Extremal Problems in Disjoint Cycles and Graph Saturation

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Final Examination
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07 April 2015

Disjoint Cycles and Corrádi-Hajnal

Corrádi-Hajnal, 1963

If G is a graph on n vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

Conjecture of Erdős

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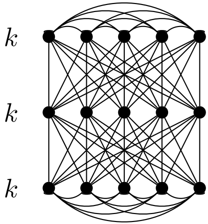
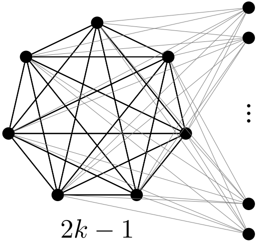
Sharpness:





Enomoto, Wang

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If G is a graph on n vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

$\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$
That is: low-degree vertices are all connected;
other vertices have higher degree to compensate

Enomoto 1998, Wang 1999

If G is a graph on n vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k disjoint cycles.

Implies Corrádi-Hajnal

Enomoto 1998, Wang 1999

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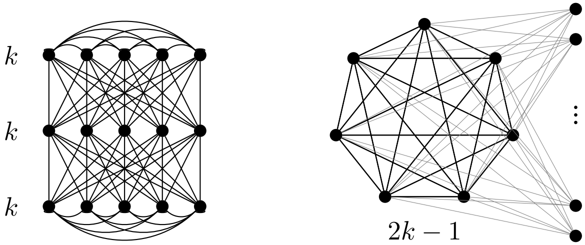


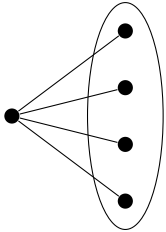
$n = 3k$ $\qquad$ $\alpha(G) > n - 2k$

H. A. Kierstead, A. V. Kostochka and Y.,
On the Corrádi-Hajnal Theorem and a question of Dirac.
submitted

A Note about Independence Number and Cycles

Observation

Any cycle has at least two vertices outside any independent set.



Corollary

Any graph G with k disjoint cycles has $\alpha(G) \leq |G| - 2k$.

Kierstead-Kostochka-Y, 2015+

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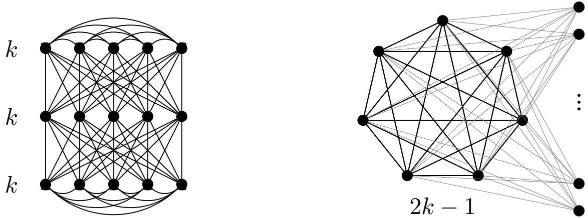

Kierstead-Kostochka-Y., 2015$^+$

For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k+1$ and $\sigma_2(G) \geq 4k-3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n-2k$.

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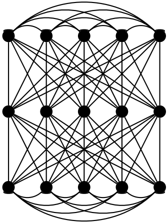
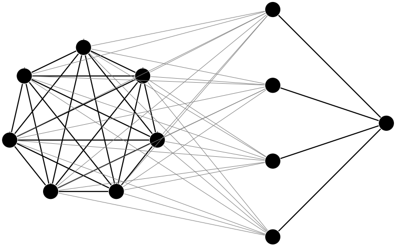
$n \geq 3k+1$

k

k

k





$2k-1$ $\qquad$ k

Kierstead-Kostochka-Y., 2015+

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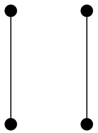
$k = 1$:



Kierstead-Kostochka-Y., 2015+

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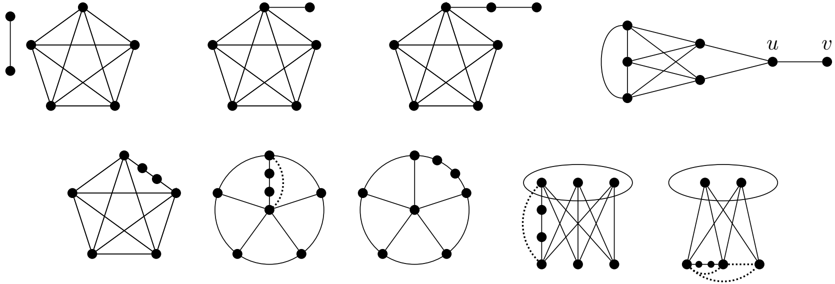

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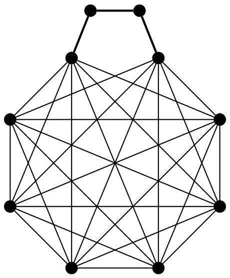

