Graph Saturation in Color

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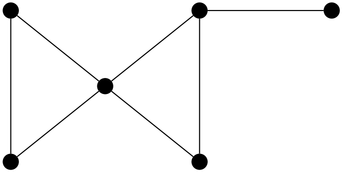
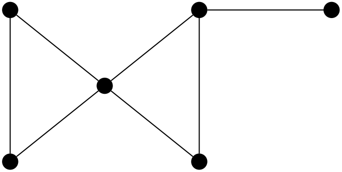
Outline

1. Introduction to Graphs
2. Graph Saturation
3. Ramsey Theory
4. Colored Graph Saturation

What Do You Know About Graphs?



Graph Basics



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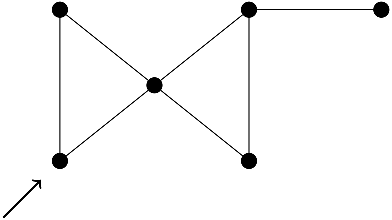

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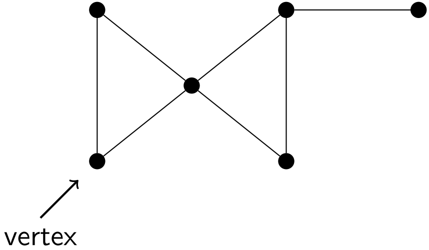

$|G|$: number of vertices in G

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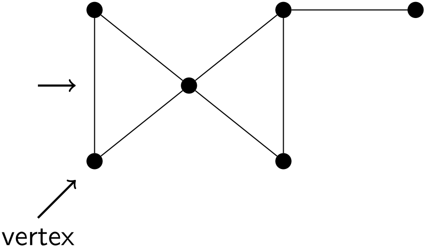

$|G|$: number of vertices in G

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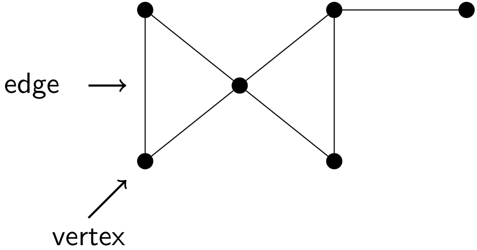

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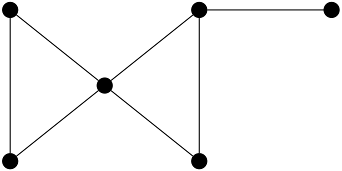

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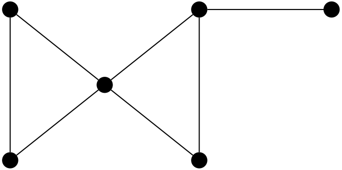

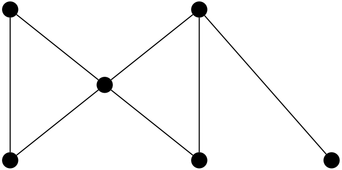

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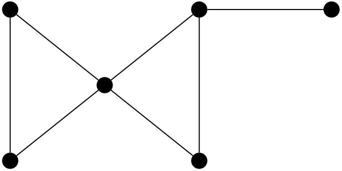

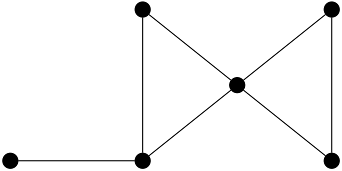

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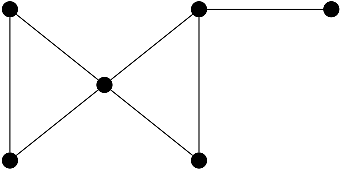

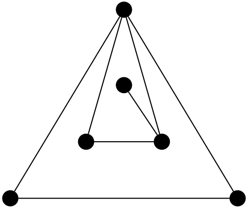

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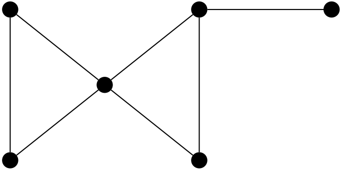

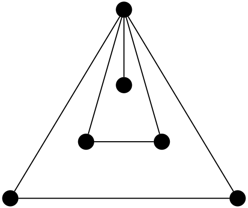

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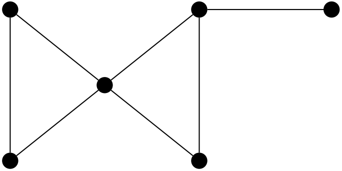
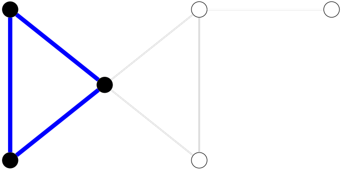


Subgraph:

Graph Basics



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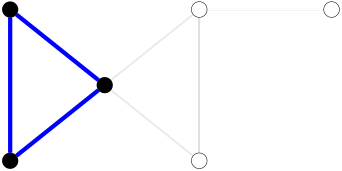

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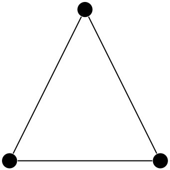

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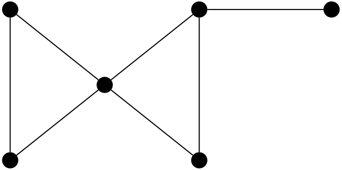

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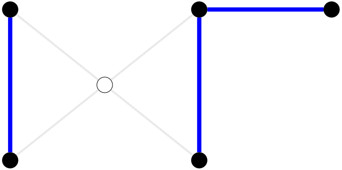

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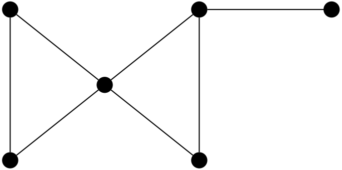

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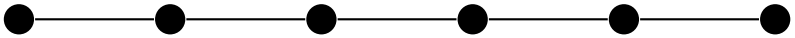

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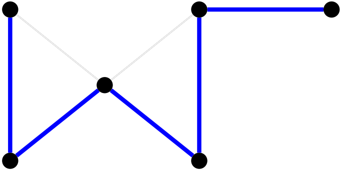

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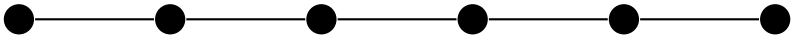

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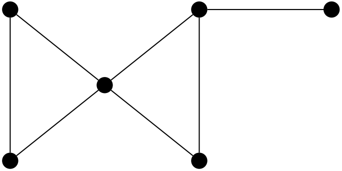

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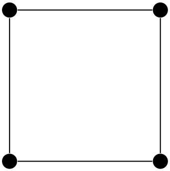

Common Graphs

Path:

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Matching:

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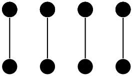

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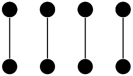

Clique:

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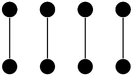

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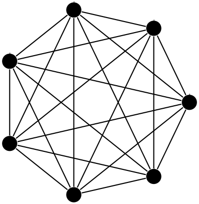

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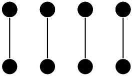

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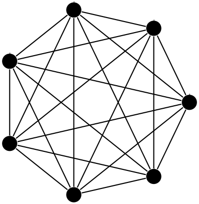

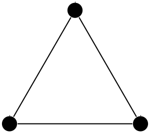

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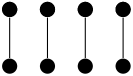

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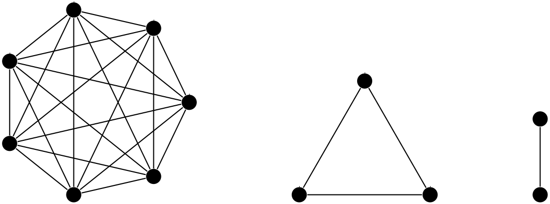

Outline

Goal:

Avoid a forbidden subgraph, but add as many edges as possible.

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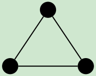
Example:

Let the triangle be forbidden.



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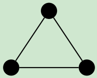

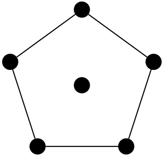

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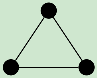
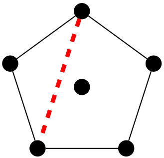
Let the triangle be forbidden. 



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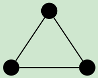

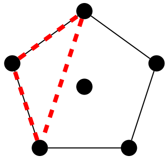

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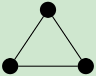
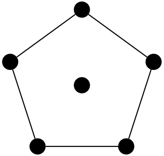
Let the triangle be forbidden. 



