Outline

Week 2: Determinants, Cross Products, Lines, and Planes

Course Notes: 2.4-2.5

Goals: Introduce determinants and cross products, computationally and with geometric interpretations. Lines and planes.

Matrices!

$$\begin{bmatrix} 8 & 15 & -4 \\ 9 & -4 & 7 \\ 6 & 1 & 1 \\ -5 & -3 & 0 \end{bmatrix}$$

Determinants in Two and Three Dimensions

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

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Tricky way: ONLY in three dimensions:

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

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$$\det \begin{bmatrix} 3 & 2 & 5 \\ 5 & 7 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

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$$\det \begin{bmatrix} 3 & 2 & 5 \\ 5 & 7 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 7 \\ 1 & 10 & 5 \end{bmatrix}$$

Geometric Interpretation: Determinant in Two Dimensions

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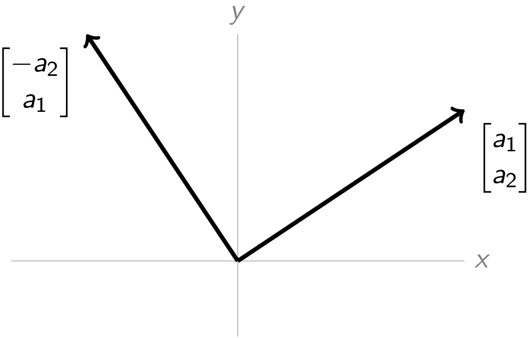

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so $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are scalar multiples of one another (parallel).

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Are the following determinants zero, or nonzero?

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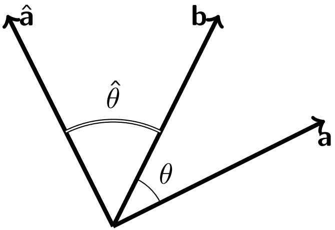

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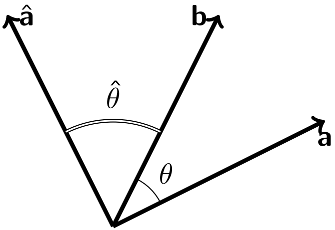

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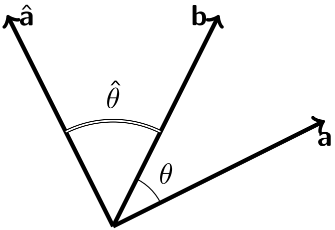

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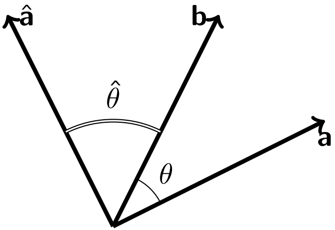

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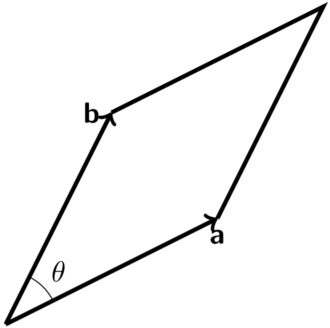

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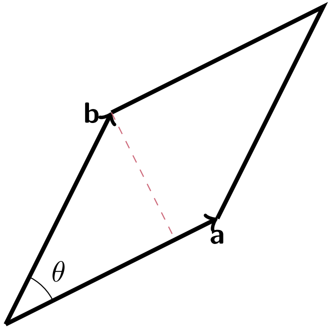

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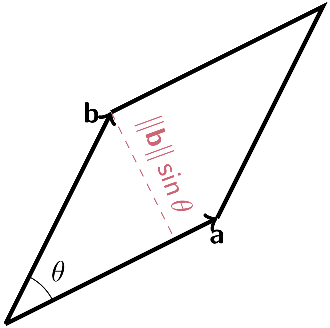

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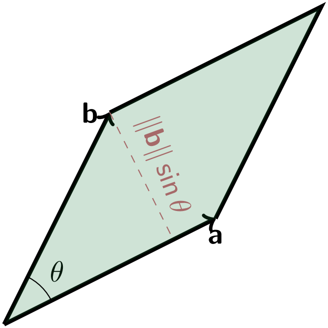

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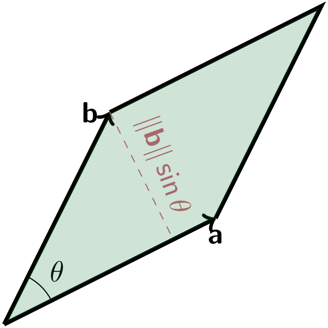

In general:

$$\left|\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right| = \text{ area of parallelogram spanned by } \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ and } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

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Example: Find the area of the parallelogram with one side given by $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and the other side $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

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$$\det\begin{bmatrix} 2 & 6 \\ -3 & 4 \end{bmatrix} = (2)(4) - (6)(-3)$$
$$= 8 + 18 = 26$$

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Silly Example: Find the area of the rectangle with corners $(0, 0)$, $(x, 0)$, $(0, y)$, and (x, y).

Cross Product

ONLY defined in three dimensions.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

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Mnemonic:

$$\begin{aligned} \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} &= \mathbf{i} \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \\ &= \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) + \mathbf{k}(a_1 b_2 - a_2 b_1) \\ &= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \end{aligned}$$

Practice

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Practice

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Not commutative!

Geometric Interpretation

1. $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and to $\mathbf{b}$.

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   Verify:

$$\begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

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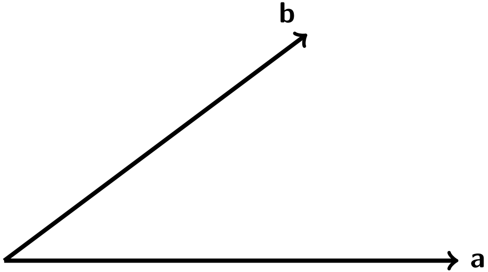
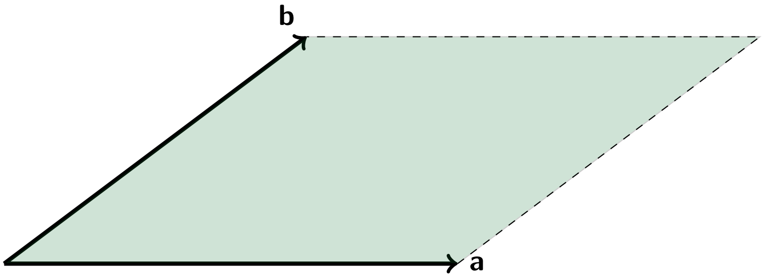
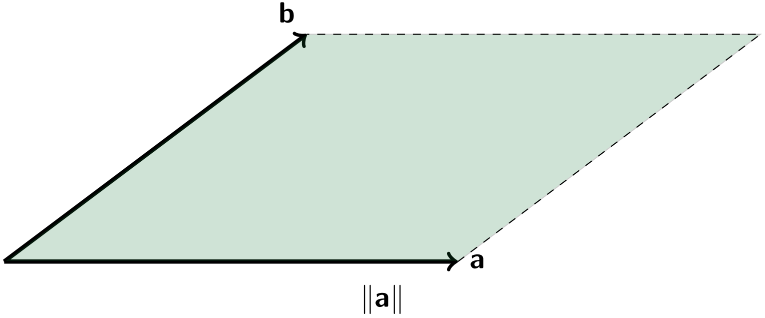
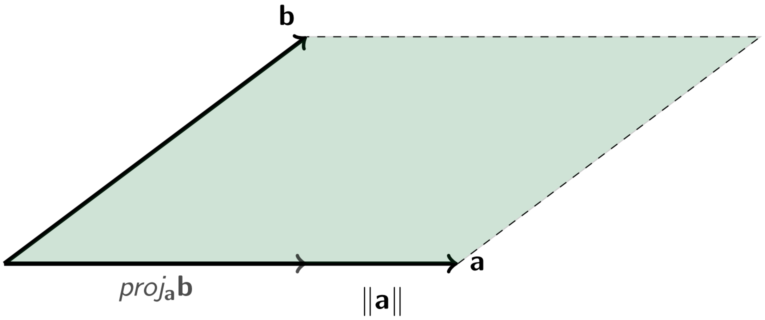
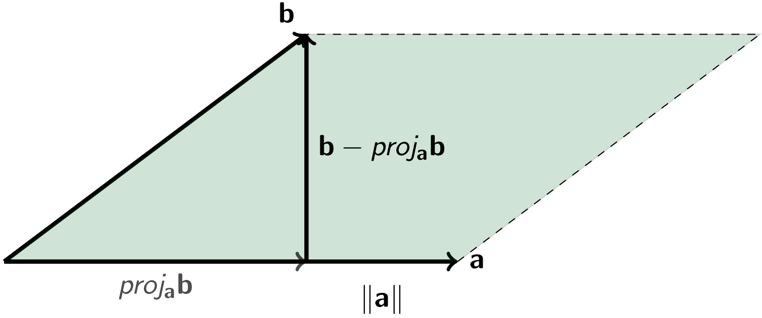
   Thus, $\sin\theta$ is positive, and $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.
3. The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ obey the *right hand rule*. That is, if you curl your fingers towards your palm from $\mathbf{a}$ to $\mathbf{b}$, your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.


b
a

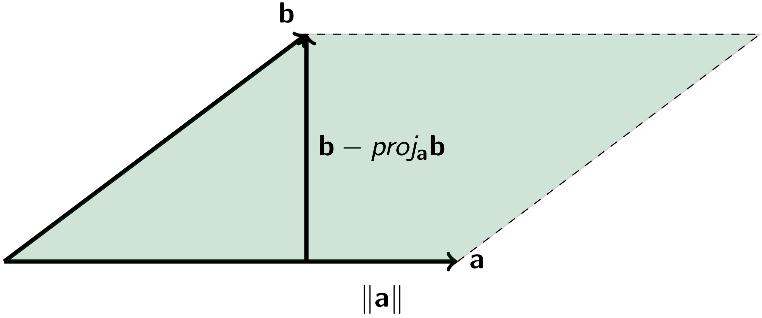

b
a


b
a
‖a‖


b
a
proj_a b
‖a‖



$\mathbf{b}$
$\mathbf{b} - proj_{\mathbf{a}}\mathbf{b}$
$\mathbf{a}$
$proj_{\mathbf{a}}\mathbf{b}$
$\|\mathbf{a}\|$



$$A = (\text{base})(\text{height}) = \|\mathbf{a}\|\|\mathbf{b} - proj_{\mathbf{a}}\mathbf{b}\|$$

$\|\mathbf{a} \times \mathbf{b}\|$ = area of parallelogram

$$A^2 = \|\mathbf{a}\|^2 \|\mathbf{b} - proj_{\mathbf{a}}\mathbf{b}\|^2$$

$\|\mathbf{a} \times \mathbf{b}\| =$ area of parallelogram

$$\begin{aligned} A^2 &= \|\mathbf{a}\|^2 \|\mathbf{b} - proj_{\mathbf{a}}\mathbf{b}\|^2 \\ &= \|\mathbf{a}\|^2 \left\| \mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right) \mathbf{a} \right\|^2 \end{aligned}$$

$\|\mathbf{a} \times \mathbf{b}\| = \text{area of parallelogram}$

$$
\begin{aligned}
A^2 &= \|\mathbf{a}\|^2 \|\mathbf{b} - proj_{\mathbf{a}}\mathbf{b}\|^2 \\
&= \|\mathbf{a}\|^2 \left\| \mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right) \mathbf{a} \right\|^2 \\
&= \|\mathbf{a}\|^2 \left(\mathbf{b} \cdot \mathbf{b} - 2(\mathbf{b} \cdot \mathbf{a}) \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right) + \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right)^2 \|\mathbf{a}\|^2 \right) \\
&= \|\mathbf{a}\|^2 \left(\|\mathbf{b}\|^2 - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2} \right)
\end{aligned}
$$

$\|\mathbf{a} \times \mathbf{b}\| =$ area of parallelogram

$$
\begin{aligned}
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&= \|\mathbf{a}\|^2 \left\| \mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right) \mathbf{a} \right\|^2 \\
&= \|\mathbf{a}\|^2 \left(\mathbf{b} \cdot \mathbf{b} - 2(\mathbf{b} \cdot \mathbf{a}) \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right) + \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right)^2 \|\mathbf{a}\|^2 \right) \\
&= \|\mathbf{a}\|^2 \left(\|\mathbf{b}\|^2 - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2} \right) \\
&= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= (a_1^2 + a_2^2 + a_3^2)^2 (b_1^2 + b_2^2 + b_3^2)^2 - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= \cdots = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= \|\mathbf{a} \times \mathbf{b}\|^2
\end{aligned}
$$

Find the Area of the Parallelograms

Find the area of the parallelogram spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$.