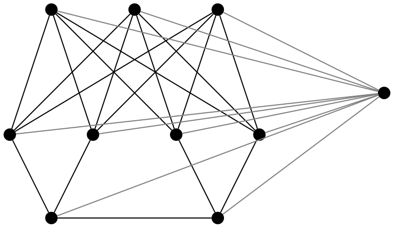

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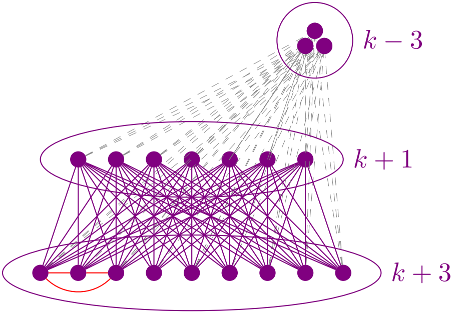
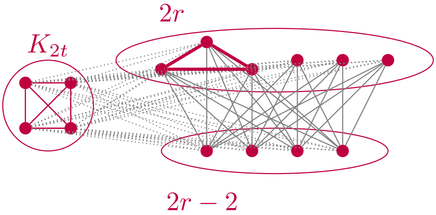
$$\sigma_2 = 4k-4:$$





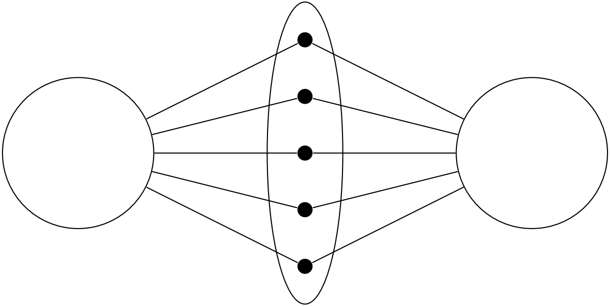
Section 1

H. A. Kierstead, A. V. Kostochka and Y.,
The $(2k-1)$-connected graphs with no k disjoint cycles.
Combinatorica, to appear.

Dirac: $(2k-1)$-connected without k disjoint cycles

Dirac, 1963

What $(2k-1)$-connected graphs do not have k disjoint cycles?



Observation:

G is $(2k-1)$ connected $\Rightarrow \delta(G) \geq 2k-1 \Rightarrow \sigma_2(G) \geq 4k-2$

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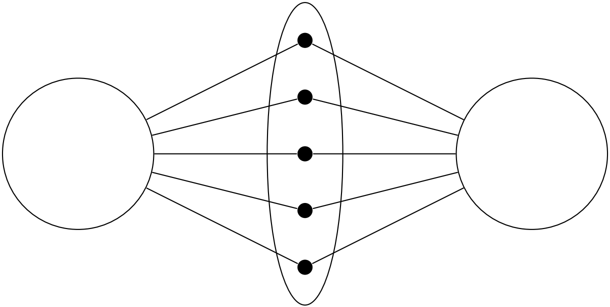

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KKMY: Holds for $\sigma_2(G) \geq 4k-3$

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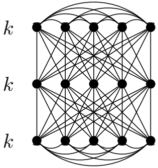
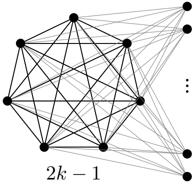
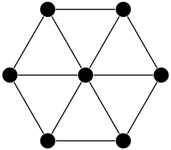
Answer to Dirac's Question for Simple Graphs

Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if

- if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$, and
- $\alpha(G) \leq |G| - 2k$, and
- if $k = 2$ then G is not a wheel.







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Further:

characterization for multigraphs

Multigraph Corrádi-Hajnal

The **simple degree** of a vertex is the number of its (distinct) neighbors.

Theorem (Extension of Corrádi-Hajnal to Multigraphs)

For $k \in \mathbb{Z}^+$, let G be a multigraph with simple degree at least $2k$. Then G has k disjoint cycles if and only if

$$|V(G)| \geq 3k - 2\ell - \alpha'$$

where $3k - 2\ell - \alpha'$ is the trivially necessary number of vertices.

