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$P = NP$ (where P and NP are sets of problems).
To disprove: find one problem that is in one set but not in the other.

A Word of Caution

A pattern: $2^n \not\equiv 3 \mod n$:

n	$\mathbf{2^n}$	$\mathbf{2^n \equiv 3 \mod n}$?
$n = 2$	$2^n = 4$	$4 \not\equiv 3 \mod 2$
$n = 3$	$2^n = 8$	$8 \not\equiv 3 \mod 3$
$n = 4$	$2^n = 16$	$16 \not\equiv 3 \mod 4$
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$n = 6$	$2^n = 64$	$64 \not\equiv 3 \mod 6$
$n = 7$	$2^n = 128$	$128 \not\equiv 3 \mod 7$
$n = 8$	$2^n = 256$	$256 \not\equiv 3 \mod 8$
$n = 9$	$2^n = 512$	$512 \not\equiv 3 \mod 9$
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$n = 1000$	$2^n = [big]$	$2^{1000} \not\equiv 3 \mod 1000$

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Result: D.H. and Emma Lehmer
Source: Richard K Guy, **The Strong Law of Small Numbers**:
`http://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/Guy697-712.pdf` (Recommended read!)

Example

Conjecture: For every $n \in \mathbb{N}$, $n^2 - n + 11$ is prime.

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The conjecture is false. 11 is a natural number, and $11^2 - 11 + 11 = 11^2$, which is not prime.

True or False?

- True or False: For every $x, y \in \mathbb{N}$, $2^{2x} + 3^{2y+1}$ is prime.

- True or False:
  For every even natural number n other than $n = 2$, $2^n - 1$ is not prime.

- True or False:
  For every $m \in \mathbb{Z}$, there exists an $n \in \mathbb{N}$ such that $\left|\frac{1}{m} - \frac{1}{n}\right| > \frac{1}{2}$.

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  False: Let $x = 3$ and $y = 1$. Then $2^{2x} + 3^{2y+1} = 2^6 + 3^3 = 91 = 7 * 13$, so $2^{2x} + 3^{2y+1}$ is not prime.

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  For every even natural number n other than $n = 2$, $2^n - 1$ is not prime.

  True: let n be an even natural number other than 2. Then $n = 2x$ for some $x \in \mathbb{N}$ (since n is even) and $x \geq 2$ (since $n \neq 2$). Then $2^n - 1 = 2^{2x} - 1 = (2^x - 1)(2^x + 1)$. Since $x \geq 2$, $2^x + 1 > 1$ and $2^x - 1 > 1$. Then $2^n - 1$ has two factors that are greater than one, hence it is not prime.

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- True or False:
  For every $m \in \mathbb{Z}$, there exists an $n \in \mathbb{N}$ such that $\left|\frac{1}{m} - \frac{1}{n}\right| > \frac{1}{2}$.

  False: consider $m = 2$. We claim that for every $n \in \mathbb{N}$, $\left|\frac{1}{2} - \frac{1}{n}\right| \leq \frac{1}{2}$.
  Case 1: $n = 1$. Then $\left|\frac{1}{2} - \frac{1}{n}\right| = \frac{1}{2}$.
  Case 2: $n = 2$. Then $\left|\frac{1}{2} - \frac{1}{n}\right| = 0 \leq \frac{1}{2}$.
  Case 3: $n \geq 33$. Then $\frac{1}{n} < \frac{1}{2}$, so $\left|\frac{1}{2} - \frac{1}{n}\right| = \frac{1}{2} - \frac{1}{n} < \frac{1}{2}$.
  So, for every $n \in \mathbb{N}$, $\left|\frac{1}{2} - \frac{1}{n}\right| \leq \frac{1}{2}$.

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we must prove its negation: $\forall x \sim P(x)$.

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- True or False: $\exists x \in \mathbb{R}$ s.t. $x^3 < x < x^2$.
  True: $-2 \in \mathbb{R}$ and $(-2)^3 < -2 < (-2)^2$.

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- True or False: $\exists x \in \mathbb{R}$ s.t. $x^4 < x < x^2$.
  False. Suppose $x \in \mathbb{R}$ and $x < x^2$. Then $x < 0$ or $x > 1$.
  Case 1: $x < 0$. Then $x^4 > x$ because $x^4 > 0$. Then it is not true that $x^4 < x < x^2$.
  Case 2: $x > 1$. Then $x^4 > x$, so it is not true that $x^4 < x < x^2$.
  So, for every real x, it is not true that both $x < x^2$ and $x^4 < x$.

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"Suppose P is true.
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Statement: There exists $x \in \mathbb{R}$ such that $x^6 + 2x^2 + 1 = 0$.

Suppose the statement is true, and let x be a real number such that $x^6 + 2x^2 + 1 = 0$. Since 6 and 2 are even, $x^6 \geq 0$ and $2x^2 \geq 0$. Then

$$0 = x^6 + 2x^2 + 1 \geq 1$$

so $0 \geq 1$. This is a contradiction. We conclude the statement is false.

Prove or disprove each of the following statements.

- Let A, B, and C be sets. If $A \times C = B \times C$, then $A = B$.
- Every even integer is the sum of three *distinct* even integers.
- There exists an irrational number p and a rational number q such that $\frac{p}{q}$ is rational.
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- There exist prime numbers p and q such that $p - q = 513$.

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- There exists a rational number p and an irrational number q such that $\frac{p}{q}$ is rational.
  True
- There exist prime numbers p and q such that $p - q = 513$.
  False

Let A, B, and C be sets. If $A \times C = B \times C$, then $A = B$.

False: Let $C = \emptyset$, $A = \emptyset$, and $B = \{1\}$. Then $A \times C = \emptyset = B \times C$, but $A \neq B$.

Every even integer is the sum of three distinct even integers.

True: let a be an even integer. If $a = 0$, then $a = 6 + (-4) + (-2)$. If $a \neq 0$, then $a = 4a + (-2a) + (-a)$, and all those integers are even and distinct.

There exists an irrational number p and a rational number q such that $\frac{p}{q}$ is rational.

False. We prove by contradiction. Suppose p is irrational, and q is rational, so $q = \frac{x}{y}$ for some nonzero integers x and y. If $\frac{p}{q}$ is rational, then $\frac{p}{q} = \frac{a}{b}$ for some nonzero integers a and b. Then $q = \frac{aq}{b}$, and both numerator and denominator are integers, contradicting that q is irrational. We conclude that, for every irrational number p and every rational number q, $\frac{p}{q}$ is irrational.

There exists a rational number p and an irrational number q such that $\frac{p}{q}$ is rational.

True: let $p = 0$ and let q be any irrational number. Then $\frac{p}{q} = 0$, which is rational.

There exist prime numbers p and q such that $p - q = 513$.

False. Suppose the statement is true. If p and q have an odd difference, then they have different parity, so one of them is even. The only even prime is 2, so p or q is equal to 2. Since $p - q$ is positive, q is smaller than p, so $q = 2$ because 2 is the smallest prime.

Then $p = 513 + 2 = 515$, but $5 | 515$, contradicting that p is prime.

We conclude that for every pair of primes p and q, $p - q \neq 513$.