

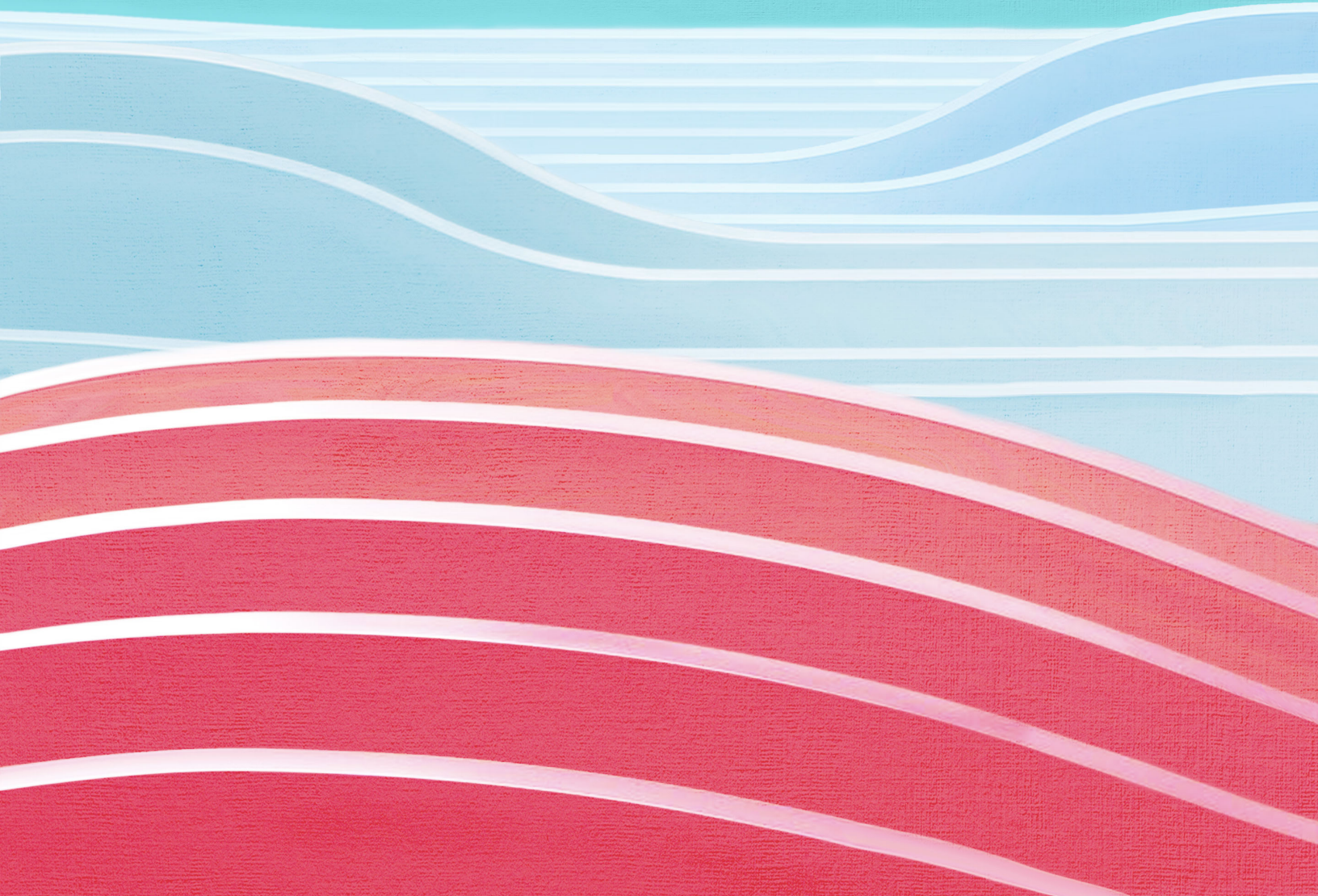
UBC Math 101: Integral Calculus

Practice book for AY 2025/26

This practice book is intended for use in the course UBC-V Math 101, Integral Calculus, in AY 2025/26. It is primarily a remix of other open-source materials.

- The vast majority of this text is from *CLP-2 Integral Calculus* by Joel Feldman, Andrew Rechnitzer, and Elyse Yeager
- Sections on probability are adapted from the probability chapter of *Optimal, Integral, Likely* compiled by Bruno Belevan, Parham Hamidi, Nisha Malhotra, and Elyse Yeager, which in turn contains content from *Introductory Statistics* by Barbara Illowski and Susan Dean.

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This document contains original content as well as content remixed from the sources listed above. More details about the relationship between this text and its primary sources are below.

- *How to use this book* is adapted from CLP-2
- Chapter 1 is adapted from CLP-2 Chapter 1, Integration.
 - Section 1.1 is adapted from 1.1, Definition of the Integral. Questions 18, 38, and 39 are original content, 2025.
 - Section 1.2 is adapted from 1.2, Basic properties of the definite integral.
 - Section 1.3 is adapted from 1.3, The Fundamental Theorem of Calculus.
 - Section 1.5 is adapted from 1.4, Substitution.
 - Section 1.6 is adapted from 1.5, Area between curves.
 - Section 1.7 is adapted from 1.8, Trigonometric Integrals.
 - Section 1.8 is adapted from 1.9 Trigonometric Substitution.
 - Section 1.9 is adapted from 1.10 Partial Fractions.
 - Section 1.10 is adapted from 1.6, Volumes.

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- Section 1.11 is adapted from 1.7, Integration by parts.
 - Section 1.12 is adapted from 1.11 Numerical Integration. Questions 10, 21, 22, and 23 are original content, 2025. Midpoint rule has been removed.
 - Section 1.13 is adapted from 1.12 Improper Integrals.
 - Section 1.14 is adapted from 1.13 More Integration Examples.
 - Chapter 2 is adapted from Optimal, Integral, Likely Chapter 4, Probability. That chapter uses some material from Introductory Statistics by Ilowksy and Dean. Mentions of discrete random variables, and the definition of the CDF, have been removed; the definition of the PDF has been changed.
 - Section 2.1 is adapted from 4.1, Introduction.
 - Section 2.2 is adapted from 4.4, Probability Density.
 - Section 2.3 is adapted from 4.5, Expected Value
 - Section 2.4 is adapted from 4.6, Variance and Standard Deviation
 - Chapter 3 is adapted from CLP–2 Chapter 3, Sequences and Series.
 - Section 3.1 is adapted from Section 3.1, Sequences.
 - Section 3.2 is adapted from 3.2, Series. Questions 9 and 10 are original content, 2025. Telescoping series have been removed, and geometric series moved to 3.3.
 - Section 3.3 is adapted from 3.3, Convergence Tests.
 - Section 3.5 is adapted from 3.5, Power Series. Interval of convergence removed.
 - Section 3.6 is adapted from 3.6, Taylor Series.

►► Errors

If you find an error in this practice book or the accompanying textbook, please first check the list of known errors, <https://personal.math.ubc.ca/~elyse/Math101Text/errata.html>. If it has not yet been reported, please email Elyse Yeager at elyse.math.ubc.ca with the location of the error (page number, and possibly location on the page) and a description of the error.

HOW TO USE THIS BOOK

▲ Introduction

First of all, welcome to Calculus!

This book is written as a companion to the Math 101 integral calculus textbook.

►► How to Work Questions

This book is organized into four sections: Questions, Hints, Answers, and Solutions. As you are working problems, resist the temptation to prematurely peek at the back! It's important to allow yourself to struggle for a time with the material. Even professional mathematicians don't always know right away how to solve a problem. The art is in gathering your thoughts and figuring out a strategy to use what you know to find out what you don't.

If you find yourself at a real impasse, go ahead and look for a hint in the Hints section. Think about it for a while, and don't be afraid to read back in the notes to look for a key idea that will help you proceed. If you still can't solve the problem, well, we included the Solutions section for a reason! As you're reading the solutions, try hard to understand why we took the steps we did, instead of memorizing step-by-step how to solve that one particular problem.

If you struggled with a question quite a lot, it's probably a good idea to return to it in a few days. That might have been enough time for you to internalize the necessary ideas, and you might find it easily conquerable. Pat yourself on the back—sometimes math makes you feel good! If you're still having troubles, read over the solution again, with an emphasis on understanding why each step makes sense.

One of the reasons so many students are required to study calculus is the hope that it will improve their problem-solving skills. In this class, you will learn lots of concepts, and be asked to apply them in a variety of situations. Often, this will involve answering one really big problem by breaking it up into manageable chunks, solving those chunks, then

putting the pieces back together. When you see a particularly long question, remain calm and look for a way to break it into pieces you can handle.

» Working with Friends

Study buddies are fantastic! If you don't already have friends in your class, you can ask your neighbours in lecture to form a group. Often, a question that you might bang your head against for an hour can be easily cleared up by a friend who sees what you've missed. Regular study times make sure you don't procrastinate too much, and friends help you maintain a positive attitude when you might otherwise succumb to frustration. Struggle in mathematics is desirable, but suffering is not.

When working in a group, make sure you try out problems on your own before coming together to discuss with others. Learning is a process, and getting answers to questions that you haven't considered on your own can rob you of the practice you need to master skills and concepts, and the tenacity you need to develop to become a competent problem-solver.

» Types of Questions

Q[1](*):

hint answer solution

In addition to original problems, this book contains problems pulled from quizzes and exams given at UBC for Math 101 and 105 (second-semester calculus) and Math 121 (honours second-semester calculus). These problems are marked with a star. The authors would like to acknowledge the contributions of the many people who collaborated to produce these exams over the years.

Instructions and other comments that are attached to more than one question are written in this font. The questions are organized into Stage 1, Stage 2, and Stage 3.

» Stage 1

The first category is meant to test and improve your understanding of basic underlying concepts. These often do not involve much calculation. They range in difficulty from very basic reviews of definitions to questions that require you to be thoughtful about the concepts covered in the section.

» Stage 2

Questions in this category are for practicing skills. It's not enough to understand the philosophical grounding of an idea: you have to be able to apply it in appropriate situations. This takes practice!

» Stage 3

The last questions in each section go a little farther than Stage 2. Often they will combine more than one idea, incorporate review material, or ask you to apply your understanding of a concept to a new situation.

In exams, as in life, you will encounter questions of varying difficulty. A good skill to practice is recognizing the level of difficulty a problem poses. Exams will have some easy questions, some standard questions, and some harder questions.

CONTENTS

0	How to use this book	4
I	The questions	10
1	Integration	11
1.1	Definition of the Integral	11
1.2	Basic properties of the definite integral	20
1.3	The Fundamental Theorem of Calculus	23
1.4	Interpretations of the definite integral	30
1.5	Substitution	32
1.6	Area between curves	35
1.7	Trigonometric Integrals	38
1.8	Trigonometric Substitution	40
1.9	Partial Fractions	44
1.10	Volumes	47
1.11	Integration by parts	52
1.12	Numerical Integration	55
1.13	Improper Integrals	62
1.14	More Integration Examples	66
2	Probability	72
2.1	Introduction	72
2.2	Probability density	72
2.3	Expected value	74
2.4	Variance and standard deviation	76
3	Sequences and Series	79
3.1	Sequences	79
3.2	Series	84
3.3	Convergence Tests	87

3.4	Absolute and Conditional Convergence	98
3.5	Power Series	100
3.6	Taylor Series	104

II Hints to problems 113

1.1	Definition of the Integral	114
1.2	Basic properties of the definite integral	117
1.3	The Fundamental Theorem of Calculus	118
1.4	Interpretations of the definite integral	120
1.5	Substitution	121
1.6	Area between curves	122
1.7	Trigonometric Integrals	122
1.8	Trigonometric Substitution	123
1.9	Partial Fractions	125
1.10	Volumes	126
1.11	Integration by parts	127
1.12	Numerical Integration	129
1.13	Improper Integrals	131
1.14	More Integration Examples	133
2.1	Introduction	134
2.2	Probability density	135
2.3	Expected value	135
2.4	Variance and standard deviation	136
3.1	Sequences	136
3.2	Series	137
3.3	Convergence Tests	138
3.4	Absolute and Conditional Convergence	141
3.5	Power Series	142
3.6	Taylor Series	143

III Answers to problems 146

1.1	Definition of the Integral	147
1.2	Basic properties of the definite integral	151
1.3	The Fundamental Theorem of Calculus	152
1.4	Interpretations of the definite integral	156
1.5	Substitution	157
1.6	Area between curves	159
1.7	Trigonometric Integrals	161
1.8	Trigonometric Substitution	163
1.9	Partial Fractions	164
1.10	Volumes	166
1.11	Integration by parts	169
1.12	Numerical Integration	171
1.13	Improper Integrals	174
1.14	More Integration Examples	175

2.1	Introduction	177
2.2	Probability density	177
2.3	Expected value	178
2.4	Variance and standard deviation	178
3.1	Sequences	179
3.2	Series	181
3.3	Convergence Tests	182
3.4	Absolute and Conditional Convergence	187
3.5	Power Series	187
3.6	Taylor Series	189

IV Solutions to problems 193

1.1	Definition of the Integral	194
1.2	Basic properties of the definite integral	230
1.3	The Fundamental Theorem of Calculus	240
1.4	Interpretations of the definite integral	260
1.5	Substitution	265
1.6	Area between curves	275
1.7	Trigonometric Integrals	287
1.8	Trigonometric Substitution	295
1.9	Partial Fractions	314
1.10	Volumes	335
1.11	Integration by parts	359
1.12	Numerical Integration	377
1.13	Improper Integrals	394
1.14	More Integration Examples	409
2.1	Introduction	440
2.2	Probability density	440
2.3	Expected value	444
2.4	Variance and standard deviation	449
3.1	Sequences	453
3.2	Series	468
3.3	Convergence Tests	472
3.4	Absolute and Conditional Convergence	507
3.5	Power Series	515
3.6	Taylor Series	530

Part I

THE QUESTIONS

INTEGRATION

1.1▲ Definition of the Integral

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

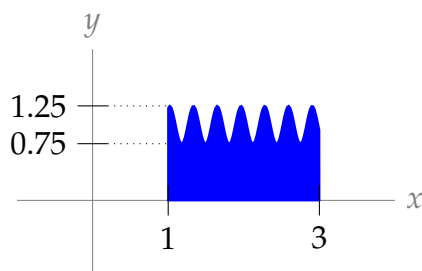
►► Stage 1

For Questions 1 through 5, we want you to develop an understanding of the model we are using to define an integral: we approximate the area under a curve by bounding it between rectangles. Later, we will learn more sophisticated methods of integration, but they are all based on this simple concept.