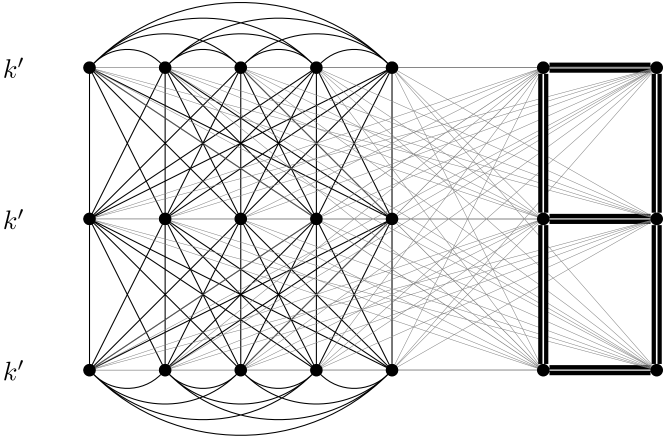
Corollary

Let G be a multigraph with simple degree at least $2k-1$ for some integer $k \geq 2$. Suppose G contains at least one loop. Then G has k disjoint cycles if and only if

$$|V(G)| \geq 3k - 2\ell - \alpha'.$$

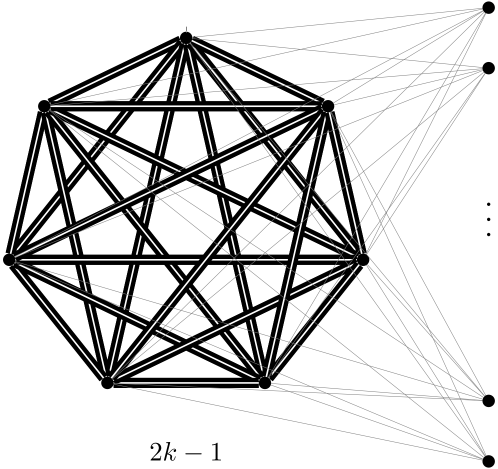
Multiple edges have a perfect matching

Example: $k = 8$



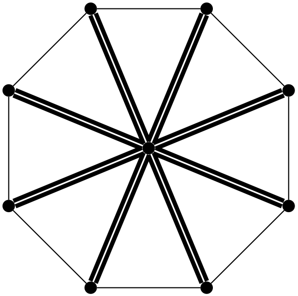
Big independent set, incident to no multiple edges

Example: $k = 4$



Wheel, with possibly some spokes multiple

Example: $k = 2$

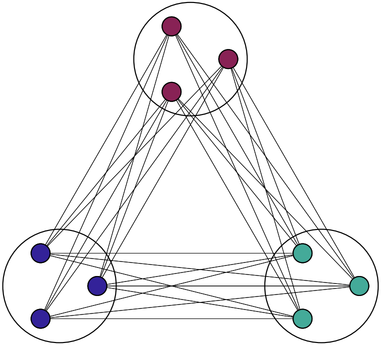


H. A. Kierstead, A. V. Kostochka, T. N. Molla, and Y.,
Sharpening an Ore-type version of the Corrádi-Hajnal Theorem.
in preparation

Equitable Coloring

Definition

An *equitable k-coloring* of a graph G is a partition of its vertices into independent sets (called color classes) such that any two color classes differ in size by at most one.



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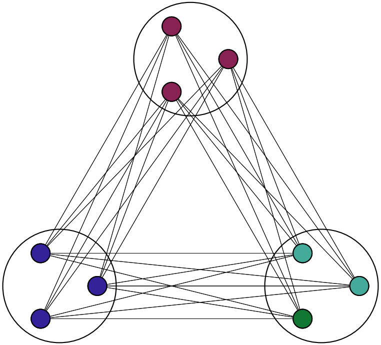

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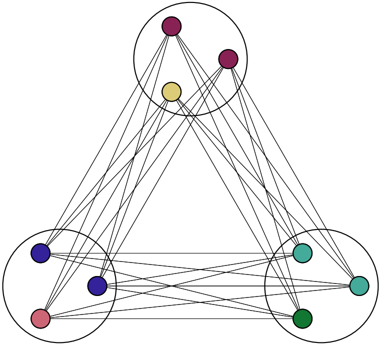
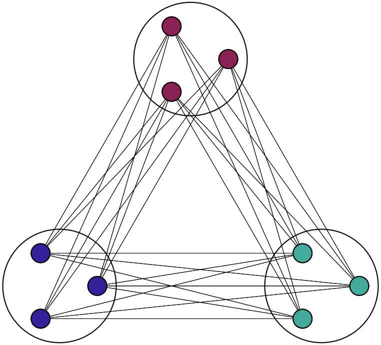


Equitable Coloring and Cycles

$n = 3k$

If G has $n = 3k$ vertices, then G has an equitable k-coloring if and only if $\overline{G}$ has k disjoint cycles (all triangles).



