Outline

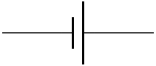
Week 12: Application of vector differential equations to electrical networks

Course Notes: 6.4

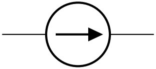
Goals: determine behaviour of LCR networks using vector differential equations

Circuit Components

Resistor



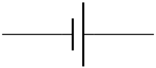
Voltage Source



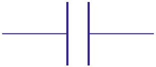
Current Source

Circuit Components

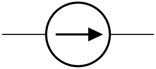
Resistor



Voltage Source



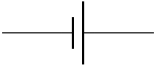
Capacitor



Current Source

Circuit Components

Resistor



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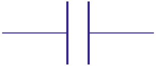

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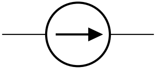
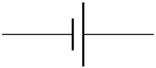


Current Source

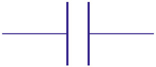
Inductor

Circuit Components

Resistor

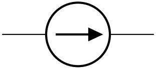


Voltage Source



Capacitor

acts like a changing voltage source

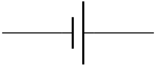


Current Source

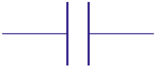
Inductor

Circuit Components

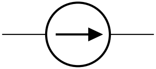
Resistor



Voltage Source



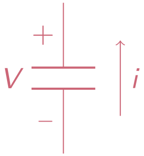
Capacitor
acts like a changing voltage source

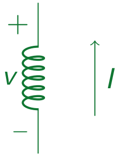


Current Source

Inductor
acts like a changing current source

Differential Equations





V: Voltage drop
i: current
C: capacitance

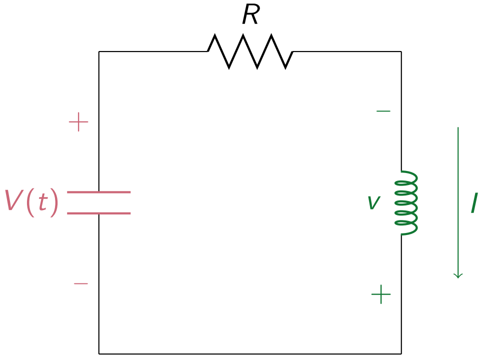
$$\frac{dV}{dt} = \frac{-i}{C}$$

v: voltage drop
I: current
L: inductance

$$\frac{dI}{dt} = \frac{-v}{L}$$

Changing:
V, v: voltage drops
I: current

Constant:
R: resistance
C: capacitance
L: inductance



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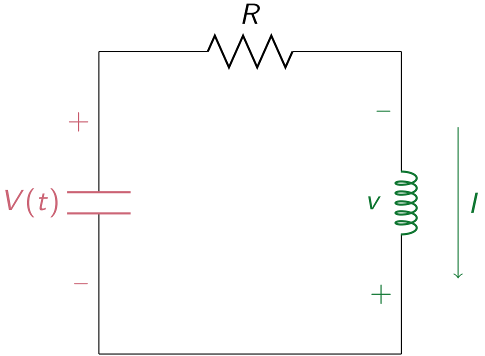
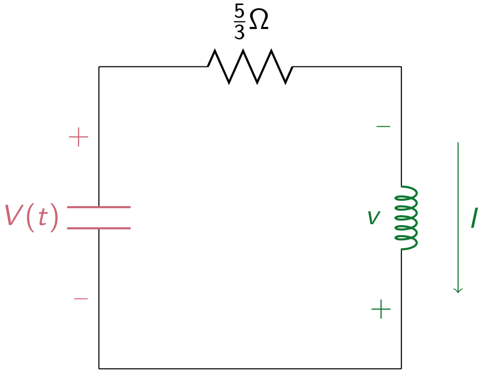


Goal: find equations for $V(t)$ (voltage across capacitor) and $I(t)$ (current through inductor).

