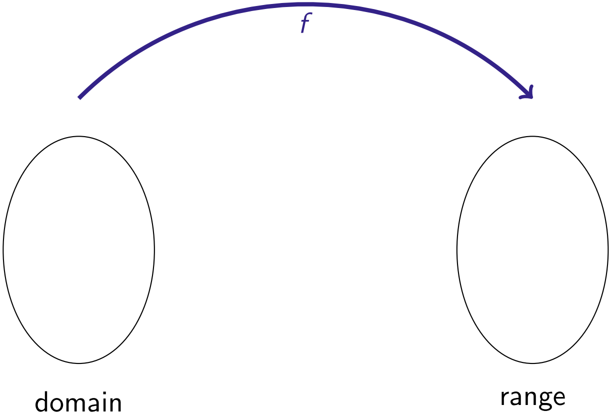
Outline

Week 7: Rotations, projections and reflections in 2D; matrix representation and composition of linear transformations; random walks; transpose.

Course Notes: 4.2, 4.3, 4.4

Goals: Understand that a linear transformation of a vector can always be achieved by matrix multiplication; use specific examples of linear transformations.

Functions and Transformations



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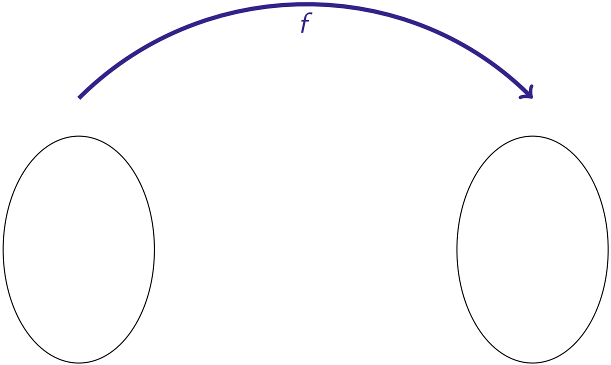
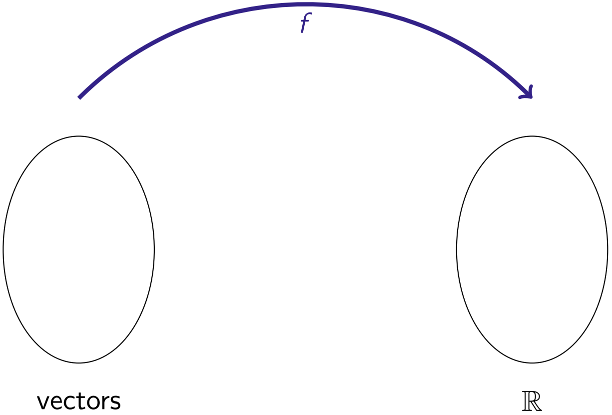


$$f(v) = \|v\|$$

Functions and Transformations



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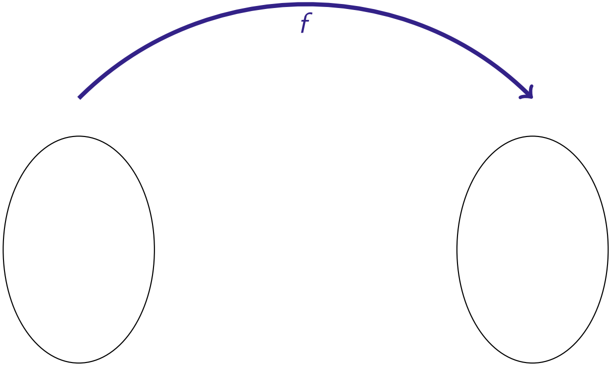

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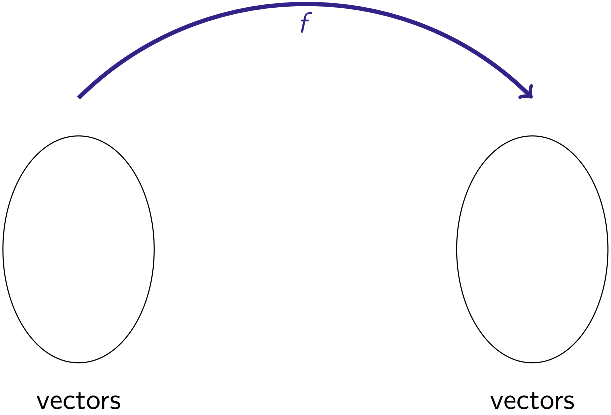

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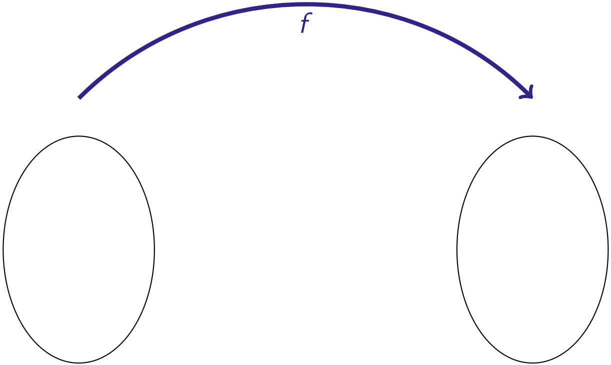

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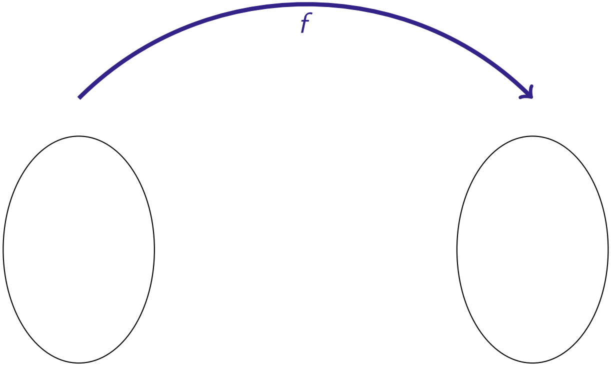

pairs of vectors in $\mathbb{R}^3$

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Linear Transformations

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A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s,

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Is every line ($f(x) = mx + b$) a linear transformation?