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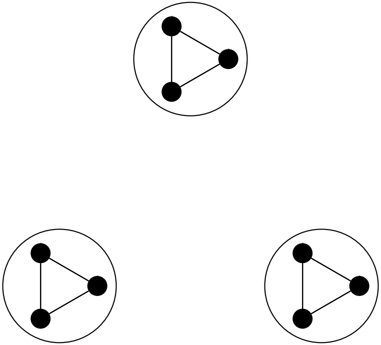

Ore Conditions

Chen-Lih-Wu Conjecture

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then G is equitably k-colorable.

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Kierstead-Kostochka-Molla-Y, 2015+

If G is a k-colorable $3k$-vertex graph such that for each edge xy, $d(x) + d(y) \leq 2k + 1$, then G is equitably k-colorable, or is one of several exceptions.

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Equivalent

If G is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles, or is one of several exceptions, or $\overline{G}$ is not k-colorable.

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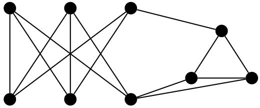
For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

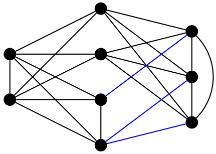
Exceptions

- $k = 3$

Equitable coloring:



Cycles:



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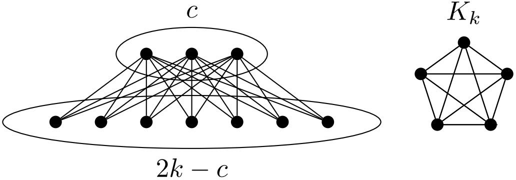

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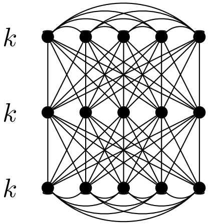

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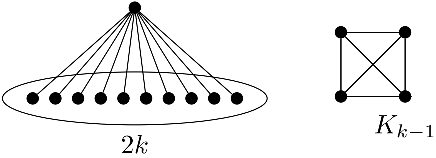

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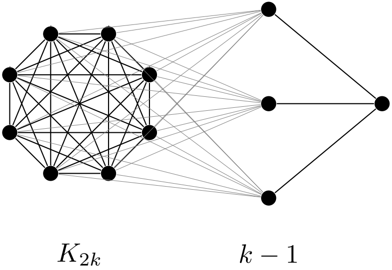
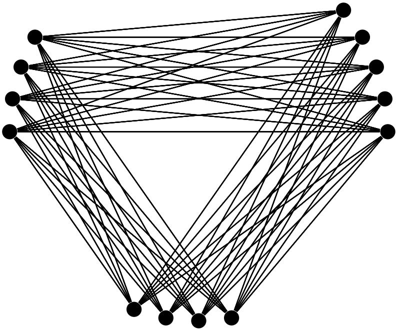


Section 2

Turán: Maximum number of edges in a graph with no forbidden subgraph.



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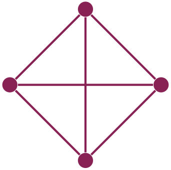
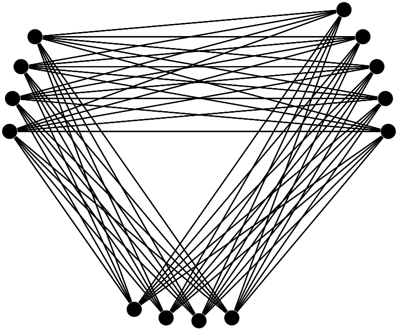


Turán: Maximum number of edges in a graph with no forbidden subgraph.
Extra property:



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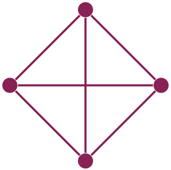

Introduction to Graph Saturation

Turán: Maximum number of edges in a graph with no forbidden subgraph.
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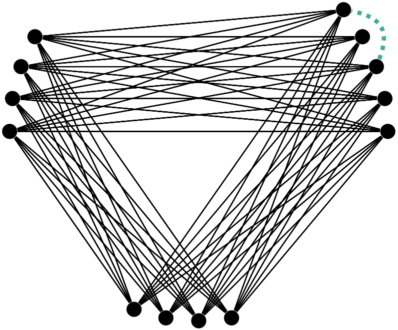