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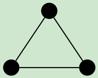

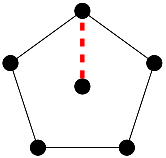

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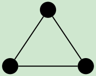

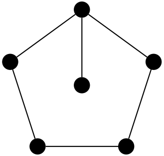

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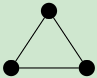

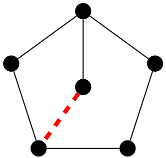

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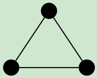

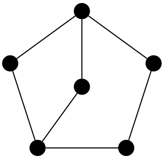

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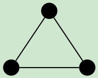

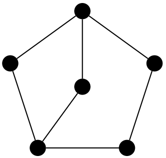

This graph is *triangle saturated*

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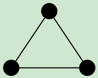
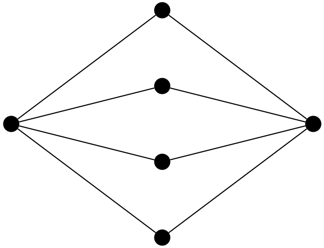
Let the triangle be forbidden. 



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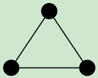

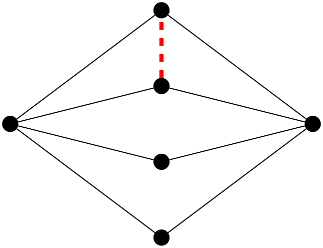

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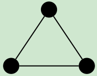

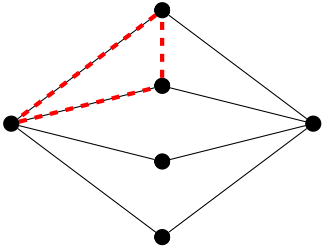

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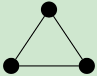

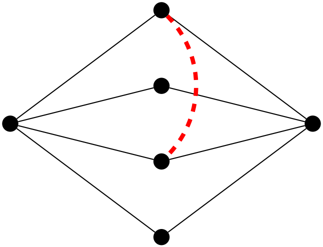

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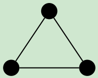

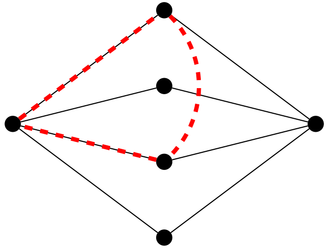

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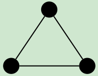

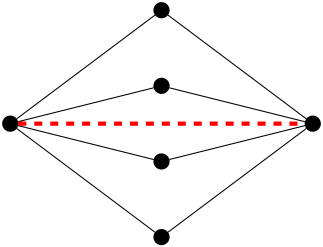

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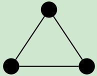

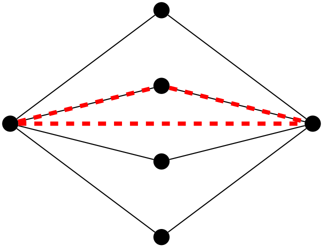

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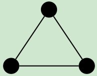

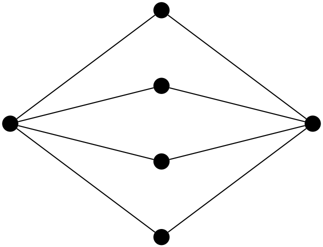

This graph is *triangle saturated*

Goal:

Avoid a forbidden subgraph, but add as many edges as possible.

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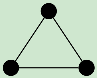

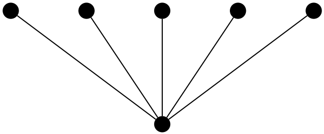

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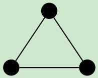

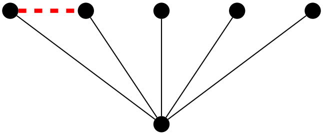

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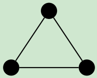

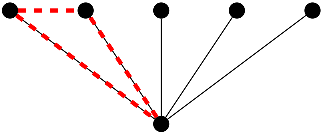

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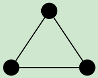

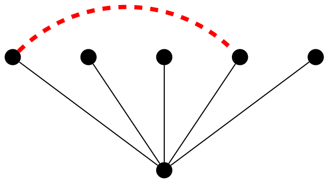

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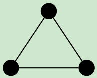

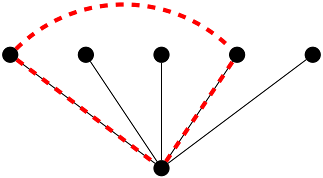

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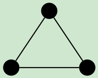

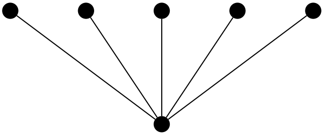

This graph is *triangle saturated*

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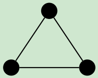

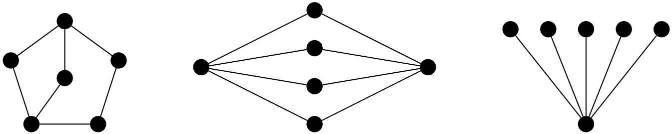

Definition

A graph G is H-*saturated* if:

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Avoid a forbidden subgraph, but add as many edges as possible.

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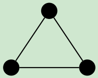

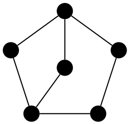

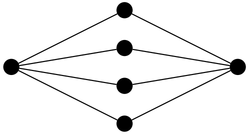

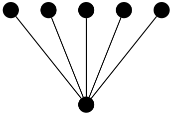

Definition

A graph G is H-*saturated* if:

1. H is *not* a subgraph of G

Goal:

Avoid a forbidden subgraph, but add as many edges as possible.

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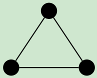
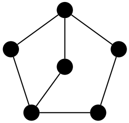
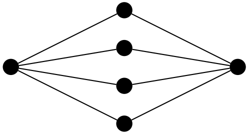
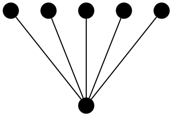






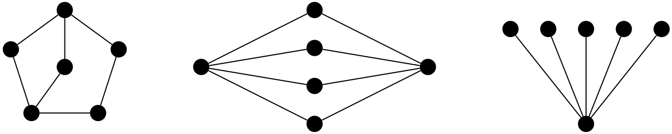


Definition

A graph G is H-*saturated* if:

1. H is *not* a subgraph of G and
2. If we add any edge to G, H *is* a subgraph of the resulting graph.

Saturation Number



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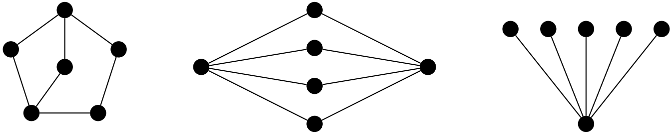

Definition: (Erdős-Hajnal-Moon)

The *saturation number* of a graph H and a number n is the minimum number of edges in a graph on n vertices that is H saturated.

$$sat(n; H) := \min\{\|G\| : |G| = n \text{ and } G \text{ is } H \text{ saturated}\}$$

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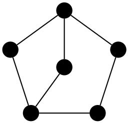
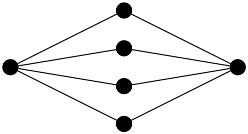
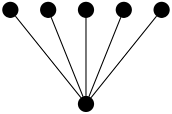

7 edges

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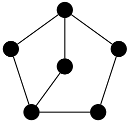

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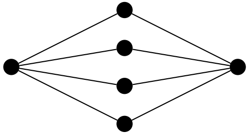
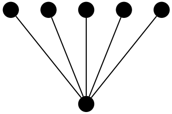


8 edges



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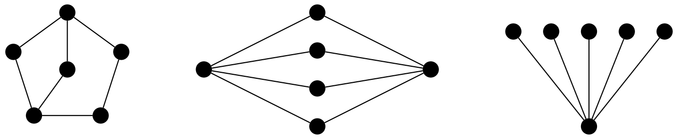

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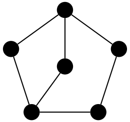

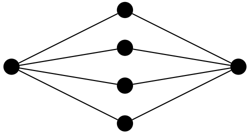

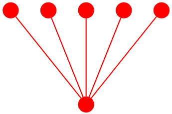

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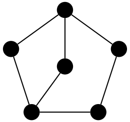

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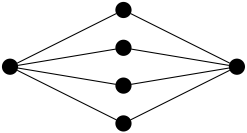

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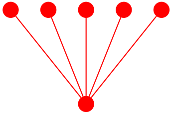

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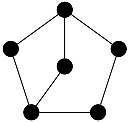
Given the above examples:

$$sat(6; \textit{triangle})$$

Saturation Number



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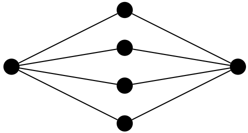

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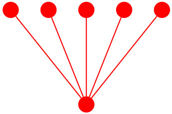

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Given the above examples:

$$sat(6; triangle) \leq 5$$

Outline

1. Introduction to Graphs
2. Graph Saturation
3. Ramsey Theory
4. Colored Graph Saturation

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Definition:

An *edge coloring* of a graph is an assignment of a color to each edge.