Find the area of the parallelogram spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

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Can you do that with a cross product, by imagining these vectors in $\mathbb{R}^3$?

Suppose a plane contains the points $P_1(3, 2, 2)$, $P_2(2, 2, 1)$, and $P_3(1, 1, 1)$. Find a normal vector to the plane. That is, find a vector that is perpendicular to every line on the plane.

Properties of Cross Product

1. $\mathbf{a} \times \mathbf{b} = \quad \mathbf{b} \times \mathbf{a}$

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   https://proofwiki.org/wiki/Lagrange's_Formula

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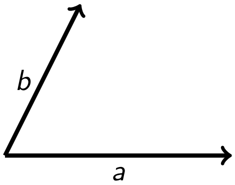

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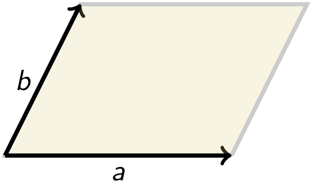

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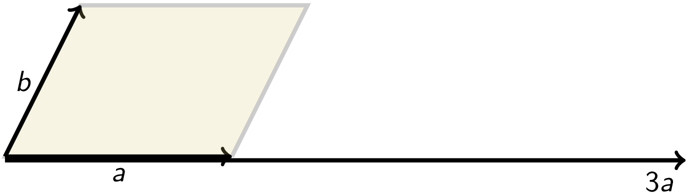

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5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ Is it also true that $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$?

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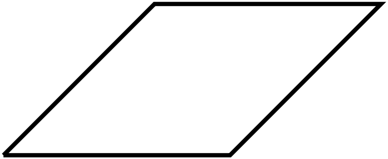
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ "triple product"

Parallelograms



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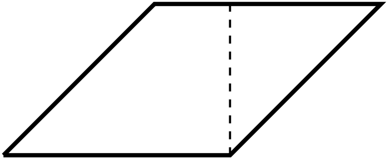

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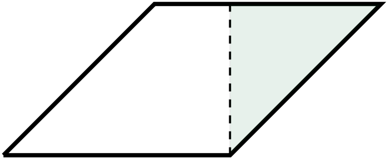

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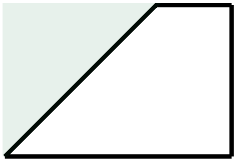

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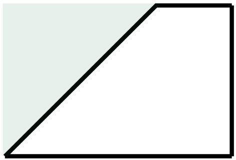
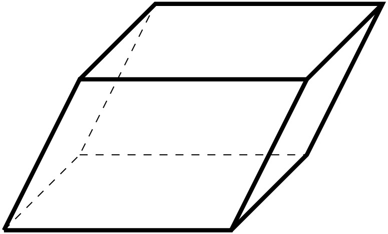


Area: (base)$\times$(height)

Parallelepipeds



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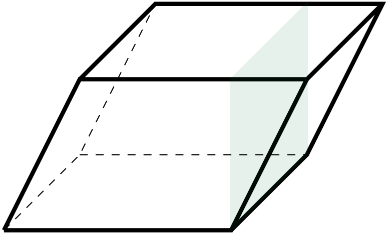

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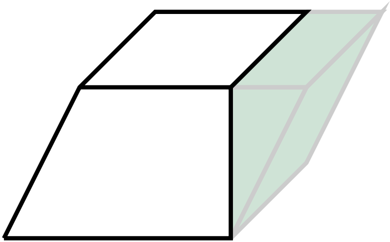
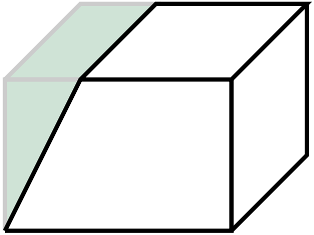
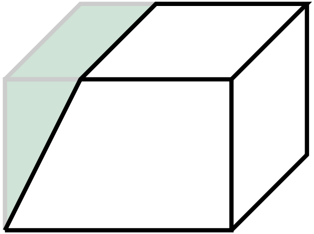


Parallelepipeds

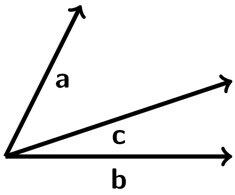


Parallelepipeds

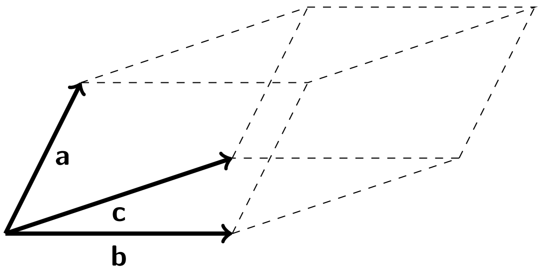


Volume: (area of base)$\times$(height)

Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$



Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$



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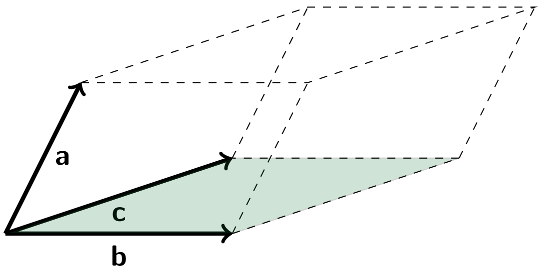

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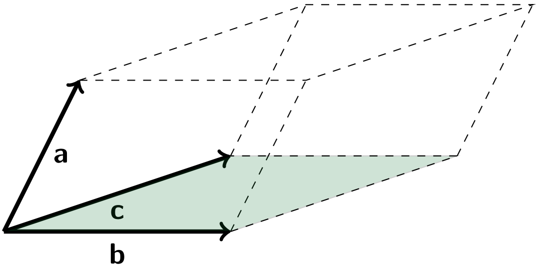
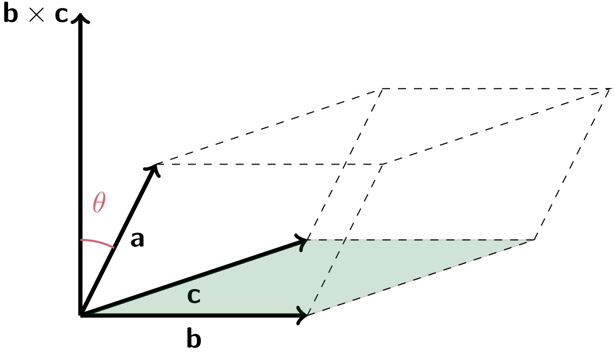


Area of base: $\|\mathbf{b} \times \mathbf{c}\|$

Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$



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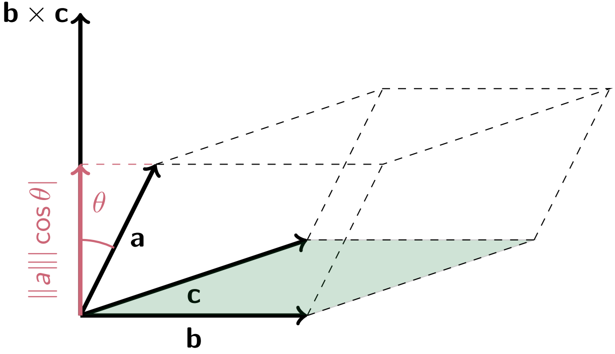

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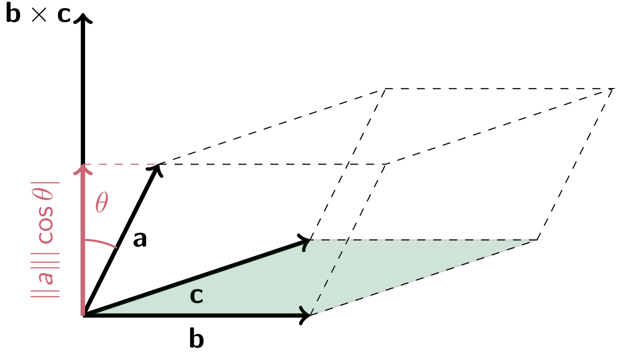
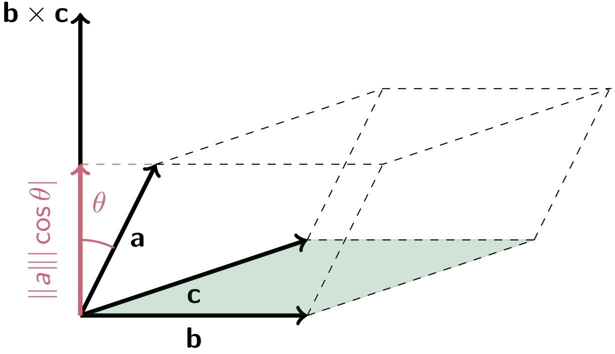


Area of base: $\|\mathbf{b} \times \mathbf{c}\|$

Height of parallelepiped: $\|\mathbf{a}\|\|\cos\theta\|$

Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$



Area of base: $\|\mathbf{b} \times \mathbf{c}\|$

Height of parallelepiped: $\|\mathbf{a}\||\cos\theta|$

Volume of parallelepiped:

(area of base)(height)$= \|\mathbf{a}\|\|\mathbf{b} \times \mathbf{c}\||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$

Calculating the Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) =$$

Calculating the Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Calculating the Triple Product

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix} = \mathbf{a}\cdot\begin{bmatrix}\det\begin{bmatrix}b_2 & b_3\\ c_2 & c_3\end{bmatrix}\\ -\det\begin{bmatrix}b_1 & b_3\\ c_1 & c_3\end{bmatrix}\\ \det\begin{bmatrix}b_1 & b_2\\ c_1 & c_2\end{bmatrix}\end{bmatrix}$$

Calculating the Triple Product

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$$= a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

Calculating the Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \mathbf{a} \cdot \begin{bmatrix} \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} \\ -\det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} \\ \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \end{bmatrix}$$

$$= a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$= \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Find the volume of the parallelepiped spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

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For positive a, b, and c, find the determinant and interpret it as a volume:

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Find the volume of the parallelepiped spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

For positive a, b, and c, find the determinant and interpret it as a volume:

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Calculate and explain geometrically:

$$\det \begin{bmatrix} 2 & 0 & 3 \\ 8 & 1 & 7 \\ 20 & 3 & 15 \end{bmatrix}$$

Right-Hand Rule

Predict the following cross products without using the cross-product calculation. Draw your results. Check using the cross-product calculation.

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}$$

Given any 3-dimensional vector $\mathbf{a}$, is there a simple expression for $\mathbf{a} \times \mathbf{a}$?

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What about $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$?

Consider $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Will this vector be in the same plane as $\mathbf{b}$ and $\mathbf{c}$, or in an orthogonal plane?