Changing:
V, v: voltage drops
I: current

Constant:
$R = \frac{5}{3}$: resistance
$C = \frac{1}{2}$: capacitance
$L = \frac{1}{3}$: inductance



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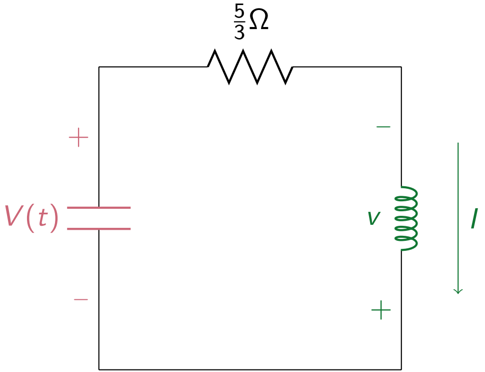

Goal: find equations for $V(t)$ (voltage across capacitor) and $I(t)$ (current through inductor).

Kirkhoff: $\quad -v - V + \frac{5}{3}I = 0 \qquad \Rightarrow \qquad v = \frac{5}{3}I - V$

Differential Equations:

$$\frac{dV}{dt} = \frac{-I}{C} = -2I \qquad \frac{dI}{dt} = \frac{-v}{L} = 3\left(V - \frac{5}{3}I\right) = 3V - 5I$$

Differential Equations: $\frac{dV}{dt} = -2I, \qquad \frac{dI}{dt} = 3V - 5I$

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$$\begin{bmatrix} V(t) \\ I(t) \end{bmatrix}' = \begin{bmatrix} -2I \\ 3V - 5I \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$$

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$\lambda_1 = -2$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ $\lambda_2 = -3$ $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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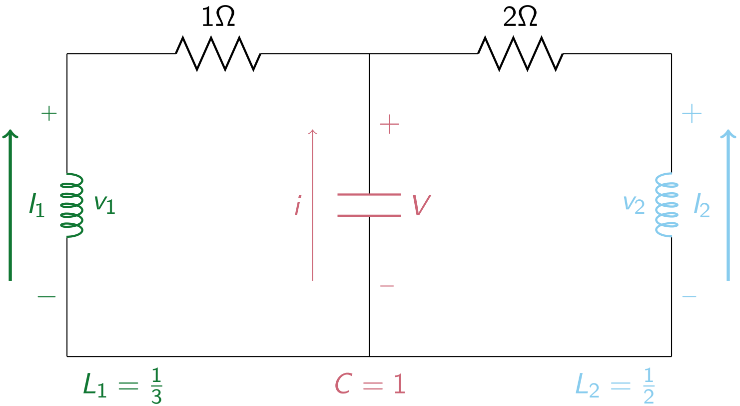
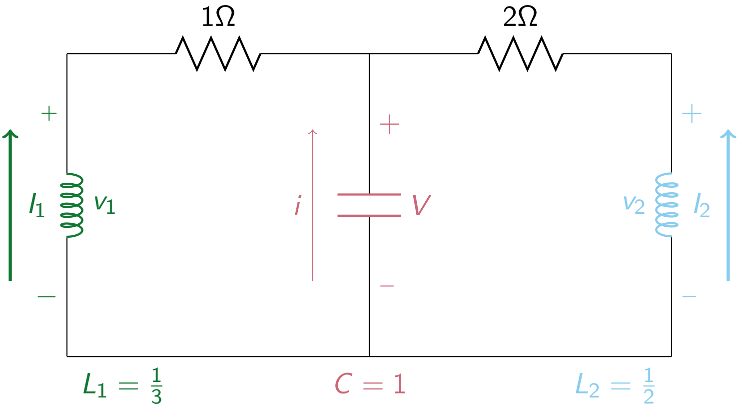
$$\begin{bmatrix} V(t) \\ I(t) \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + 2c_2 e^{-3t} \\ c_1 e^{-2t} + 3c_2 e^{-3t} \end{bmatrix}$$



Want to find: $I_1(t)$, $I_2(t)$, and $V(t)$.