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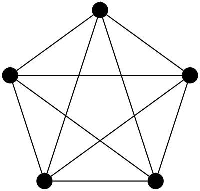

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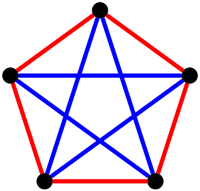

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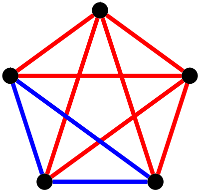

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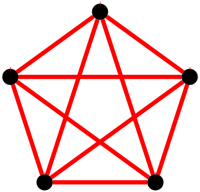

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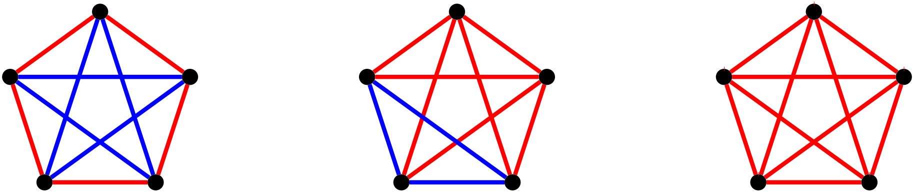

Definition:

A graph is *monochromatic* if all its edges are assigned the same color.

Ramsey Problems

Goal:

An edge-coloring with no forbidden monochromatic subgraph.

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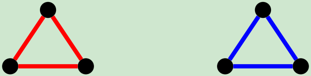
Color red and blue; forbid red triangle and blue triangle.



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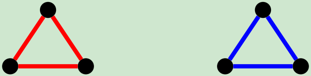

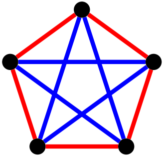

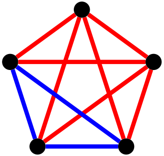

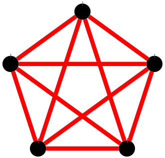

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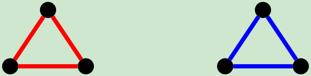
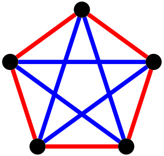
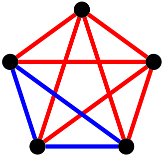
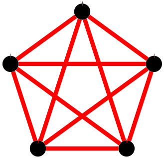




good coloring





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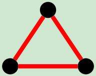

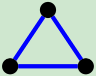

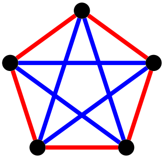

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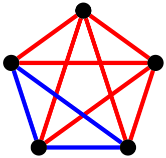
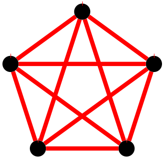


bad coloring



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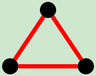
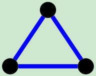
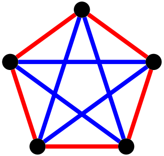
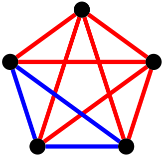


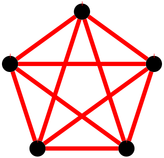




good coloring



bad coloring



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Ramsey Problems

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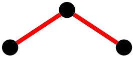
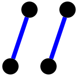
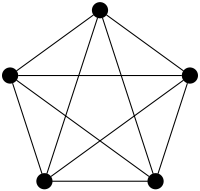
Example: Any red-blue coloring of the edges of K_5 contains a monochromatic red  or a monochromatic blue 



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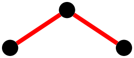
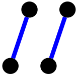
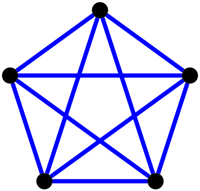
Example: Any red-blue coloring of the edges of K_5 contains a monochromatic red  or a monochromatic blue 



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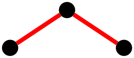
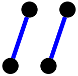
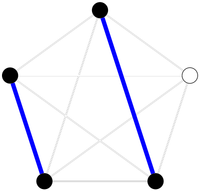
monochromatic red  or a monochromatic blue 



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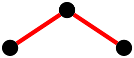
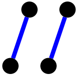
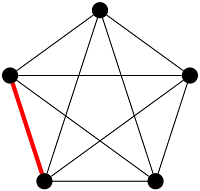
monochromatic red  or a monochromatic blue 



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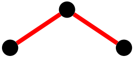
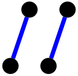
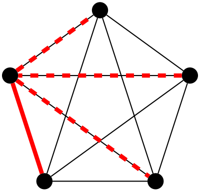
monochromatic red  or a monochromatic blue 



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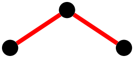
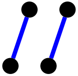
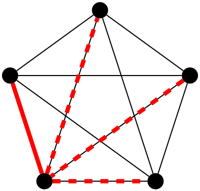
monochromatic red  or a monochromatic blue 



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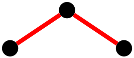
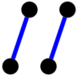
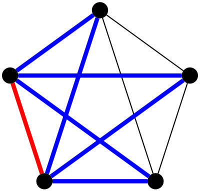
monochromatic red  or a monochromatic blue 



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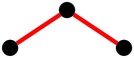
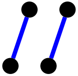
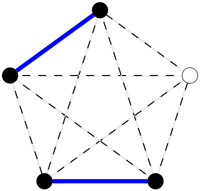
monochromatic red  or a monochromatic blue 



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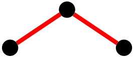
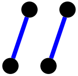
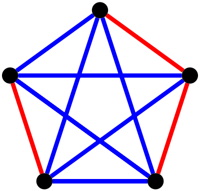
Example: Any red-blue coloring of the edges of K_5 contains a monochromatic red  or a monochromatic blue 



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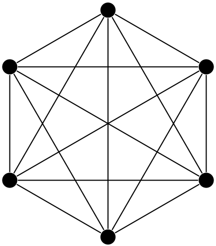

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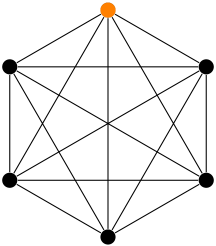

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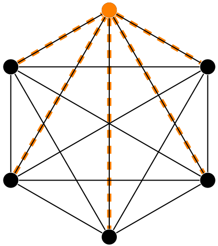

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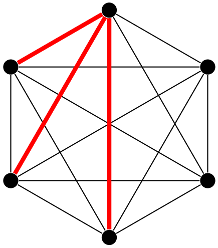

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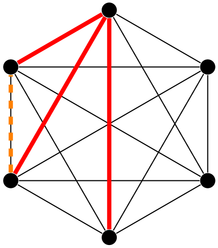

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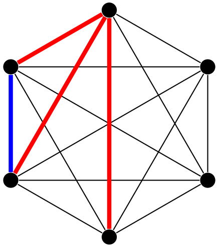

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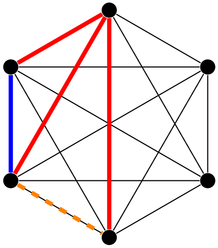

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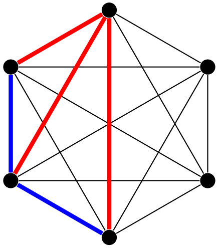

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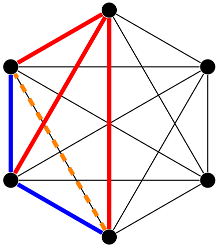

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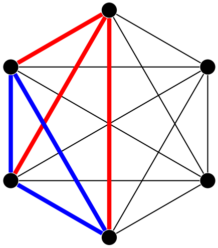

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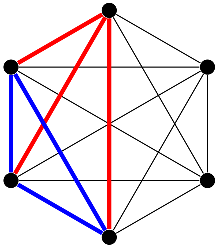

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Given **any** collection of forbidden subgraphs, and **any** sufficiently large clique, no good coloring exists.