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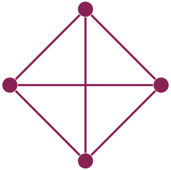

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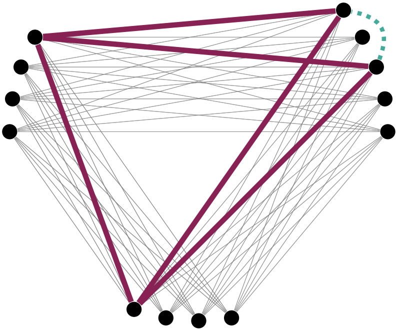

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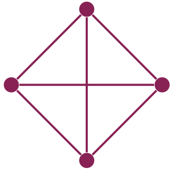
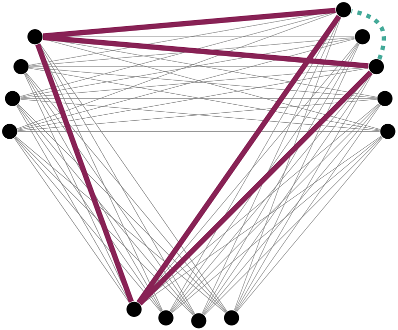


Introduction to Graph Saturation

Turán: Maximum number of edges in a graph with no forbidden subgraph.
Extra property: adding any edge results in a forbidden subgraph



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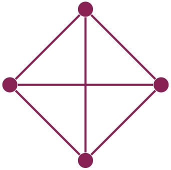

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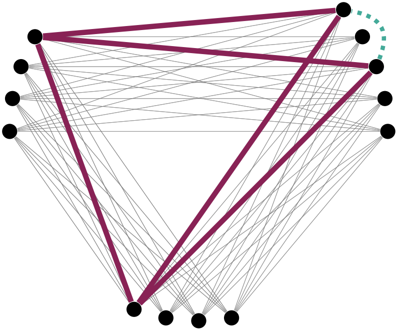

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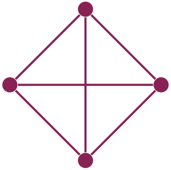
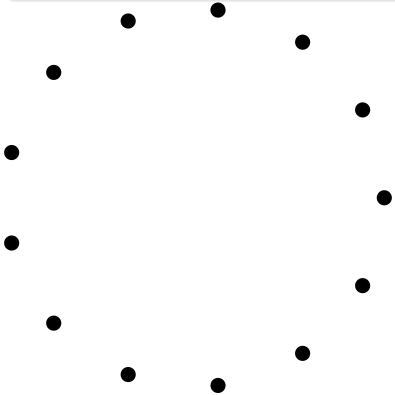


Graph Saturation

A graph **G** is *H*-saturated if **G** contains no forbidden *H* subgraph, but adding any edge to **G** gives rise to an *H* subgraph.

Easy Question

What is the *minimum* number of edges in a graph **G** with no forbidden subgraph H?



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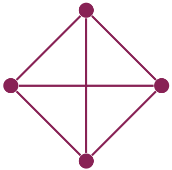

Saturation Number

Easy Question

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Interesting Question: Saturation Number (Erdős-Hajnal-Moon)

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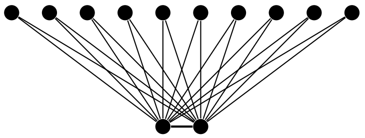

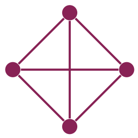

Forbidden

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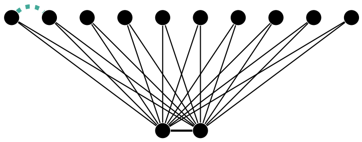

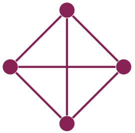

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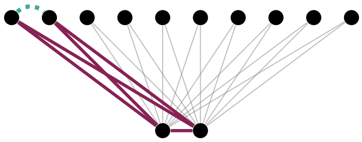

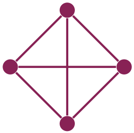

Forbidden

M. Ferrara, J. Kim, and Y.,
Ramsey-minimal saturation numbers for matchings.
Discrete Mathematics 322(2014), 26-30.

Edge-Colored Graph Saturation

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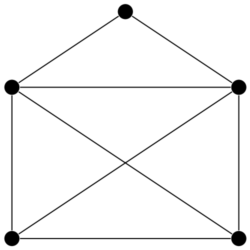

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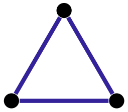