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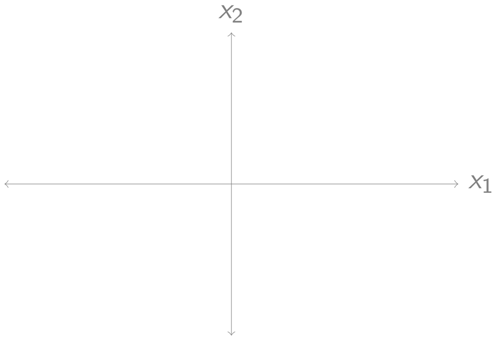
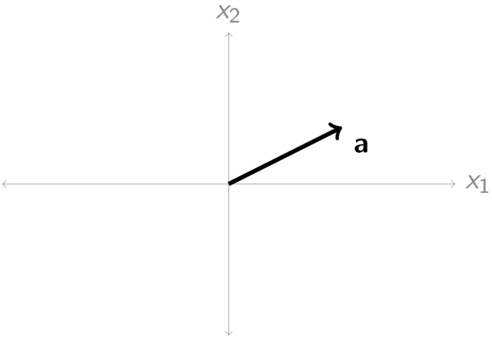
Consider $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Will this vector be in the same plane as $\mathbf{b}$ and $\mathbf{c}$, or in an orthogonal plane?

Notice $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$: a linear combination of $\mathbf{b}$ and $\mathbf{c}$.

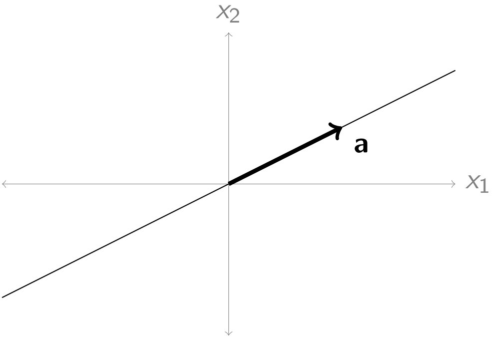
Parametric Equations of Lines



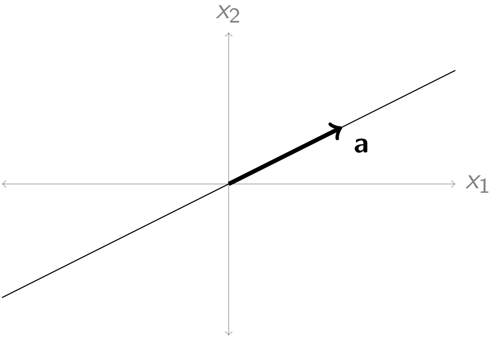
Parametric Equations of Lines



Parametric Equations of Lines

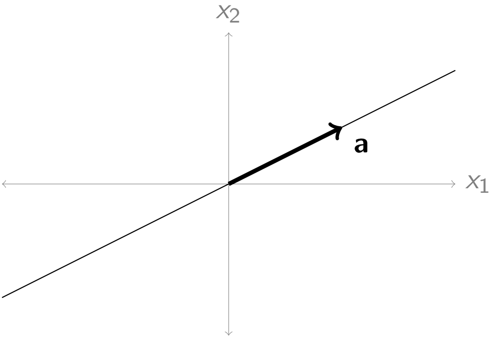


Parametric Equations of Lines



$$\mathbf{x} = s\mathbf{a}$$

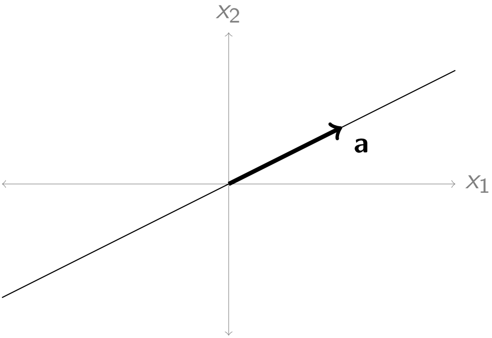
Parametric Equations of Lines



Line passing through the origin:

$$\mathbf{x} = s\mathbf{a}$$

Parametric Equations of Lines

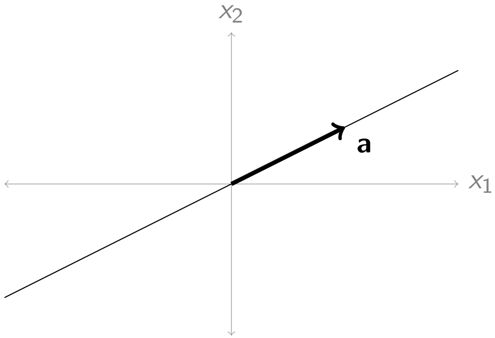


Line passing through the origin:

$$\mathbf{x} = s\mathbf{a}$$

Question: is this the only such equation for the line?

Parametric Equations of Lines

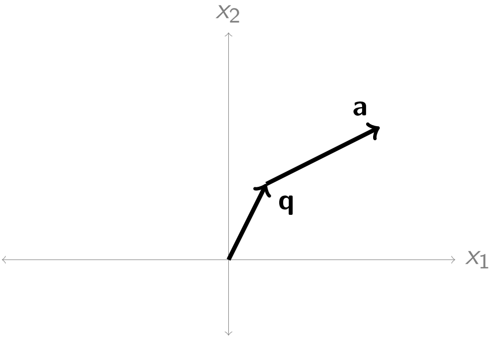


Line passing through the origin:

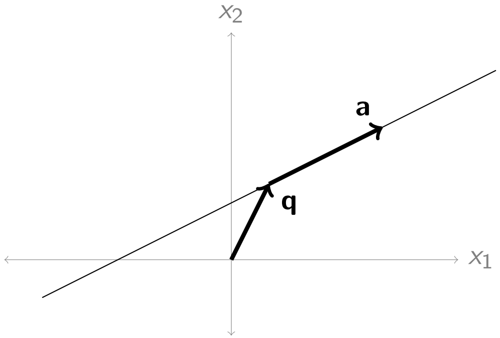
$$\mathbf{x} = s\mathbf{a}$$

Can we use this equation with a line not passing through the origin?

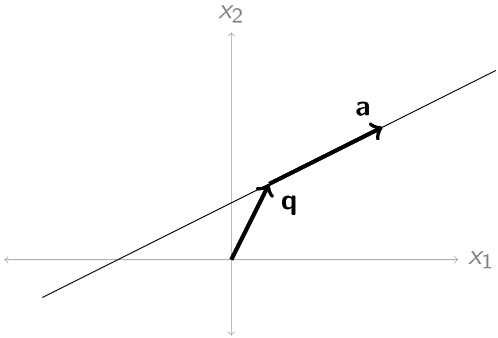
Parametric Equations of Lines



Parametric Equations of Lines

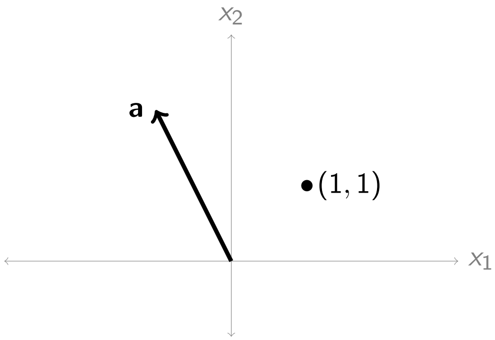


Parametric Equations of Lines

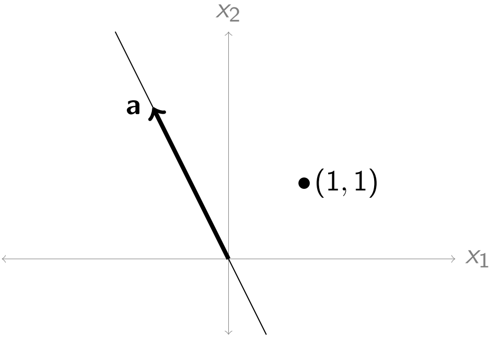


General equation of a line:

$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$



Find a parametric equation describing the line in the direction of $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, passing through the point $(1, 1)$.



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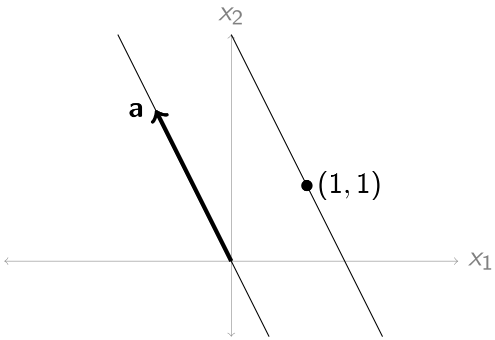

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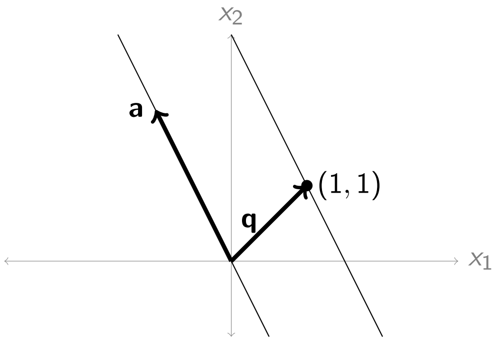

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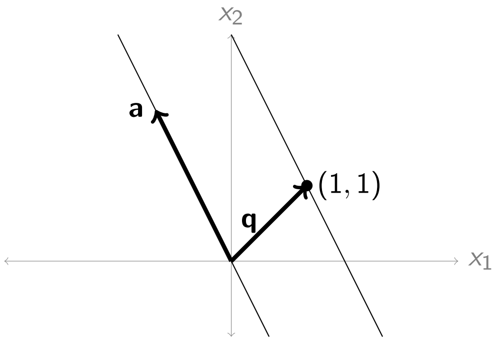
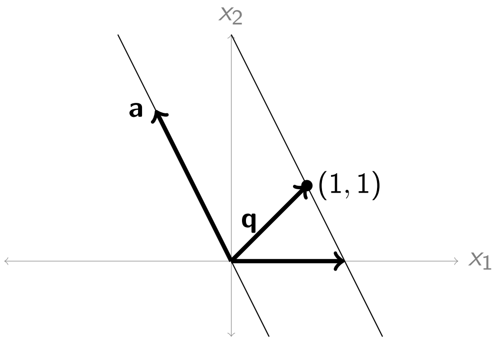


Find a parametric equation describing the line in the direction of $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, passing through the point $(1, 1)$.

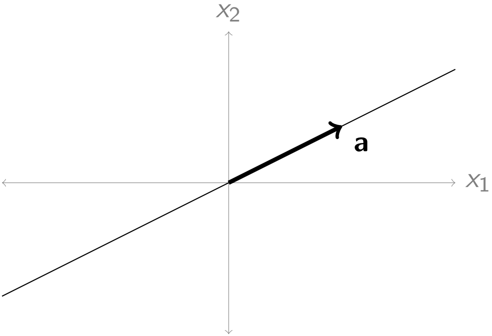
Can you find another?



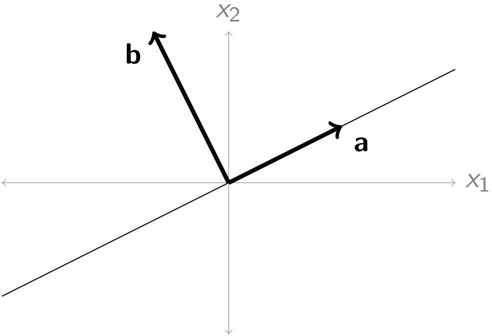
Find a parametric equation describing the line in the direction of $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, passing through the point $(1,1)$.

Can you find another?