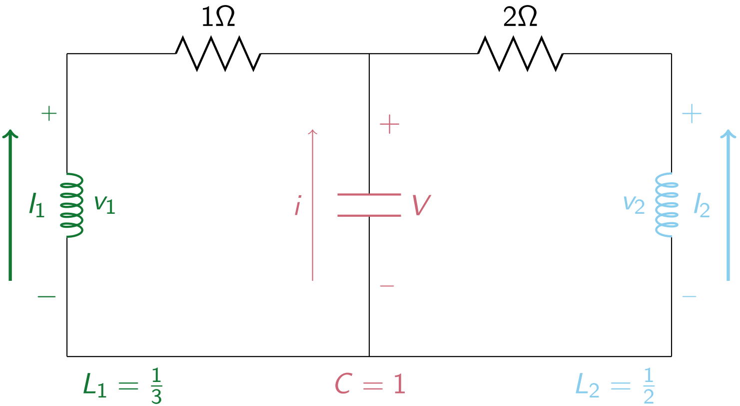
Want to find: $I_1(t)$, $I_2(t)$, and $V(t)$.

Differential:

$\frac{dI_1}{dt} = \frac{-v_1}{L_1} = -3v_1$

$\frac{dI_2}{dt} = \frac{-v_2}{L_2} = -2v_2$

$\frac{dV}{dt} = \frac{-i}{C} = -i$



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$$\frac{dI_1}{dt} = \frac{-v_1}{L_1} = -3v_1$$
$$\frac{dI_2}{dt} = \frac{-v_2}{L_2} = -2v_2$$
$$\frac{dV}{dt} = \frac{-i}{C} = -i$$

Kirkhoff:

$$-v_1 + 1I_1 + V = 0$$
$$-v_2 + 2I_2 + V = 0$$
$$i = -I_1 - I_2$$

Differential:

$\frac{dI_1}{dt} = \frac{-v_1}{L_1} = -3v_1$

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Kirkhoff:

$-v_1 + 1I_1 + V = 0$

$-v_2 + 2I_2 + V = 0$

$i = -I_1 - I_2$

Combined:

$\frac{dI_1}{dt} = -3v_1 = -3(I_1 + V) = -3I_1 - 3V$

$\frac{dI_2}{dt} = -2v_2 = -2(2I_1 + V) = -4I_2 - 2V$

$\frac{dV}{dt} = -I = I_1 + I_2$

$$\begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}$$

Now we need the eigenvalues and eigenvectors of the matrix.

$$\begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}$$

$$\lambda_1 \approx -1.6 - 1.5i, \qquad \mathbf{x}_1 \approx \begin{bmatrix} 1 \\ 0.46 - 0.13i \\ -.46 + .49i \end{bmatrix}$$

$$\lambda_2 \approx -1.6 + 1.5i, \qquad \mathbf{x}_2 \approx \begin{bmatrix} 1 \\ 0.46 + 0.13i \\ -.46 - .49i \end{bmatrix}$$

$$\lambda_3 \approx -3.7, \qquad \mathbf{x}_3 \approx \begin{bmatrix} 1 \\ -1.9 \\ 0.25 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}$$

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$$c_1 e^{-1.6t} e^{-1.5it} \mathbf{x}_1 + c_2 e^{-1.6t} e^{1.5.it} \mathbf{x}_2 + c_3 e^{-3.7t} \mathbf{x}_3$$

$$\begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}$$

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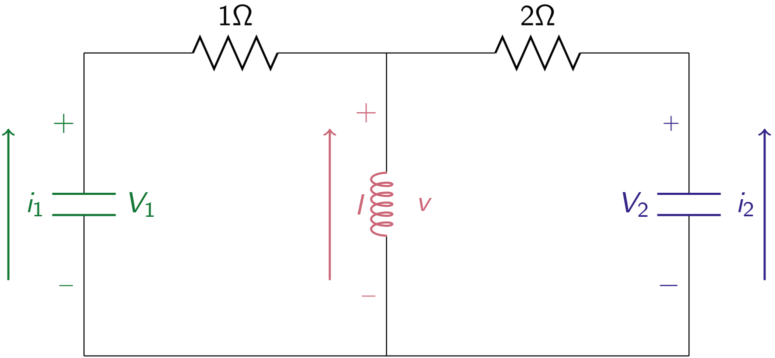
$$c_1 e^{-1.6t} e^{-1.5it} \mathbf{x}_1 + c_2 e^{-1.6t} e^{1.5.it} \mathbf{x}_2 + c_3 e^{-3.7t} \mathbf{x}_3$$

