Outline

Week 4: Solving Linear Systems

Course Notes: 3.2, 3.3, 3.4

Goals: Learn the method of Gaussian elimination to efficiently solve linear systems; describe infinite families of solutions as parametric equations; use properties of associated homogeneous systems.

Row Operations

- Multiply one row by a scalar
- Add a multiple of one row to another
- Interchange rows

These operations will change the system of equations, but will not change the *solutions* of the system.

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$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 34 \\ 1 & 1 & 0 & 13 \\ 0 & 1 & 2 & 11 \end{array}\right]$$

Quick Notation

Row Echelon Form: the position of the first non-zero entry in a row strictly increases from one row to the row below it.

$$\left[\begin{array}{cccccc|c} 2 & 3 & 5 & 2 & 5 & 9 & 13 \\ 0 & 5 & 4 & 3 & 4 & 0 & 11 \\ 0 & 0 & 0 & 3 & 2 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced Row Echelon Form: the first non-zero entry in every row is a 1, and is the only non-zero entry in its column; the position of the first non-zero entry in a row strictly increases with every row.

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 2 & 0 & 0 & 13 \\ 0 & 1 & 4 & 5 & 0 & 0 & 11 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Gaussian Elimination

Use row operations to change this system to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 2 & 2 & -6 & -10 \\ 3 & -2 & 1 & -5 \\ -2 & 6 & 2 & 0 \end{array}\right]$$

Gaussian Elimination

Use row operations to change this system to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 8 \\ 5 & 3 & 6 & 10 \end{array}\right]$$

(Try to do it using at most 6 row operations!)

Gaussian Elimination

Use row operations to change this system to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 15 \\ 4 & 4 & 0 & 36 \\ 3 & -1 & 6 & -31 \end{array}\right]$$

(Try to do it using at most 8 row operations!)

Gaussian Elimination

Use row operations to solve this system.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 6 \\ 5 & 10 & 3 & 3 & 28 \\ 2 & 4 & 1 & 2 & 15 \\ 3 & 6 & 2 & 3 & 21 \end{array}\right]$$

Gaussian Elimination

Use row operations to solve this system.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 6 \\ 5 & 10 & 3 & 3 & 28 \\ 2 & 4 & 1 & 2 & 15 \\ 3 & 6 & 2 & 3 & 21 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_2 = 5$, $x_3 = -3$, $x_4 = 4$.

Gaussian Elimination

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$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 6 \\ 5 & 10 & 3 & 3 & 28 \\ 2 & 4 & 1 & 2 & 15 \\ 3 & 6 & 2 & 3 & 21 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_2 = 5,\ x_3 = -3,\ x_4 = 4.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 - 2s \\ s \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Give a parametric equation for the solutions of this augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 17 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give a parametric equation for the solutions of this augmented matrix.

$$\left[\begin{array}{cccccc|c} 1 & 5 & 7 & 0 & 2 & 0 & 17 \\ 0 & 0 & 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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Equations: $x_1 + 5x_2 + 7x_3 + 2x_5 = 17$, $x_4 + 3x_5 = -4$, and $x_6 = -4$.

Give a parametric equation for the solutions of this augmented matrix.

$$\left[\begin{array}{cccccc|c} 1 & 5 & 7 & 0 & 2 & 0 & 17 \\ 0 & 0 & 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Equations: $x_1 + 5x_2 + 7x_3 + 2x_5 = 17$, $x_4 + 3x_5 = -4$, and $x_6 = -4$.
Parameters: $x_2 = r$, $x_3 = s$, and $x_5 = t$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 17 - 5r - 7s - 2t \\ r \\ s \\ -4 - 3t \\ t \\ -4 \end{bmatrix} = \begin{bmatrix} 17 \\ 0 \\ 0 \\ -4 \\ 0 \\ -4 \end{bmatrix} + r \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

Give a parametric equation for the solutions of this augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give a parametric equation for the solutions of this augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

no solution

Give a parametric equation for the solutions of this augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 5 & 5 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

Fill in the Table

	more variables $\left[\begin{array}{ccc\|c} * & * & * & * \\ * & * & * & * \end{array}\right]$	square $\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \end{array}\right]$	more equations $\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$
No Solutions			
One Solution			
Infinitely Many			

Fill in the Table

	more variables $\left[\begin{array}{ccc\|c} * & * & * & * \\ * & * & * & * \end{array}\right]$	square $\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \end{array}\right]$	more equations $\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$
No Solutions	$\left[\begin{array}{ccc\|c} 0 & 0 & 0 & 1 \end{array}\right]$		
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Fill in the Table

	more variables	square	more equations
	$\left[\begin{array}{ccc\|c} * & * & * & * \\ * & * & * & * \end{array}\right]$	$\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \end{array}\right]$	$\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$
No Solutions	$\left[\begin{array}{ccc\|c} 0 & 0 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{c\|c} 0 & 1 \end{array}\right]$	
One Solution			
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	more variables	square	more equations
	$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \end{array}\right]$	$\left[\begin{array}{cc|c} * & * & * \\ * & * & * \end{array}\right]$	$\left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$
No Solutions	$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{c|c} 0 & 1 \end{array}\right]$	$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$
One Solution			
Infinitely Many			