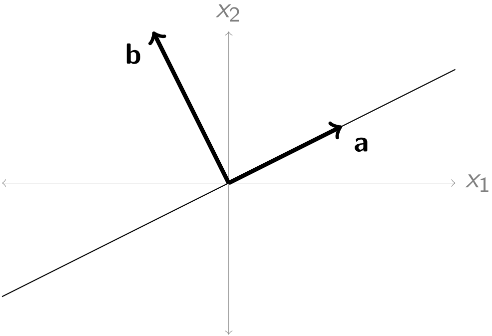
Equations of Lines



Equations of Lines

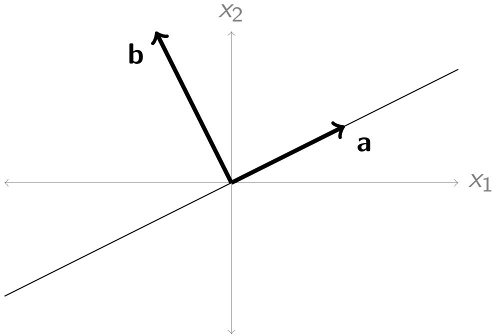


Equations of Lines



Line passing through the origin:

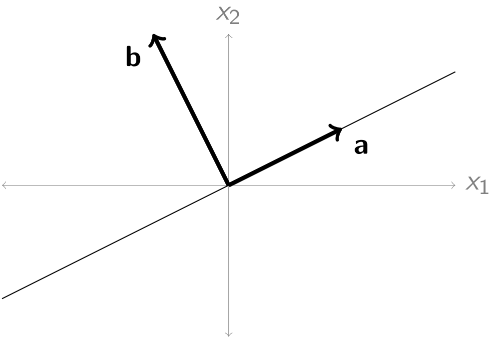
Equations of Lines



Line passing through the origin:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Equations of Lines

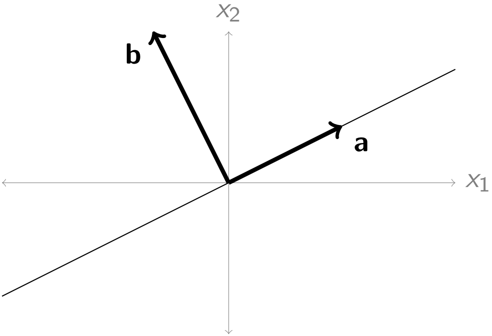


Line passing through the origin:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \boxed{b_1 x_1 + b_2 x_2 = 0}$$

Equations of Lines



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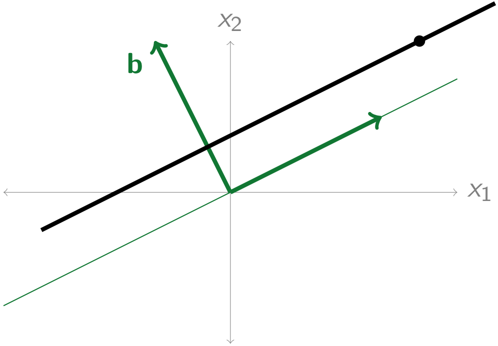
$$\Rightarrow \boxed{b_1 x_1 + b_2 x_2 = 0}$$

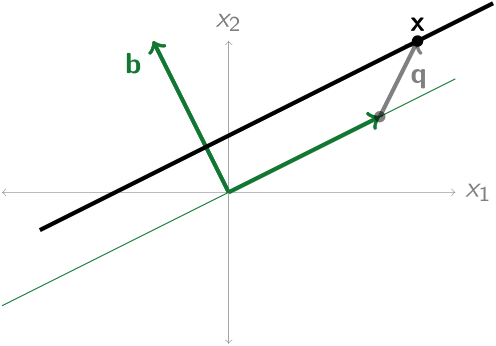
$$\Rightarrow x_2 = (-b_2/b_1) x_1$$

Equations of Lines



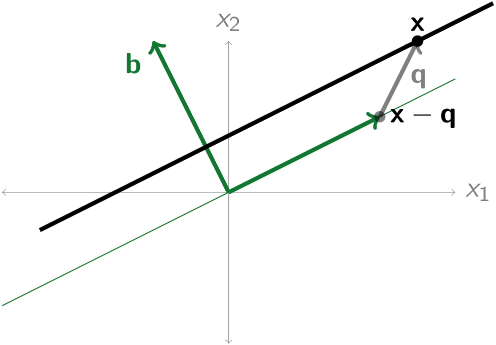
General Line in $\mathbb{R}^2$

Equations of Lines



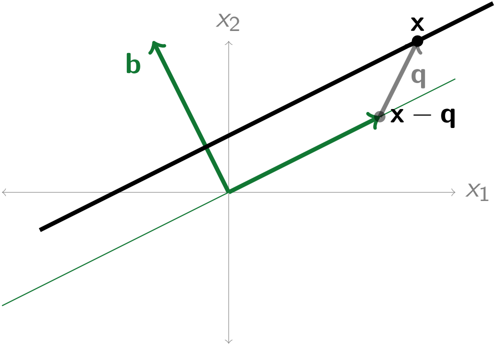
General Line in $\mathbb{R}^2$

Equations of Lines



General Line in $\mathbb{R}^2$

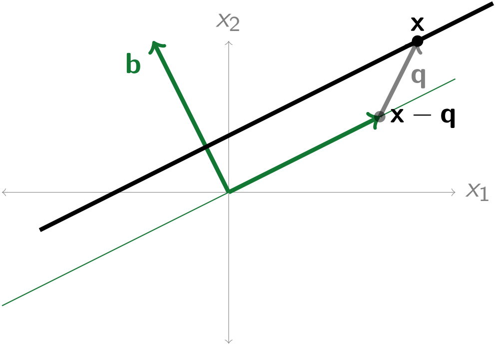
Equations of Lines



General Line in $\mathbb{R}^2$

$$(\mathbf{x}-\mathbf{q})\cdot\mathbf{b}=0$$

Equations of Lines

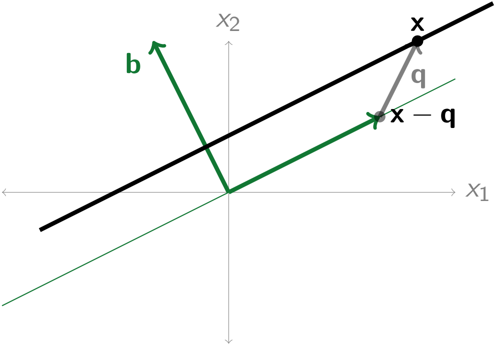


General Line in $\mathbb{R}^2$

$$(\mathbf{x} - \mathbf{q}) \cdot \mathbf{b} = 0$$

$$\mathbf{x} \cdot \mathbf{b} = \mathbf{q} \cdot \mathbf{b}$$

Equations of Lines



General Line in $\mathbb{R}^2$

$$(\mathbf{x} - \mathbf{q}) \cdot \mathbf{b} = 0$$

$$\mathbf{x} \cdot \mathbf{b} = \mathbf{q} \cdot \mathbf{b}$$

$$\boxed{b_1 x_1 + b_2 x_2 = c}$$

Suppose the parametric equation of a line is given by
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Convert this to an equation of the form
$ax_1 + bx_2 = c$.

Suppose the parametric equation of a line is given by
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 7 \end{bmatrix}$. Convert this to an equation of the form
$ax_1 + bx_2 = c$.

Suppose the parametric equation of a line is given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Convert this to an equation of the form $ax_1 + bx_2 = c$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\Leftrightarrow \begin{cases} x_1 = 3 + s \\ x_2 = -1 + s \end{cases}$$

Suppose the parametric equation of a line is given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Convert this to an equation of the form $ax_1 + bx_2 = c$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} x_1 = 3 + s \\ x_2 = -1 + s \end{cases}$$

$$\Leftrightarrow \begin{cases} s = x_1 - 3 \\ s = x_2 + 1 \end{cases}$$

Suppose the parametric equation of a line is given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Convert this to an equation of the form $ax_1 + bx_2 = c$.

$$\begin{aligned} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \Leftrightarrow & \begin{cases} x_1 = 3 + s \\ x_2 = -1 + s \end{cases} \\ \Leftrightarrow & \begin{cases} s = x_1 - 3 \\ s = x_2 + 1 \end{cases} \\ \Rightarrow & x_1 - 3 = x_2 + 1 \\ \Leftrightarrow & x_1 - x_2 = 2 \end{aligned}$$

Suppose the parametric equation of a line is given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Convert this to an equation of the form $ax_1 + bx_2 = c$.

$$\begin{aligned} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \Leftrightarrow & \begin{cases} x_1 = 3 + s \\ x_2 = -1 + s \end{cases} \\ \Leftrightarrow & \begin{cases} s = x_1 - 3 \\ s = x_2 + 1 \end{cases} \\ \Rightarrow & x_1 - 3 = x_2 + 1 \\ \Leftrightarrow & x_1 - x_2 = 2 \end{aligned}$$

Suppose the parametric equation of a line is given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 7 \end{bmatrix}$. Convert this to an equation of the form $ax_1 + bx_2 = c$.

Suppose the parametric equation of a line is given by
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Convert this to an equation of the form
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Suppose the parametric equation of a line is given by
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$ax_1 + bx_2 = c$.

$$\begin{cases} x_1 = 3 + 2s \\ x_2 = -1 + 7s \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 - 3 = 2s \\ x_2 + 1 = 7s \end{cases}$$

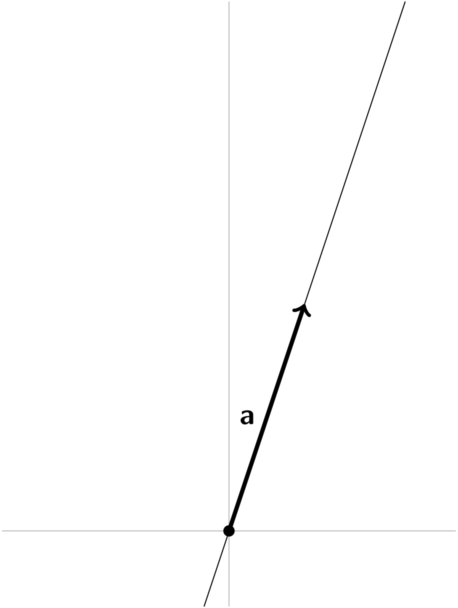
$$\Leftrightarrow \begin{cases} 7x_1 - 21 = 14s \\ 2x_2 + 2 = 14s \end{cases}$$

$$\Rightarrow 7x_1 - 12 = 2x_2 + 2$$

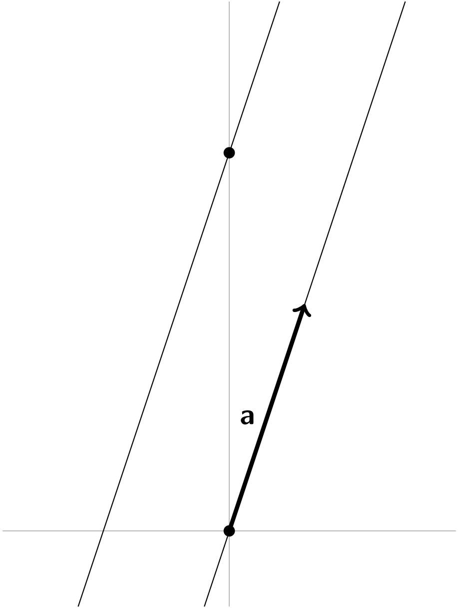
$$\Leftrightarrow 7x_1 - 2x_2 = 23$$

Give a parametric equation for the line $x_2 = 3x_1 + 5$.

Give a parametric equation for the line $x_2 = 3x_1 + 5$.



Give a parametric equation for the line $x_2 = 3x_1 + 5$.