Q[1]:

[hint](#) [answer](#) [solution](#)

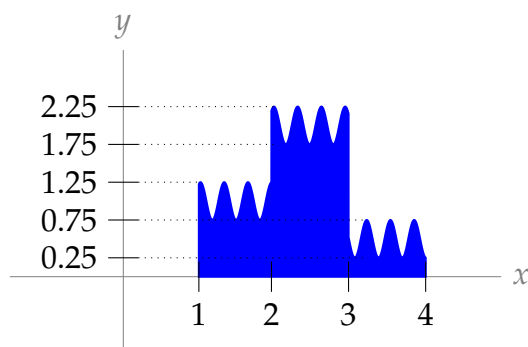
Give a range of possible values for the shaded area in the picture below.



Q[2]:

[hint](#) [answer](#) [solution](#)

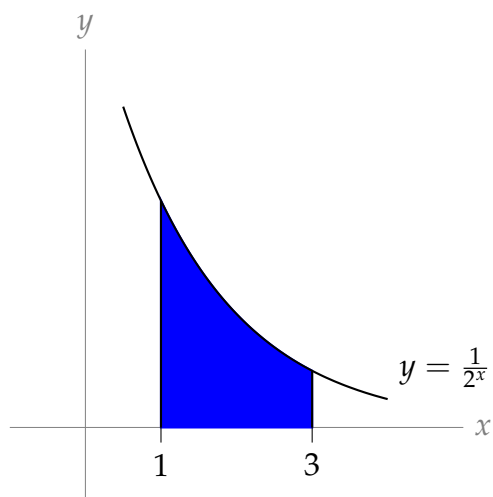
Give a range of possible values for the shaded area in the picture below.



Q[3]:

[hint](#) [answer](#) [solution](#)

Using rectangles, find a lower and upper bound for $\int_1^3 \frac{1}{2^x} dx$ that differ by at most 0.2 square units.



Q[4]:

[hint](#) [answer](#) [solution](#)

Let $f(x)$ be a function that is *decreasing* from $x = 0$ to $x = 5$. Which Riemann sum approximation of $\int_0^5 f(x) dx$ is the largest—left, right, or midpoint?

Q[5]:

[hint](#) [answer](#) [solution](#)

Give an example of a function $f(x)$, an interval $[a, b]$, and a number n such that the midpoint Riemann sum of $f(x)$ over $[a, b]$ using n intervals is *larger than* both the left and right Riemann sums of $f(x)$ over $[a, b]$ using n intervals.

In Questions 6 through 10, we practice using sigma notation. There are many ways to write a given sum in sigma notation. You can practice finding several, and deciding which looks the clearest.

Q[6]:

[hint](#) [answer](#) [solution](#)

Express the following sums in sigma notation:

(a) $3 + 4 + 5 + 6 + 7$

(b) $6 + 8 + 10 + 12 + 14$

(c) $7 + 9 + 11 + 13 + 15$

(d) $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$

Q[7]:

[hint](#) [answer](#) [solution](#)

Express the following sums in sigma notation:

(a) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$

(b) $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}$

(c) $-\frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81}$

(d) $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{81}$

Q[8]:

[hint](#) [answer](#) [solution](#)

Express the following sums in sigma notation:

(a) $\frac{1}{3} + \frac{1}{3} + \frac{5}{27} + \frac{7}{81} + \frac{9}{243}$

(b) $\frac{1}{5} + \frac{1}{11} + \frac{1}{29} + \frac{1}{83} + \frac{1}{245}$

(c) $1000 + 200 + 30 + 4 + \frac{1}{2} + \frac{3}{50} + \frac{7}{1000}$

Q[9]:

[hint](#) [answer](#) [solution](#)

Evaluate the following sums. You might want to use the formulas from Theorems 1.1.6 and 1.1.7 in the text.

(a) $\sum_{i=0}^{100} \left(\frac{3}{5}\right)^i$

(b) $\sum_{i=50}^{100} \left(\frac{3}{5}\right)^i$

(c) $\sum_{i=1}^{10} (i^2 - 3i + 5)$

(d) $\sum_{n=1}^b \left[\left(\frac{1}{e}\right)^n + en^3 \right]$, where b is some integer greater than 1.

Q[10]:

[hint](#) [answer](#) [solution](#)

Evaluate the following sums. You might want to use the formulas from Theorem 1.1.7 in the text.

(a) $\sum_{i=50}^{100} (i - 50) + \sum_{i=0}^{50} i$

(b) $\sum_{i=10}^{100} (i - 5)^3$

(c) $\sum_{n=1}^{11} (-1)^n$

(d) $\sum_{n=2}^{11} (-1)^{2n+1}$

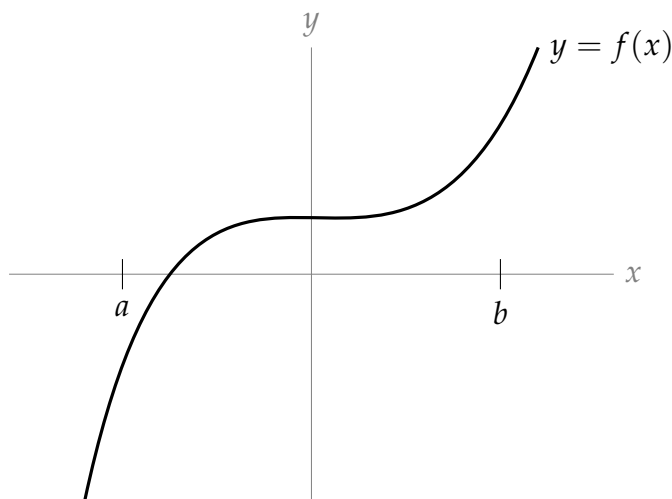
Questions 11 through 15 are meant to give you practice interpreting the formulas in Definition 1.1.12 of the text. The formulas might look complicated at first, but if you understand what each piece means, they are easy to learn.

Q[11]:

[hint](#) [answer](#) [solution](#)

In the picture below, draw in the rectangles whose (signed) area is being computed by

the midpoint Riemann sum $\sum_{i=1}^4 \frac{b-a}{4} \cdot f\left(a + \left(i - \frac{1}{2}\right) \frac{b-a}{4}\right)$.



Q[12](*):

[hint](#) [answer](#) [solution](#)

$\sum_{k=1}^4 f(1+k) \cdot 1$ is a left Riemann sum for a function $f(x)$ on the interval $[a, b]$ with n subintervals. Find the values of a , b and n .

Q[13]:

[hint](#) [answer](#) [solution](#)

Draw a picture illustrating the area given by the following Riemann sum.

$$\sum_{i=1}^3 2 \cdot (5 + 2i)^2$$

Q[14]:

[hint](#) [answer](#) [solution](#)

Draw a picture illustrating the area given by the following Riemann sum.

$$\sum_{i=1}^5 \frac{\pi}{20} \cdot \tan\left(\frac{\pi(i-1)}{20}\right)$$

Q[15](*):

[hint](#) [answer](#) [solution](#)

Fill in the blanks with right, left, or midpoint; an interval; and a value of n .

$\sum_{k=0}^3 f(1.5 + k) \cdot 1$ is a _____ Riemann sum for f on the interval $[\_, \_]$ with $n = \_$.

Q[16]:

[hint](#) [answer](#) [solution](#)

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_0^5 x \, dx$$

Q[17]:

[hint](#) [answer](#) [solution](#)

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_{-2}^5 x \, dx$$

Q[18]:

[hint](#) [answer](#) [solution](#)

Suppose you're setting up a right Riemann sum calculation in a spreadsheet, where column A holds the x -values where you're evaluating your function. If you're approximating

$$\int_a^b f(x) \, dx$$

with a right Riemann sum using n rectangles (where a , b , n , and f are given), you can double-check that column A is set up correctly with an easy trick: what values should be in rows 1 and n ?

► Stage 2

Q[19](*):

[hint](#) [answer](#) [solution](#)

Use sigma notation to write the midpoint Riemann sum for $f(x) = x^8$ on $[5, 15]$ with $n = 50$. Do not evaluate the Riemann sum.

Q[20](*):

[hint](#) [answer](#) [solution](#)

Estimate $\int_{-1}^5 x^3 \, dx$ using three approximating rectangles and left hand end points.

Q[21](*):

[hint](#) [answer](#) [solution](#)

Let f be a function on the whole real line. Express $\int_{-1}^7 f(x) \, dx$ as a limit of Riemann sums, using the right endpoints.

Q[22](*):

[hint](#) [answer](#) [solution](#)

The value of the following limit is equal to the area below a graph of $y = f(x)$, integrated over the interval $[0, b]$:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4}{n} \left[\sin \left(2 + \frac{4i}{n} \right) \right]^2$$

Find $f(x)$ and b .

Q[23](*):

[hint](#) [answer](#) [solution](#)

For a certain function $f(x)$, the following equation holds:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2} \sqrt{1 - \frac{k^2}{n^2}} = \int_0^1 f(x) \, dx$$

Find $f(x)$.

Q[24](*):

[hint](#) [answer](#) [solution](#)

Express $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} e^{-i/n} \cos\left(\frac{3i}{n}\right)$ as a definite integral.

Q[25](*):

[hint](#) [answer](#) [solution](#)

Let $R_n = \sum_{i=1}^n \frac{ie^{i/n}}{n^2}$. Express $\lim_{n \rightarrow \infty} R_n$ as a definite integral. *Do not evaluate this integral.*

Q[26](*):

[hint](#) [answer](#) [solution](#)

Express $\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-1-2i/n} \cdot \frac{2}{n} \right)$ as an integral in three different ways.

Questions 27 and 28 use the formula for a geometric sum, Equation 1.1.3 in the text.

Q[27]:

[hint](#) [answer](#) [solution](#)

Evaluate the sum $1 + r^3 + r^6 + r^9 + \cdots + r^{3n}$.

Q[28]:

[hint](#) [answer](#) [solution](#)

Evaluate the sum $r^5 + r^6 + r^7 + \cdots + r^{100}$.

Remember that a definite integral is a signed area between a curve and the x -axis. We'll spend a lot of time learning strategies for evaluating definite integrals, but we already know lots of ways to find area of geometric shapes. In Questions 29 through 34, use your knowledge of geometry to find the signed areas described by the integrals given.

Q[29](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{-1}^2 |2x| \, dx$.

Q[30]:

[hint](#) [answer](#) [solution](#)

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_{-3}^5 |t - 1| \, dt$$

Q[31]:

[hint](#) [answer](#) [solution](#)

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_a^b x \, dx$$

where $0 \leq a \leq b$.

Q[32]:

[hint](#) [answer](#) [solution](#)

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_a^b x \, dx$$

where $a \leq b \leq 0$.

Q[33]:

[hint](#) [answer](#) [solution](#)

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_0^4 \sqrt{16 - x^2} \, dx$$

Q[34](*):

[hint](#) [answer](#) [solution](#)

Use elementary geometry to calculate $\int_0^3 f(x) \, dx$, where

$$f(x) = \begin{cases} x, & \text{if } x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Q[35](*):

[hint](#) [answer](#) [solution](#)

A car's gas pedal is applied at $t = 0$ seconds and the car accelerates continuously until $t = 2$ seconds. The car's speed at half-second intervals is given in the table below. Find the best possible **upper** estimate for the distance that the car traveled during these two seconds.

t (s)	0	0.5	1.0	1.5	2
v (m/s)	0	14	22	30	40

Q[36]:

[hint](#) [answer](#) [solution](#)

True or false: the answer you gave for Question 35 is definitely greater than or equal to the distance the car travelled during the two seconds in question.

Q[37]:

[hint](#) [answer](#) [solution](#)

An airplane's speed at one-hour intervals is given in the table below. Approximate the distance travelled by the airplane from noon to 4pm using a midpoint Riemann sum.

time	12:00 pm	1:00 pm	2:00 pm	3:00 pm	4:00 pm
speed (km/hr)	800	700	850	900	750

Q[38]:

[hint](#) [answer](#) [solution](#)

Use a spreadsheet to approximate

$$\int_{-10}^{10} \frac{1}{1 + x^2} \, dx$$

using a right Riemann sum with $n = 100$ rectangles.

Q[39]:

[hint](#) [answer](#) [solution](#)

Use a spreadsheet to approximate

$$\int_{-\pi}^{\pi} \sin^2 x \, dx$$

using a right Riemann sum with $n = 75$ rectangles.

►► Stage 3

Q[40](*):

[hint](#) [answer](#) [solution](#)

(a) Express

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \sqrt{4 - \left(-2 + \frac{2i}{n}\right)^2}$$

as a definite integral.

(b) Evaluate the integral of part (a).

Q[41](*):

[hint](#) [answer](#) [solution](#)

Consider the integral:

$$\int_0^3 (7 + x^3) \, dx. \quad (*)$$

(a) Approximate this integral using the left Riemann sum with $n = 3$ intervals.

(b) Write down the expression for the right Riemann sum with n intervals and calculate the sum. Now take the limit $n \rightarrow \infty$ in your expression for the Riemann sum, to evaluate the integral (*) exactly.

You may use the identity

$$\sum_{i=1}^n i^3 = \frac{n^4 + 2n^3 + n^2}{4}$$

Q[42](*):

[hint](#) [answer](#) [solution](#)

Using a limit of right-endpoint Riemann sums, evaluate $\int_2^4 x^2 \, dx$.

You may use the formulas $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

Q[43](*):

[hint](#) [answer](#) [solution](#)

Find $\int_0^2 (x^3 + x) \, dx$ using the definition of the definite integral. You may use the summation formulas $\sum_{i=1}^n i^3 = \frac{n^4 + 2n^3 + n^2}{4}$ and $\sum_{i=1}^n i = \frac{n^2 + n}{2}$.

Q[44](*):

[hint](#) [answer](#) [solution](#)

Using a limit of right-endpoint Riemann sums, evaluate $\int_1^4 (2x - 1) \, dx$. Do not use anti-differentiation, except to check your answer.¹ You may use the formula $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

Q[45]:

[hint](#) [answer](#) [solution](#)

Give a function $f(x)$ that has the following expression as a right Riemann sum when $n = 10$, $\Delta(x) = 10$ and $a = -5$:

$$\sum_{i=1}^{10} 3(7 + 2i)^2 \sin(4i).$$

Q[46]:

[hint](#) [answer](#) [solution](#)

Using the method of Example 1.1.2 in the text, evaluate

$$\int_0^1 2^x \, dx$$

Q[47]:

[hint](#) [answer](#) [solution](#)

(a) Using the method of Example 1.1.2 in the text, evaluate

$$\int_a^b 10^x \, dx$$

(b)

Using your answer from above, make a guess for

$$\int_a^b c^x \, dx$$

where c is a positive constant. Does this agree with Question 46?