The lowest "sufficiently large" number for a given collection of subgraphs is called the Ramsey Number.

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- $43 \leq R(5,5) \leq 49$

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Erdős-Szekeres: Given N, any sufficiently large collection of points in general position contains a subset forming the vertices of a convex N-gon

Van der Waerden: Any coloring of the natural numbers contains arbitrarily long monochromatic arithmetic sequences.

Green-Tao: The sequence of prime numbers contains arbitrarily long arithmetic progressions.

Outline

Marrying The Two

Recall:

A graph G is H-*saturated* if G contains no H subgraph, but adding any edge to G creates an H subgraph.

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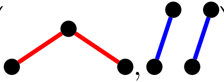
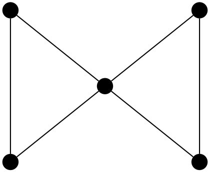
Ramsey version of saturation:

Given forbidden graphs $H_1, \ldots, H_k$ (in colors $1, \ldots, k$ respectively), we say a graph G is $(H_1, \ldots, H_k)$-*saturated* if a good edge-coloring of G exists (using colors $1, \ldots, k$), but if we add *any* edge to G, all colorings are bad.

Marrying The Two

Example: the graph below is $\left(\right.$  $\left.\right)$-saturated.



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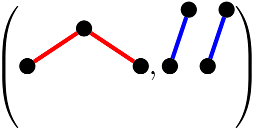
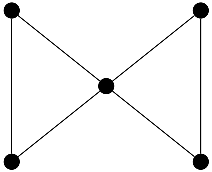
Example: the graph below is  -saturated.



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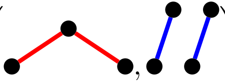
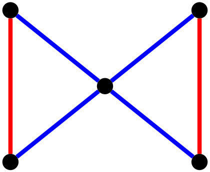
Example: the graph below is ()-saturated.



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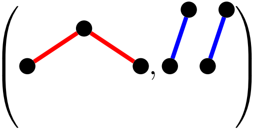
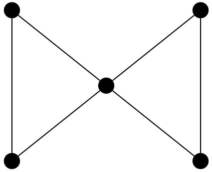
Example: the graph below is  -saturated.



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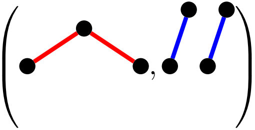
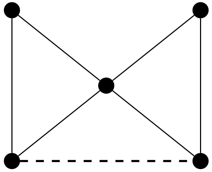
Example: the graph below is  -saturated.



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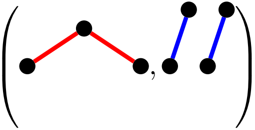
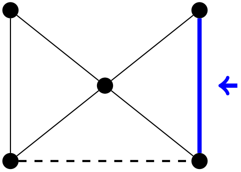
Example: the graph below is  -saturated.



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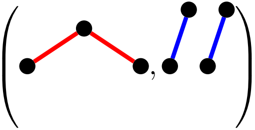
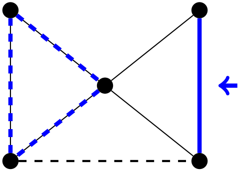
Example: the graph below is  -saturated.



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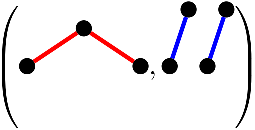

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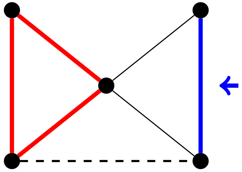

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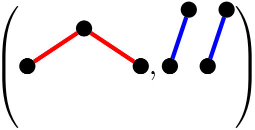
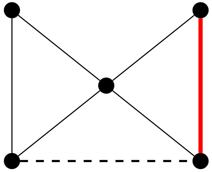
Example: the graph below is  -saturated.



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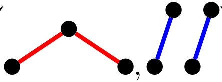
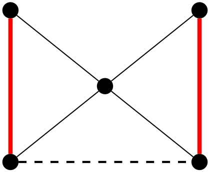
Example: the graph below is ()-saturated.



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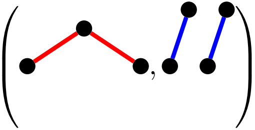
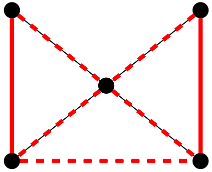
Example: the graph below is  -saturated.



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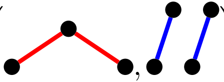
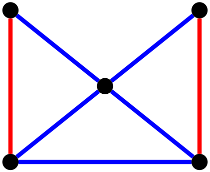
Example: the graph below is $\Bigg($  $\Bigg)$-saturated.



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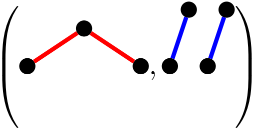
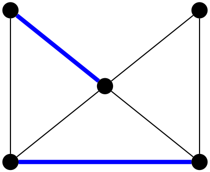
Example: the graph below is  -saturated.



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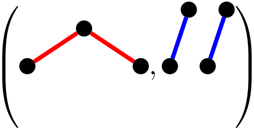
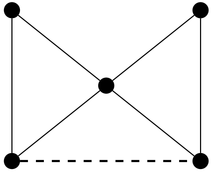
Example: the graph below is -saturated.



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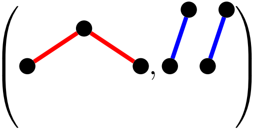
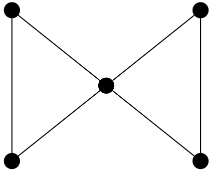
Example: the graph below is  -saturated.



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$rsat(n; H_1, \dots, H_k)$

Again, we are interested in $(H_1, \dots, H_k)$-saturated graphs with as few edges as possible.

Definition:

For a number n and forbidden subgraphs $H_1, \dots, H_k$, we define

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to be the minimum number of edges over all n-vertex graphs that are $(H_1, \dots, H_k)$-saturated.

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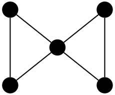
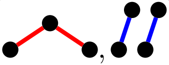
 is ()-saturated

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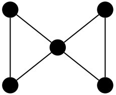
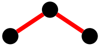
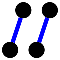
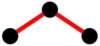
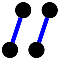
 is (, )-saturated, so *rsat* (5; , ) ≤ 6.

Forbidding Cliques: Ramsey Numbers

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Let $R = R(c_1, \ldots, c_t)$ be the smallest natural number so that, for forbidden cliques $K_{c_1}, \ldots, K_{c_t}$, no good coloring of K_R exists.

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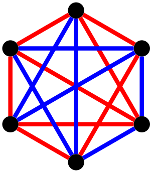

no good coloring of K_6 exists

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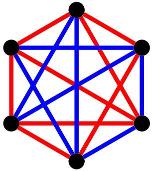

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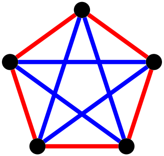

a good coloring of K_5 exists

Forbidding Cliques: Hanson-Toft

Let $R = R(c_1, \ldots, c_t)$. The construction below is $(K_{c_1}, \ldots, K_{c_t})$-saturated.

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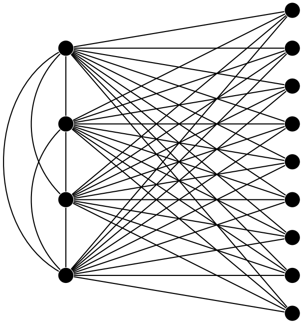

K_{R-2} $\qquad$ $n - (R-2)$

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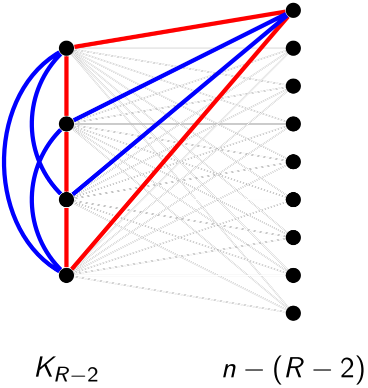

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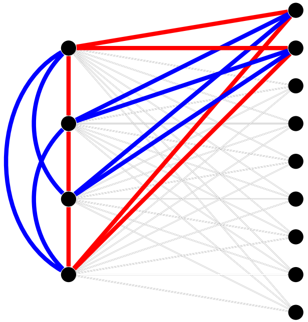