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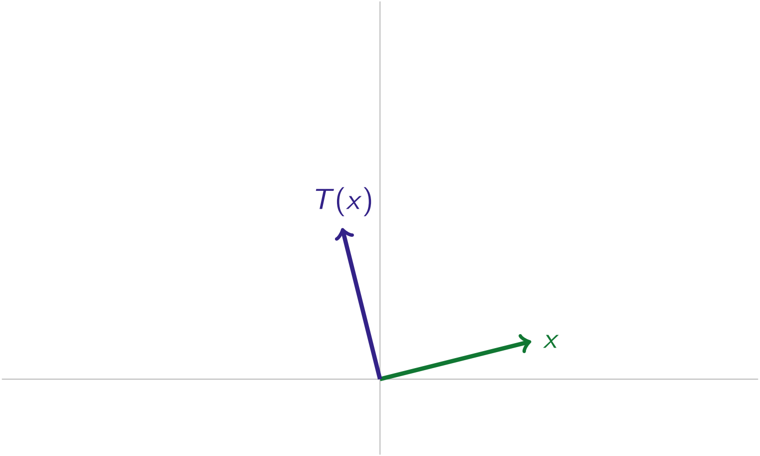

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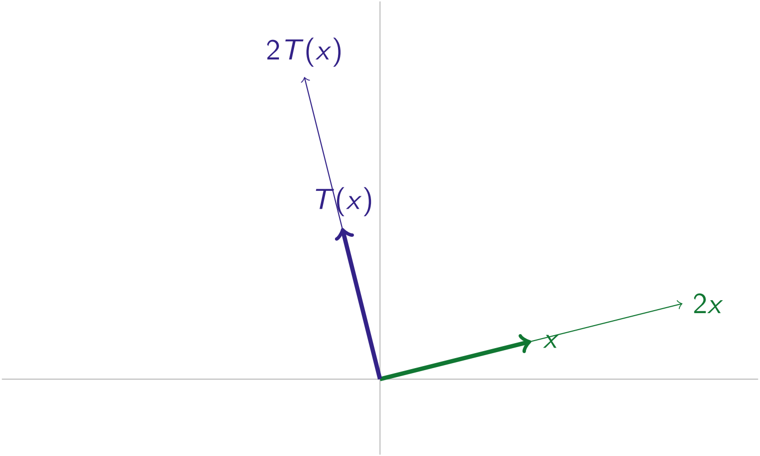

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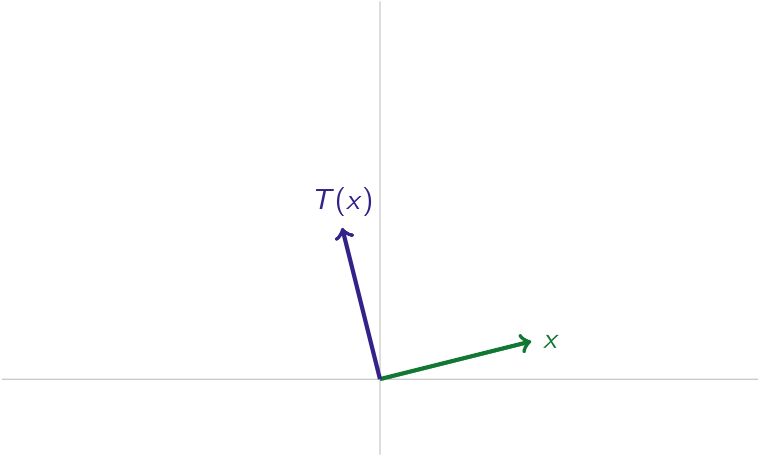

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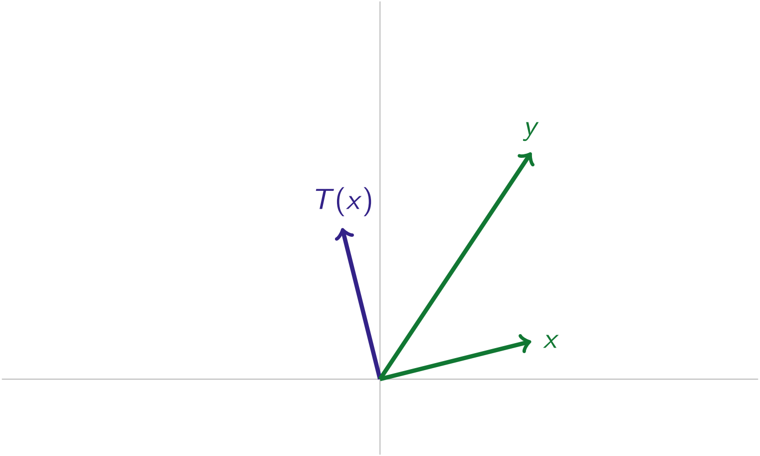