Q[48]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^a \sqrt{1 - x^2} \, dx$ using geometry, if $0 \leq a \leq 1$.

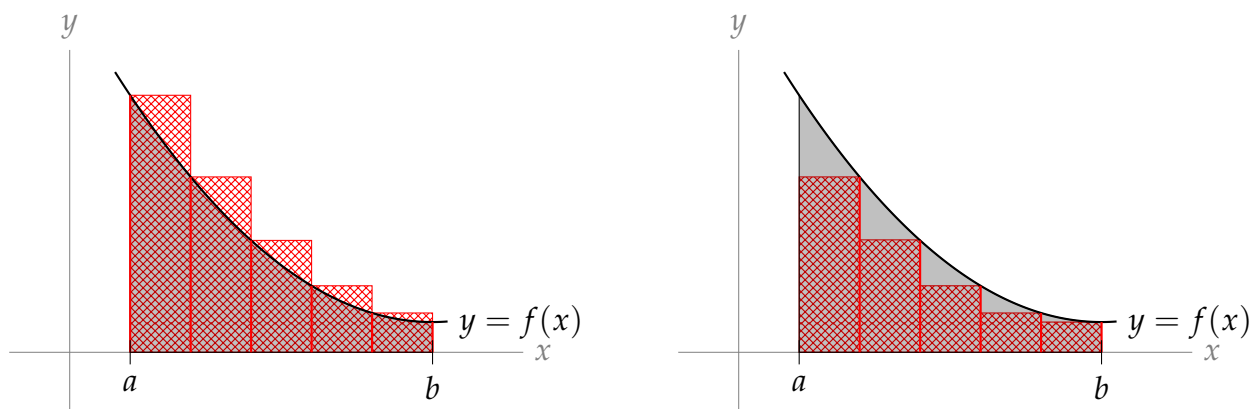
Q[49]:

[hint](#) [answer](#) [solution](#)

Suppose $f(x)$ is a positive, decreasing function from $x = a$ to $x = b$. You give an upper and lower bound on the area under the curve $y = f(x)$ using n rectangles and a left and right Riemann sum, respectively, as in the picture below.



¹ You'll learn about this method starting in Section 1.3 of the text. You can also check this answer using geometry.



- (a) What is the difference between the lower bound and the upper bound? (That is, if we subtract the smaller estimate from the larger estimate, what do we get?) Give your answer in terms of f , a , b , and n .
- (b) If you want to approximate the area under the curve to within 0.01 square units using this method, how many rectangles should you use? That is, what should n be?

Q[50]:

[hint](#) [answer](#) [solution](#)

Let $f(x)$ be a linear function, let $a < b$ be integers, and let n be a whole number. True or false: if we average the left and right Riemann sums for $\int_a^b f(x) dx$ using n rectangles, we get the same value as the midpoint Riemann sum using n rectangles.

1.2▲ Basic properties of the definite integral

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

For each of the following properties of definite integrals, draw a picture illustrating the concept, interpreting definite integrals as areas under a curve.

For simplicity, you may assume that $a \leq c \leq b$, and that $f(x), g(x)$ give positive values.

- (a) $\int_a^a f(x) dx = 0$ (Theorem 1.2.3.a in the text)
- (b) $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ (Theorem 1.2.3.c in the text)
- (c) $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$ (Theorem 1.2.1.a in the text)

Q[2]:

[hint](#) [answer](#) [solution](#)

If $\int_0^b \cos x dx = \sin b$, then what is $\int_a^b \cos x dx$?

Q[3](*):

[hint](#) [answer](#) [solution](#)

Decide whether each of the following statements is true or false. If false, provide a counterexample. If true, provide a brief justification.

(a) $\int_{-3}^{-2} f(x) dx = - \int_3^2 f(x) dx.$

(b) If $f(x)$ is an odd function, then $\int_{-3}^{-2} f(x) dx = \int_2^3 f(x) dx.$

(c) $\int_0^1 f(x) \cdot g(x) dx = \int_0^1 f(x) dx \cdot \int_0^1 g(x) dx.$

Q[4]:

[hint](#) [answer](#) [solution](#)

Suppose we want to make a right Riemann sum with 100 intervals to approximate

$\int_0^5 f(x) dx$, where $f(x)$ is a function that gives only positive values.

(a) What is Δx ?

(b) Are the heights of our rectangles positive or negative?

(c) Is our Riemann sum positive or negative?

(d) Is the signed area under the curve $y = f(x)$ from $x = 0$ to $x = 5$ positive or negative?

► Stage 2

Q[5](*):

[hint](#) [answer](#) [solution](#)

Suppose $\int_2^3 f(x) dx = -1$ and $\int_2^3 g(x) dx = 5$. Evaluate $\int_2^3 (6f(x) - 3g(x)) dx.$

Q[6](*):

[hint](#) [answer](#) [solution](#)

If $\int_0^2 f(x) dx = 3$ and $\int_0^2 g(x) dx = -4$, calculate $\int_0^2 (2f(x) + 3g(x)) dx.$

Q[7](*):

[hint](#) [answer](#) [solution](#)

The functions $f(x)$ and $g(x)$ obey

$$\int_0^{-1} f(x) dx = 1 \quad \int_0^2 f(x) dx = 2 \quad \int_{-1}^0 g(x) dx = 3 \quad \int_0^2 g(x) dx = 4$$

Find $\int_{-1}^2 [3g(x) - f(x)] dx.$

Q[8]:

[hint](#) [answer](#) [solution](#)

In Question 48, Section 1.1, we found that

$$\int_0^a \sqrt{1-x^2} dx = \frac{\pi}{4} - \frac{1}{2} \arccos(a) + \frac{1}{2} a \sqrt{1-a^2}$$

when $0 \leq a \leq 1$.

Using this fact, evaluate the following:

(a) $\int_a^0 \sqrt{1-x^2} \, dx$, where $-1 \leq a \leq 0$

(b) $\int_a^1 \sqrt{1-x^2} \, dx$, where $0 \leq a \leq 1$

Q[9](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{-1}^2 |2x| \, dx$.

You may use the result from Example 1.2.6 in the text that $\int_a^b x \, dx = \frac{b^2-a^2}{2}$.

Q[10]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{-5}^5 x|x| \, dx$.

Q[11]:

[hint](#) [answer](#) [solution](#)

Suppose $f(x)$ is an even function and $\int_{-2}^2 f(x) \, dx = 10$. What is $\int_{-2}^0 f(x) \, dx$?

► Stage 3

Q[12](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{-2}^2 (5 + \sqrt{4-x^2}) \, dx$.

Q[13](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{-2012}^{+2012} \frac{\sin x}{\log(3+x^2)} \, dx$.

Q[14](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{-2012}^{+2012} x^{1/3} \cos x \, dx$.

Q[15]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^6 (x-3)^3 \, dx$.

Q[16]:

[hint](#) [answer](#) [solution](#)

We want to compute the area of an ellipse, $(ax)^2 + (by)^2 = 1$ for some (let's say positive) constants a and b .

(a) Solve the equation for the upper half of the ellipse. It should have the form " $y = \dots$ "

(b) Write an integral for the area of the upper half of the ellipse. Using properties of integrals, make the integrand look like the upper half of a circle.

(c) Using geometry and your answer to part (b), find the area of the ellipse.

Q[17]:

[hint](#) [answer](#) [solution](#)

Fill in the following table: the product of an (even/odd) function with an (even/odd) function is an (even/odd) function. You may assume that both functions are defined for all real numbers.

$\times$	even	odd
even		
odd		

Q[18]: [hint](#) [answer](#) [solution](#)
 Suppose $f(x)$ is an odd function and $g(x)$ is an even function, both defined at $x = 0$. What are the possible values of $f(0)$ and $g(0)$?

Q[19]: [hint](#) [answer](#) [solution](#)
 Suppose $f(x)$ is a function defined on all real numbers that is both even and odd. What could $f(x)$ be?

Q[20]: [hint](#) [answer](#) [solution](#)
 Is the derivative of an even function even or odd? Is the derivative of an odd function even or odd?

1.3▲ The Fundamental Theorem of Calculus

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

► Stage 1

Q[1](*) [hint](#) [answer](#) [solution](#)
 Suppose that $f(x)$ is a function and $F(x) = e^{(x^2-3)} + 1$ is an antiderivative of $f(x)$. Evaluate the definite integral $\int_1^{\sqrt{5}} f(x) dx$.

Q[2](*) [hint](#) [answer](#) [solution](#)
 For the function $f(x) = x^3 - \sin 2x$, find its antiderivative $F(x)$ that satisfies $F(0) = 1$.

Q[3](*) [hint](#) [answer](#) [solution](#)
 Decide whether each of the following statements is true or false. Provide a brief justification.

(a) If $f(x)$ is continuous on $[1, \pi]$ and differentiable on $(1, \pi)$, then

$$\int_1^{\pi} f'(x) dx = f(\pi) - f(1).$$

(b) $\int_{-1}^1 \frac{1}{x^2} dx = 0$.

(c) If f is continuous on $[a, b]$ then $\int_a^b xf(x) dx = x \int_a^b f(x) dx$.

Q[4]: [hint](#) [answer](#) [solution](#)

True or false: an antiderivative of $\frac{1}{x^2}$ is $\log(x^2)$ (where by $\log x$ we mean logarithm base e).

Q[5]:

[hint](#) [answer](#) [solution](#)

True or false: an antiderivative of $\cos(e^x)$ is $\frac{\sin(e^x)}{e^x}$.

Q[6]:

[hint](#) [answer](#) [solution](#)

Suppose $F(x) = \int_7^x \sin(t^2) dt$. What is the instantaneous rate of change of $F(x)$ with respect to x ?

Q[7]:

[hint](#) [answer](#) [solution](#)

Suppose $F(x) = \int_2^x e^{1/t} dt$. What is the slope of the tangent line to $y = F(x)$ when $x = 3$?

Q[8]:

[hint](#) [answer](#) [solution](#)

Suppose $F'(x) = f(x)$. Give two different antiderivatives of $f(x)$.

Q[9]:

[hint](#) [answer](#) [solution](#)

In Question 48, Section 1.1, we found that

$$\int_0^a \sqrt{1-x^2} dx = \frac{\pi}{4} - \frac{1}{2} \arccos(a) + \frac{1}{2} a \sqrt{1-a^2}.$$

(a) Verify that $\frac{d}{da} \left\{ \frac{\pi}{4} - \frac{1}{2} \arccos(a) + \frac{1}{2} a \sqrt{1-a^2} \right\} = \sqrt{1-a^2}$.

(b) Find a function $F(x)$ that satisfies $F'(x) = \sqrt{1-x^2}$ and $F(0) = \pi$.

Q[10]:

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals using the Fundamental Theorem of Calculus Part 2, or explain why it does not apply.

(a) $\int_{-\pi}^{\pi} \cos x dx$.

(b) $\int_{-\pi}^{\pi} \sec^2 x dx$.

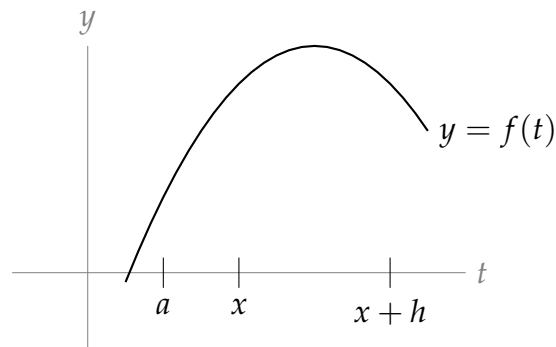
(c) $\int_{-2}^0 \frac{1}{x+1} dx$.

Questions 11 through 14 are meant to help reinforce key ideas in the Fundamental Theorem of Calculus and its proof.

Q[11]:

[hint](#) [answer](#) [solution](#)

As in the proof of the Fundamental Theorem of Calculus, let $F(x) = \int_a^x f(t) dt$. In the diagram below, shade the area corresponding to $F(x+h) - F(x)$.



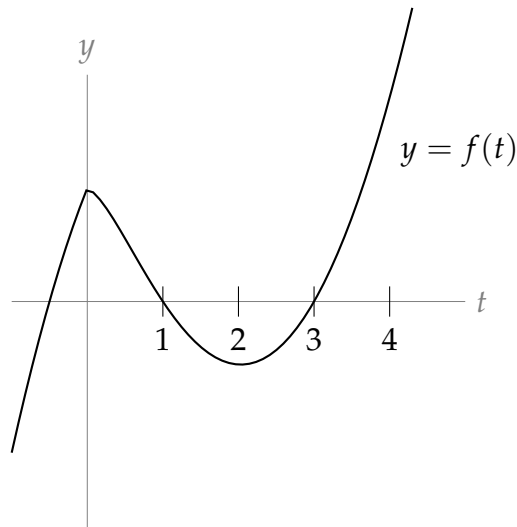
Q[12]:

[hint](#) [answer](#) [solution](#)

Let $F(x) = \int_0^x f(t) dt$, where $f(t)$ is shown in the graph below, and $0 \leq x \leq 4$.

(a) Is $F(0)$ positive, negative, or zero?

(b) Where is $F(x)$ increasing and where is it decreasing?



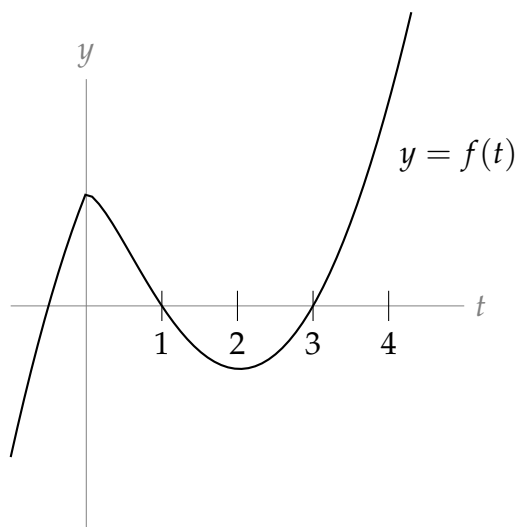
Q[13]:

[hint](#) [answer](#) [solution](#)

Let $G(x) = \int_x^0 f(t) dt$, where $f(t)$ is shown in the graph below, and $0 \leq x \leq 4$.

(a) Is $G(0)$ positive, negative, or zero?

(b) Where is $G(x)$ increasing and where is it decreasing?



Q[14]:

[hint](#) [answer](#) [solution](#)

Let $F(x) = \int_a^x t \, dt$. Using the definition of the derivative, find $F'(x)$.

Q[15]:

[hint](#) [answer](#) [solution](#)

Give a continuous function $f(x)$ so that $F(x) = \int_0^x f(t) \, dt$ is a constant.