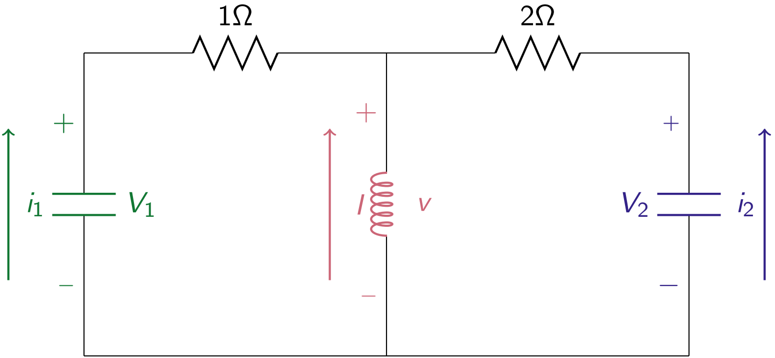
Regardless of initial conditions, the solutions will decay.
They may or may not oscillate while decaying.



Capacitances: $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{3}$

Inductance: $L = \frac{1}{3}$

Find $V_1(t)$, $V_2(t)$, and $I(t)$.



Capacitances: $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{3}$
Inductance: $L = \frac{1}{3}$
Find $V_1(t)$, $V_2(t)$, and $I(t)$.
Differential Equations:

$$\frac{dV_1}{dt} = -\frac{i_1}{C_1} \qquad \frac{dV_2}{dt} = -\frac{i_2}{C_2} \qquad \frac{dI}{dt} = -\frac{v}{L}$$

We use Kirkhoff's Laws to solve for i_1, i_2, and v in terms of V_1, V_2, and I.

Left Loop: $-V_1 + i_1 + v = 0$

Right Loop: $-V_2 + 2i_2 + v = 0$

Inductor (like a current source): $i_1 + i_2 = -I_1$

$$\overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}}^{i_1 \quad i_2 \quad v}\left|\begin{matrix} V_1 \\ V_2 \\ -I \end{matrix}\right. \xrightarrow{\text{row reduce}} \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\;\middle|\; \begin{matrix} \frac{1}{3}(V_1 - V_2 - 2I) \\ \frac{1}{3}(-V_1 + V_2 - I) \\ \frac{1}{3}(2V_1 + V_2 + 2I) \end{matrix}\right]$$

$$\frac{dV_1}{dt} = -\frac{i_1}{C_1} = -6\left(\frac{1}{3}\right)(V_1 - V_2 - 2I) = -2V_1 + 2V_2 + 4I$$

$$\frac{dV_2}{dt} = -\frac{i_2}{C_2} = -3\left(\frac{1}{3}\right)(-V_1 + V_2 - I) = V_1 - V_2 + I$$

$$\frac{dI}{dt} = -\frac{v}{L} = -3\left(\frac{1}{3}\right)(2V_1 + V_2 + 2I) = -2V_1 - V_2 - 2I$$

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$$\frac{dI}{dt} = -\frac{v}{L} = -3\left(\frac{1}{3}\right)(2V_1 + V_2 + 2I) = -2V_1 - V_2 - 2I$$

In vector differential equation form:

$$\begin{bmatrix} V_1 \\ V_2 \\ I \end{bmatrix}' = \begin{bmatrix} -2 & 2 & 4 \\ 1 & -1 & 1 \\ -2 & -1 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I \end{bmatrix}$$

Eigenvalues and Eigenvectors:

$$\lambda_1 = -2,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \lambda_2 = \overline{\lambda_3} = \frac{-3 + 3\sqrt{3}i}{2},\ \mathbf{x}_2 = \overline{\mathbf{x}_3} = \begin{bmatrix} 4 \\ 1 - \sqrt{3}i \\ 2\sqrt{3}i \end{bmatrix}$$

Solutions all decay; some oscillate.