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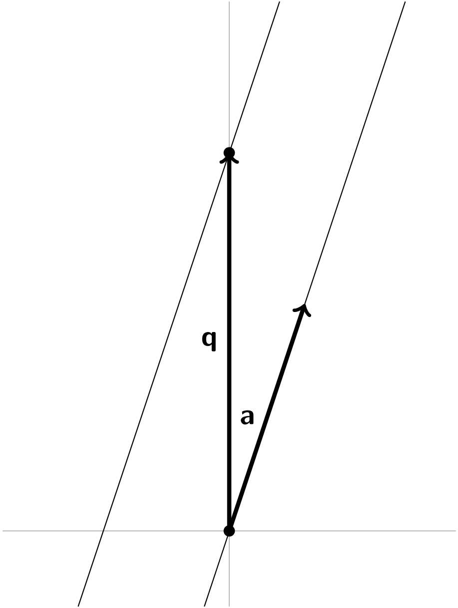

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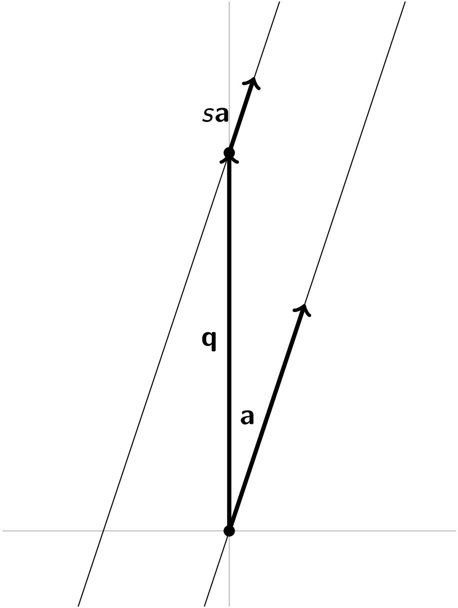
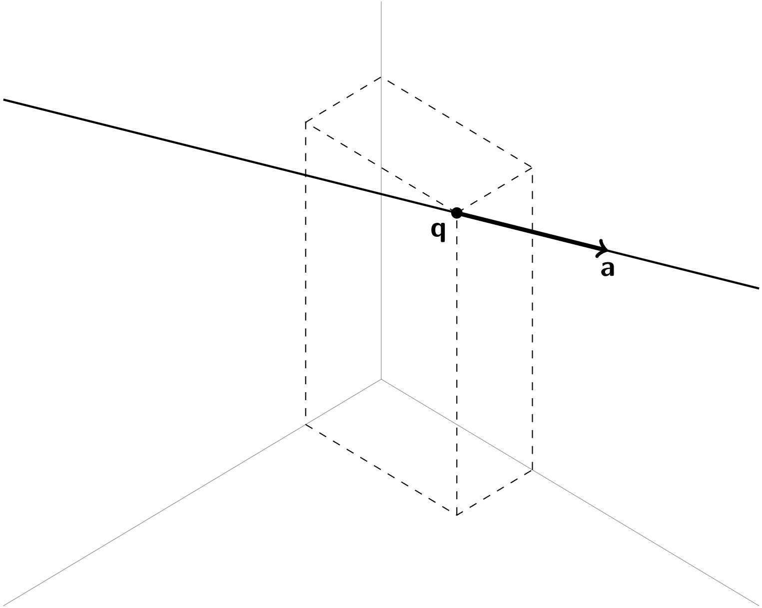
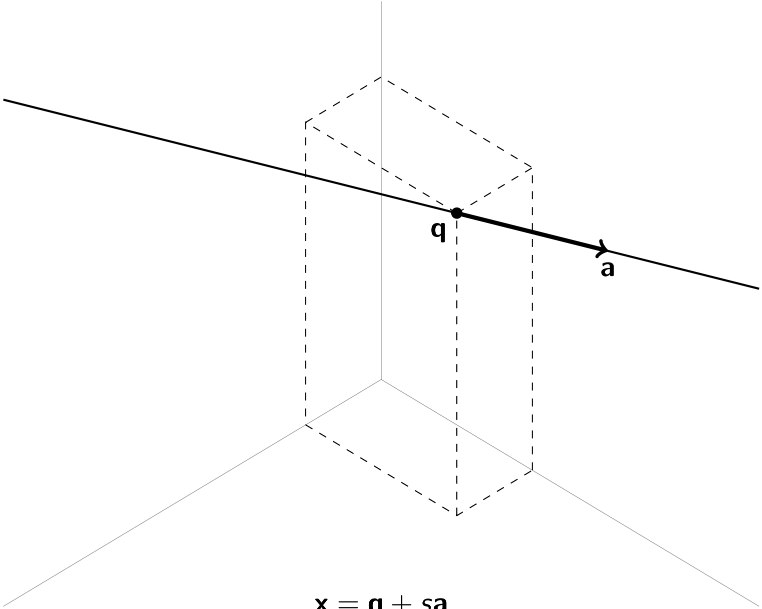
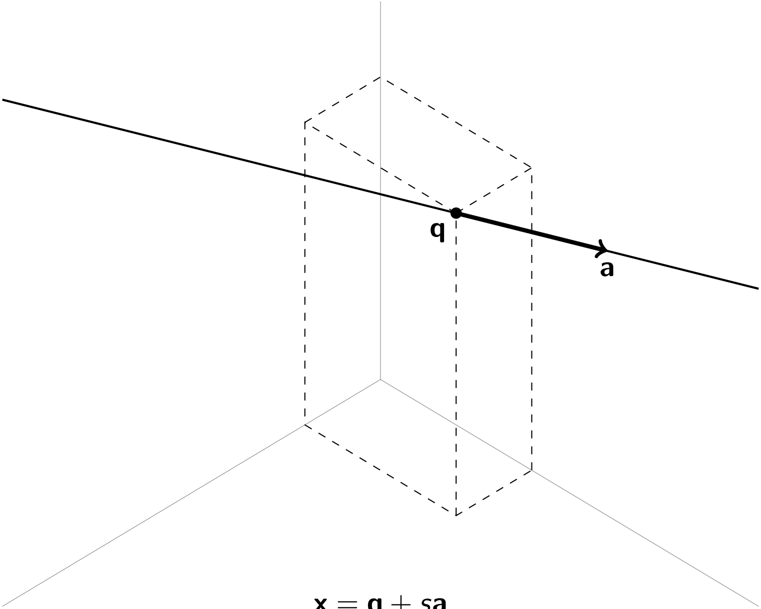



q
a

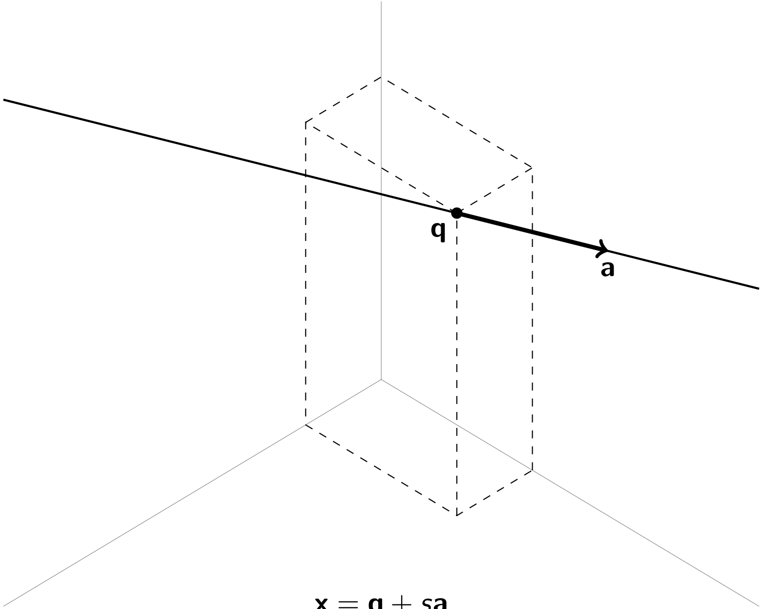


$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$



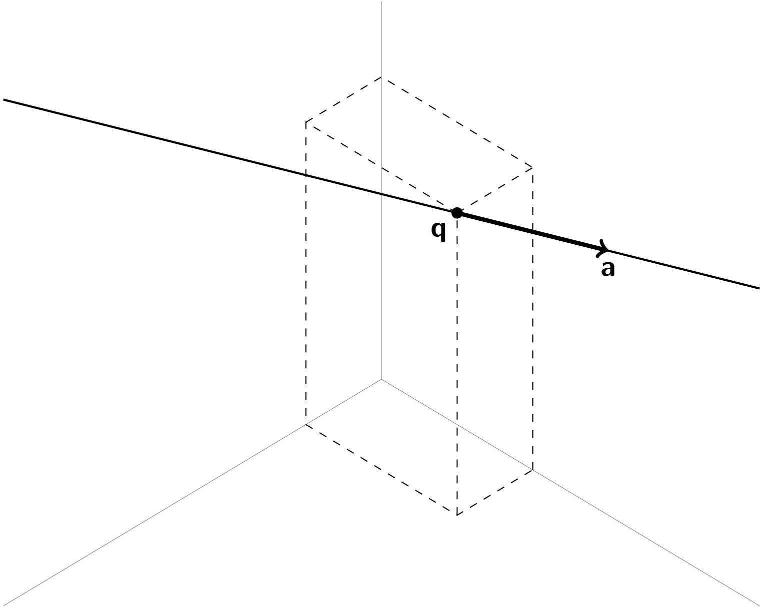
$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$

How many components do the vectors have?

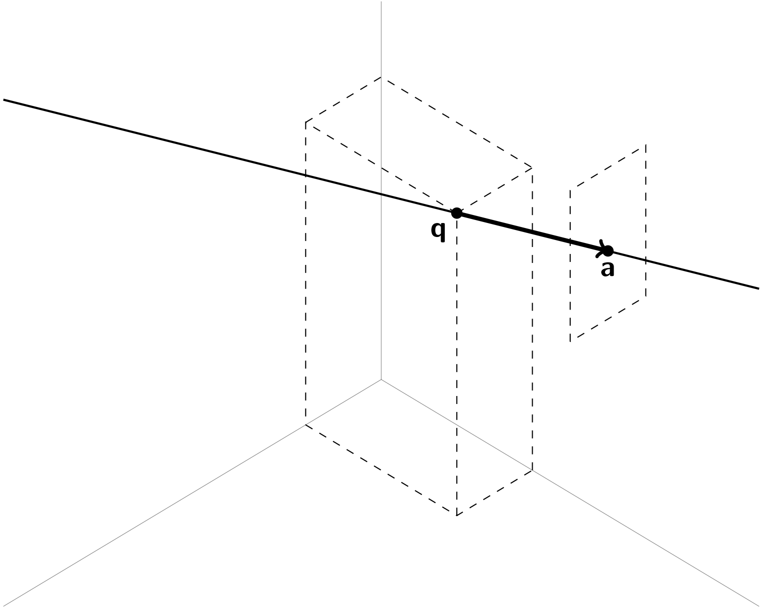


$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$

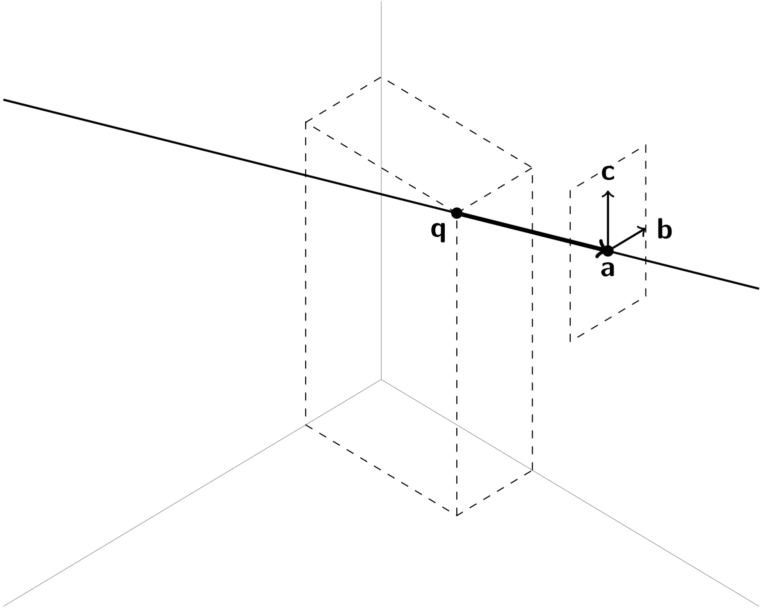
How many dimensions does a vector have?