So far, we have been able to guess many antiderivatives. Often, however, antiderivatives are very difficult to guess. In Questions 16 through 19, we will find some antiderivatives that might appear in a table of integrals. Coming up with the antiderivative might be quite difficult (strategies to do just that will form a large part of this semester), but verifying that your antiderivative is correct is as simple as differentiating.

Q[16]:

[hint](#) [answer](#) [solution](#)

Evaluate and simplify $\frac{d}{dx} \{x \log(ax) - x\}$, where a is some constant and $\log(x)$ is the logarithm base e . What *antiderivative* does this tell you?

Q[17]:

[hint](#) [answer](#) [solution](#)

Evaluate and simplify $\frac{d}{dx} \{e^x (x^3 - 3x^2 + 6x - 6)\}$. What *antiderivative* does this tell you?

Q[18]:

[hint](#) [answer](#) [solution](#)

Evaluate and simplify $\frac{d}{dx} \left\{ \log \left| x + \sqrt{x^2 + a^2} \right| \right\}$, where a is some constant. What *antiderivative* does this tell you?

Q[19]:

[hint](#) [answer](#) [solution](#)

Evaluate and simplify $\frac{d}{dx} \left\{ \sqrt{x(a+x)} - a \log(\sqrt{x} + \sqrt{a+x}) \right\}$, where a is some constant. What *antiderivative* does this tell you?

►► Stage 2

Q[20](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^2 (x^3 + \sin x) \, dx$.

Q[21](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_1^2 \frac{x^2 + 2}{x^2} dx$.

Q[22]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{1 + 25x^2} dx$.

Q[23]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{\sqrt{2 - x^2}} dx$.

Q[24]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^2 x dx$.

Q[25]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int 3 \sin x \cos x dx$.

Q[26]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \cos^2 x dx$.

Q[27](*):

[hint](#) [answer](#) [solution](#)

If

$$F(x) = \int_0^x \log(2 + \sin t) dt \quad \text{and} \quad G(y) = \int_y^0 \log(2 + \sin t) dt$$

find $F'(\frac{\pi}{2})$ and $G'(\frac{\pi}{2})$.

Q[28](*):

[hint](#) [answer](#) [solution](#)

Let $f(x) = \int_1^x 100(t^2 - 3t + 2)e^{-t^2} dt$. Find the interval(s) on which f is increasing.

Q[29](*):

[hint](#) [answer](#) [solution](#)

If $F(x) = \int_0^{\cos x} \frac{1}{t^3 + 6} dt$, find $F'(x)$.

Q[30](*):

[hint](#) [answer](#) [solution](#)

Compute $f'(x)$ where $f(x) = \int_0^{1+x^4} e^{t^2} dt$.

Q[31](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\frac{d}{dx} \left\{ \int_0^{\sin x} (t^6 + 8) dt \right\}$.

Q[32](*):

[hint](#) [answer](#) [solution](#)

Let $F(x) = \int_0^{x^3} e^{-t} \sin\left(\frac{\pi t}{2}\right) dt$. Calculate $F'(1)$.

Q[33](*):

[hint](#) [answer](#) [solution](#)

Find $\frac{d}{du} \left\{ \int_{\cos u}^0 \frac{dt}{1 + t^3} \right\}$.

Q[34](*):

[hint](#) [answer](#) [solution](#)

Find $f(x)$ if $x^2 = 1 + \int_1^x f(t) dt$.

Q[35](*):

[hint](#) [answer](#) [solution](#)

If $x \sin(\pi x) = \int_0^x f(t) dt$ where f is a continuous function, find $f(4)$.

Q[36](*):

[hint](#) [answer](#) [solution](#)

Consider the function $F(x) = \int_0^{x^2} e^{-t} dt + \int_{-x}^0 e^{-t^2} dt$.

(a) Find $F'(x)$.

(b) Find the value of x for which $F(x)$ takes its minimum value.

Q[37](*):

[hint](#) [answer](#) [solution](#)

If $F(x)$ is defined by $F(x) = \int_{x^4-x^3}^x e^{\sin t} dt$, find $F'(x)$.

Q[38](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\frac{d}{dx} \left\{ \int_{x^5}^{-x^2} \cos(e^t) dt \right\}$.

Q[39](*):

[hint](#) [answer](#) [solution](#)

Differentiate $\int_x^{e^x} \sqrt{\sin t} dt$ for $0 < x < \log \pi$.

Q[40](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_1^5 f(x) dx$, where $f(x) = \begin{cases} 3 & \text{if } x \leq 3 \\ x & \text{if } x \geq 3 \end{cases}$.

► Stage 3

Q[41](*):

[hint](#) [answer](#) [solution](#)

If $f'(1) = 2$ and $f'(2) = 3$, find $\int_1^2 f'(x)f''(x) dx$.

Q[42](*):

[hint](#) [answer](#) [solution](#)

A car traveling at 30 m/s applies its brakes at time $t = 0$, its velocity (in m/s) decreasing according to the formula $v(t) = 30 - 10t$. How far does the car go before it stops?

Q[43](*):

[hint](#) [answer](#) [solution](#)

Compute $f'(x)$ where $f(x) = \int_0^{2x-x^2} \log(1+e^t) dt$. Does $f(x)$ have an absolute maximum? Explain.

Q[44](*):

[hint](#) [answer](#) [solution](#)

Find the minimum value of $\int_0^{x^2-2x} \frac{dt}{1+t^4}$. Express your answer as an integral.

Q[45](*):

[hint](#) [answer](#) [solution](#)

Define the function $F(x) = \int_0^{x^2} \sin(\sqrt{t}) \, dt$ on the interval $0 < x < 4$. On this interval, where does $F(x)$ have a maximum?

Q[46](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\lim_{n \rightarrow \infty} \frac{\pi}{n} \sum_{j=1}^n \sin\left(\frac{j\pi}{n}\right)$ by interpreting it as a limit of Riemann sums.

Q[47](*):

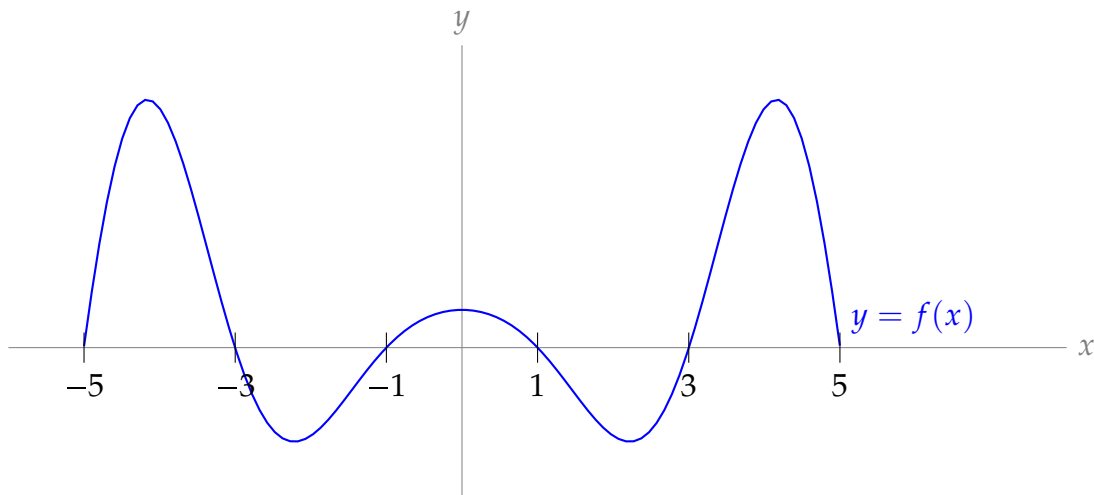
[hint](#) [answer](#) [solution](#)

Use Riemann sums to evaluate the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{1}{1 + \frac{j}{n}}$.

Q[48]:

[hint](#) [answer](#) [solution](#)

Below is the graph of $y = f(t)$, $-5 \leq t \leq 5$. Define $F(x) = \int_0^x f(t) \, dt$ for any x in $[-5, 5]$. Sketch $F(x)$.



Q[49](*):

[hint](#) [answer](#) [solution](#)

Define $f(x) = x^3 \int_0^{x^3+1} e^{t^3} \, dt$.

- Find a formula for the derivative $f'(x)$. (Your formula may include an integral sign.)
- Find the equation of the tangent line to the graph of $y = f(x)$ at $x = -1$.

Q[50]:

[hint](#) [answer](#) [solution](#)

Two students calculate $\int f(x) \, dx$ for some function $f(x)$.

- Student A calculates $\int f(x) \, dx = \tan^2 x + x + C$
- Student B calculates $\int f(x) \, dx = \sec^2 x + x + C$
- It is a fact that $\frac{d}{dx} \{\tan^2 x\} = f(x) - 1$

Who ended up with the correct answer?

Q[51]:

[hint](#) [answer](#) [solution](#)

Let $F(x) = \int_0^x x^3 \sin(t) \, dt$.

(a) Evaluate $F(3)$.

(b) What is $F'(x)$?

Q[52]:

[hint](#) [answer](#) [solution](#)

Let $f(x)$ be an even function, defined everywhere, and let $F(x)$ be an antiderivative of $f(x)$. Is $F(x)$ even, odd, or not necessarily either one? (You may use your answer from Section 1.2, Question 20.)

1.4▲ Interpretations of the definite integral

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

In each part of this question, set up an expression (possibly involving an integral) that represents the desired quantity.

- A torrent is uploading at a speed of $s(t)$ gigabytes per second, where t is time (measured in seconds). How many GB of data are uploaded from $t = 0$ to $t = 1$?
- A population changes over time due to births, deaths, and migration. Let $N(t)$ be the population at time t , $b(t)$ be the rate at which new individuals are born, $d(t)$ be the rate at which individuals are dying, $i(t)$ be the rate at which new individuals are emigrating from elsewhere, and $o(t)$ be the rate at which members of the population are immigrating away. What is the net change in population from $t = 0$ to $t = 1$?
- Let $L(w)$ be the amount of light (in lumens) emitted from a bulb when it is given w watts of power. How much more light is emitted by a bulb supplied with 2 watts than a bulb with 1 watt?
- Let $W(x)$ be the average weight in kg of a 6-week-old lamb, when fed with x litres of formula per day. A farm generally feeds their lamb 1 litre of formula per day. If they were to change the amount of formula just a little, what would be the effect (measured in kg of 6-week-old lamb per litre of formula per day)?

Q[2]:

[hint](#) [answer](#) [solution](#)

In each part of this question, interpret the integral in terms of the model. Your answer should be written descriptions that would be understandable by someone without a knowledge of calculus.

- A farm investigates the rate at which its corn grows throughout the growing season. Suppose $v(t)$ is the rate at which the corn's height is changing at time t . Interpret $\int_0^1 v(t) \, dt$.

- (b) The number of calls per hour to a customer service line is modelled as $c(t)$, where t is hours past midnight. Interpret $\int_0^{24} c(t) dt$.
- (c) A real estate agency investigates the average price per area of homes in Vancouver. Suppose $f(a)$ gives the average cost difference between a home that is $a + 1$ square cm and a home that is a square centimetres. Interpret $\int_{750\,000}^{1\,000\,000} f(a) da$.

►► Stage 2

Q[3]:

[hint](#) [answer](#) [solution](#)

The velocity of a particle moving along a straight line at time t is given by $v(t) = 3t - 2t^2$.

- (a) What is the net change in position of the particle from $t = 1$ to $t = 2$?
- (b) What is the total distance travelled by the particle from $t = 1$ to $t = 2$?

Q[4]:

[hint](#) [answer](#) [solution](#)

Water is dripping out of a hole in a bag at a rate of $\frac{1}{1+x}$ mL per minute, starting at $t = 0$. How many litres of water are lost in the first hour?

Q[5]:

[hint](#) [answer](#) [solution](#)

A particle is moving along the x -axis. At time 0, its position is 2, and its velocity is 1. At time t , its acceleration is $a(t) = 6t - 6$.

- (a) Find the velocity of the particle at times $t = 0$, $t = 1$, $t = 2$, and $t = 3$. Of those four times, when it is moving most quickly to the left, and when it is moving most quickly to the right?
- (b) Find a formula for the position of the particle at time t .

►► Stage 3

Q[6]:

[hint](#) [answer](#) [solution](#)

A basin shaped like a right circular cone with height 1 m and top diameter $r = 5$ m is filling with water. At time $t = 0$, the basin is empty. From $t = 0$ to $t = 1$ minute, water enters the basin at a rate of $f(t) = \frac{1}{2\sqrt{t}}$ cubic m per minute. How much did the height of the water in the basin rise from $t = 0$ to $t = 1$?

Q[7]:

[hint](#) [answer](#) [solution](#)

Two particles are racing along the x -axis. The race starts at time $t = 0$, with the particles in the same position, and ends after 500 metres. Their speeds at time t during the race are given by

$$A(t) = t^3 + 10t$$

and

$$B(t) = 7t^2$$

metres per second, respectively. Call the particles A and B, in the obvious way.

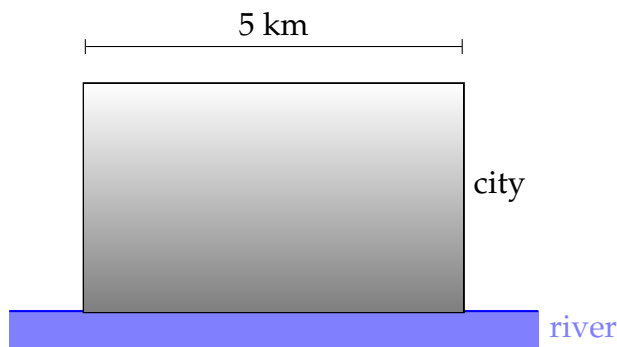
- (a) Which particle is leading just after the race starts?
- (b) Which particle wins the race?

(c) At what time(s) during the race are the particles in the same position?

Q[8]:

[hint](#) [answer](#) [solution](#)

A certain city is build on one bank of a straight river, in the shape of a rectangle with width 5 km. At a distance of x km from the river, the population density is $d(x) = \frac{20}{1+x^2} - 4$ people per square km. The border of the city is where the population density drops to 0.



How many people live in the city?

1.5▲ Substitution

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

Recall that we are using $\log x$ to denote the logarithm of x with base e . In other courses it is often denoted $\ln x$.

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

(a) True or False: $\int \sin(e^x) \cdot e^x \, dx = \int \sin(u) \, du \Big|_{u=e^x} = -\cos(e^x) + C$

(b) True or False: $\int_0^1 \sin(e^x) \cdot e^x \, dx = \int_0^1 \sin(u) \, du = 1 - \cos(1)$

Q[2]:

[hint](#) [answer](#) [solution](#)

Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\int (2x + 1)^2 dx$.