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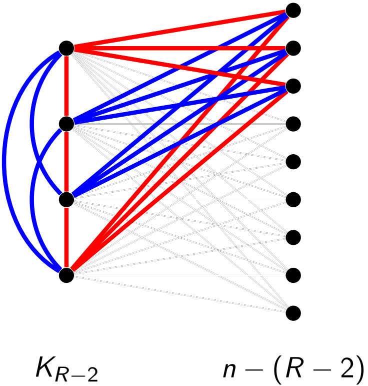

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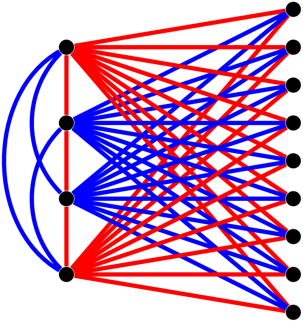

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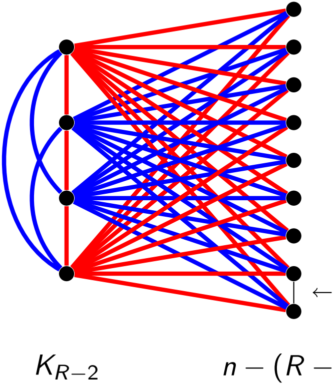

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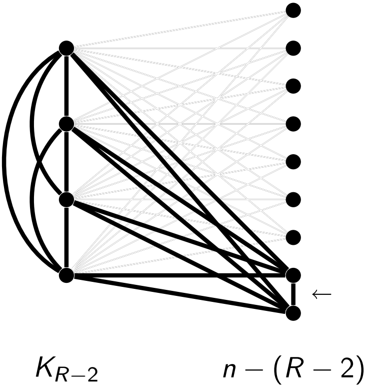

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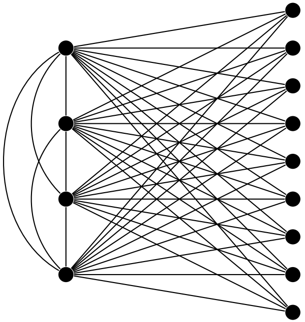

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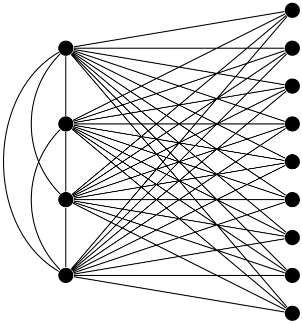

K_{R-2} $\qquad$ $n - (R-2)$

Hanson-Toft

Conjecture: The construction above has the fewest possible edges.

Hanson-Toft

Hanson-Toft Conjecture

$$sat(n; \mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r \end{cases}$$

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Chen, Ferrara, Gould, Magnant, Schmitt; 2011

$$sat(n; \mathcal{R}_{min}(K_3, K_3)) = \begin{cases} \binom{n}{2} & n < 6 = r \\ 4n - 10 & n \geq 56 \end{cases}$$

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Matching Saturation: Mader, 1973

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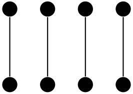
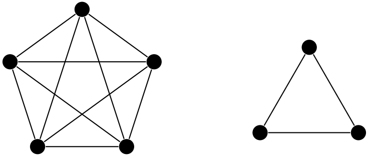


Graph:



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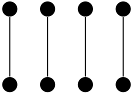

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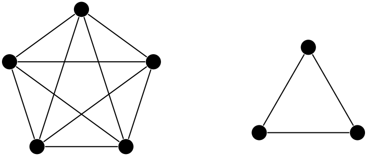

- No forbidden subgraph

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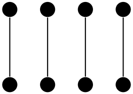

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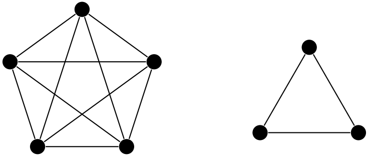

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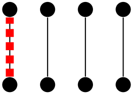

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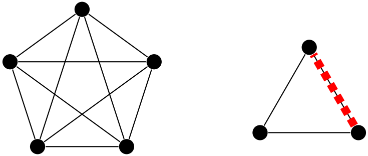

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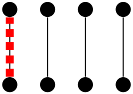

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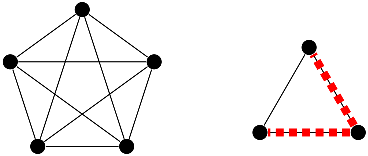

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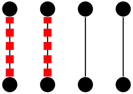

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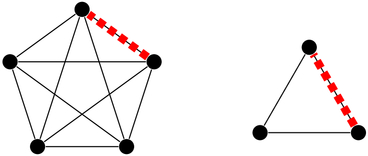

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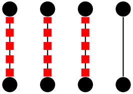

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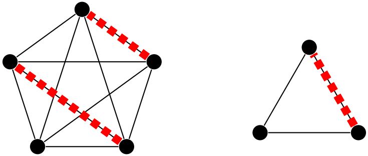

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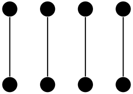

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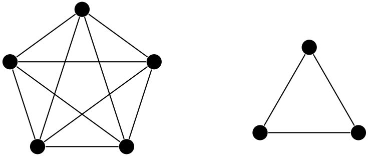

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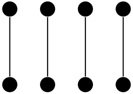

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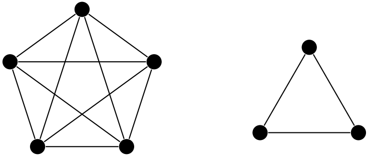

- No forbidden subgraph
- Adding any edge creates forbidden subgraph

Matching Saturation: Mader, 1973

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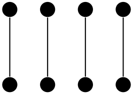

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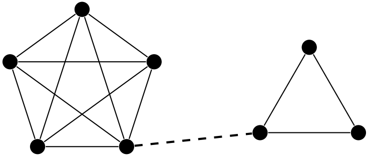

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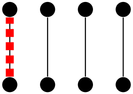
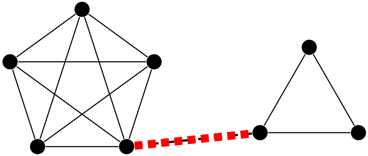
Matching Saturation: Mader, 1973

Forbidden:



Graph:



- No forbidden subgraph
- Adding any edge creates forbidden subgraph

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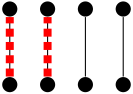

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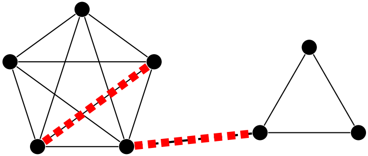

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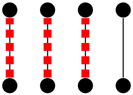
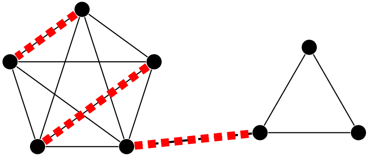


Graph:



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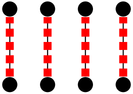

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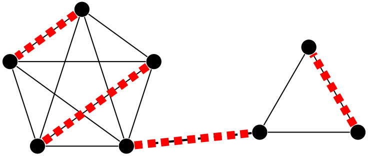

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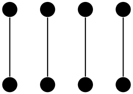
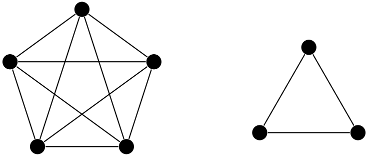


Graph:



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Forbidding Matchings: Ferrara-Kim-Y

Example: Forbidden matching on 10 edges.