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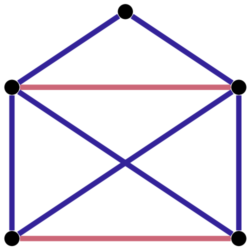

(i)

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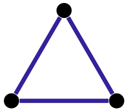

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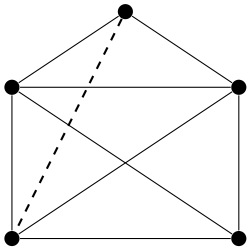

(ii)

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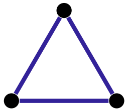

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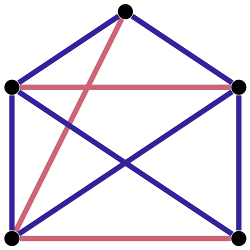

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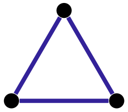

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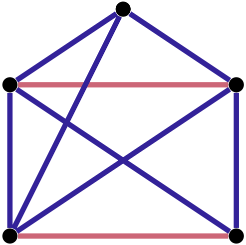

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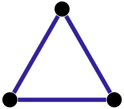

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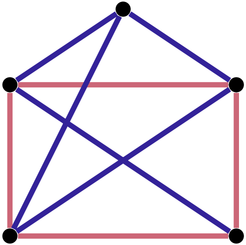

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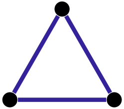

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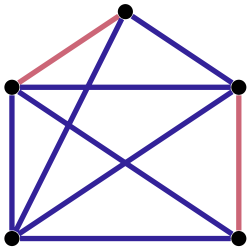

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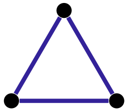

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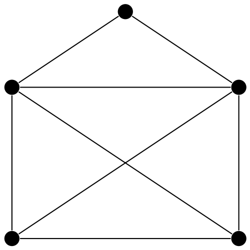

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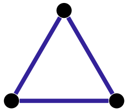
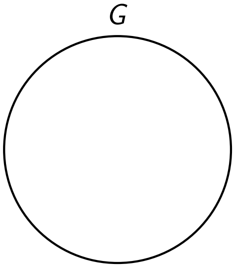


Forbidden in Blue

Ferrara-Kim-Y

Ferrara-Kim-Y, 2014

Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.



Forbidden: H_1, H_2, H_3, H_4

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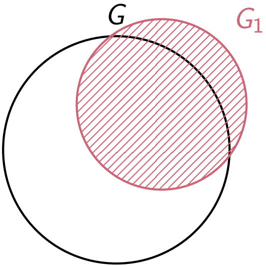

Forbidden: H_1, H_2, H_3, H_4

G_1 is H_1-saturated

Ferrara-Kim-Y

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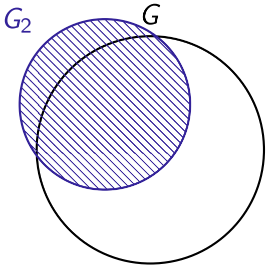

Forbidden: H_1, H_2, H_3, H_4

G_1 is H_1-saturated

G_2 is H_2-saturated

Ferrara-Kim-Y

Ferrara-Kim-Y, 2014

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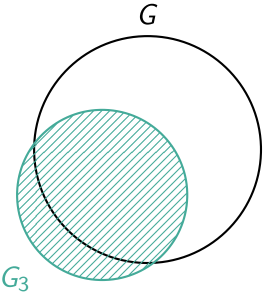


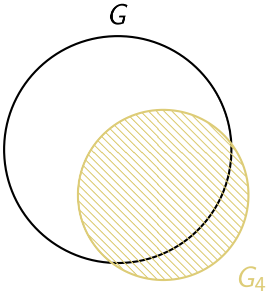
Forbidden: H_1, H_2, H_3, H_4

G_1 is H_1-saturated
G_2 is H_2-saturated
G_3 is H_3-saturated

Ferrara-Kim-Y

Ferrara-Kim-Y, 2014

Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.



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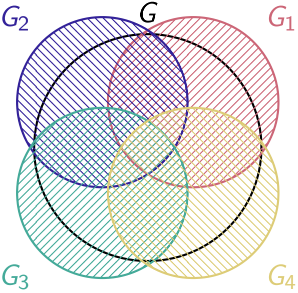

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Ferrara-Kim-Y

Ferrara-Kim-Y, 2014

Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.