Work: We use the substitution $u = 2x + 1$. Then:

$$\begin{aligned}\int (2x + 1)^2 dx &= \int u^2 du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(2x + 1)^3 + C\end{aligned}$$

Q[3]:

[hint](#) [answer](#) [solution](#)

Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\int_1^\pi \frac{\cos(\log t)}{t} dt$.

Work: We use the substitution $u = \log t$, so $du = \frac{1}{t} dt$. Then:

$$\begin{aligned}\int_1^\pi \frac{\cos(\log t)}{t} dt &= \int_1^\pi \cos(u) du \\ &= \sin(\pi) - \sin(1) = -\sin(1).\end{aligned}$$

Q[4]:

[hint](#) [answer](#) [solution](#)

Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\int_0^{\pi/4} x \tan(x^2) dx$.

Work: We begin with the substitution $u = x^2$, $du = 2x dx$:

$$\begin{aligned}\int_0^{\pi/4} x \tan(x^2) dx &= \int_0^{\pi/4} \frac{1}{2} \tan(x^2) \cdot 2x dx \\ &= \int_0^{\pi^2/16} \frac{1}{2} \tan u du \\ &= \frac{1}{2} \int_0^{\pi^2/16} \frac{\sin u}{\cos u} du\end{aligned}$$

Now we use the substitution $v = \cos u$, $dv = -\sin u du$:

$$\begin{aligned}&= \frac{1}{2} \int_{\cos 0}^{\cos(\pi^2/16)} -\frac{1}{v} dv \\ &= -\frac{1}{2} \int_1^{\cos(\pi^2/16)} \frac{1}{v} dv \\ &= -\frac{1}{2} [\log |v|]_1^{\cos(\pi^2/16)} \\ &= -\frac{1}{2} (\log(\cos(\pi^2/16)) - \log(1)) \\ &= -\frac{1}{2} \log(\cos(\pi^2/16))\end{aligned}$$

Q[5](*):

[hint](#) [answer](#) [solution](#)

What is the integral that results when the substitution $u = \sin x$ is applied to the integral

$$\int_0^{\pi/2} f(\sin x) dx?$$

Q[6]:

[hint](#) [answer](#) [solution](#)

Let f and g be functions that are continuous and differentiable everywhere. Simplify

$$\int f'(g(x))g'(x) dx - f(g(x)).$$

►► Stage 2

Q[7](*):

[hint](#) [answer](#) [solution](#)

Use substitution to evaluate $\int_0^1 xe^{x^2} \cos(e^{x^2}) dx$.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Let $f(t)$ be any function for which $\int_1^8 f(t) dt = 1$. Calculate the integral $\int_1^2 x^2 f(x^3) dx$.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{x^2}{(x^3 + 1)^{101}} dx$.

Q[10](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_e^{e^4} \frac{dx}{x \log x}$.

Q[11](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} dx$.

Q[12](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^{\pi/2} \cos x \cdot (1 + \sin^2 x) dx$.

Q[13](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_1^3 (2x - 1)e^{x^2 - x} dx$.

Q[14](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{(x^2 - 4)x}{\sqrt{4 - x^2}} dx$.

Q[15]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{e^{\sqrt{\log x}}}{2x\sqrt{\log x}} dx$.

►► Stage 3

Q[16](*):

[hint](#) [answer](#) [solution](#)

Calculate $\int_{-2}^2 x e^{x^2} dx$.

Q[17](*):

[hint](#) [answer](#) [solution](#)

Calculate $\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{j}{n^2} \sin \left(1 + \frac{j^2}{n^2} \right)$.

Questions 18 through 22 can be solved by substitution, but it may not be obvious which substitution will work. In general, when evaluating integrals, it is not always immediately clear which methods are appropriate. If this happens to you, don't despair, and definitely don't give up! Just guess a method and try it. Even if it fails, you'll probably learn something that you can use to make a better guess.²

Q[18]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^1 \frac{u^3}{u^2 + 1} du$.

Q[19]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^3 \theta d\theta$.

Q[20]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{e^x + e^{-x}} dx$

Q[21]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^1 (1 - 2x) \sqrt{1 - x^2} dx$

Q[22]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan x \cdot \log(\cos x) dx$

Q[23](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{j}{n^2} \cos \left(\frac{j^2}{n^2} \right)$.

Q[24](*):

[hint](#) [answer](#) [solution](#)

Calculate $\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{j}{n^2} \sqrt{1 + \frac{j^2}{n^2}}$.

Q[25]:

[hint](#) [answer](#) [solution](#)

Using Riemann sums, prove that

$$\int_a^b 2f(2x)dx = \int_{2a}^{2b} f(x)dx$$

1.6▲ Area between curves

Exercises

² This is also pretty decent life advice.

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

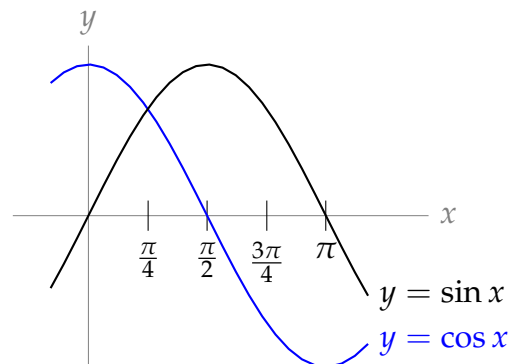
► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

We want to approximate the area between the graphs of $y = \cos x$ and $y = \sin x$ from $x = 0$ to $x = \pi$ using a left Riemann sum with $n = 4$ rectangles.

- On the graph below, sketch the four rectangles.
- Calculate the Riemann approximation.



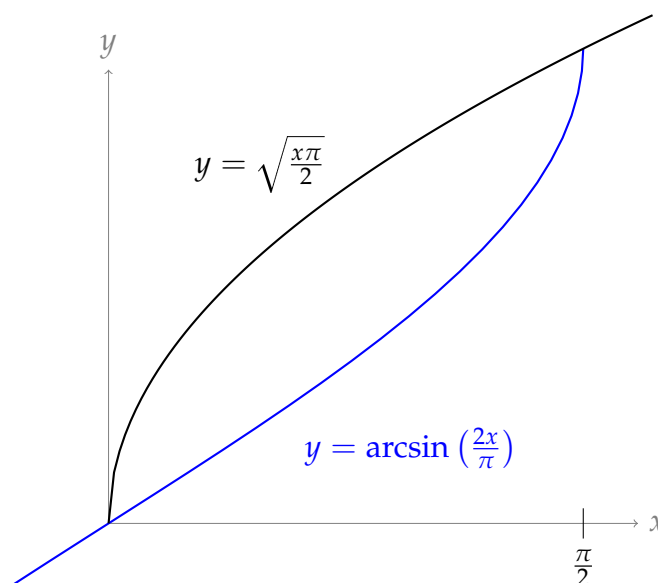
Q[2]:

[hint](#) [answer](#) [solution](#)

We want to approximate the bounded area between the curves $y = \arcsin\left(\frac{2x}{\pi}\right)$ and

$y = \sqrt{\frac{\pi x}{2}}$ using $n = 5$ rectangles.

- Draw the five (vertical) rectangles on the picture below corresponding to a right Riemann sum.
- Draw five rectangles on the picture below we might use if we were using horizontal rectangles.



Q[3](*):

[hint](#) [answer](#) [solution](#)

Write down a definite integral that represents the finite area bounded by the curves $y = x^3 - x$ and $y = x$ for $x \geq 0$. *Do not evaluate the integral explicitly.*

Q[4](*):

[hint](#) [answer](#) [solution](#)

Write down a definite integral that represents the area of the finite region bounded by the line $y = -\frac{x}{2}$ and the parabola $y^2 = 6 - \frac{5x}{4}$. *Do not evaluate the integral explicitly.*

Q[5](*):

[hint](#) [answer](#) [solution](#)

Write down a definite integral that represents the area of the finite plane region bounded by $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$ is a constant. *Do not evaluate the integral explicitly.*

Q[6](*):

[hint](#) [answer](#) [solution](#)

Write down a definite integral that represents the area of the finite region bounded between the line $x + 12y + 5 = 0$ and the curve $x = 4y^2$. *Do not evaluate the integral explicitly.*

► Stage 2

Q[7](*):

[hint](#) [answer](#) [solution](#)

Find the area of the region bounded by the graph of $f(x) = \frac{1}{(2x-4)^2}$ and the x -axis between $x = 0$ and $x = 1$.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Find the area of the finite region between the curves $y = x$ and $y = 3x - x^2$, by first identifying the points of intersection and then integrating.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Calculate the area of the finite region enclosed by $y = 2^x$ and $y = \sqrt{x} + 1$.

Q[10](*):

[hint](#) [answer](#) [solution](#)

Find the area of the finite region bounded between the two curves $y = \sqrt{2} \cos(\pi x/4)$ and $y = |x|$.

Q[11](*):

[hint](#) [answer](#) [solution](#)

Find the area of the finite region that is bounded by the graphs of $f(x) = x^2\sqrt{x^3+1}$ and $g(x) = 3x^2$.

Q[12](*):

[hint](#) [answer](#) [solution](#)

Find the area to the left of the y -axis and to the right of the curve $x = y^2 + y$.

Q[13]:

[hint](#) [answer](#) [solution](#)

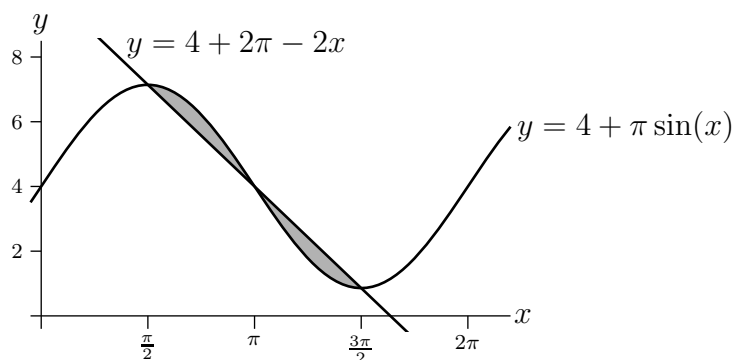
Find the area of the finite region below $y = \sqrt{9-x^2}$ and above both $y = |x|$ and $y = \sqrt{1-x^2}$.

► Stage 3

Q[14](*):

[hint](#) [answer](#) [solution](#)

The graph below shows the finite region between $y = 4 + \pi \sin x$ and $y = 4 + 2\pi - 2x$.



Find the area of this region.

Q[15](*):

[hint](#) [answer](#) [solution](#)

Compute the area of the finite region bounded by the curves $x = 0$, $x = 3$, $y = x + 2$ and $y = x^2$.

Q[16](*):

[hint](#) [answer](#) [solution](#)

Find the total area between the curves $y = x\sqrt{25 - x^2}$ and $y = 3x$, on the interval $0 \leq x \leq 4$.

Q[17]:

[hint](#) [answer](#) [solution](#)

Find the area of the finite region below $y = \sqrt{9 - x^2}$ and $y = x$, and above $y = \sqrt{1 - (x - 1)^2}$.

Q[18]:

[hint](#) [answer](#) [solution](#)

Find the area of the finite region bounded by the curve $y = x(x^2 - 4)$ and the line $y = x - 2$.

1.7▲ Trigonometric Integrals

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

Recall that we are using $\log x$ to denote the logarithm of x with base e . In other courses it is often denoted $\ln x$.

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Suppose you want to evaluate $\int_0^{\pi/4} \sin x \cos^n x \, dx$ using the substitution $u = \cos x$.

Which of the following need to be true for your substitution to work?

- (a) n must be even
- (b) n must be odd
- (c) n must be an integer
- (d) n must be positive
- (e) n can be any real number

Q[2]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sec^n x \tan x dx$, where n is a strictly positive integer.

Q[3]:

[hint](#) [answer](#) [solution](#)

Derive the identity $\tan^2 x + 1 = \sec^2 x$ from the easier-to-remember identity $\sin^2 x + \cos^2 x = 1$.

► Stage 2

Questions 4 through 10 deal with powers of sines and cosines. Review Section 1.7.1 in the text for integration strategies.

Q[4](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \cos^3 x dx$.

Q[5](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^\pi \cos^2 x dx$.

Q[6](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sin^{36} t \cos^3 t dt$.

Q[7]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{\sin^3 x}{\cos^4 x} dx$.

Q[8]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^{\pi/3} \sin^4 x dx$.

Q[9]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sin^5 x dx$.

Q[10]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sin^{1.2} x \cos x dx$.

Q[11]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan x \sec^2 x dx$.

Q[12](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^3 x \sec^5 x dx$.

Q[13]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^3 x \sec^{1.5} x dx$.

Q[14]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^3 x \sec^2 x \, dx$.

Q[15]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^4 x \sec^2 x \, dx$.

Q[16]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^3 x \sec^{-0.7} x \, dx$.

Q[17]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^5 x \, dx$.

Q[18]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan^5 x \cos^2 x \, dx$.

Q[19]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \tan x \sqrt{\sec x} \, dx$.

Q[20]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{\cos^2 \theta} d\theta$.

Q[21]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \cot x \, dx$.

Q[22]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int e^x \sin(e^x) \cos(e^x) \, dx$.

►► Stage 3

Q[23](*):

[hint](#) [answer](#) [solution](#)

A reduction formula.

(a) Let n be a positive integer with $n \geq 2$. Derive the reduction formula

$$\int \tan^n(x) \, dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x) \, dx.$$

(b) Calculate $\int_0^{\pi/4} \tan^6(x) \, dx$.

1.8▲ Trigonometric Substitution

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

Recall that we are using $\log x$ to denote the logarithm of x with base e . In other courses it is often denoted $\ln x$.

►► Stage 1

Q[1](*):

[hint](#) [answer](#) [solution](#)

For each of the following integrals, choose the substitution that is most beneficial for evaluating the integral.

(a) $\int \frac{2x^2}{\sqrt{9x^2 - 16}} dx$

(b) $\int \frac{x^4 - 3}{\sqrt{1 - 4x^2}} dx$

(c) $\int (25 + x^2)^{-5/2} dx$

Q[2]:

[hint](#) [answer](#) [solution](#)

For each of the following integrals, choose a trigonometric substitution that will eliminate the roots.

(a) $\int \frac{1}{\sqrt{x^2 - 4x + 1}} dx$

(b) $\int \frac{(x - 1)^6}{(-x^2 + 2x + 4)^{3/2}} dx$

(c) $\int \frac{1}{\sqrt{4x^2 + 6x + 10}} dx$

(d) $\int \sqrt{x^2 - x} dx$

Q[3]:

[hint](#) [answer](#) [solution](#)

In each part of this question, assume θ is an angle in the interval $[0, \pi/2]$.