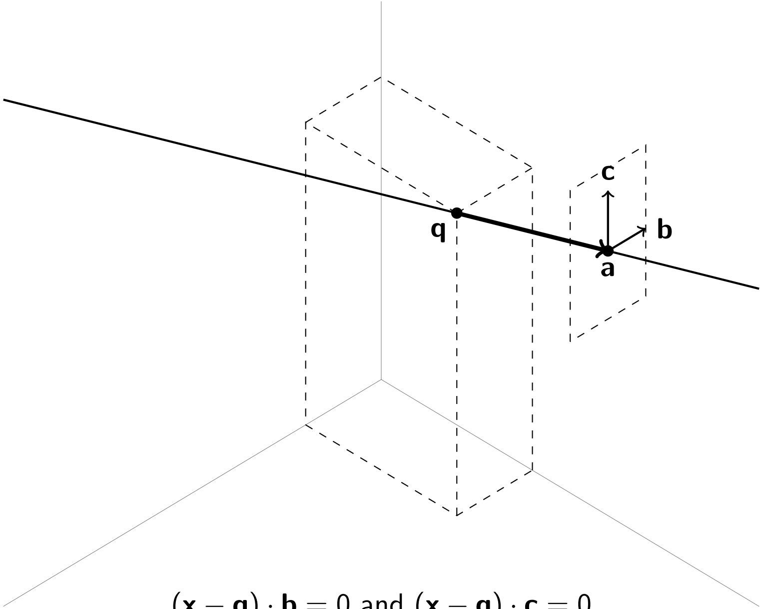
q
a



q
a



c
q
b
a



$$(\mathbf{x}-\mathbf{q})\cdot\mathbf{b}=0 \text{ and } (\mathbf{x}-\mathbf{q})\cdot\mathbf{c}=0$$

Equation of a Line in $\mathbb{R}^3$

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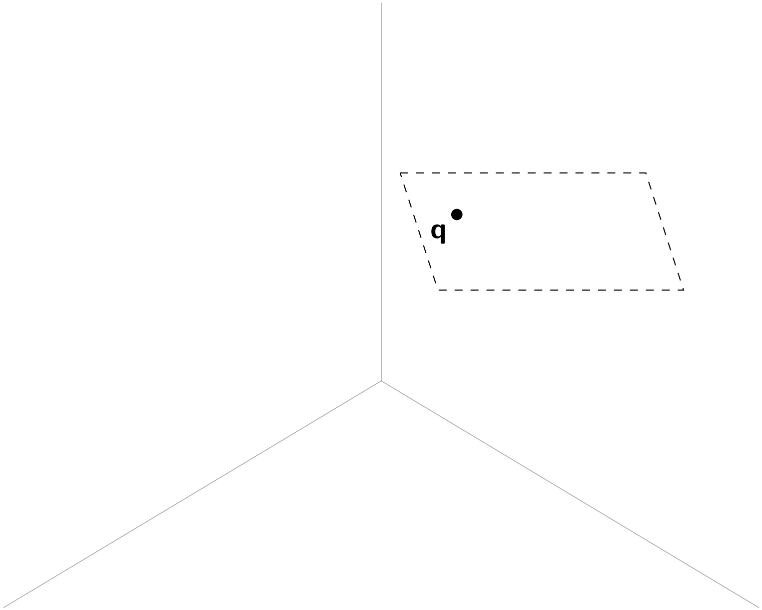
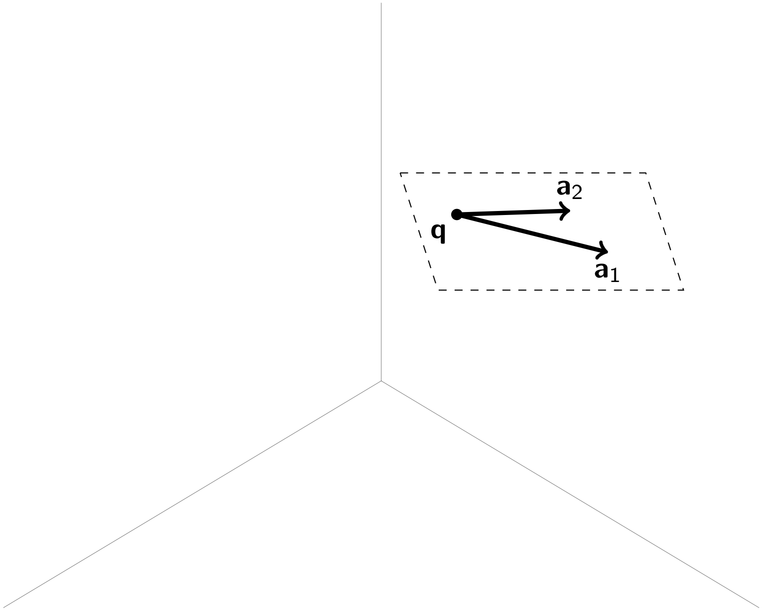
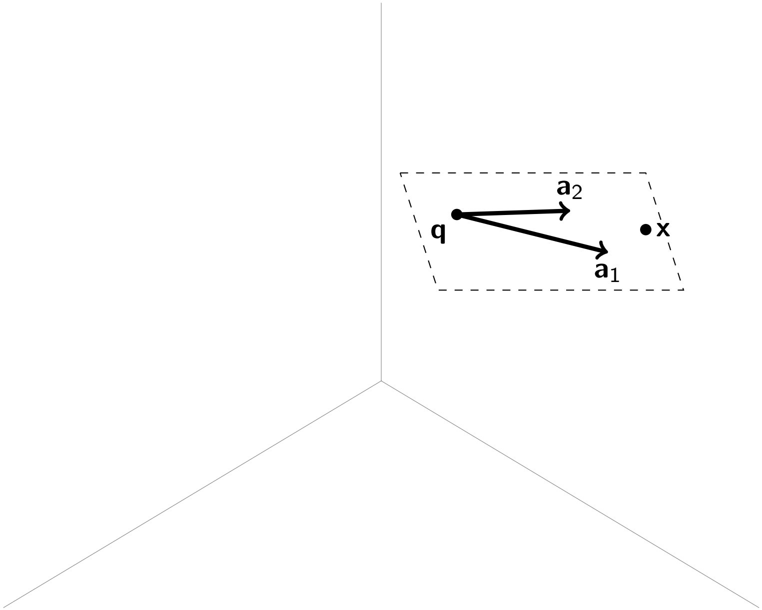
To define a line in $\mathbb{R}^3$, we need a *system* of equations:

$$\begin{cases} x_1 b_1 + x_2 b_2 + x_3 b_3 & = & s_1 \\ x_1 c_1 + x_2 c_2 + x_3 c_3 & = & s_2 \end{cases}$$

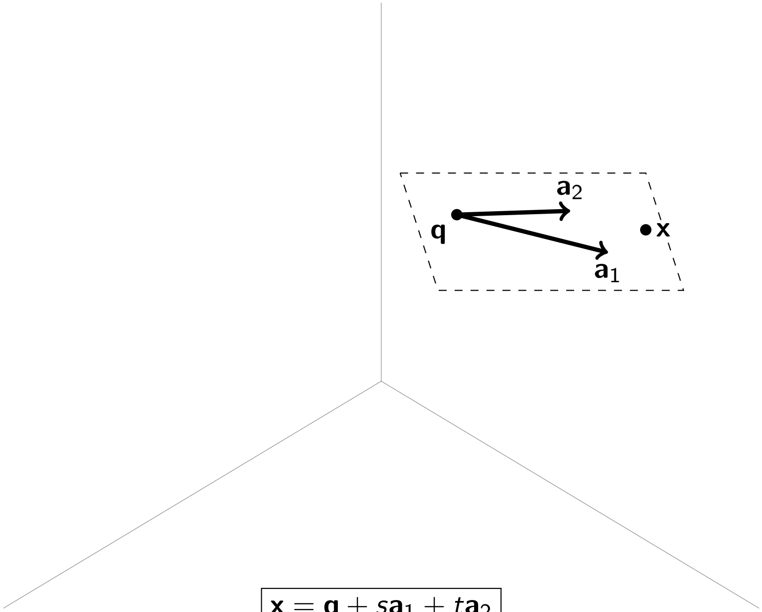

q



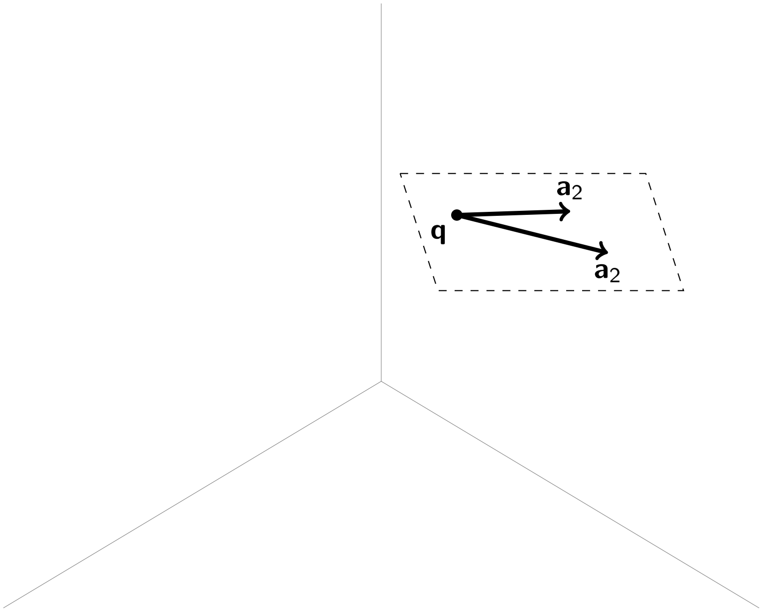
$\mathbf{a}_2$
$\mathbf{q}$
$\mathbf{a}_1$