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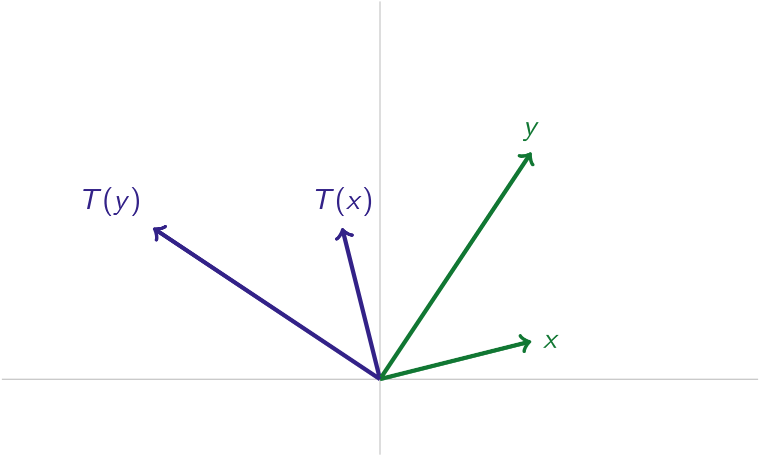

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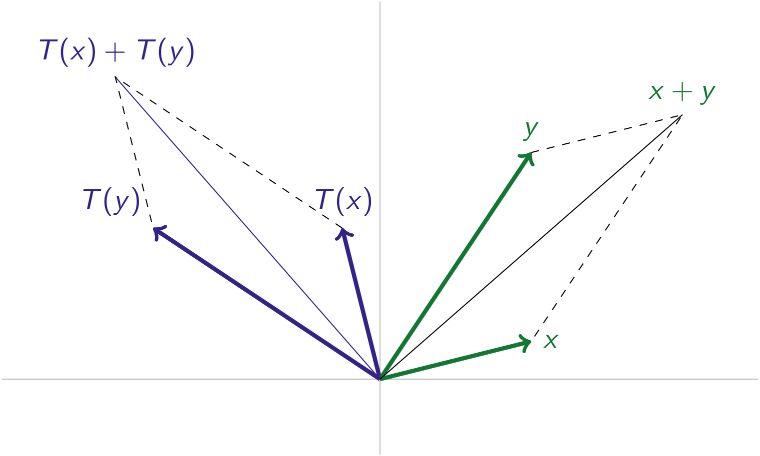

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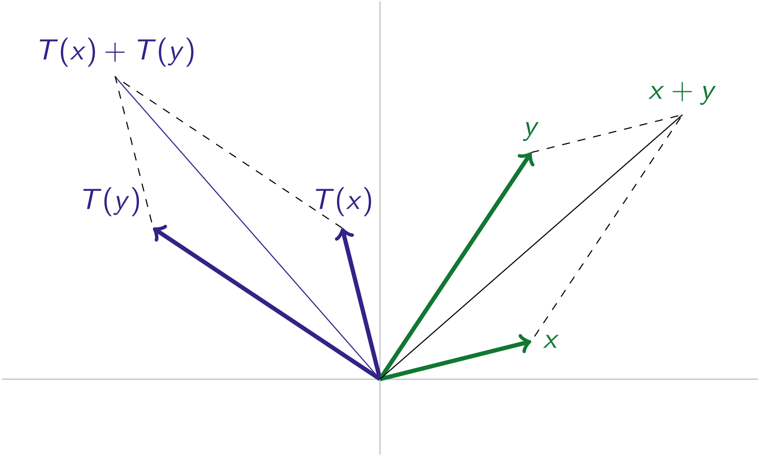
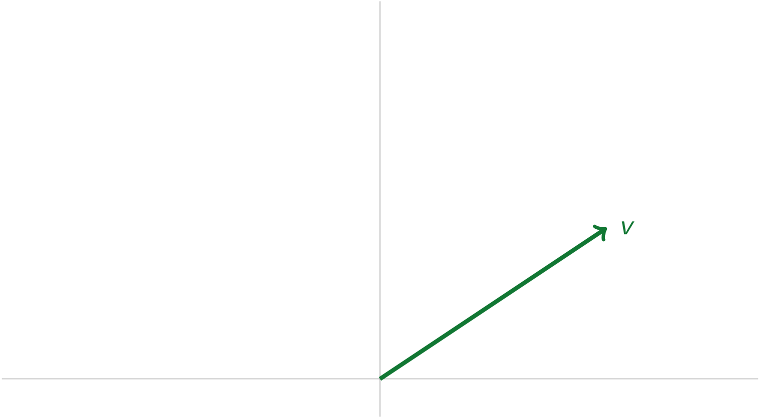


Rotation by a fixed angle is a linear transformation.