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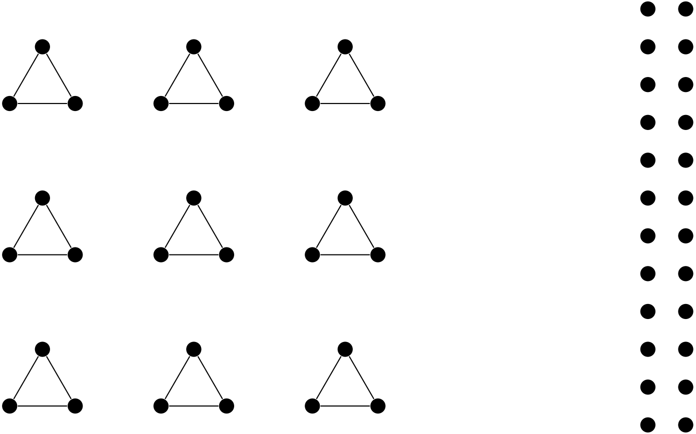

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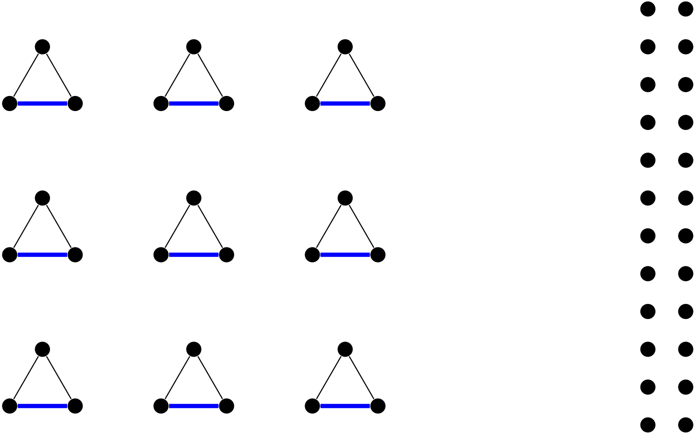

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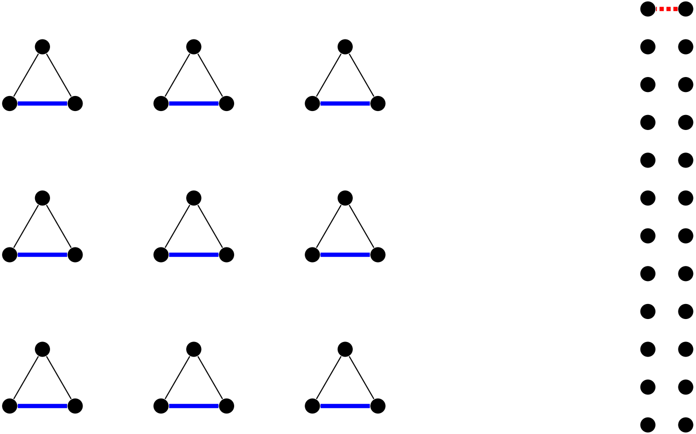

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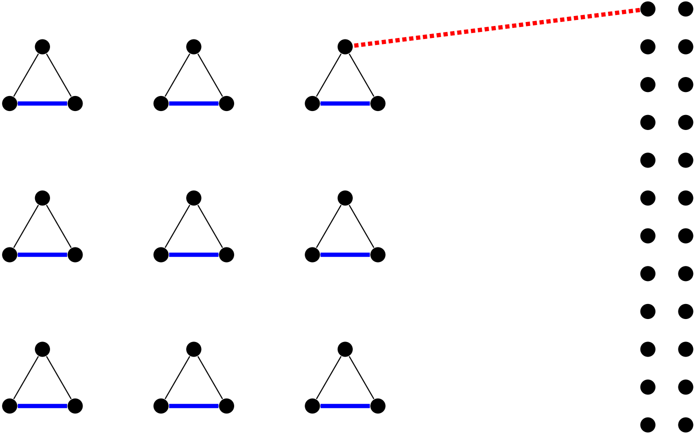

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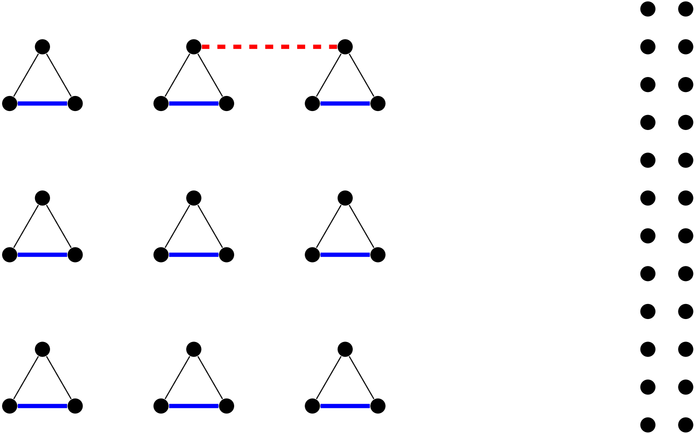

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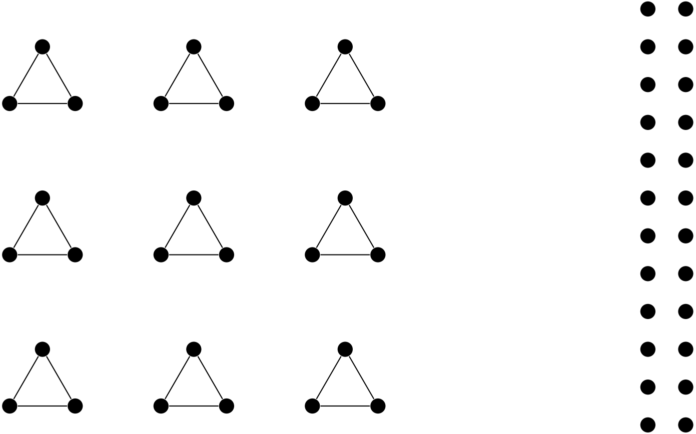
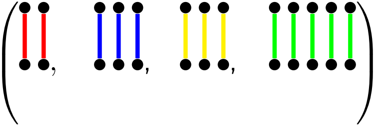
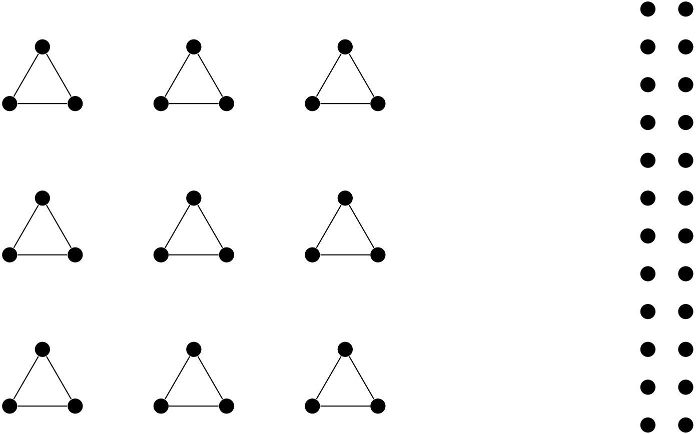


Forbidding Matchings: Ferrara-Kim-Y

Colored Example: Forbidden  .



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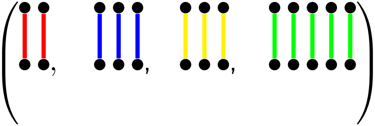
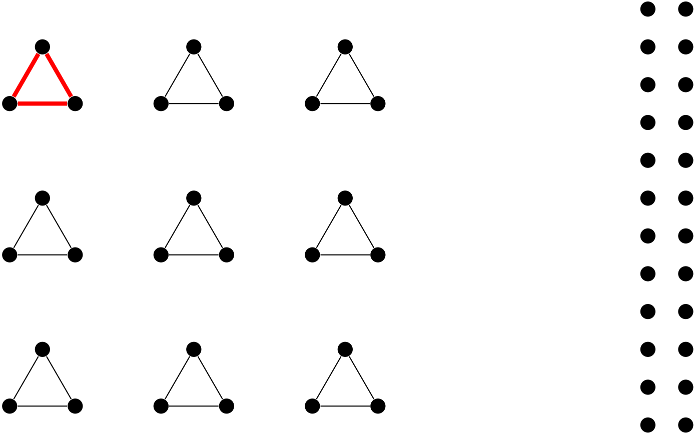
Colored Example: Forbidden  .



