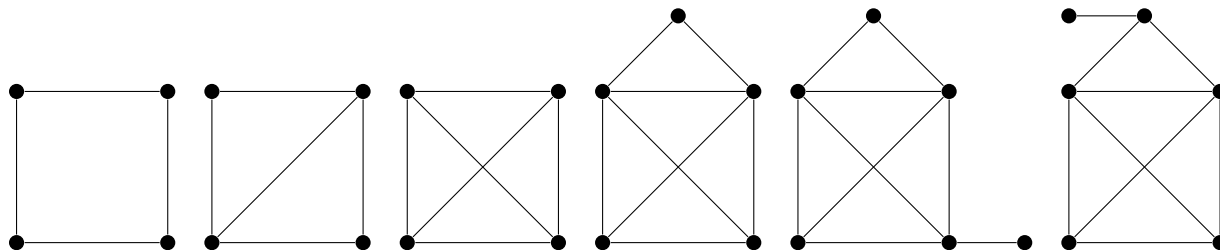Graph Theory Workshop, 20 November 2013

**Problem 1.** For each of the graphs below, determine using any method whether it is possible to walk through the graph, using every edge precisely once. (Such a route is called an *Eulerian path*. If it starts and ends at the same place, it's called an *Eulerian circuit*.)

**Problem 2.** Look again at the figures with Eulerian paths. Can you start and stop wherever you like?

**Problem 3.** You knew this was coming, right? Find a general rule–what's an easy way to tell whether an Eulerian path exists, and where you have to start and stop?

**Problem 4.** Let's go back to Königsberg, and develop a little story. (The ensuing text is shamelessly stolen from Wikipedia. Partly because it's a nice problem, and partly because it's adorably written.)

The northern bank of the river is occupied by the Schloß, or castle, of the Blue Prince; the southern by that of the Red Prince. The east bank is home to the Bishop's Kirche, or church; and on the small island in the center is a Gasthaus, or inn.
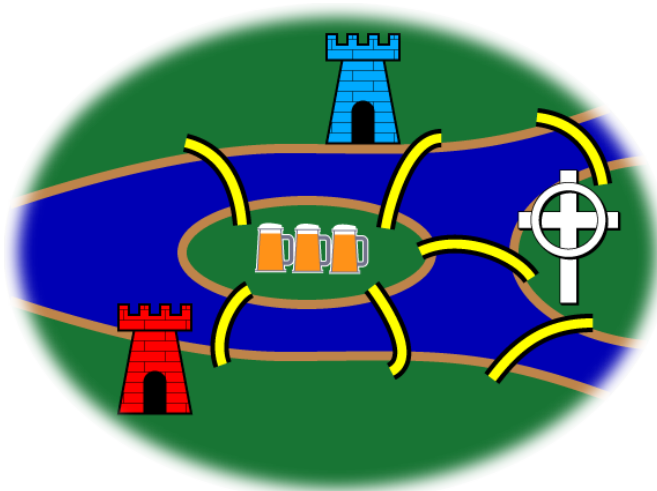


It being customary among the townsmen, after some hours in the Gasthaus, to attempt to walk the bridges (each precisely once), many have returned for more refreshment claiming success. However, none have been able to repeat the feat by the light of day.

(a) The Blue Prince, having analyzed the town's bridge system by means of graph theory, concludes that the bridges cannot be walked. He contrives a stealthy plan to build an eighth bridge so that he can begin in the evening at his Schloß, walk the bridges, and end at the Gasthaus to brag of his victory. Of course, he wants the Red Prince to be unable to duplicate the feat from the Red Castle. Where does the Blue Prince build the eighth bridge?

(b) The Red Prince, infuriated by his brother's Gordian solution to the problem, wants to build a ninth bridge, enabling him to begin at his Schloß, walk the bridges, and end at the Gasthaus to rub dirt in his brother's face. As an extra bit of revenge, his brother should then no longer be able to walk the bridges starting at his Schloßand ending at the Gasthaus as before. Where does the Red Prince build the ninth bridge?

(c) The Bishop has watched this furious bridge-building with dismay. It upsets the town's Weltanschauung and, worse, contributes to excessive drunkenness. He wants to build a tenth bridge that allows all the inhabitants to walk the bridges and return to their own beds. Where does the Bishop build the tenth bridge?

