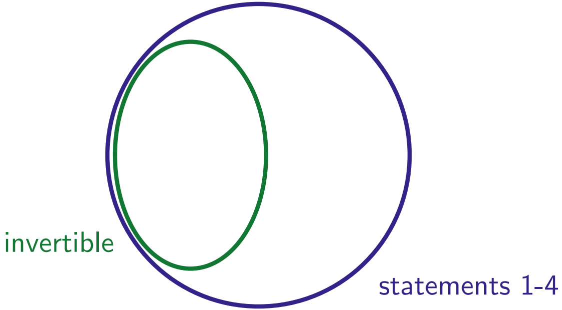
And now we've shown that if the statements hold, then A is invertible



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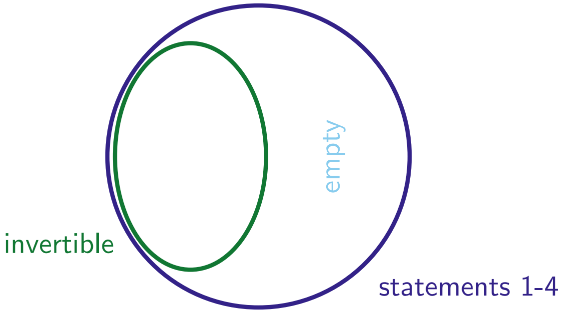

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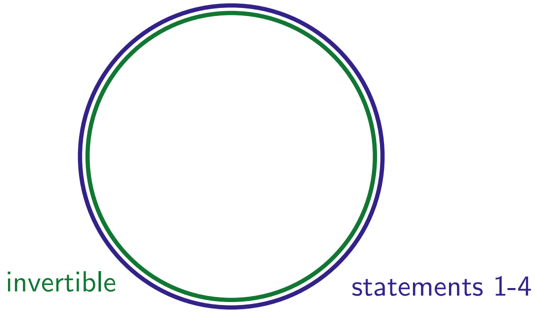

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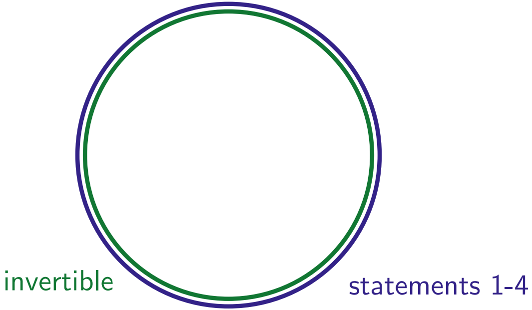
5) A is invertible



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A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$.

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Suppose A is a matrix with the following reduced form. Is A invertible?

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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Calculate the inverse of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 2 & 0 & 7 \end{bmatrix}$

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Calculate the inverse of $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Inverses and Products

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Simplify:

$$\left[(AC)^{-1}A(AB)^{-1}\right]^{-1}$$

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In general:

$$\det \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = a_{1,1} D_{1,1} - a_{1,2} D_{1,2} + a_{1,3} D_{1,3} \cdots \pm a_{1,n} D_{1,n}$$

where $D_{i,j}$ is the determinant of the matrix obtained from A by deleting row i and column j.

Calculate

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

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$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

Calculate

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} = 6$$

$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = 210$$

Determinants of Triangular Matrices

Calculate, where $*$ is any number:

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 2 & 0 & 0 & 0 \\ * & * & 3 & 0 & 0 \\ * & * & * & 4 & 0 \\ * & * & * & * & 5 \end{bmatrix}$$

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$$\det \begin{bmatrix} 1 & * & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & 4 & * \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

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$$\det \begin{bmatrix} 1 & * & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & 4 & * \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Fact: for any square matrix A, $\det(A) = \det(A^T)$

Determinants of Upper Triangular Matrices

Is the determinant of ANY triangular matrix the product of the diagonal entries?