Matchings, Ferrara-Kim-Y

If $m_1, \ldots, m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, then

$$sat(n, \mathcal{R}_{min}(m_1 K_2, \ldots, m_k K_2)) = 3(m_1 + \ldots + m_k - k).$$

Section 2

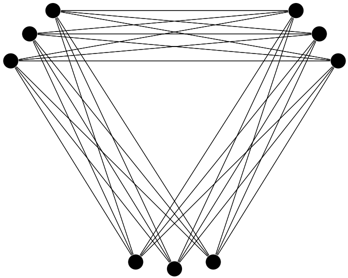
1. Extremal Problems in Disjoint Cycles
   - Background: Corrádi-Hajnal
   - A Refinement of Corrádi-Hajnal
   - Dirac's Question
   - Equitable Coloring
2. Variations on Graph Saturation
   - Background: Graph Saturation
   - Saturation of Ramsey-Minimal Families
   - Induced Saturation

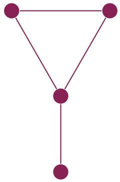
Citation

S. Behrens, C. Erbes, M. Santana, D. Yager, Y.
Graphs with induced-saturation number zero.
submitted

Idea for Induced Saturation:

Adding *or deleting* any edge produces a forbidden induced subgraph.





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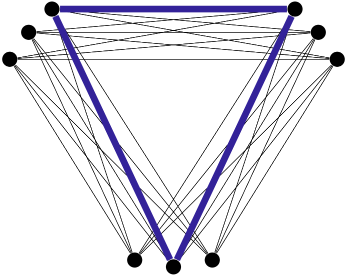

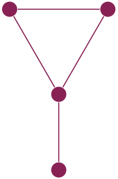

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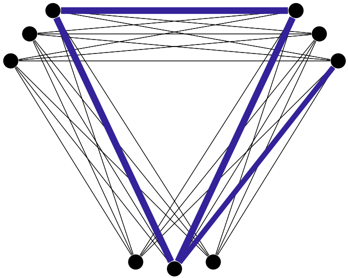

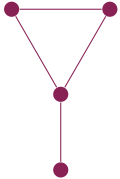

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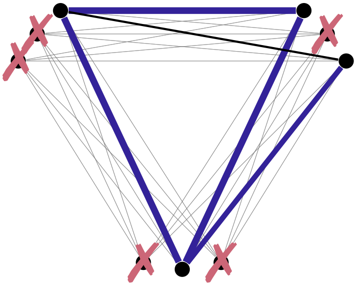

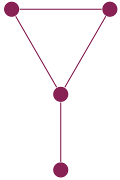

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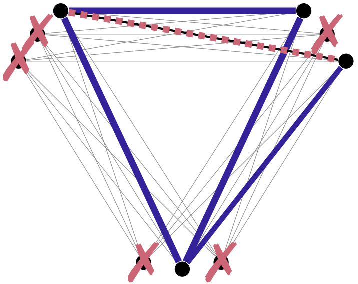

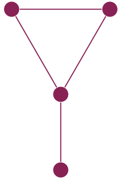

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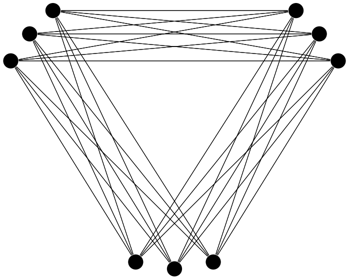

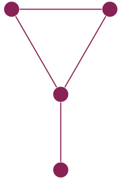

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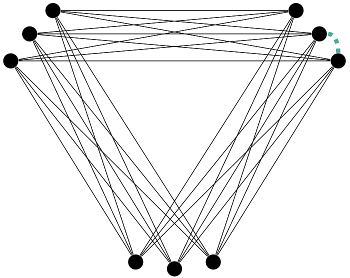

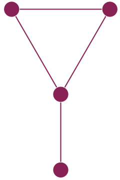

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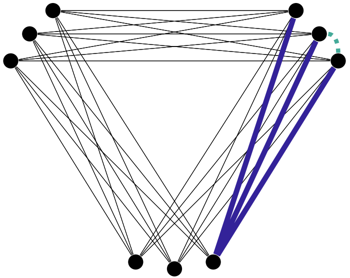

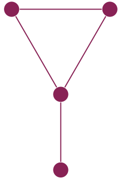

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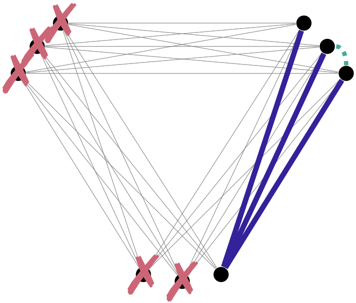