Computing Rotations



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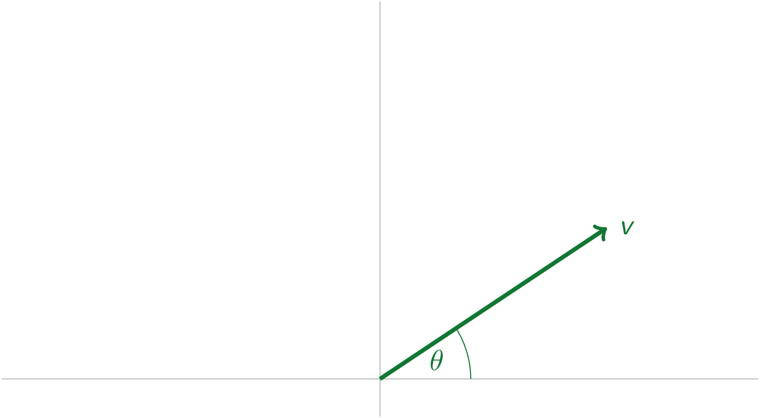

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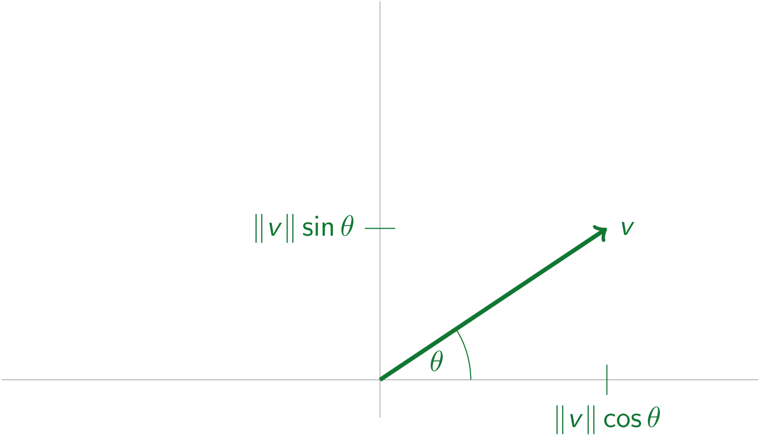

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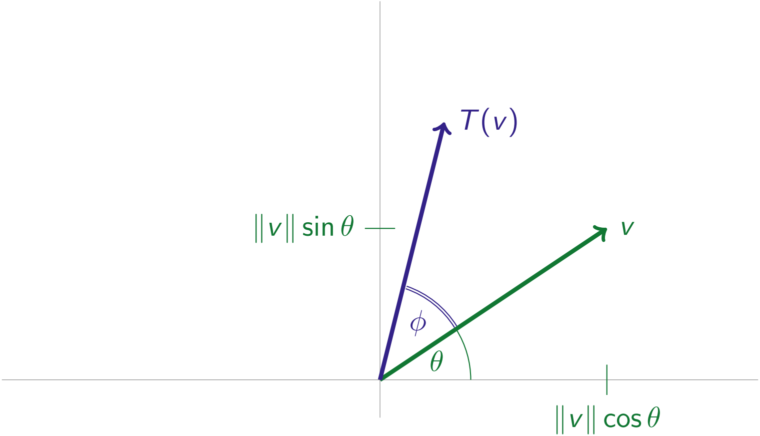

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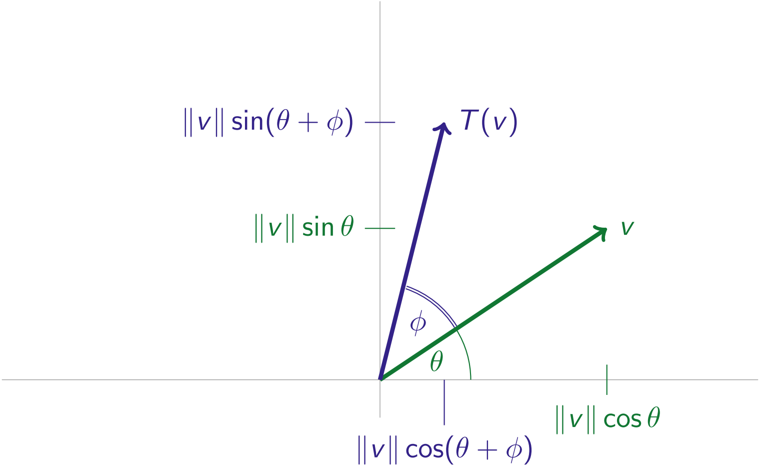

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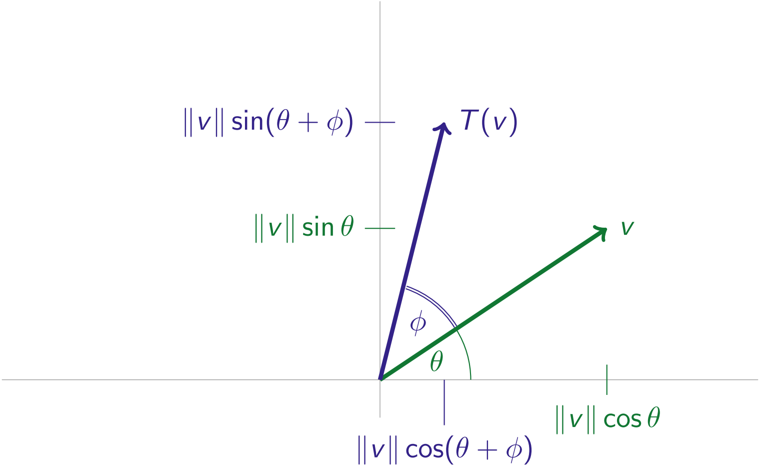
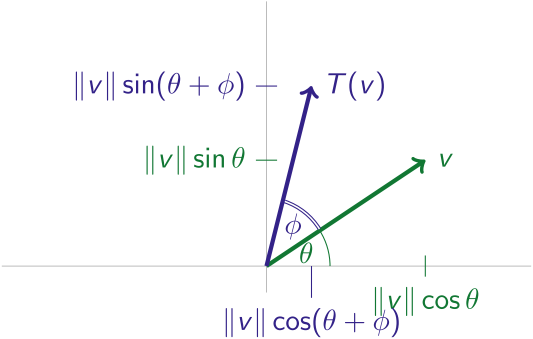