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$$\det \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} = a \det \begin{bmatrix} b & * \\ 0 & c \end{bmatrix} - * \det \begin{bmatrix} 0 & * \\ 0 & c \end{bmatrix} + * \det \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

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$$\begin{aligned} \det \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} &= a \det \underbrace{\begin{bmatrix} b & * \\ 0 & c \end{bmatrix}}_{\textit{triangular}} - * \det \underbrace{\begin{bmatrix} 0 & * \\ 0 & c \end{bmatrix}}_{\textit{triangular}} + * \det \underbrace{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}}_{\textit{triangular}} \\ &= a(bc) - *(0 \cdot c) + *(0 \cdot 0) \\ &= abc \end{aligned}$$

Determinants of Upper Triangular Matrices

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The determinant of any triangular matrix (upper or lower) is the product of the diagonal entries.

$$\det \begin{bmatrix} 10 & 9 & 8 & 4 & 12 \\ 0 & 5 & 9 & 7 & 15 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{2}{7} \\ 0 & 0 & 0 & 2 & 32 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\det \begin{bmatrix} 10 & 9 & 8 & 4 & 12 \\ 0 & 5 & 9 & 7 & 15 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{2}{7} \\ 0 & 0 & 0 & 2 & 32 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = (10)(5)\left(\frac{1}{2}\right)(2)(5) = 250$$

$$\det \begin{bmatrix} 10 & 9 & 8 & 4 & 12 \\ 0 & 5 & 9 & 7 & 15 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{2}{7} \\ 0 & 0 & 0 & 2 & 32 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = (10)(5)\left(\frac{1}{2}\right)(2)(5) = 250$$

Careful: this ONLY works with triangular matrices!

More Determinant Tricks

Helpful Facts for Calculating Determinants:

1. If B is obtained from A by multiplying *one* row of A by the constant c then $\det B = c \det A$.
2. If B is obtained from A by switching two rows of A then $\det B = -\det A$.
3. If B is obtained from A by adding a multiple of one row to another then $\det B = \det A$.
4. $\det(A) = 0$ if and only if A is not invertible
5. For all square matrices B of the same size as A, $\det(AB) = \det(A)\det(B)$.
6. $\det(A^T) = \det(A)$

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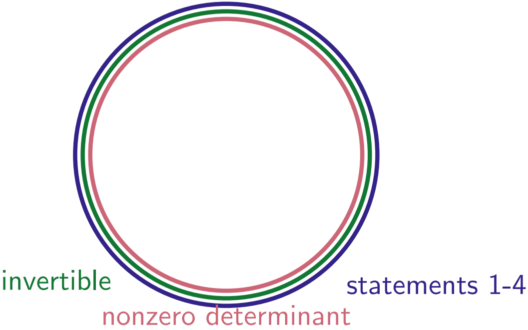
Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
2) $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
3) The rank of A is n.
4) The reduced form of A has no zeroes along the main diagonal.
5) A is invertible
6) $\det(A) \neq 0$



Is A invertible?

$$A = \begin{bmatrix} 72 & 9 & 8 & 16 \\ 0 & 4 & 3 & -9 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 21 \end{bmatrix}$$

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$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 1 & 2 & 0 \\ 10 & 5 & 0 & 1 \\ 10 & 0 & 5 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} = ?$$

Is A invertible?

$$A = \begin{bmatrix} 72 & 9 & 8 & 16 \\ 0 & 4 & 3 & -9 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 21 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 1 & 2 & 0 \\ 10 & 5 & 0 & 1 \\ 10 & 0 & 5 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} = ?$$