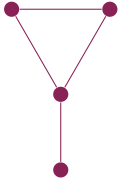

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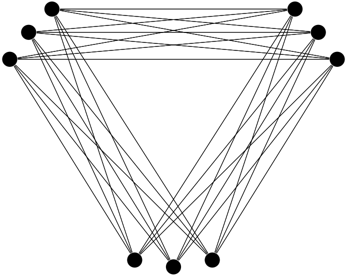

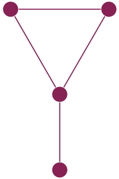

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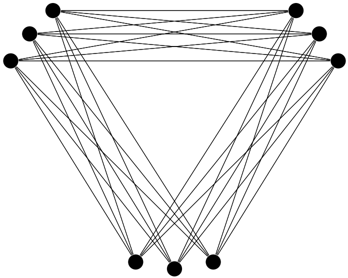
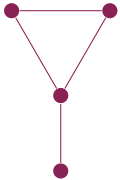




Forbidden

Warning: not defined for all forbidden subgraphs!

Induced Saturation

Martin-Smith

Definition that works for all forbidden subgraphs.

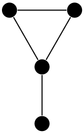
Induced Saturation

Martin-Smith

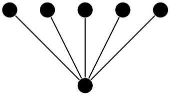
Definition that works for all forbidden subgraphs.

Behrens-Erbes-Santana-Yager-Y, 2015+

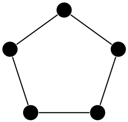
- A large number of common families fit into the simpler definition



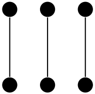
paw



stars



odd cycles



matchings

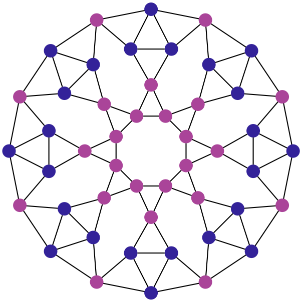
Induced Saturation

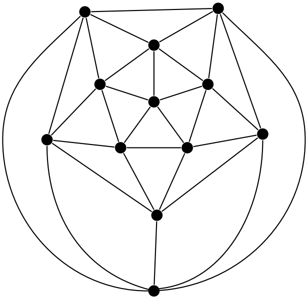
Martin-Smith

Definition that works for all forbidden subgraphs.

Behrens-Erbes-Santana-Yager-Y, 2015+

- A large number of common families fit into the simpler definition
- Using simpler definition, minimize number of edges

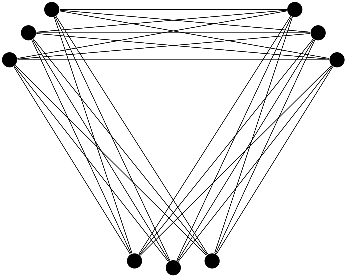


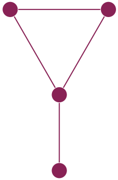


Induced Saturation, Behrens-Erbes-Santana-Yager-Y

Paw

Every component of a paw-induced-saturated graph is a complete multipartite graph.

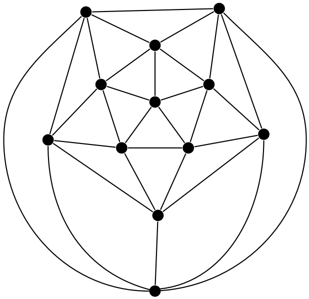


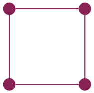


Induced Saturation, BESYY

C_4

The icosahedron is C_4-induced saturated.
For all $n \geq 12$, there exists a graph that is a generalized version of an icosahedron that is C_4-induced-saturated.



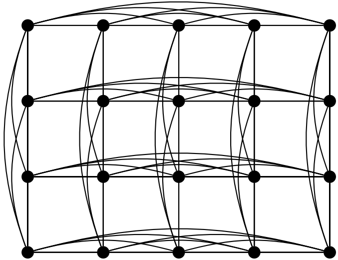


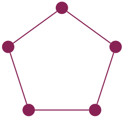
Forbidden

Induced Saturation, BESYY

Odd Cyces

For $k \geq 3$, the product of (appropriate) cliques is C_{2k-1}-induced-saturated.





Thanks for Listening!