$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ $\qquad$ $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

Computing Rotations



$$v = [v_1, v_2]; \qquad T(v) = [x, y]$$

$$\begin{aligned} x &= \|v\| \cos(\theta+\phi) \\ &= \|v\|(\cos\theta\cos\phi - \sin\phi\sin\theta) \\ &= v_1\cos\phi - v_2\sin\phi \end{aligned} \qquad \begin{aligned} y &= \|v\| \sin(\theta+\phi) \\ &= \|v\|(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= v_1\sin\phi + v_2\cos\phi \end{aligned}$$

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The matrix is called a rotation matrix, Rot_ϕ

Computing Rotations

$$\mathsf{Rot}_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

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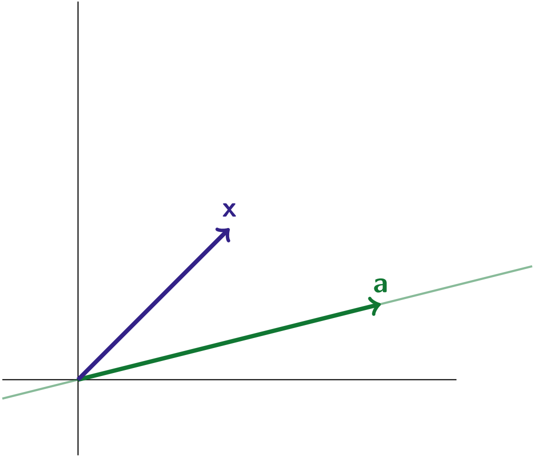
Are rotations commutative?

Projections

For a fixed vector $\mathbf{a}$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



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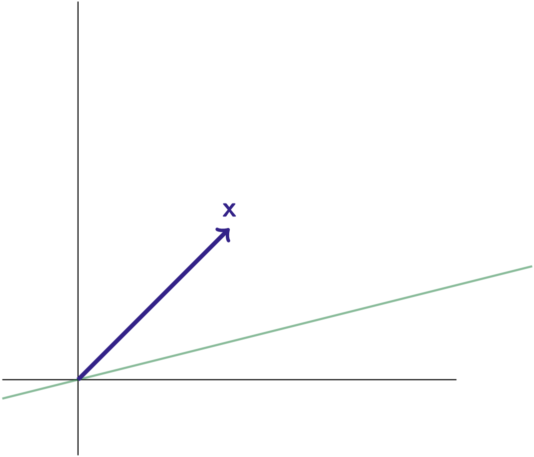

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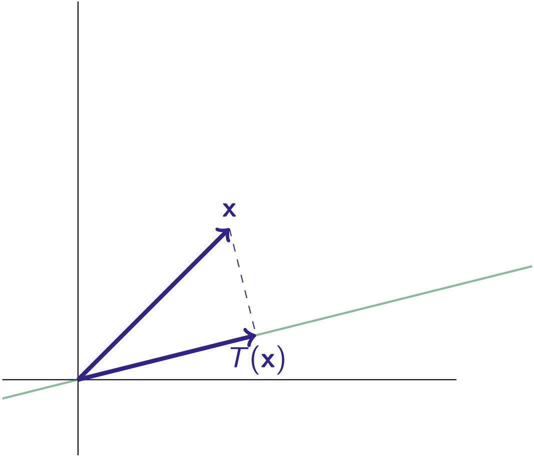

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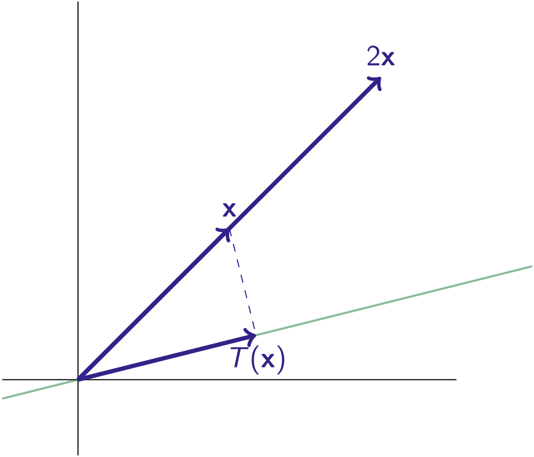

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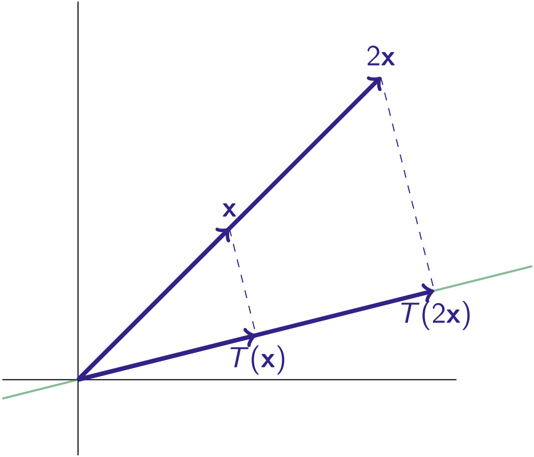