(a) If $\sin \theta = \frac{1}{20}$, what is $\cos \theta$?

(b) If $\tan \theta = 7$, what is $\csc \theta$?

(c) If $\sec \theta = \frac{\sqrt{x-1}}{2}$, what is $\tan \theta$?

Q[4]:

[hint](#) [answer](#) [solution](#)

Simplify the following expressions.

(a) $\sin \left(\arccos \left(\frac{x}{2} \right) \right)$

(b) $\sin \left(\arctan \left(\frac{1}{\sqrt{3}} \right) \right)$

(c) $\sec \left(\arcsin \left(\sqrt{x} \right) \right)$

► Stage 2

Q[5](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{(x^2 + 4)^{3/2}} dx$.

Q[6](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^4 \frac{1}{(4 + x^2)^{3/2}} dx$. Your answer may not contain inverse trigonometric functions.

Q[7](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^{5/2} \frac{dx}{\sqrt{25 - x^2}}$.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{dx}{\sqrt{x^2 + 25}}$. You may use that $\int \sec x dx = \log |\sec x + \tan x| + C$.

Q[9]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{x + 1}{\sqrt{2x^2 + 4x}} dx$.

Q[10](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{dx}{x^2 \sqrt{x^2 + 16}}$.

Q[11](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{dx}{x^2 \sqrt{x^2 - 9}}$ for $x \geq 3$. Do not include any inverse trigonometric functions in your answer.

Q[12](*):

[hint](#) [answer](#) [solution](#)

(a) Show that $\int_0^{\pi/4} \cos^4 \theta d\theta = (8 + 3\pi)/32$.

(b) Evaluate $\int_{-1}^1 \frac{dx}{(x^2 + 1)^3}$.

Q[13]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{-\pi/12}^{\pi/12} \frac{15x^3}{(x^2 + 1)(9 - x^2)^{5/2}} dx$.

Q[14](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sqrt{4 - x^2} dx$.

Q[15](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{\sqrt{25x^2 - 4}}{x} dx$ for $x > \frac{2}{5}$.

Q[16]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{\sqrt{10}}^{\sqrt{17}} \frac{x^3}{\sqrt{x^2 - 1}} dx$.

Q[17](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{dx}{\sqrt{3-2x-x^2}}.$

Q[18]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{(2x-3)^3 \sqrt{4x^2-12x+8}} dx$ for $x > 2$.

Q[19]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^1 \frac{x^2}{(x^2+1)^{3/2}} dx.$

You may use that $\int \sec x dx = \log |\sec x + \tan x| + C.$

Q[20]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{(x^2+1)^2} dx.$

►► Stage 3

Q[21]:

[hint](#) [answer](#) [solution](#)

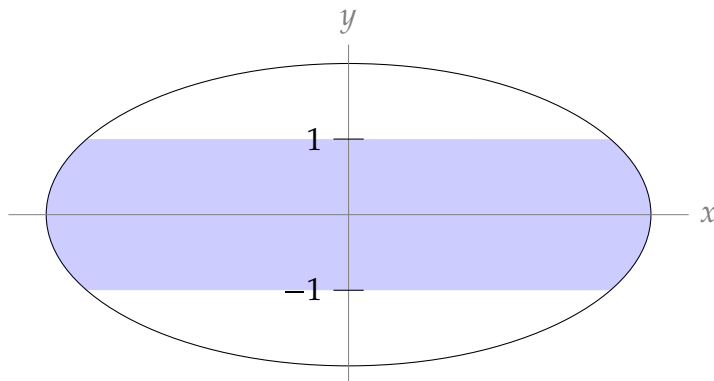
Evaluate $\int \frac{1}{\sqrt{3x^2+5x}} dx.$

You may use that $\int \sec x dx = \log |\sec x + \tan x| + C.$

Q[22]:

[hint](#) [answer](#) [solution](#)

Below is the graph of the ellipse $(\frac{x}{4})^2 + (\frac{y}{2})^2 = 1$. Find the area of the shaded region using the ideas from this section.



Q[23]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sqrt{1+e^x} dx$. You may use the antiderivative $\int \csc \theta d\theta = \log |\cot \theta - \csc \theta| + C.$

Q[24]:

[hint](#) [answer](#) [solution](#)

(a) Suppose we are evaluating an integral that contains the term $\sqrt{a^2-x^2}$, where a is a positive constant, and we use the substitution $x = a \sin u$ (with inverse $u = \arcsin(x/a)$), so that

$$\sqrt{a^2-x^2} = \sqrt{a^2 \cos^2 u} = |a \cos u|$$

Under what circumstances is $|a \cos u| \neq a \cos u$?

- (b) Suppose we are evaluating an integral that contains the term $\sqrt{a^2 + x^2}$, where a is a positive constant, and we use the substitution $x = a \tan u$ (with inverse $u = \arctan(x/a)$), so that

$$\sqrt{a^2 + x^2} = \sqrt{a^2 \sec^2 u} = |a \sec u|$$

Under what circumstances is $|a \sec u| \neq a \sec u$?

- (c) Suppose we are evaluating an integral that contains the term $\sqrt{x^2 - a^2}$, where a is a positive constant, and we use the substitution $x = a \sec u$ (with inverse $u = \operatorname{arcsec}(x/a) = \arccos(a/x)$), so that

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2 u} = |a \tan u|$$

Under what circumstances is $|a \tan u| \neq a \tan u$?

1.9▴ Partial Fractions

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

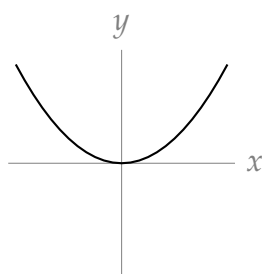
Recall that we are using $\log x$ to denote the logarithm of x with base e . In other courses it is often denoted $\ln x$.

► Stage 1

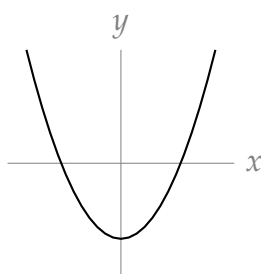
Q[1]:

[hint](#) [answer](#) [solution](#)

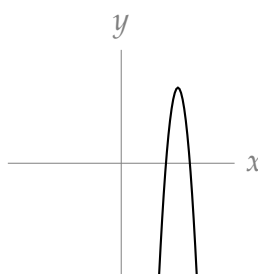
Below are the graphs of four different quadratic functions. For each quadratic function, decide whether it is: (i) irreducible, (ii) the product of two distinct linear factors, or (iii) the product of a repeated linear factor (and possibly a constant).



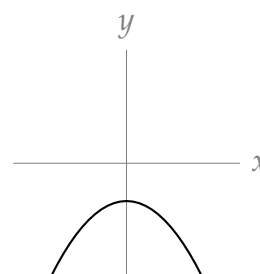
(a)



(b)



(c)



(d)

Q[2](*):

[hint](#) [answer](#) [solution](#)

Find the coefficient of $\frac{1}{x-1}$ in the partial fraction decomposition of $\frac{3x^3 - 2x^2 + 11}{x^2(x-1)(x^2+3)}$.

Q[3]:

[hint](#) [answer](#) [solution](#)

Re-write the following rational functions as the sum of a polynomial and a rational function whose numerator has a strictly smaller degree than its denominator. (Remember

our method of partial fraction decomposition of a rational function only works when the degree of the numerator is strictly smaller than the degree of the denominator.)

(a) $\frac{x^3 + 2x + 2}{x^2 + 1}$

(b) $\frac{15x^4 + 6x^3 + 34x^2 + 4x + 20}{5x^2 + 2x + 8}$

(c) $\frac{2x^5 + 9x^3 + 12x^2 + 10x + 30}{2x^2 + 5}$

Q[4]:

[hint](#) [answer](#) [solution](#)

Factor the following polynomials into linear and irreducible factors.

(a) $5x^3 - 3x^2 - 10x + 6$

(b) $x^4 - 3x^2 - 5$

(c) $x^4 - 4x^3 - 10x^2 - 11x - 6$

(d) $2x^4 + 12x^3 - x^2 - 52x + 15$

Q[5]:

[hint](#) [answer](#) [solution](#)

Here is a fact:

Suppose we have a rational function with a repeated linear factor $(ax + b)^n$ in the denominator, and the degree of the numerator is strictly less than the degree of the denominator. In the partial fraction decomposition, we can replace the terms

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_n}{(ax + b)^n} \quad (1)$$

with the single term

$$\frac{B_1 + B_2x + B_3x^2 + \cdots + B_nx^{n-1}}{(ax + b)^n} \quad (2)$$

and still be guaranteed to find a solution.

Why do we use the sum in (1), rather than the single term in (2), in partial fraction decomposition?

► Stage 2

Q[6](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_1^2 \frac{dx}{x + x^2}$.

Q[7](*):

[hint](#) [answer](#) [solution](#)

Calculate $\int \frac{1}{x^4 + x^2} dx$.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Calculate $\int \frac{12x + 4}{(x - 3)(x^2 + 1)} dx$.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following indefinite integral using partial fraction:

$$F(x) = \int \frac{3x^2 - 4}{(x - 2)(x^2 + 4)} dx.$$

Q[10](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{x - 13}{x^2 - x - 6} dx$.

Q[11](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{5x + 1}{x^2 + 5x + 6} dx$.

Q[12]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{5x^2 - 3x - 1}{x^2 - 1} dx$.

Q[13]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{4x^4 + 14x^2 + 2}{4x^4 + x^2} dx$.

Q[14]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{3x^2 - 4x - 10}{2x^3 - x^2 - 8x + 4} dx$.

Q[15]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^1 \frac{10x^2 + 24x + 8}{2x^3 + 11x^2 + 6x + 5} dx$.

►► Stage 3

In Questions 16 and 17, we use partial fraction to find the antiderivatives of two important functions: cosecant, and cosecant cubed.

Q[16]:

[hint](#) [answer](#) [solution](#)

Using the method of Example 1.9.5 in the text, integrate $\int \csc x \, dx$.

Q[17]:

[hint](#) [answer](#) [solution](#)

Using the method of Example 1.9.6 in the text, integrate $\int \csc^3 x \, dx$.

Q[18]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_1^2 \frac{3x^3 + 15x^2 + 35x + 10}{x^4 + 5x^3 + 10x^2} dx$.

In Questions 19 through 21, we use substitution to turn a non-rational integrand into a rational integrand, then evaluate the resulting integral using partial fraction. Till now, the partial fraction problems you've seen

have all looked largely the same, but keep in mind that a partial fraction decomposition can be a small step in a larger problem.

Q[19]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{\cos \theta}{3 \sin \theta + \cos^2 \theta - 3} d\theta$.

Q[20]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{e^{2t} + e^t + 1} dt$.

Q[21]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sqrt{1 + e^x} dx$ using partial fraction.

Q[22]:

[hint](#) [answer](#) [solution](#)

Find the area of the finite region bounded by the curves $y = \frac{4}{3 + x^2}$, $y = \frac{2}{x(x + 1)}$, $x = \frac{1}{4}$, and $x = 3$.

Q[23]:

[hint](#) [answer](#) [solution](#)

Let $F(x) = \int_1^x \frac{1}{t^2 - 9} dt$.

(a) Give a formula for $F(x)$ that does not involve an integral.

(b) Find $F'(x)$.

1.10▲ Volumes

Exercises

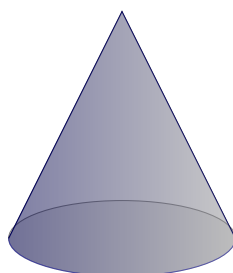
Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

►► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Consider a right circular cone.



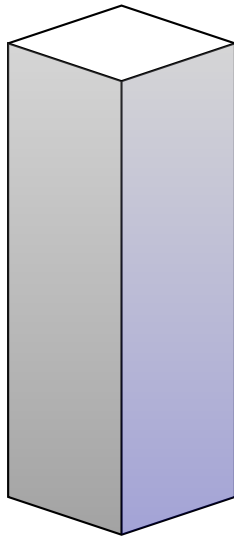
What shape are horizontal cross-sections? Are the vertical cross-sections the same?

Q[2]:

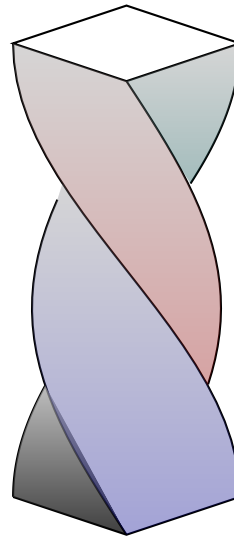
[hint](#) [answer](#) [solution](#)

Two potters start with a block of clay h units tall, and identical square cookie cutters. They form columns by pushing the square cookie cutter straight down over the clay, so that its cross-section is the same square as the cookie cutter. Potter A pushes their cookie

cutter down while their clay block is sitting motionless on a table; Potter B pushes their cookie cutter down while their clay block is rotating on a potter's wheel, so their column looks twisted. Which column has greater volume?



Column A

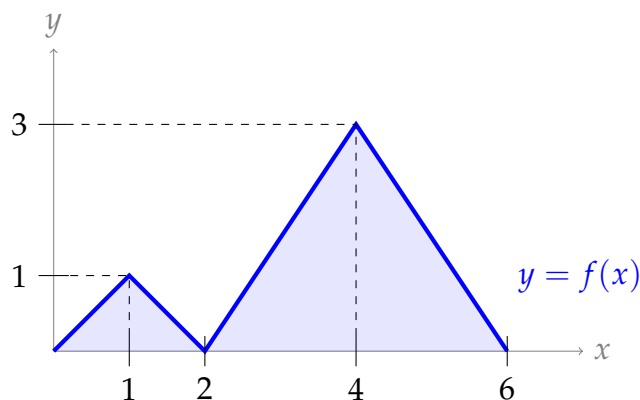


Column B

Q[3]:

[hint](#) [answer](#) [solution](#)

Let R be the region bounded above by the graph of $y = f(x)$ shown below and bounded below by the x -axis, from $x = 0$ to $x = 6$. Sketch the washers that are formed by rotating R about the y -axis. In your sketch, label all the radii in terms of y , and label the thickness.



Q[4](*):

[hint](#) [answer](#) [solution](#)

Write down definite integrals that represent the following quantities. *Do not evaluate the integrals explicitly.*

- The volume of the solid obtained by rotating around the x -axis the region between the x -axis and $y = \sqrt{x}e^{x^2}$ for $0 \leq x \leq 3$.
- The volume of the solid obtained by revolving the region bounded by the curves $y = x^2$ and $y = x + 2$ about the line $x = 3$.