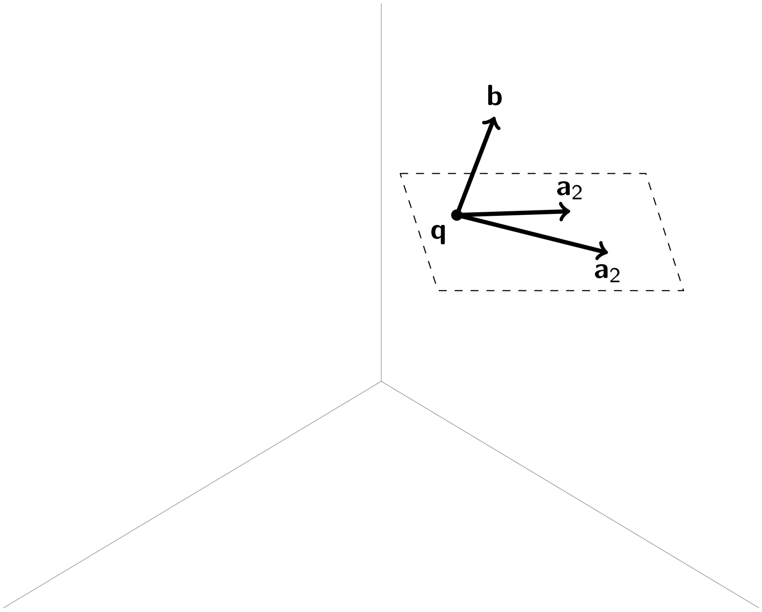


$\mathbf{a}_2$
$\mathbf{q}$
$\mathbf{x}$
$\mathbf{a}_1$

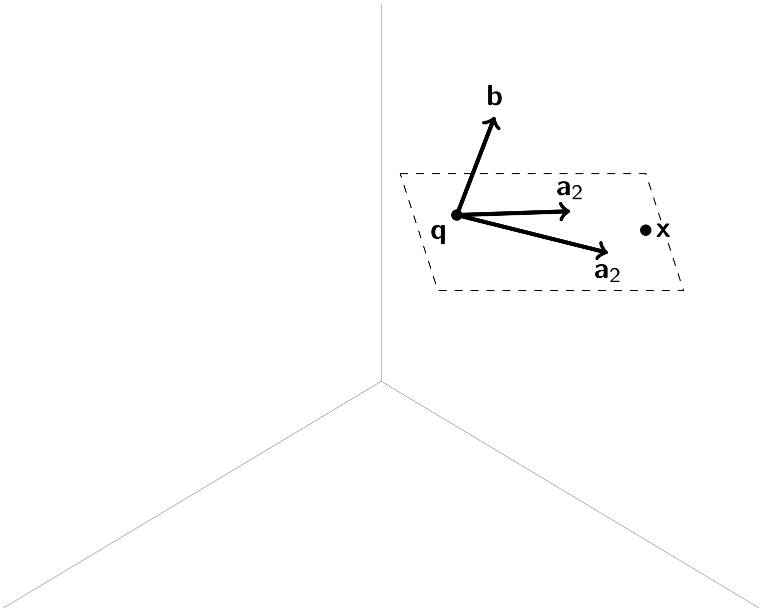


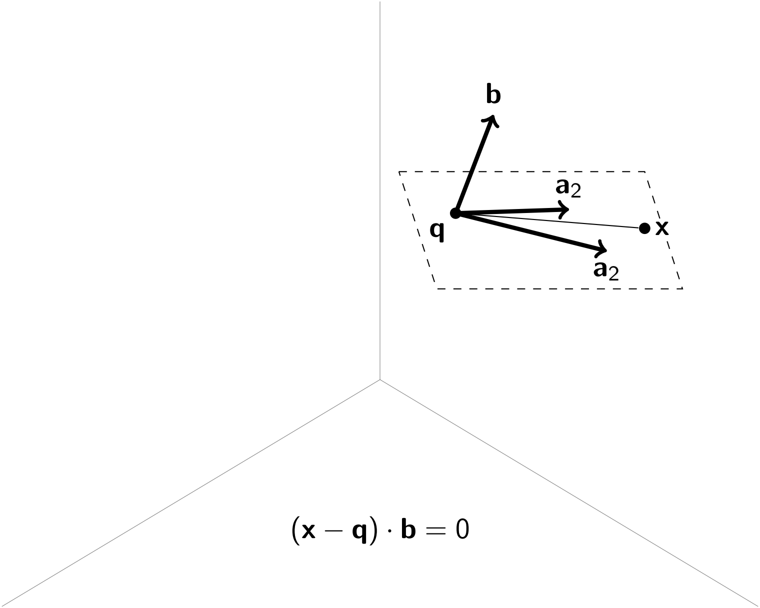
$$\mathbf{x} = \mathbf{q} + s\mathbf{a}_1 + t\mathbf{a}_2$$


$\mathbf{a}_2$
$\mathbf{q}$
$\mathbf{a}_2$

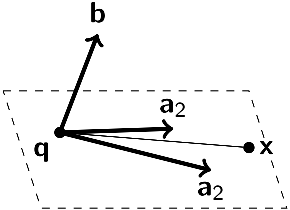


b
a2
q
a2


b
a_2
q
x
a_2

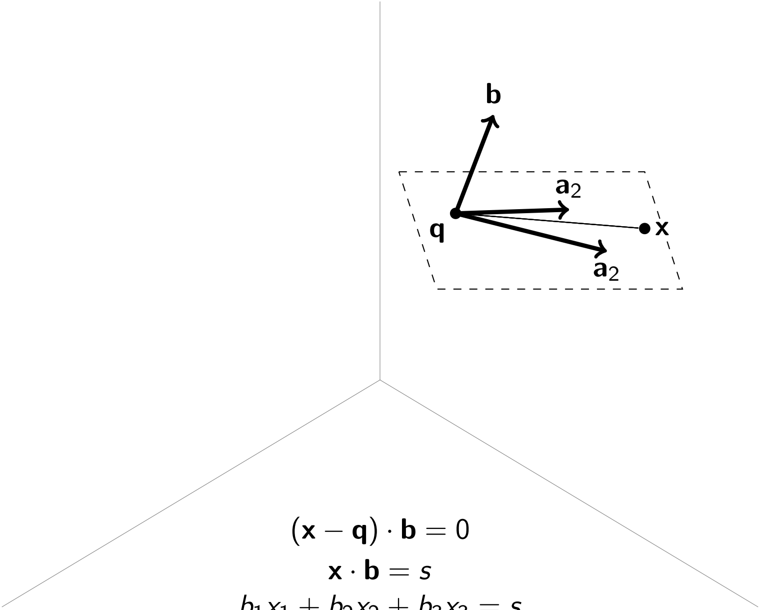


$$(\mathbf{x} - \mathbf{q}) \cdot \mathbf{b} = 0$$



$$(\mathbf{x} - \mathbf{q}) \cdot \mathbf{b} = 0$$

$$\mathbf{x} \cdot \mathbf{b} = s$$



$$(\mathbf{x}-\mathbf{q})\cdot\mathbf{b}=0$$

$$\mathbf{x}\cdot\mathbf{b}=s$$

$$b_1x_1+b_2x_2+b_3x_3=s$$

Equations

	Parametric	Component
Line in $\mathbb{R}^2$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$b_1x_1 + b_2x_2 = s$
Line in $\mathbb{R}^3$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$\begin{cases} b_1x_1 + b_2x_2 + b_3x_3 = s \\ c_1x_1 + c_2x_2 + c_3x_3 = t \end{cases}$
Plane in $\mathbb{R}^3$	$\mathbf{x} = \mathbf{q} + s\mathbf{a} + t\mathbf{b}$	$b_1x_1 + b_2x_2 + b_3x_3 = s$

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Suppose $\mathbf{q}$ and $\mathbf{a}$ are vectors in $\mathbb{R}^{18}$. What would you call the geometric object resulting from the equation $\mathbf{x} = \mathbf{q} + s\mathbf{a}$?

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Suppose P and Q are planes. What is the intersection of P and Q?

Equations

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Line in $\mathbb{R}^3$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$\begin{cases} b_1 x_1 + b_2 x_2 + b_3 x_3 = s \\ c_1 x_1 + c_2 x_2 + c_3 x_3 = t \end{cases}$
Plane in $\mathbb{R}^3$	$\mathbf{x} = \mathbf{q} + s\mathbf{a} + t\mathbf{b}$	$b_1 x_1 + b_2 x_2 + b_3 x_3 = s$

Suppose $\mathbf{q}$ and $\mathbf{a}$ are vectors in $\mathbb{R}^{18}$. What would you call the geometric object resulting from the equation $\mathbf{x} = \mathbf{q} + s\mathbf{a}$?

Suppose P and Q are planes. What is the intersection of P and Q?

Are there any vectors $\mathbf{q}$ and $\mathbf{a}$ in $\mathbb{R}^3$ for which the equation $\mathbf{x} = \mathbf{q} + s\mathbf{a}$ is **not** a line?

Equations

	Parametric	Component
Line in $\mathbb{R}^2$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$b_1x_1 + b_2x_2 = s$
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Suppose **q** and **a** are vectors in $\mathbb{R}^{18}$. What would you call the geometric object resulting from the equation $\mathbf{x} = \mathbf{q} + s\mathbf{a}$?

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Are there any vectors **q** and **a** in $\mathbb{R}^3$ for which the equation $\mathbf{x} = \mathbf{q} + s\mathbf{a}$ is **not** a line?

Are there any vectors **q**, **a**, and **b** in $\mathbb{R}^3$ for which the equation $\mathbf{x} = \mathbf{q} + s\mathbf{a} + t\mathbf{b}$ is **not** a plane?

Equations

	Parametric	**Component**
Line in $\mathbb{R}^2$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$b_1x_1 + b_2x_2 = s$
Line in $\mathbb{R}^3$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$\begin{cases} b_1x_1 + b_2x_2 + b_3x_3 = s \\ c_1x_1 + c_2x_2 + c_3x_3 = t \end{cases}$
Plane in $\mathbb{R}^3$	$\mathbf{x} = \mathbf{q} + s\mathbf{a} + t\mathbf{b}$	$b_1x_1 + b_2x_2 + b_3x_3 = s$

Are there any constants b_1, b_2, and s for which the equation $b_1x_1 + b_2x_2 = s$ is **not** a line?

Equations

	Parametric	Component
Line in $\mathbb{R}^2$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$b_1x_1 + b_2x_2 = s$
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Are there any constants b_1, b_2, and s for which the equation $b_1x_1 + b_2x_2 = s$ is **not** a line?

Recall: $\mathbf{b}$ was the normal vector to the plane $b_1x_1 + b_2x_2 + b_3x_3 = s$.

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Are there any constants b_1, b_2, and s for which the equation $b_1x_1 + b_2x_2 = s$ is **not** a line?

Recall: **b** was the normal vector to the plane $b_1x_1 + b_2x_2 + b_3x_3 = s$.

True or False: for point P on the plane $5x_1 + 7x_2 + 11x_3 = 22$, the vector with head at P and tail at the origin is orthogonal to the vector $[5, 7, 11]$.

Equations

	Parametric	Component
Line in $\mathbb{R}^2$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$b_1x_1 + b_2x_2 = s$
Line in $\mathbb{R}^3$	$\mathbf{x} = \mathbf{q} + s\mathbf{a}$	$\begin{cases} b_1x_1 + b_2x_2 + b_3x_3 = s \\ c_1x_1 + c_2x_2 + c_3x_3 = t \end{cases}$
Plane in $\mathbb{R}^3$	$\mathbf{x} = \mathbf{q} + s\mathbf{a} + t\mathbf{b}$	$b_1x_1 + b_2x_2 + b_3x_3 = s$

Are there any constants b_1, b_2, and s for which the equation $b_1x_1 + b_2x_2 = s$ is **not** a line?

Recall: $\mathbf{b}$ was the normal vector to the plane $b_1x_1 + b_2x_2 + b_3x_3 = s$.

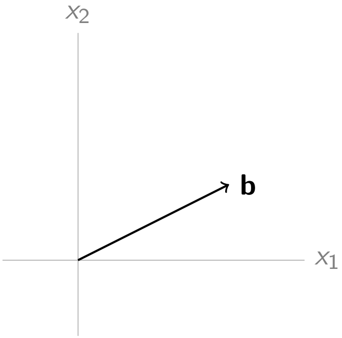
True or False: for point P on the plane $5x_1 + 7x_2 + 11x_3 = 22$, the vector with head at P and tail at the origin is orthogonal to the vector $[5, 7, 11]$.

True or False: for any two distinct points P_1 and P_2 on the plane $5x_1 + 7x_2 + 11x_3 = 22$, the vector with head at P_1 and tail at P_2 is orthogonal to the vector $[5, 7, 11]$.

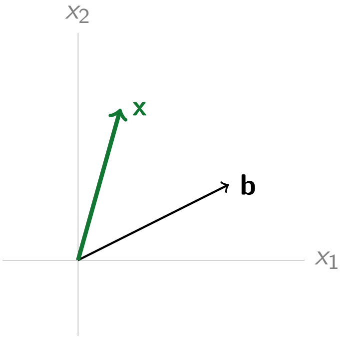
Prove It

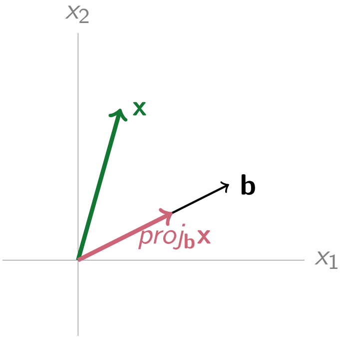
Suppose a plane has equation $b_1x_1 + b_2x_2 + b_3x_3 = s$.