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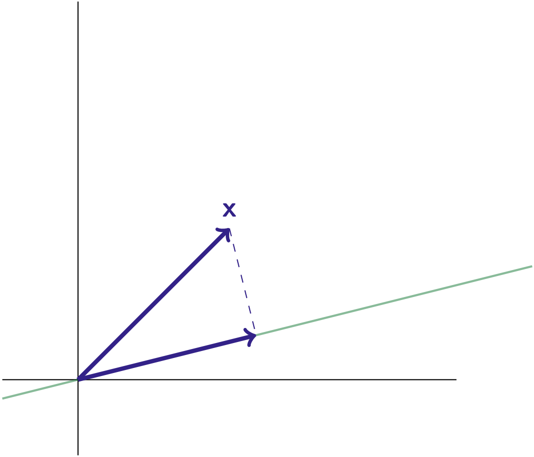

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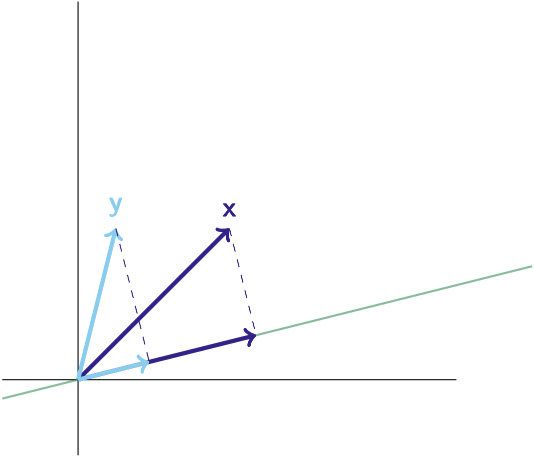

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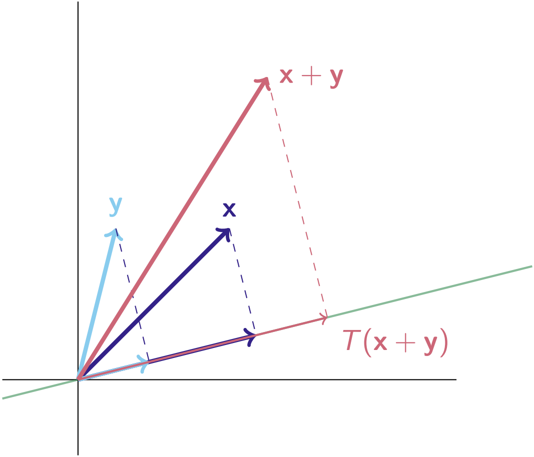

Computing Projections

Let $\mathbf{a} = [a_1, a_2]$ and $\mathbf{x} = [x_1, x_2]$.

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2}\begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Let $\mathbf{a} = [1, 1]$ and $\mathbf{x} = [2, 3]$. Calculate $proj_{\mathbf{a}}\mathbf{x}$ two ways.

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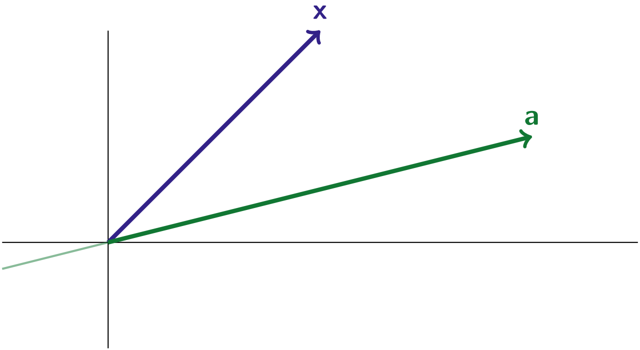
$$T(\mathbf{x}) = proj_{\mathbf{b}}\left(proj_{\mathbf{a}}\mathbf{x}\right)$$

Is the projection of a projection a projection?
(Is there a vector $\mathbf{c}$ so that $T(\mathbf{x}) = proj_{\mathbf{c}}\mathbf{x}$?)

Example: $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Reflections

For a fixed vector $\mathbf{a}$, let $Ref(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.



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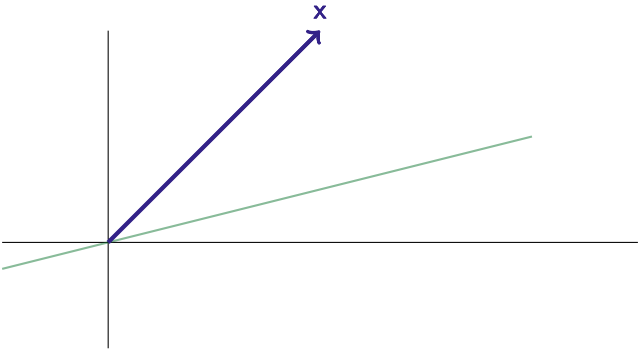