Q[5](*):

[hint](#) [answer](#) [solution](#)

Write down definite integrals that represent the following quantities. *Do not evaluate the integrals explicitly.*

- (a) The volume of the solid obtained by rotating the finite plane region bounded by the curves $y = 1 - x^2$ and $y = 4 - 4x^2$ about the line $y = -1$.
- (b) The volume of the solid obtained by rotating the finite plane region bounded by the curve $y = x^2 - 1$ and the line $y = 0$ about the line $x = 5$.

Q[6](*):

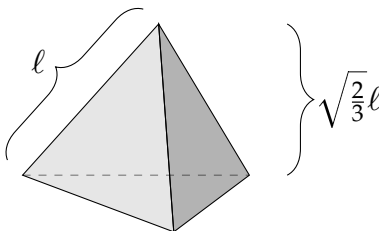
[hint](#) [answer](#) [solution](#)

Write down a definite integral that represents the volume of the solid obtained by rotating around the line $y = -1$ the region between the curves $y = x^2$ and $y = 8 - x^2$. *Do not evaluate the integrals explicitly.*

Q[7]:

[hint](#) [answer](#) [solution](#)

A tetrahedron is a three-dimensional shape with four faces, each of which is an equilateral triangle. (You might have seen this shape as a 4-sided die; think of a pyramid with a triangular base.) Using the methods from this section, calculate the volume of a tetrahedron with side-length ℓ . You may assume without proof that the height of a tetrahedron with side-length ℓ is $\sqrt{\frac{2}{3}}\ell$.



► Stage 2

Q[8](*):

[hint](#) [answer](#) [solution](#)

Let $a > 0$ be a constant. Let R be the finite region bounded by the graph of $y = 1 + \sqrt{x}e^{x^2}$, the line $y = 1$, and the line $x = a$. Using vertical slices, find the volume generated when R is rotated about the line $y = 1$.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Find the volume of the solid generated by rotating the finite region bounded by $y = 1/x$ and $3x + 3y = 10$ about the x -axis.

Q[10](*):

[hint](#) [answer](#) [solution](#)

Let R be the region inside the circle $x^2 + (y - 2)^2 = 1$. Let S be the solid obtained by rotating R about the x -axis.

- (a) Write down an integral representing the volume of S .
- (b) Evaluate the integral you wrote down in part (a).

Q[11](*):

[hint](#) [answer](#) [solution](#)

The region R is the portion of the first quadrant which is below the parabola $y^2 = 8x$ and above the hyperbola $y^2 - x^2 = 15$.

- (a) Sketch the region R .
- (b) Find the volume of the solid obtained by revolving R about the x axis.

Q[12](*):

[hint](#) [answer](#) [solution](#)

The region R is bounded by $y = \log x$, $y = 0$, $x = 1$ and $x = 2$. (Recall that we are using $\log x$ to denote the logarithm of x with base e . In other courses it is often denoted $\ln x$.)

- (a) Sketch the region R .
- (b) Find the volume of the solid obtained by revolving this region about the y axis.

Q[13](*):

[hint](#) [answer](#) [solution](#)

The finite region between the curves $y = \cos(\frac{x}{2})$ and $y = x^2 - \pi^2$ is rotated about the line $y = -\pi^2$. Using vertical slices (disks and/or washers), find the volume of the resulting solid.

Q[14](*):

[hint](#) [answer](#) [solution](#)

The solid V is 2 meters high and has square horizontal cross sections. The length of the side of the square cross section at height x meters above the base is $\frac{2}{1+x}$ m. Find the volume of this solid.

Q[15](*):

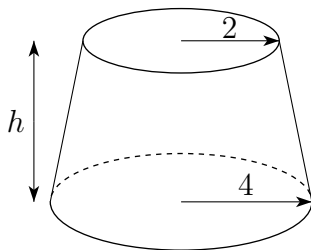
[hint](#) [answer](#) [solution](#)

Consider a solid whose base is the finite portion of the xy -plane bounded by the curves $y = x^2$ and $y = 8 - x^2$. The cross-sections perpendicular to the x -axis are squares with one side in the xy -plane. Compute the volume of this solid.

Q[16](*):

[hint](#) [answer](#) [solution](#)

A frustum of a right circular cone (as shown below) has height h . Its base is a circular disc with radius 4 and its top is a circular disc with radius 2. Calculate the volume of the frustum.



► Stage 3

Q[17]:

[hint](#) [answer](#) [solution](#)

The shape of the earth is often approximated by an oblate spheroid, rather than a sphere. An *oblate spheroid* is formed by rotating an ellipse about its *minor axis* (its shortest diameter).

- (a) Find the volume of the oblate spheroid obtained by rotating the upper (positive) half of the ellipse $(ax)^2 + (by)^2 = 1$ about the x -axis, where a and b are positive constants with $a \geq b$.

- (b) Suppose³ the earth has radius at the equator of 6378.137 km, and radius at the poles of 6356.752 km. If we model the earth as an oblate spheroid formed by rotating the upper half of the ellipse $(ax)^2 + (by)^2 = 1$ about the x -axis, what are a and b ?
- (c) What is the volume of this model of the earth? (Use a calculator.)
- (d) Suppose we had calculated the volume of the earth by modelling it as a sphere with radius 6378.137 km. What would our absolute and relative errors be, compared to our oblate spheroid calculation?

Q[18](*):

[hint](#) [answer](#) [solution](#)

Let R be the bounded region that lies between the curve $y = 4 - (x - 1)^2$ and the line $y = x + 1$.

- (a) Sketch R and find its area.
- (b) Write down a definite integral giving the volume of the region obtained by rotating R about the line $y = 5$. *Do not evaluate this integral.*

Q[19](*):

[hint](#) [answer](#) [solution](#)

Let $\mathcal{R} = \{(x, y) : (x - 1)^2 + y^2 \leq 1 \text{ and } x^2 + (y - 1)^2 \leq 1\}$.

- (a) Sketch $\mathcal{R}$ and find its area.
- (b) If $\mathcal{R}$ rotates around the y -axis, what volume is generated?

Q[20](*):

[hint](#) [answer](#) [solution](#)

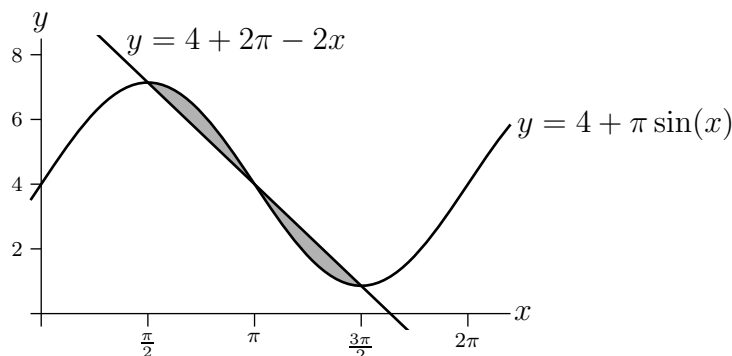
Let $\mathcal{R}$ be the plane region bounded by $x = 0$, $x = 1$, $y = 0$ and $y = c\sqrt{1 + x^2}$, where $c \geq 0$ is a constant.

- (a) Find the volume V_1 of the solid obtained by revolving $\mathcal{R}$ about the x -axis.
- (b) Find the volume V_2 of the solid obtained by revolving $\mathcal{R}$ about the y -axis.
- (c) If $V_1 = V_2$, what is the value of c ?

Q[21](*):

[hint](#) [answer](#) [solution](#)

The graph below shows the region between $y = 4 + \pi \sin x$ and $y = 4 + 2\pi - 2x$.



The region is rotated about the line $y = -1$. Express in terms of definite integrals the volume of the resulting solid. Do not evaluate the integrals.

3 Earth Fact Sheet, NASA, <https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html>, accessed 2 July 2017

Q[22]:

[hint](#) [answer](#) [solution](#)

On a particular, highly homogeneous⁴ planet, we observe that the density of the atmosphere h kilometres above the surface is given by the equation $\rho(h) = c2^{-h/6} \frac{\text{kg}}{\text{m}^3}$, where c is the density on the planet's surface.

- What is the mass of the atmosphere contained in a vertical column with radius one metre, sixty kilometres high?
- What height should a column be to contain $\frac{3000c\pi}{\log 2}$ kilograms of air?

Q[23]:

[hint](#) [answer](#) [solution](#)

Let $f(x) = \frac{|x|}{\sqrt[4]{1-x^2}}$, and let R be the region between $f(x)$ and the x -axis over the interval $[-\frac{1}{2}, \frac{1}{2}]$.

- Find the area of R .
- Find the volume of the solid formed by rotating R about the x -axis.

Q[24](*):

[hint](#) [answer](#) [solution](#)

The region R is the portion of the first quadrant where $3 \leq x \leq 4$ and $0 \leq y \leq \frac{10}{\sqrt{25-x^2}}$.

- Sketch the region R .
- Determine the volume of the solid obtained by revolving R around the x -axis.
- Determine the volume of the solid obtained by revolving R around the y -axis.

1.11▴ Integration by parts

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

The method of integration by substitution comes from the _____ rule for differentiation.

The method of integration by parts comes from the _____ rule for differentiation.

Q[2]:

[hint](#) [answer](#) [solution](#)

Suppose you want to evaluate an integral using integration by parts. You choose part of your integrand to be u , and part to be dv . The part chosen as u will be: (differentiated, antiderivativated). The part chosen as dv will be: (differentiated, antiderivativated).

4 This is clearly a simplified model: air density changes all the time, and depends on lots of complicated factors aside from altitude. However, the equation we're using is not so far off from an idealized model of the earth's atmosphere, taken from *Pressure and the Gas Laws* by H.P. Schmid, http://www.indiana.edu/~geog109/topics/10_Forces&Winds/GasPressWeb/PressGasLaws.html, accessed 3 July 2017.

Q[3]:

[hint](#) [answer](#) [solution](#)

Let $f(x)$ and $g(x)$ be differentiable functions. Using the quotient rule for differentiation, give an equivalent expression to $\int \frac{f'(x)}{g(x)} dx$.

Q[4]:

[hint](#) [answer](#) [solution](#)

Suppose we want to use integration by parts to evaluate $\int u(x) \cdot v'(x) dx$ for some differentiable functions u and v . We need to find an antiderivative of $v'(x)$, but there are infinitely many choices. Show that every antiderivative of $v'(x)$ gives an equivalent final answer.

Q[5]:

[hint](#) [answer](#) [solution](#)

Suppose you want to evaluate $\int f(x) dx$ using integration by parts. Explain why $dv = f(x) dx$, $u = 1$ is generally a bad choice.

Note: compare this to Example 1.11.8 of the text, where we chose $u = f(x)$, $dv = 1 dx$.

► Stage 2

Q[6](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int x \log x dx$.

Q[7](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{\log x}{x^7} dx$.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^\pi x \sin x dx$.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^{\frac{\pi}{2}} x \cos x dx$.

Q[10]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int x^3 e^x dx$.

Q[11]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int x \log^3 x dx$.

Q[12]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int x^2 \sin x dx$.

Q[13]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int (3t^2 - 5t + 6) \log t dt$.

Q[14]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sqrt{s} e^{\sqrt{s}} ds$.

Q[15]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \log^2 x dx$.

Q[16]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int 2xe^{x^2+1} dx$.

Q[17](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \arccos y dy$.

Q[18](*):

[hint](#) [answer](#) [solution](#)

Let R be the part of the first quadrant that lies below the curve $y = \arctan x$ and between the lines $x = 0$ and $x = 1$.

(a) Sketch the region R and determine its area.

(b) Find the volume of the solid obtained by rotating R about the y -axis.

Q[19](*):

[hint](#) [answer](#) [solution](#)

Let R be the region between the curves $T(x) = \sqrt{x}e^{3x}$ and $B(x) = \sqrt{x}(1 + 2x)$ on the interval $0 \leq x \leq 3$. (It is true that $T(x) \geq B(x)$ for all $0 \leq x \leq 3$.) Compute the volume of the solid formed by rotating R about the x -axis.

► Stage 3

Q[20](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int 4y \arctan(2y) dy$.

Q[21]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int x^2 \arctan x dx$.

Q[22]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int 2^{x+\log_2 x} dx$.

Q[23]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int e^{\cos x} \sin(2x) dx$.

Q[24]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{xe^{-x}}{(1-x)^2} dx$.

Q[25](*):

[hint](#) [answer](#) [solution](#)

A reduction formula.

(a) Derive the reduction formula

$$\int \sin^n(x) \, dx = -\frac{\sin^{n-1}(x) \cos(x)}{n} + \frac{n-1}{n} \int \sin^{n-2}(x) \, dx.$$

(b) Calculate $\int_0^{\pi/2} \sin^8(x) \, dx$.

Q[26](*):

[hint](#) [answer](#) [solution](#)

Let $f(0) = 1$, $f(2) = 3$ and $f'(2) = 4$. Calculate $\int_0^4 f''(\sqrt{x}) \, dx$.

Q[27]:

[hint](#) [answer](#) [solution](#)

Evaluate $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left(\frac{2}{n}i - 1 \right) e^{\frac{2}{n}i-1}$.

Q[28]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int x \sin x \cos x \, dx$.

Q[29]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sin(\cos x) \sin^3 x \, dx$.

1.12▲ Numerical Integration

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

Recall that we are using $\log x$ to denote the logarithm of x with base e . In other courses it is often denoted $\ln x$.

► Stage 1

Q[1]:

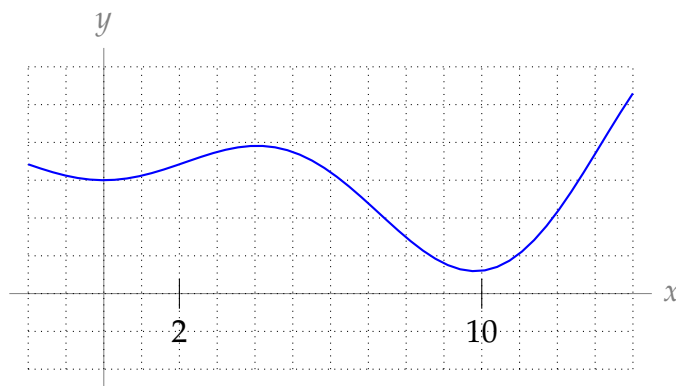
[hint](#) [answer](#) [solution](#)

Suppose we approximate an object to have volume 1.5m^3 , when its exact volume is 1.387m^3 . Give the relative error, absolute error, and percent error of our approximation.