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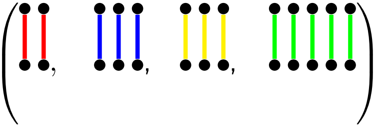

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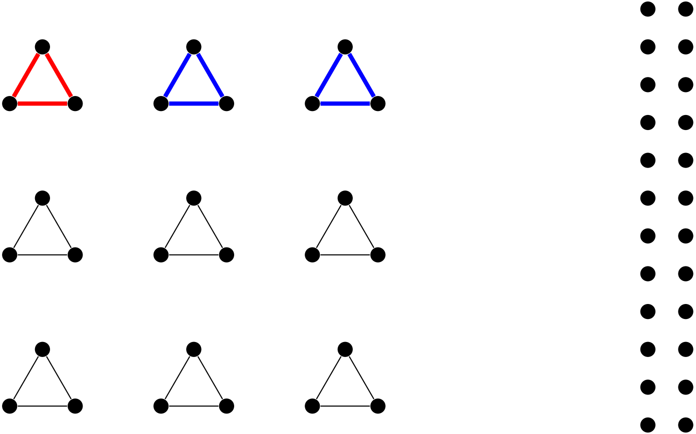

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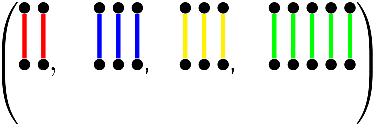
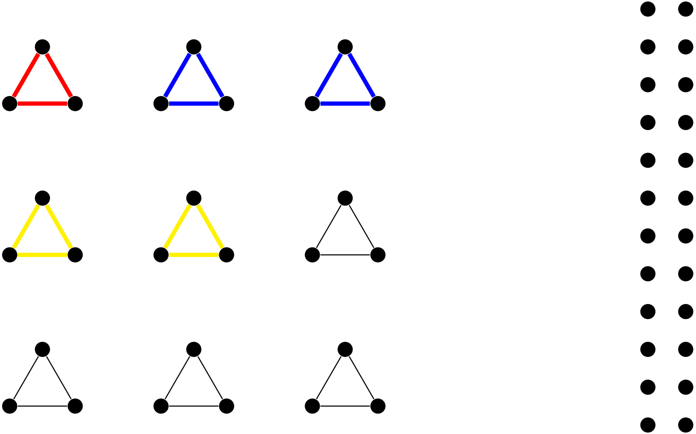
Colored Example: Forbidden .



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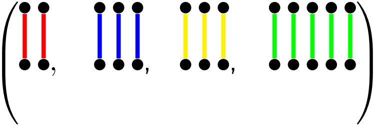
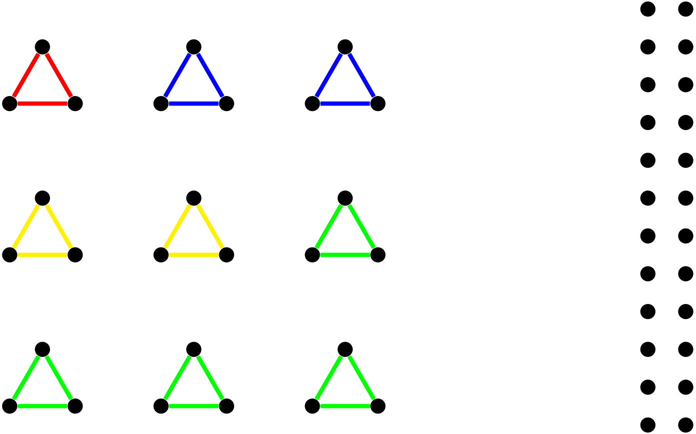
Colored Example: Forbidden  .



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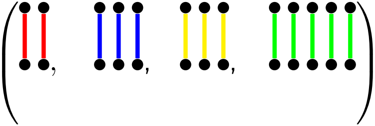
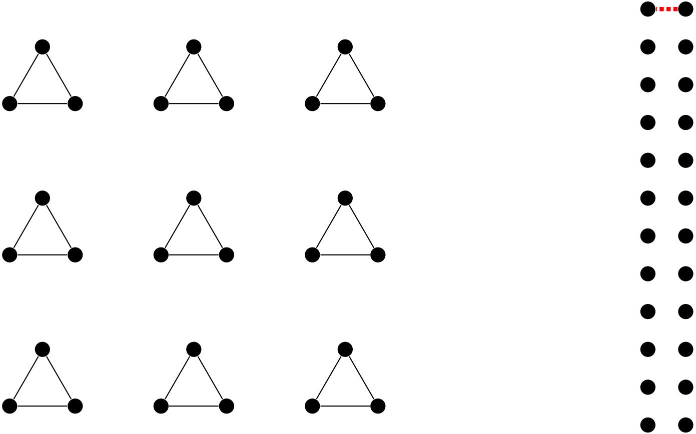
Colored Example: Forbidden  .



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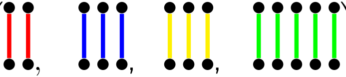
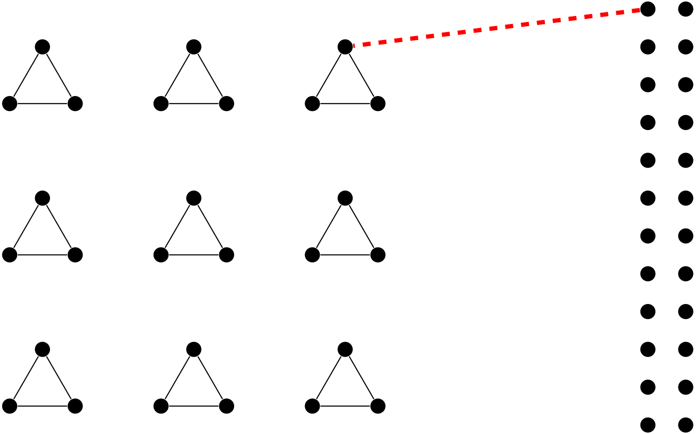
Colored Example: Forbidden $\Bigg($  $\Bigg)$.



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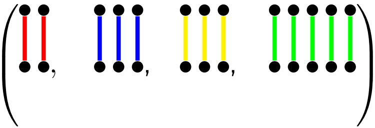
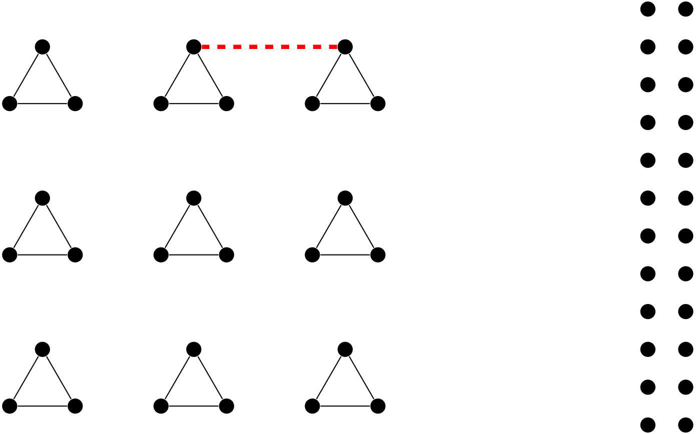
Colored Example: Forbidden  .



Iterated Recoloring: Ferrara-Kim-Yeager

Goal:

Use results from (uncolored) saturation in the Ramsey version.

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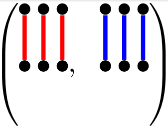
Example: Forbidden graphs  .

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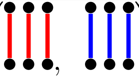
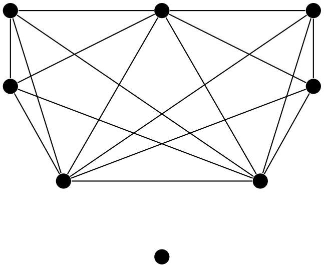
Example: Forbidden graphs ().



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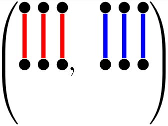
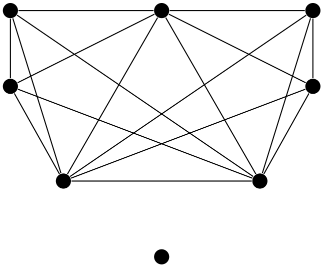
Example: Forbidden graphs  .



good coloring

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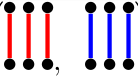
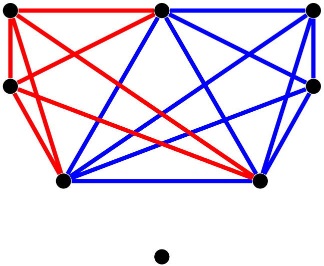
Example: Forbidden graphs $\Bigg($  $\Bigg)$.



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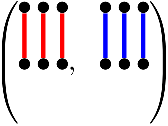
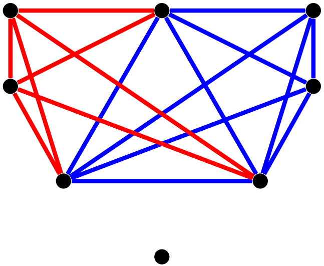
Example: Forbidden graphs  .