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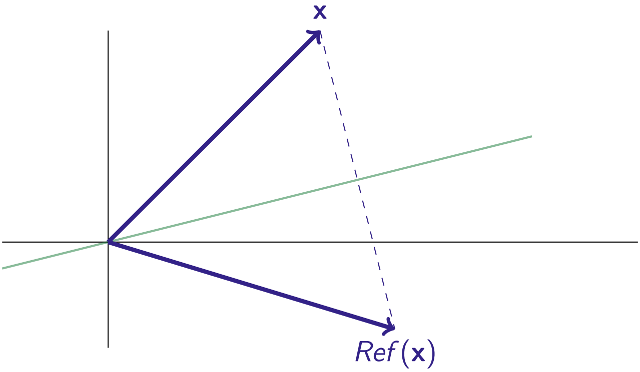

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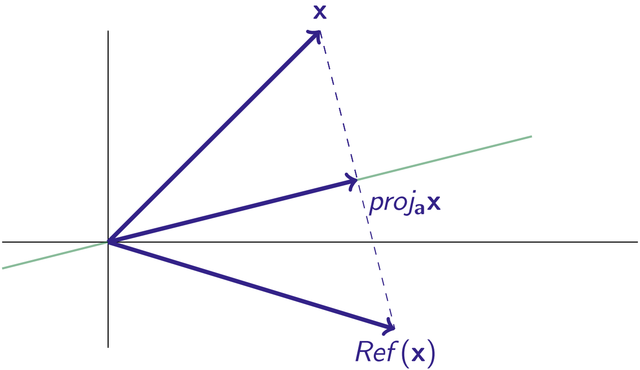

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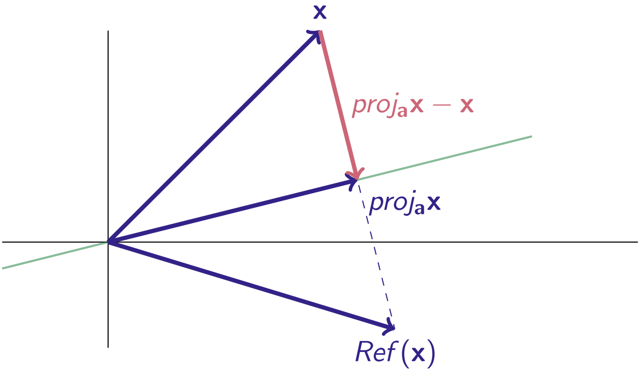

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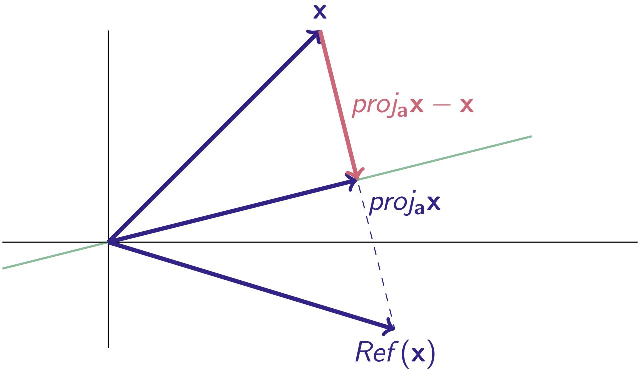

$$Ref(\mathbf{x}) = \mathbf{x} + 2(proj_{\mathbf{a}}\mathbf{x} - \mathbf{x}) = 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x}$$

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Identity:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$\begin{aligned} Ref(\mathbf{x}) &= 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x} \\ &= \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1 a_2}{a_1^2+a_2^2} \\ \frac{2a_1 a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Cleanup

$$Ref(\mathbf{x}) = \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1a_2}{a_1^2+a_2^2} \\ \frac{2a_1a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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If **a** is a unit vector, then $a_1^2 + a_2^2 = 1$. Then:

$$Ref(\mathbf{x}) = \begin{bmatrix} 2a_1^2 - 1 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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And if **a** makes angle θ with the x-axis, then $a_1 = \cos\theta$ and $a_2 = \sin\theta$, so:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Example: find the reflection of the vector $[2, 4]$ across the line through the origin that makes an angle of 15 degrees with the x-axis.

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Example: find the reflection of the vector $[2, 4]$ across the line through the origin that makes an angle of 15 degrees with the x-axis.

$$\begin{bmatrix} \cos(2(\pi/12)) & \sin(2(\pi/12)) \\ \sin(2(\pi/12)) & -\cos(2(\pi/12)) \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \cos(\pi/6) & \sin(\pi/6) \\ \sin(\pi/6) & -\cos(\pi/6)) \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \approx \begin{bmatrix} 3.7 \\ -2.4 \end{bmatrix}$$

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$$\begin{aligned} &\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi)) \end{bmatrix} = Rot_{2(\theta-\phi)} \end{aligned}$$

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Are reflections commutative?

Are reflections commutative with rotations?

Reflections and Rotations

Try the following with a cell phone or book:

1. Rotate 90 degrees clockwise
2. Flip 180 degrees vertically

Alternately:

1. Flip 180 degrees vertically
2. Rotate 90 degrees clockwise

Summary: Examples of Linear Transformations

To compute the rotation of the vector $\mathbf{x}$ by θ, multiply $\mathbf{x}$ by the matrix

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Theorem

Every linear transformation T that takes a vector as an input, and gives a vector as an output, is equivalent to a matrix multiplication.

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Extended Theorem

Suppose T is a linear transformation that transforms vectors of $\mathbb{R}^n$ into vectors of $\mathbb{R}^m$. If $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, then:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

General Linear Transformations

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + z\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

General Linear Transformations

$T : \mathbb{R}^n \to \mathbb{R}^n$ linear

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$$T : \mathbb{R}^n \to \mathbb{R}^n \qquad\qquad \text{linear}$$

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Examples

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}7\\7\end{bmatrix}$$

Give a matrix A so that $T(x) = Ax$ for every vector x in $\mathbb{R}^2$.