Calculate:

$$\det \begin{bmatrix} 1 & 5 & 10 & 15 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\det\begin{bmatrix}0 & 10 & 10 & 0\\ 1 & 5 & 0 & 2\\ 2 & 0 & 5 & 1\\ 0 & 1 & 3 & 1\end{bmatrix} = -210; \qquad \det\begin{bmatrix}0 & 20 & 20 & 0\\ 1 & 5 & 0 & 2\\ 2 & 0 & 5 & 1\\ 0 & 1 & 3 & 1\end{bmatrix} = ?$$

$$\det\begin{bmatrix}2 & 0 & 5 & 1\\ 1 & 5 & 0 & 2\\ 0 & 10 & 10 & 0\\ 0 & 1 & 3 & 1\end{bmatrix} = ? \qquad \det\begin{bmatrix}2 & 0 & 5 & 1\\ 0 & 10 & 10 & 0\\ 1 & 5 & 0 & 2\\ 0 & 1 & 3 & 1\end{bmatrix} = ?$$

$$\det\begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det\begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = ?$$

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$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -420$$

$$\det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 10 & 10 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} = ? \qquad \det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix} = ?$$

$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -420$$

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$$\det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 10 & 10 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} = 210 \qquad \det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix} = ?$$

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Suppose $\det A = 5$ for an invertible matrix A. What is $\det(A^{-1})$?

Suppose A is an n-by-n matrix with determinant 5. What is the determinant of $3A$?

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

det : -1

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xleftarrow{R_3 + R_2} \qquad \det : -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{det} : -1 \qquad \xleftarrow{R_3 + R_2} \qquad \text{det} : -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xleftarrow{2R_2} \quad \det: -1 \quad \xleftarrow{R_3 + R_2} \quad \det: -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1 & 0 & 4\\ 1 & 2 & 8\\ 0 & 1 & 1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1 & 0 & 4\\ 0 & 2 & 4\\ 0 & 1 & 1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & 1 & 1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & 0 & -1\end{bmatrix}$$

$$\text{det}: -2 \quad \xleftarrow{2R_2} \quad \text{det}: -1 \quad \xleftarrow{R_3+R_2} \quad \text{det}: -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1 & 0 & 4\\ 1 & 2 & 8\\ 0 & 1 & 1\end{bmatrix} \underset{R_2+R_1}{\overset{R_2-R_1}{\rightleftarrows}} \begin{bmatrix}1 & 0 & 4\\ 0 & 2 & 4\\ 0 & 1 & 1\end{bmatrix} \underset{2R_2}{\overset{\frac{1}{2}R_2}{\rightleftarrows}} \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & 1 & 1\end{bmatrix} \underset{R_3+R_2}{\overset{R_3-R_2}{\rightleftarrows}} \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & 0 & -1\end{bmatrix}$$

det : -2 det : -1 det : -1

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1 & 0 & 4\\ 1 & 2 & 8\\ 0 & 1 & 1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1 & 0 & 4\\ 0 & 2 & 4\\ 0 & 1 & 1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & 1 & 1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & 0 & -1\end{bmatrix}$$

$$\text{det} : -2 \xleftarrow{R_2+R_1} \text{det} : -2 \xleftarrow{2R_2} \text{det} : -1 \xleftarrow{R_3+R_2} \text{det} : -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \xleftarrow{R_2+R_1} \text{det}: -2 \xleftarrow{2R_2} \text{det}: -1 \xleftarrow{R_3+R_2} \text{det}: -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det} : -2 \xleftarrow{R_2+R_1} \text{det} : -2 \xleftarrow{2R_2} \text{det} : -1 \xleftarrow{R_3+R_2} \text{det} : -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{det} : -2 \xleftarrow{R_2 + R_1} \text{det} : -2 \xleftarrow{2R_2} \text{det} : -1 \xleftarrow{R_3 + R_2} \text{det} : -1$$