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	more variables	square	more equations
	$\left[\begin{array}{ccc\|c} * & * & * & * \\ * & * & * & * \end{array}\right]$	$\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \end{array}\right]$	$\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$
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One Solution	X		
Infinitely Many			

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	$\left[\begin{array}{ccc\|c} * & * & * & * \\ * & * & * & * \end{array}\right]$	$\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \end{array}\right]$	$\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$
No Solutions	$\left[\begin{array}{ccc\|c} 0 & 0 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{c\|c} 0 & 1 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$
One Solution	X	$\left[\begin{array}{cc\|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$	
Infinitely Many			

Fill in the Table

	more variables	square	more equations
	$\left[\begin{array}{ccc\|c} * & * & * & * \\ * & * & * & * \end{array}\right]$	$\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \end{array}\right]$	$\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$
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Rank and Solutions

How many solutions?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

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The *rank* of a matrix is the number of non-zero rows in the matrix obtained after reducing it.

Rank and Solutions

How many solutions?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

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The *rank* of a matrix is the number of non-zero rows in the matrix obtained after reducing it. If a matrix has rank r and n rows, with $n > r$, then the solution will require $\boxed{}$ parameters (provided a solution exists at all).

Rank and Solutions

How many solutions?

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Homogeneous Systems

System of equations:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Associated homogeneous system of equations:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Give a solution to this equation.

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Give a solution to this equation.

Suppose $\mathbf{a}$ and $\mathbf{b}$ are solutions to a homogeneous system of equations.

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Give a solution to this equation.

Suppose $\mathbf{a}$ and $\mathbf{b}$ are solutions to a homogeneous system of equations.

Then $\mathbf{a} + \mathbf{b}$ is also a solution.

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Give a solution to this equation.

Suppose $\mathbf{a}$ and $\mathbf{b}$ are solutions to a homogeneous system of equations.

Then $\mathbf{a} + \mathbf{b}$ is also a solution.

Also, $c\mathbf{a}$ is a solution for any scalar c.

Connection to Non-homogeneous Systems

Suppose A is an augmented matrix, and A_0 is its associated homogeneous system.
If **a** and **b** are solutions to A, then $\boxed{}$ is a solution to A_0.

$$A = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 13 \\ 11 & 3 & -1 & 13 & 42 \\ -7 & -2 & 1 & -8 & -26 \\ 4 & 1 & 0 & 5 & 16 \end{array}\right]$$

$$A_0 = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 0 \\ 11 & 3 & -1 & 13 & 0 \\ -7 & -2 & 1 & -8 & 0 \\ 4 & 1 & 0 & 5 & 0 \end{array}\right]$$

Connection to Non-homogeneous Systems

Suppose A is an augmented matrix, and A_0 is its associated homogeneous system.
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One solution: $[1, 2, 1, 2]$.

$$A_0 = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 0 \\ 11 & 3 & -1 & 13 & 0 \\ -7 & -2 & 1 & -8 & 0 \\ 4 & 1 & 0 & 5 & 0 \end{array}\right]$$

One solution: $[1, 1, 1, -1]$.

Connection to Non-homogeneous Systems

Suppose A is an augmented matrix, and A_0 is its associated homogeneous system.

If $\mathbf{a}$ and $\mathbf{b}$ are solutions to A, then $\boxed{\mathbf{a} - \mathbf{b}}$ is a solution to A_0.

Given one solution $\mathbf{q}$ to A, every solution to A can be written in the form $\mathbf{q} + \mathbf{a}$ for some solution $\mathbf{a}$ of A_0.

Connection to Non-homogeneous Systems

$$A = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 13 \\ 11 & 3 & -1 & 13 & 42 \\ -7 & -2 & 1 & -8 & -26 \\ 4 & 1 & 0 & 5 & 16 \end{array}\right]$$

One solution: $[1, 2, 1, 2]$.

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Solutions: $s[1, 1, 1, -1]$.

Connection to Non-homogeneous Systems

$$A = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 13 \\ 11 & 3 & -1 & 13 & 42 \\ -7 & -2 & 1 & -8 & -26 \\ 4 & 1 & 0 & 5 & 16 \end{array}\right]$$

One solution: $[1, 2, 1, 2]$.

$$A_0 = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 0 \\ 11 & 3 & -1 & 13 & 0 \\ -7 & -2 & 1 & -8 & 0 \\ 4 & 1 & 0 & 5 & 0 \end{array}\right]$$

Solutions: $s[1, 1, 1, -1]$.