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Examples

Suppose T is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, where $T(x) = Ax$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Which vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has $T(x) = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$?

Which vector $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ has $T(y) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$?

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Characterize vectors that can come out ot T.

Random Walks: Another Use of Matrix Multiplication

- n states
- Fixed probability $p_{i,j}$ of moving to state i if you are in state j.

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use sampling to estimate properties of a large system

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3



Sleeping: `https://pixabay.com/en/penguin-linux-sleeping-animal-159784/`
Fishing: By Mimooh (Own work), via Wikimedia Commons
Playing: By Silvermoonlight217 `http://silvermoonlight217.deviantart.com/art/Penguin-Sledding-262107547`

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P: "transition matrix"

Random Walk Example: Falling Down

Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground.

Rob, https://www.flickr.com/photos/rh1985/22218233156

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Right ground			

Random Walk Example: Falling Down

Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1		
Rope			
Right ground			

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$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1		
Rope	0		
Right ground			

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$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1		
Rope	0		
Right ground	0		

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$\frac{\text{from}}{\text{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	
Rope	0		
Right ground	0		

Random Walk Example: Falling Down

Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	
Rope	0	0.94	
Right ground	0		

Random Walk Example: Falling Down

Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	
Rope	0	0.94	
Right ground	0	0.01	

Random Walk Example: Falling Down

Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground.

$\frac{from}{to}$	Left ground	Rope	Right ground
Left ground	1	0.05	0
Rope	0	0.94	
Right ground	0	0.01	

Random Walk Example: Falling Down

Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	0
Rope	0	0.94	0
Right ground	0	0.01	

Random Walk Example: Falling Down

Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	0
Rope	0	0.94	0
Right ground	0	0.01	1

Random Walk Example: Falling Down

Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	0
Rope	0	0.94	0
Right ground	0	0.01	1

Where are you after 100 steps?

Random Walk Example: Error Messages

Suppose you are using a buggy program. You start up without a problem.

- If you have never encountered an error message, your odds of encountering an error message with your next click are 0.01.
- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.
- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.
- After the third error message, you uninstall the program.

Random Walk Example: Error Messages

Suppose you are using a buggy program. You start up without a problem.

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- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.
- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.
- After the third error message, you uninstall the program.

Possible states: no errors; one error; two errors; three errors; uninstalled.

Random Walk Example

Suppose you are using a buggy program. You start up without a problem.

- If you have never encountered an error message, your odds of encountering an error message with your next click are 0.01.
- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.
- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.
- After the third error message, you uninstall the program.

Possible states: no errors; one error; two errors; three errors; uninstalled.

from / *to*	0	1	2	3	*u*
0	.99	0	0	0	0
1	.01	.95	0	0	0
2	0	.05	.9	0	0
3	0	0	.1	0	0
u	0	0	0	1	1

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 12 & 18 \\ 15 & 30 & 45 \end{bmatrix} \qquad BA = DNE$$

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 12 & 18 \\ 15 & 30 & 45 \end{bmatrix} \qquad BA = DNE$$

$$B^T A^T = \begin{bmatrix} 6 & 15 \\ 12 & 30 \\ 18 & 45 \end{bmatrix} \qquad AB = (B^T A^T)^T$$

Transpose

Previous example of noncommutativity of matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$$

Transpose

Previous example of noncommutativity of matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 5 & 0 \end{bmatrix}$$

Transpose and Dot Product

$$\mathbf{y} \cdot (A\mathbf{x}) = (A^T\mathbf{y}) \cdot \mathbf{x}$$

where A is an m-by-n matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

Transpose and Dot Product

$$\mathbf{y} \cdot (A\mathbf{x}) = (A^T\mathbf{y}) \cdot \mathbf{x}$$

where A is an m-by-n matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 9 \\ 1 \end{bmatrix} = 8 + 18 + 3 = 29$$

$$\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 9 \end{bmatrix} = -16 + 45 = 29$$

True or False?

Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $\mathbf{y} \cdot (A\mathbf{x}) = (A^T\mathbf{y}) \cdot \mathbf{x}$

- $\left(A^T\right)^T = A$
- $\left(\left(\left(\left(A^T\right)^T\right)^T\right)^T\right)^T = A$
- $(AB)\mathbf{x} = \left(\mathbf{x}^T B^T\right)^T A$
- $\mathbf{y} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (A^T\mathbf{y})$

True or False?

Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $\mathbf{y} \cdot (A\mathbf{x}) = (A^T \mathbf{y}) \cdot \mathbf{x}$

- $\left(A^T\right)^T = A$ true
- $\left(\left(\left(\left(A^T\right)^T\right)^T\right)^T\right)^T = A$
- $(AB)\mathbf{x} = \left(\mathbf{x}^T B^T\right)^T A$
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True or False?

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True or False?

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True or False?

Summary

- Transpose swaps rows and columns
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- $\left(A^T\right)^T = A$ — true
- $\left(\left(\left(\left(A^T\right)^T\right)^T\right)^T\right)^T = A$ — false
- $(AB)\mathbf{x} = \left(\mathbf{x}^T B^T\right)^T A$ — false
- $\mathbf{y} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (A^T \mathbf{y})$ — true