Show that, for any two points on this plane, the vector with head at one and tail at the other is orthogonal to $[b_1, b_2, b_3]$.

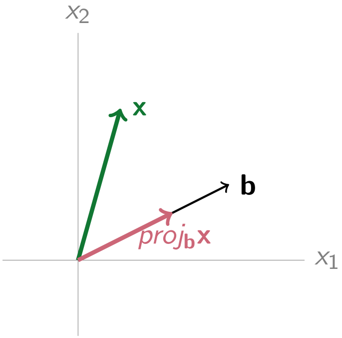


x_2
b
x_1

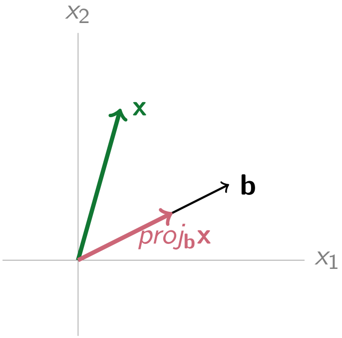

x_2
$\mathbf{x}$
$\mathbf{b}$
x_1



x2
x
b
projb x
x1

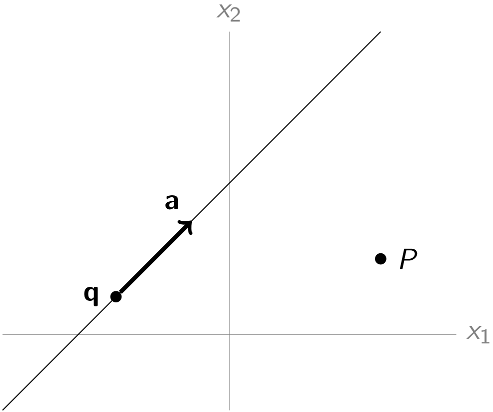


Draw lots of other vectors whose projections onto **b** are also the pink vector.

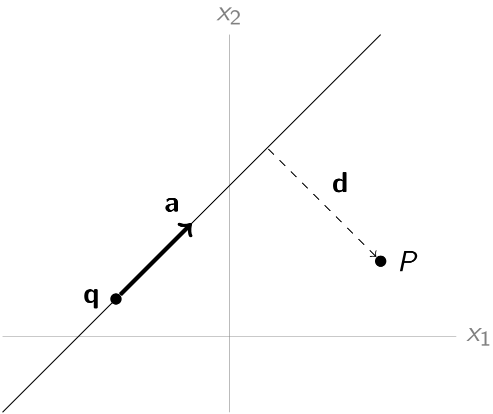


Draw lots of other vectors whose projections onto **b** are also the pink vector.

Find a simplified expression for the collection of vectors $[x_1, x_2]$ that all have the same projection onto **b** in.



How can you find the distance from the point P to the line $\mathbf{x} = \mathbf{q} + s\mathbf{a}$?



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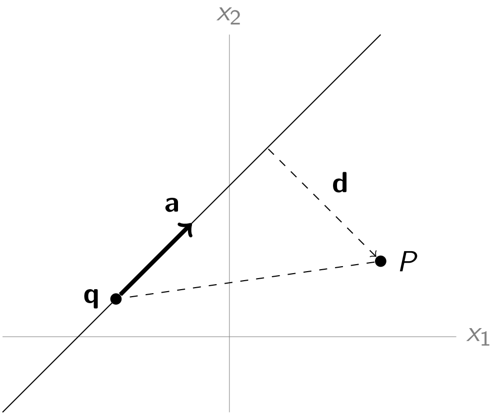

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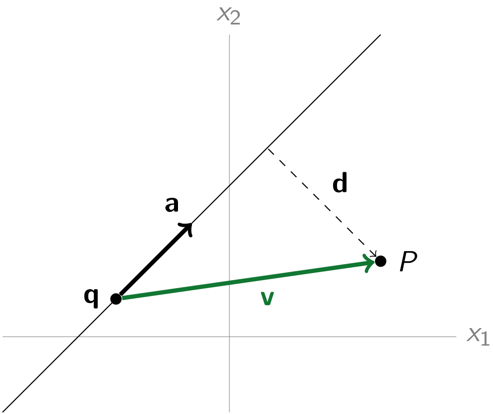
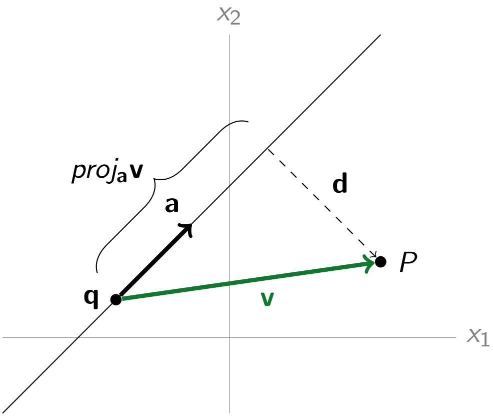


How can you find the distance from the point P to the line $\mathbf{x} = \mathbf{q} + s\mathbf{a}$?



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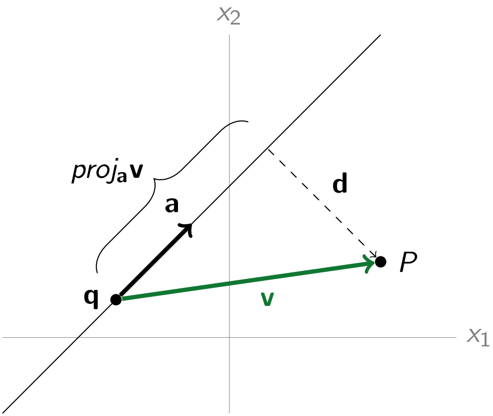
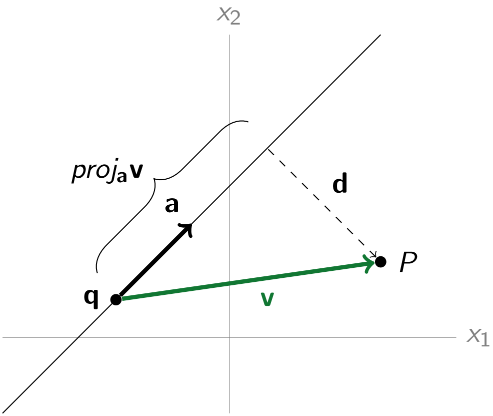


How can you find the distance from the point P to the line $\mathbf{x} = \mathbf{q} + s\mathbf{a}$?
$\mathbf{d} + proj_{\mathbf{a}}\mathbf{v} =$



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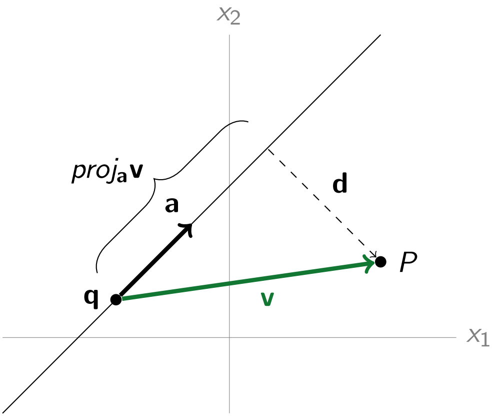

How can you find the distance from the point P to the line $\mathbf{x} = \mathbf{q} + s\mathbf{a}$?
$\mathbf{d} + proj_{\mathbf{a}}\mathbf{v} = \mathbf{v}$, so $\|\mathbf{d}\| = \|\mathbf{v} - proj_{\mathbf{a}}\mathbf{v}\|$

Find the distance from the point P to the plane Q.

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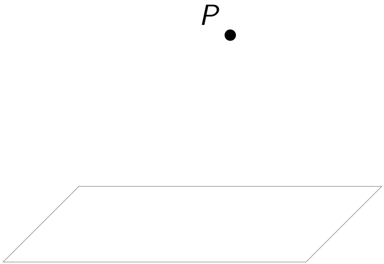
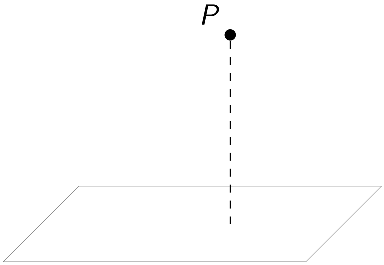
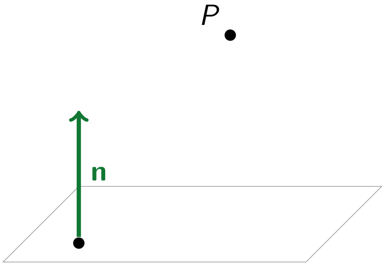


Find the distance from the point P to the plane Q.



Find the distance from the point P to the plane Q.



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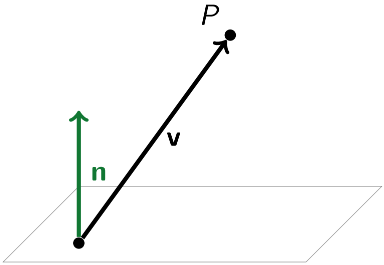
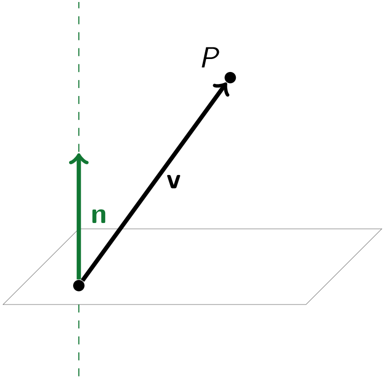
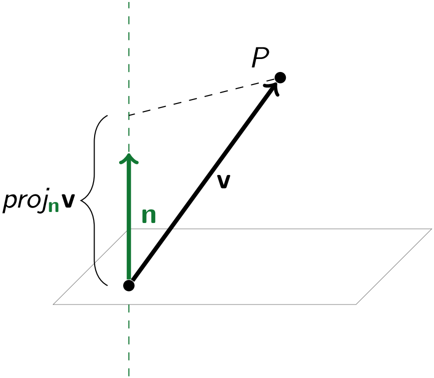


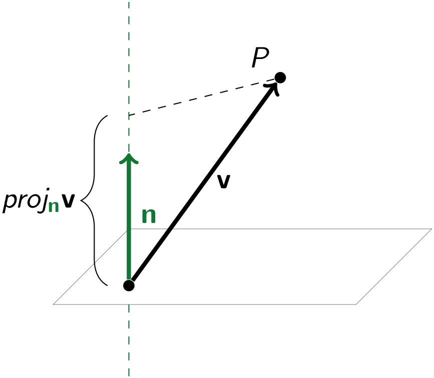
Find the distance from the point P to the plane Q.



Find the distance from the point P to the plane Q.



Find the distance from the point P to the plane Q.



Find the distance from the point $(3, 5, 1)$ to the plane

$$\mathbf{x} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Let P be the plane with equation $2x + 2y + 2z = 1$, and let Q be the plane with equation $x + y + z = 1$.

What will their intersection be: a plane, a line, a point, or nothing?

Let P be the plane with equation $2x + 2y + 2z = 1$, and let Q be the plane with equation $x + y + z = 1$.

What will their intersection be: a plane, a line, a point, or nothing?

Let P be the plane with equation $2x + y - z = 1$, and let Q be the plane with equation $x + 2y + 3z = 0$.

What will their intersection be: a plane, a line, a point, or nothing?

Let P be the plane with equation $2x + 2y + 2z = 1$, and let Q be the plane with equation $x + y + z = 1$.

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Let P be the plane with equation $2x + y - z = 1$, and let Q be the plane with equation $x + 2y + 3z = 0$.

What will their intersection be: a plane, a line, a point, or nothing?

Find it in parametric form.