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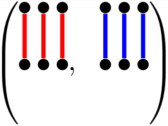
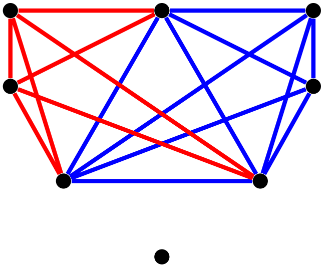
Example: Forbidden graphs  .



good coloring

↓

make red-heavy

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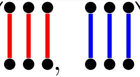
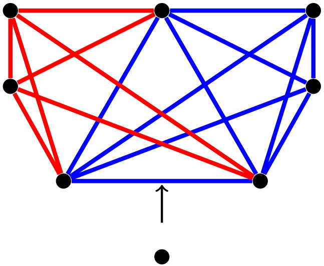
Example: Forbidden graphs ().



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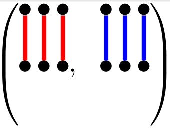
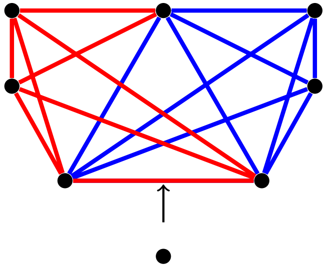
Example: Forbidden graphs  .



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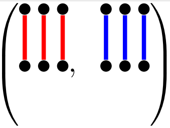
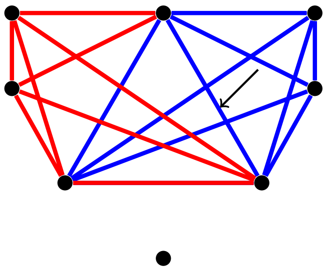
Example: Forbidden graphs  .



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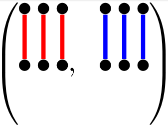
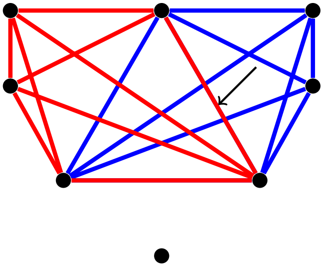
Example: Forbidden graphs  .



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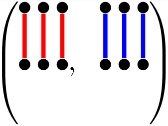
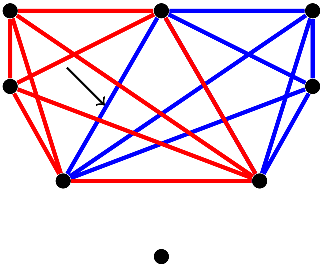
Example: Forbidden graphs  .



good coloring

$\downarrow$

make red-heavy

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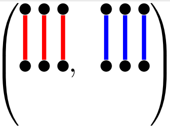
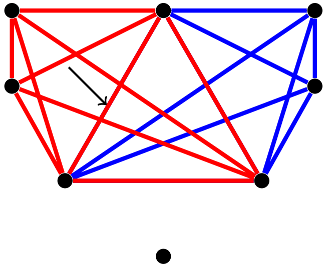
Example: Forbidden graphs .



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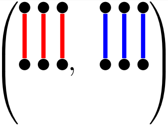
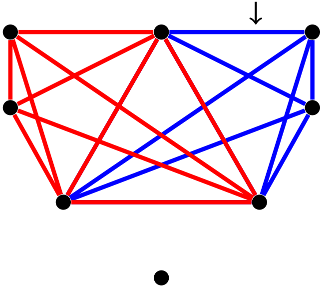
Example: Forbidden graphs  .



good coloring
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make red-heavy

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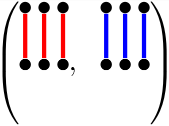
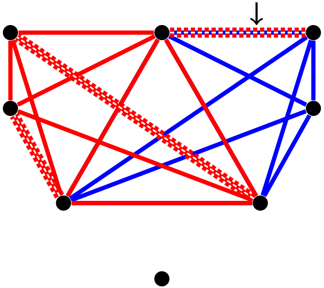
Example: Forbidden graphs .



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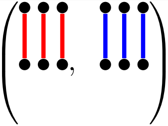
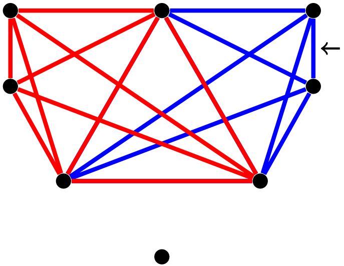
Example: Forbidden graphs  .



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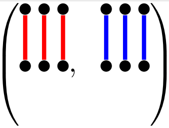
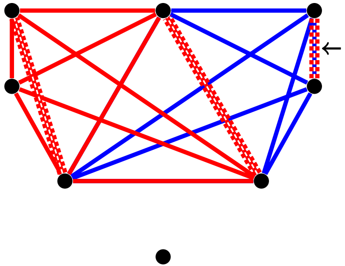
Example: Forbidden graphs  .



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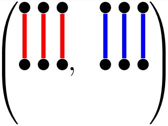
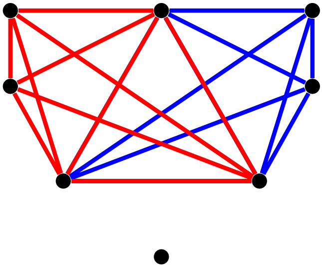
Example: Forbidden graphs  .



good coloring

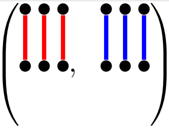
↓

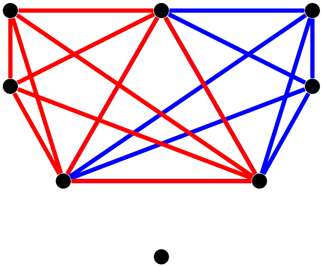
make red-heavy

Iterated Recoloring: Ferrara-Kim-Yeager

Goal:

Use results from (uncolored) saturation in the Ramsey version.

Example: Forbidden graphs .



good coloring

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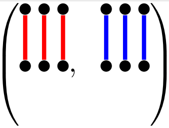
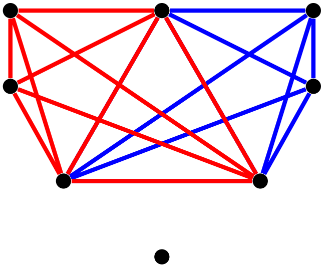
Example: Forbidden graphs  .



good coloring

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make red-heavy

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take red subgraph

Goal:

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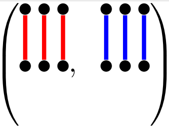
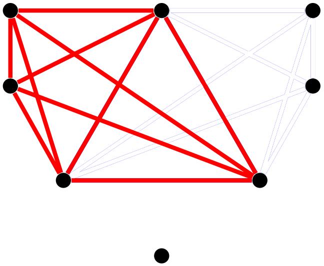
Example: Forbidden graphs  .



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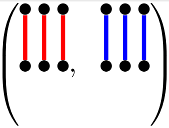
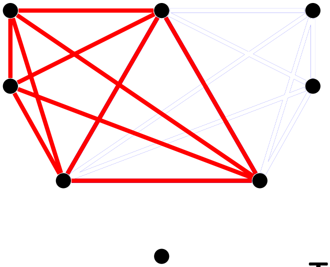
Example: Forbidden graphs .



good coloring

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take red subgraph

This (uncolored!) subgraph is saturated.

Iterated Recoloring: Ferrara-Kim-Yeager

Goal:

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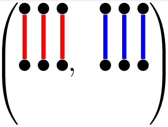
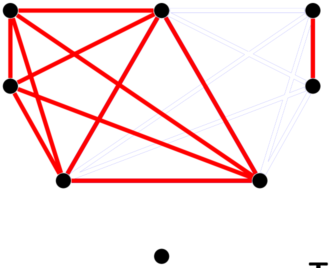
Example: Forbidden graphs  .



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Iterated Recoloring: Ferrara-Kim-Yeager

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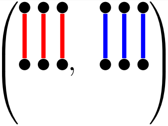
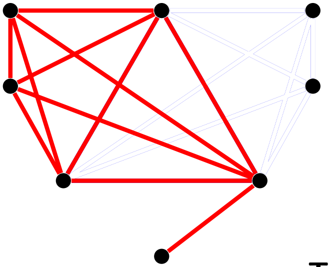
Example: Forbidden graphs .



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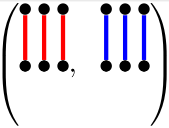
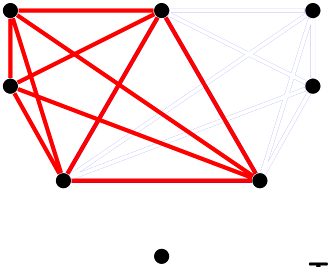
Example: Forbidden graphs  .



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Thanks for Listening!