What is the rank of the top matrix?

Connection to Non-homogeneous Systems

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One solution: $[1, 2, 1, 2]$.

$$A_0 = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 0 \\ 11 & 3 & -1 & 13 & 0 \\ -7 & -2 & 1 & -8 & 0 \\ 4 & 1 & 0 & 5 & 0 \end{array}\right]$$

Solutions: $s[1, 1, 1, -1]$.

What is the rank of the top matrix?

Are the rows of the top matrix linearly independent?

Rank and Linear Independence

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 1 & -1 & -3 & 0 \end{array}\right] \quad R_4 \to R_4 - R_3 + R_2 + 2R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank and Linear Independence

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 1 & -1 & -3 & 0 \end{array}\right] \quad R_4 \to R_4 - R_3 + R_2 + 2R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_4 = R_3 - R_2 - 2R_2$

Rank and Linear Independence

Suppose this matrix has rows that are linearly independent:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

What is its reduced form going to be?

Rank and Linear Independence

Suppose this matrix has rows that are linearly independent:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

What is its reduced form going to be?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank and Linear Independence

Suppose this augmented matrix has rows that are linearly independent:

$$\left[\begin{array}{ccccc|c} * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \end{array}\right]$$

What will its solutions set be? (Point, line, plane, etc.)

Rank and Linear Independence

Suppose this augmented matrix has rows that are linearly independent:

$$\left[\begin{array}{ccccc|c} * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \end{array}\right]$$

What will its solutions set be? (Point, line, plane, etc.)

A line: $s\mathbf{a}$.

Rank and Linear Independence

Suppose this augmented matrix has rows that are linearly independent:

$$\left[\begin{array}{ccccc|c} * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \end{array}\right]$$

What will its solutions set be? (Point, line, plane, etc.)

A line: $s\mathbf{a}$.

Why not a line like this: $\mathbf{q} + s\mathbf{a}$

Rank and Linear Independence

Suppose this matrix has rows that are linearly independent:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

What will its reduced form be?

Rank and Linear Independence

Suppose this matrix has rows that are linearly independent:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

What will its reduced form be?

Trick question: you can't have four linearly independent vectors in $\mathbf{R}^3$

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What will be the intersection of n linearly-independent $(n-1)$-dimensional spaces of this type in $\mathbb{R}^n$?

Linear Independence

Check this collection of vectors for linear independence:
$\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and $\mathbf{c} = [c_1, c_2, c_3]$.

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Recall

Vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly independent if the equation

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n = \mathbf{0}$$

has only one solution, $s_1 = s_2 = \cdots = s_n = 0$.

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$$\begin{cases} a_1 s_1 + b_1 s_2 + c_1 s_3 & = 0 \\ a_2 s_1 + b_2 s_2 + c_2 s_3 & = 0 \\ a_3 s_1 + b_3 s_2 + c_3 s_3 & = 0 \end{cases}$$

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Vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly independent if the equation

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$$\left\{ \begin{array}{ll} a_1 s_1 + b_1 s_2 + c_1 s_3 & = 0 \\ a_2 s_1 + b_2 s_2 + c_2 s_3 & = 0 \\ a_3 s_1 + b_3 s_2 + c_3 s_3 & = 0 \end{array} \right. \rightarrow \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \end{array} \right]$$

$$\mathbf{a} = [1, 1, 1, 1], \qquad \mathbf{b} = [2, 1, 1, 1]$$

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$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

Write $\begin{bmatrix} 16 \\ 5 \\ 11 \end{bmatrix}$ as a linear combination of vectors from the set

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} \right\}$$

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$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 16 \\ 1 & 9 & 4 & 5 \\ 5 & 8 & 3 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 16 \\ 5 \\ 11 \end{bmatrix}$$

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Is the set linearly independent? Is it a basis of $\mathbb{R}^3$?

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 16 \\ 1 & 9 & 4 & 5 \\ 5 & 8 & 3 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

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Is the set linearly independent? Is it a basis?

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ -13 \end{bmatrix} \right\}$$

Is the set linearly independent? Is it a basis?

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ -13 \end{bmatrix} \right\}$$

Is the set linearly independent? Is it a basis?

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 6 \\ 9 \\ 0 \end{bmatrix} \right\}$$

Two points determine a line. How many points are needed to determine an n-th degree polynomial?

Remove one vector to make this set linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix} \right\}$$

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Suppose a vector $\mathbf{a}$ can be written as a linear combination of vectors in the set. Can you still write $\mathbf{a}$ as a linear combination of vectors in the set WITHOUT using the first vector?

NEXT TOPIC

Circuits!