**Problem 5.** Look at the graphs shown in Problem 1. Find the degree of each vertex (landmass) in a graph, and add them up. What do you notice about the relationship between these sums and the number of edges (bridges)? Why does this relationship hold?

**Problem 6.** Suppose we want to schedule games between a large number of teams. It isn't practical to have every team play every other team. Rather, every team will play the same number of games. We can represent this using graphs: every team gets a vertex, and a game between two teams is an edge. For example, in the first graph of Problem 1, we can think of four teams, each playing two games.

Team 1                    Team 2

*T1 vs T2*

*T1 vs T4*                *T2 vs T3*

*T3 vs T4*

Team 4                    Team 3
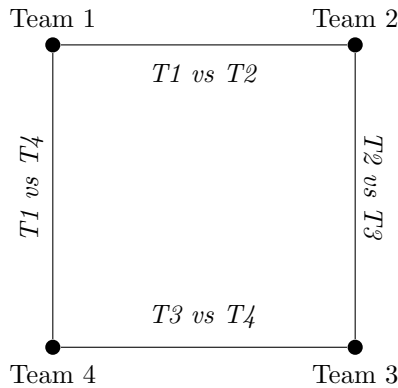
For each of the scenarios below, create a graph to plan out games between the teams.

(a) 6 teams, each plays two games

(b) 6 teams, each plays three games

(c) 7 teams, each plays three games

(d) 7 teams, each plays four games

**Problem 7.** We can also use graphs to display information about romantic partnerships. Suppose the vertices are people, and there is an edge between them if they have had a relationship. Below are three examples. We can suppose that the top bubbles represent people who identify as female, and the bottom represent people who identify as male.
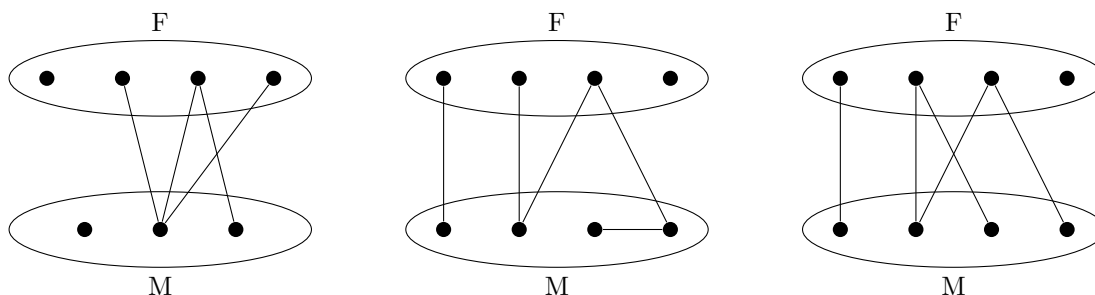


(a) Look at the sum of degrees of vertices in a bubble. What do you notice about the relationship between the sums of degrees in the top and the sums in the bottom? Why?

(b) In 2005, Mosher, Chandra, and Jones published the following results. (I can't find the paper, but I assume it's about Americans.)

- Males 30-44 report an average of 6-8 female partners in their lifetime
- Females 30-44 report an average of 4 male partners in their lifetime

Discuss the discrepancies between male and female partners, using what you know about graph theory and about life. Assume that the average number of partners was calculated by adding up all the partners every man/woman reported, then dividing by the number of men/women surveyed.

**Problem 8.** 4 couples are at a dinner party, and every pair of people who've never met before shake hands.
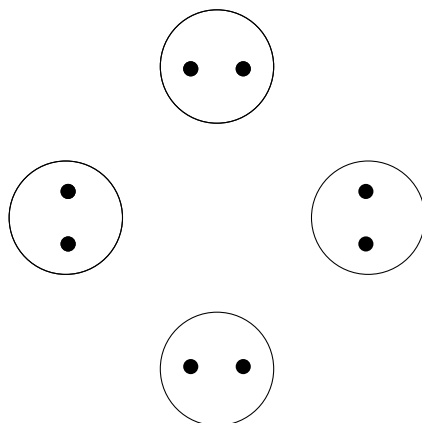


You ask every person at the party (not yourself!) how many people they shook hands with, and everyone gives you a different number. Put edges in the graph above to show which people might have shaken hands.

Bonus Homework: you throw a similar dinner party with 20 couples. How many people's hands did your partner shake? You should show that there is only possible answer.