$$\det\left(\begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 8 & 7 \\ 9 & 6 & 1 & 4 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 8 & 7 \\ 9 & 6 & 1 & 4 \end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 8 \\ 9 & 6 & 1 \end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \quad \xleftarrow{R_2+R_1} \quad \text{det}: -2 \quad \xleftarrow{2R_2} \quad \text{det}: -1 \quad \xleftarrow{R_3+R_2} \quad \text{det}: -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}1&1&1\\0&2&5\\0&-3&-8\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix}\xrightarrow{R_2-R_1}\begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix}\xrightarrow{\frac{1}{2}R_2}\begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix}\xrightarrow{R_3-R_2}\begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \xleftarrow{R_2+R_1} \text{det}: -2 \xleftarrow{2R_2} \text{det}: -1 \xleftarrow{R_3+R_2} \text{det}: -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)=\det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)=-(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

$$=\det\left(\begin{bmatrix}1&1&1\\0&2&5\\0&-3&-8\end{bmatrix}\right)=1\det\left(\begin{bmatrix}2&5\\-3&-8\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1 & 0 & 4\\ 1 & 2 & 8\\ 0 & 1 & 1\end{bmatrix} \xrightarrow[\xleftarrow{R_2+R_1}]{R_2-R_1} \begin{bmatrix}1 & 0 & 4\\ 0 & 2 & 4\\ 0 & 1 & 1\end{bmatrix} \xrightarrow[\xleftarrow{2R_2}]{\frac{1}{2}R_2} \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & 1 & 1\end{bmatrix} \xrightarrow[\xleftarrow{R_3+R_2}]{R_3-R_2} \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & 0 & -1\end{bmatrix}$$

det : −2   det : −2   det : −1   det : −1

$$\det\left(\begin{bmatrix}2 & 2 & 2 & 1\\ 1 & 1 & 1 & 1\\ 3 & 5 & 8 & 7\\ 9 & 6 & 1 & 4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0 & 0 & 0 & -1\\ 1 & 1 & 1 & 1\\ 3 & 5 & 8 & 7\\ 9 & 6 & 1 & 4\end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix}1 & 1 & 1\\ 3 & 5 & 8\\ 9 & 6 & 1\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 5\\ 0 & -3 & -8\end{bmatrix}\right) = 1\det\left(\begin{bmatrix}2 & 5\\ -3 & -8\end{bmatrix}\right) = -16 + 15 = -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{det} : -2 \xleftarrow{R_2 + R_1} \text{det} : -2 \xleftarrow{2R_2} \text{det} : -1 \xleftarrow{R_3 + R_2} \text{det} : -1$$

$$\det\left(\begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 8 & 7 \\ 9 & 6 & 1 & 4 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 8 & 7 \\ 9 & 6 & 1 & 4 \end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 8 \\ 9 & 6 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 0 & -3 & -8 \end{bmatrix}\right) = 1\det\left(\begin{bmatrix} 2 & 5 \\ -3 & -8 \end{bmatrix}\right) = -16 + 15 = -1$$

Is the original 4-by-4 matrix invertible?

Suppose a matrix has the following **reduced** form. Is the matrix invertible? What is its determinant?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose a matrix has the following **reduced** form. Is the matrix invertible? What is its determinant?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Invertible; determinant unknowable but nonzero

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Invertible; determinant unknowable but nonzero

Determinant Expansion across Alternate Lines

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Determinant Expansion across Alternate Lines

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 9 & 8 & 5 & 6 & 10 \\ 1 & 0 & 0 & 0 & 1 \\ 7 & 0 & 1 & 1 & 1 \\ 8 & 0 & 1 & 1 & 1 \\ 4 & 3 & 5 & 6 & 7 \end{bmatrix}\right)$$

Determinant Expansion across Alternate Lines

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 9 & 8 & 5 & 6 & 10 \\ 1 & 0 & 0 & 0 & 1 \\ 7 & 0 & 1 & 1 & 1 \\ 8 & 0 & 1 & 1 & 1 \\ 4 & 3 & 5 & 6 & 7 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 8 & 9 & 5 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 7 & 1 & 1 \\ 0 & 8 & 1 & 1 \end{bmatrix}\right)$$