Q[2]:

[hint](#) [answer](#) [solution](#)

Consider approximating $\int_2^{10} f(x) \, dx$, where $f(x)$ is the function in the graph below.



Draw the trapezoids associated with the trapezoidal rule approximation and $n = 4$. You don't have to give an approximation.

Q[3]:

[hint](#) [answer](#) [solution](#)

Let $f(x) = -\frac{1}{12}x^4 + \frac{7}{6}x^3 - 3x^2$.

(a) Find a reasonable value M such that $|f''(x)| \leq M$ for all $1 \leq x \leq 6$.

(b) Find a reasonable value L such that $|f^{(4)}(x)| \leq L$ for all $1 \leq x \leq 6$.

Q[4]:

[hint](#) [answer](#) [solution](#)

Let $f(x) = x \sin x + 2 \cos x$. Find a reasonable value M such that $|f'''(x)| \leq M$ for all $-3 \leq x \leq 2$.

Q[5]:

[hint](#) [answer](#) [solution](#)

Consider the quantity $A = \int_{-\pi}^{\pi} \cos x \, dx$.

(a) Find the upper bound on the error using Simpson's rule with $n = 4$ to approximate A using Theorem 1.12.7 in the text.

(b) Find the Simpson's rule approximation of A using $n = 4$.

(c) What is the (actual) absolute error in the Simpson's rule approximation of A with $n = 4$?

Q[6]:

[hint](#) [answer](#) [solution](#)

Give a function $f(x)$ such that:

- $f''(x) \leq 3$ for every x in $[0, 1]$, and
- the error using the trapezoidal rule approximating $\int_0^1 f(x) \, dx$ with $n = 2$ intervals is exactly $\frac{1}{16}$.

Q[7]:

[hint](#) [answer](#) [solution](#)

Suppose my mother is under 100 years old, and I am under 200 years old.⁵ Who is older?

Q[8](*):

[hint](#) [answer](#) [solution](#)

5 We're going somewhere with this.

Decide whether the following statement is true or false. If false, provide a counterexample. If true, provide a brief justification.

When $f(x)$ is positive and concave up, any trapezoidal rule approximation for $\int_a^b f(x) dx$ will be an upper estimate for $\int_a^b f(x) dx$.

Q[9]:

[hint](#) [answer](#) [solution](#)

Give a polynomial $f(x)$ with the property that the Simpson's rule approximation of $\int_a^b f(x) dx$ is exact for all a, b , and n .

Q[10]:

[hint](#) [answer](#) [solution](#)

Suppose you're setting up a spreadsheet to approximate an integral using Simpson's rule. Row 1 contains labels:

	A	B	C	D
1	x_i	$f(x_i)$	a_i	$a_i f(x_i)$

So, the x_i values are stored in column A, the function values are stored in column B, the coefficients (1,4,2,4,2,4,...) are stored in column C, and the products of the coefficients and the function values are stored in column D.

- What should you write in D2, to then copy down column D?
- Which two cells in column C should just have the number 1 in them? (Your answer may depend on n , the number of intervals you're using).
- What should you write into cell C3, to copy down the column?

► Stage 2

Questions 11 through 15 ask you to approximate a quantity based on observed data.

Q[11](*):

[hint](#) [answer](#) [solution](#)

The solid V is 40 cm high and the horizontal cross sections are circular disks. The table below gives the diameters of the cross sections in centimeters at 10 cm intervals. Use the trapezoidal rule to estimate the volume of V .

height	0	10	20	30	40
diameter	24	16	10	6	4

Q[12](*):

[hint](#) [answer](#) [solution](#)

A 6 metre long cedar log has cross sections that are approximately circular. The diameters of the log, measured at one metre intervals, are given below:

metres from left end of log	0	1	2	3	4	5	6
diameter in metres	1.2	1	0.8	0.8	1	1	1.2

Use Simpson's Rule to estimate the volume of the log.

Q[13](*):

[hint](#) [answer](#) [solution](#)

The circumference of an 8 metre high tree at different heights above the ground is given in the table below. Assume that all horizontal cross-sections of the tree are circular disks.

height (metres)	0	2	4	6	8
circumference (metres)	1.2	1.1	1.3	0.9	0.2

Use Simpson's rule to approximate the volume of the tree.

Q[14](*):

[hint](#) [answer](#) [solution](#)

By measuring the areas enclosed by contours on a topographic map, a geologist determines the cross sectional areas A in m^2 of a 60 m high hill. The table below gives the cross sectional area $A(h)$ at various heights h . The volume of the hill is $V = \int_0^{60} A(h) dh$.

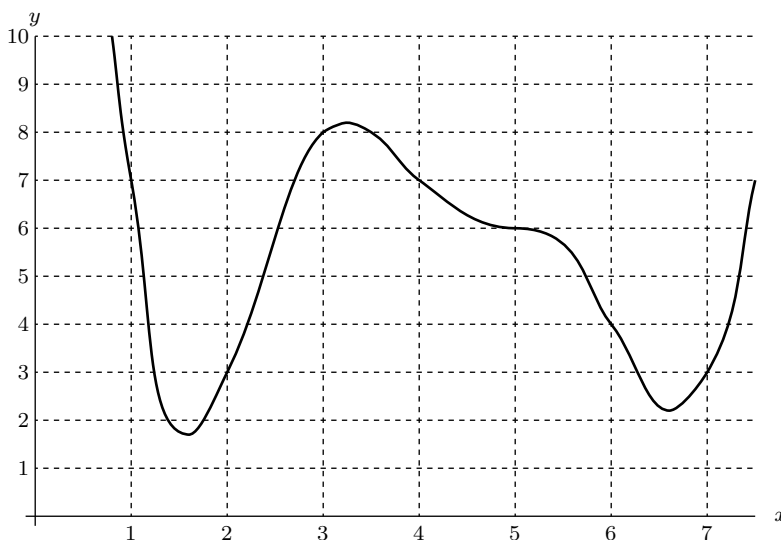
h	0	10	20	30	40	50	60
A	10,200	9,200	8,000	7,100	4,500	2,400	100

- If the geologist uses the Trapezoidal Rule to estimate the volume of the hill, what will be their estimate, to the nearest $1,000\text{m}^3$?
- What will be the geologist's estimate of the volume of the hill if they use Simpson's Rule instead of the Trapezoidal Rule?

Q[15](*):

[hint](#) [answer](#) [solution](#)

The graph below applies to both parts (a) and (b).



- Use the Trapezoidal Rule, with $n = 4$, to estimate the area under the graph between $x = 2$ and $x = 6$. Simplify your answer completely.
- Use Simpson's Rule, with $n = 4$, to estimate the area under the graph between $x = 2$ and $x = 6$.

In Questions 16 through 19, we practice finding error bounds for our approximations.

Q[16](*):

[hint](#) [answer](#) [solution](#)

Find a bound for the error in approximating $\int_1^5 \frac{1}{x} dx$ using Simpson's rule with $n = 4$.

Do not write down the Simpson's rule approximation S_4 .

In general the error in approximating $\int_a^b f(x) \, dx$ using Simpson's rule with n steps is bounded by $\frac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $L \geq |f^{(4)}(x)|$ for all $a \leq x \leq b$.

Q[17](*):

[hint](#) [answer](#) [solution](#)

Find a bound for the error in approximating

$$\int_0^1 (e^{-2x} + 3x^3) \, dx$$

using Simpson's rule with $n = 6$. Do not write down the Simpson's rule approximation S_n .

In general, the error in approximating $\int_a^b f(x) \, dx$ using Simpson's rule with n steps is bounded by $\frac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $L \geq |f^{(4)}(x)|$ for all $a \leq x \leq b$.

Q[18](*):

[hint](#) [answer](#) [solution](#)

Let $I = \int_1^2 (1/x) \, dx$.

- Write down the trapezoidal approximation T_4 for I . *You do not need to simplify your answer.*
- Write down the Simpson's approximation S_4 for I . *You do not need to simplify your answer.*
- Without computing I , find an upper bound for $|I - S_4|$. You may use the fact that if $|f^{(4)}(x)| \leq L$ on the interval $[a, b]$, then the error in using S_n to approximate $\int_a^b f(x) \, dx$ has absolute value less than or equal to $L(b-a)^5/180n^4$.

Since $\frac{1}{x^5}$ is a decreasing function when $x > 0$, look for its maximum value when x is as small as possible.

Q[19](*):

[hint](#) [answer](#) [solution](#)

A function $s(x)$ satisfies $s(0) = 1.00664$, $s(2) = 1.00543$, $s(4) = 1.00435$, $s(6) = 1.00331$, $s(8) = 1.00233$. Also, it is known to satisfy $|s^{(k)}(x)| \leq \frac{k}{1000}$ for $0 \leq x \leq 8$ and all positive integers k .

- Find the best Trapezoidal Rule and Simpson's Rule approximations that you can for

$$I = \int_0^8 s(x) \, dx.$$

- Determine the maximum possible sizes of errors in the approximations you gave in part (a). Recall that if a function $f(x)$ satisfies $|f^{(k)}(x)| \leq K_k$ on $[a, b]$, then

$$\left| \int_a^b f(x) \, dx - T_n \right| \leq \frac{K_2(b-a)^3}{12n^2} \quad \text{and} \quad \left| \int_a^b f(x) \, dx - S_n \right| \leq \frac{K_4(b-a)^5}{180n^4}$$

Q[20](*):

[hint](#) [answer](#) [solution](#)

Consider the trapezoidal rule for making numerical approximations to $\int_a^b f(x) \, dx$. The error for the trapezoidal rule satisfies $|E_T| \leq \frac{M(b-a)^3}{12n^2}$, where $|f''(x)| \leq M$ for $a \leq x \leq b$. If $-2 < f''(x) < 0$ for $1 \leq x \leq 4$, find a value of n to guarantee the trapezoidal rule will give an approximation for $\int_1^4 f(x) \, dx$ with absolute error, $|E_T|$, less than 0.001.

Q[21]:

[hint](#) [answer](#) [solution](#)

Use a spreadsheet to approximate

$$\int_{\pi/2}^{2\pi/3} \sin x \, dx$$

using a right Riemann sum and $n = 50$ rectangles.

Q[22]:

[hint](#) [answer](#) [solution](#)

Use a spreadsheet to approximate

$$\int_{\frac{1}{2}}^1 \frac{1}{x} \, dx$$

using the trapezoid rule and $n = 100$ rectangles.

Q[23]:

[hint](#) [answer](#) [solution](#)

Use a spreadsheet to approximate

$$\int_0^1 e^{x^2} \, dx$$

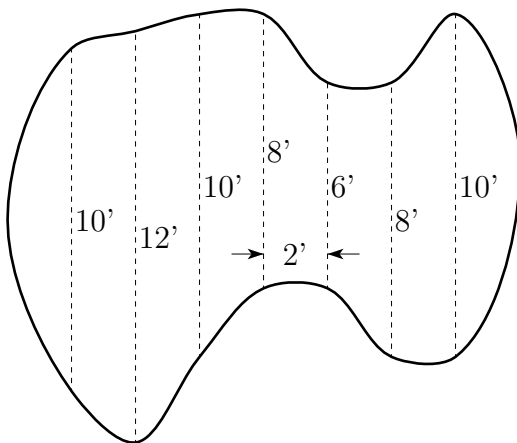
using Simpson's rule and $n = 500$ rectangles.

►► Stage 3

Q[24](*):

[hint](#) [answer](#) [solution](#)

A swimming pool has the shape shown in the figure below. The vertical cross-sections of the pool are semi-circular disks. The distances in feet across the pool are given in the figure at 2-foot intervals along the sixteen-foot length of the pool. Use Simpson's Rule to estimate the volume of the pool.



Q[25](*):

[hint](#) [answer](#) [solution](#)

A piece of wire 1m long with radius 1mm is made in such a way that the density varies in its cross-section, but is radially symmetric (that is, the local density $g(r)$ in kg/m^3 depends only on the distance r in mm from the centre of the wire). Take as given that the total mass W of the wire in kg is given by

$$W = 2\pi 10^{-6} \int_0^1 r g(r) \, dr$$

Data from the manufacturer is given below:

r	0	1/4	1/2	3/4	1
$g(r)$	8051	8100	8144	8170	8190

- (a) Find the best Trapezoidal Rule approximation that you can for W based on the data in the table.
- (b) Suppose that it is known that $|g'(r)| < 200$ and $|g''(r)| < 150$ for all values of r . Determine the maximum possible size of the error in the approximation you gave in part (a). Recall that if a function $f(x)$ satisfies $|f''(x)| \leq M$ on $[a, b]$, then

$$|I - T_n| \leq \frac{M(b-a)^3}{12n^2}$$

where $I = \int_a^b f(x) \, dx$ and T_n is the Trapezoidal Rule approximation to I using n subintervals.

Q[26](*):

[hint](#) [answer](#) [solution](#)

Simpson's rule can be used to approximate $\log 2$, since $\log 2 = \int_1^2 \frac{1}{x} \, dx$.

- (a) Use Simpson's rule with 6 subintervals to approximate $\log 2$.
- (b) How many subintervals are required in order to guarantee that the absolute error is less than 0.00001?

Note that if E_n is the error using n subintervals, then $|E_n| \leq \frac{L(b-a)^5}{180n^4}$ where L is the maximum absolute value of the fourth derivative of the function being integrated and a and b are the end points of the interval.

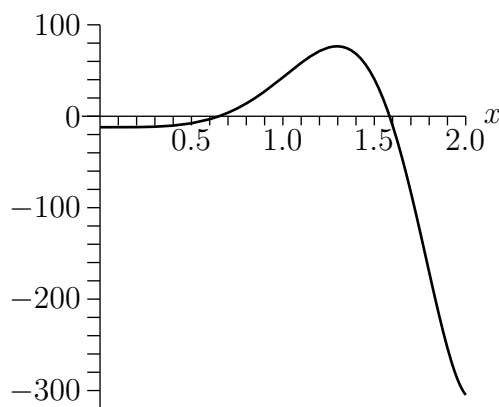
Q[27](*):

[hint](#) [answer](#) [solution](#)

Let $I = \int_0^2 \cos(x^2) \, dx$ and let S_n be the Simpson's rule approximation to I using n subintervals.

- (a) Estimate the maximum absolute error in using S_8 to approximate I .
- (b) How large should n be in order to ensure that $|I - S_n| \leq 0.0001$?

Note: The graph of $f'''(x)$, where $f(x) = \cos(x^2)$, is shown below. The absolute error in the Simpson's rule approximation is bounded by $\frac{L(b-a)^5}{180n^4}$ when $|f'''(x)| \leq L$ on the interval $[a, b]$.



Q[28](*):

[hint](#) [answer](#) [solution](#)

Define a function $f(x)$ and an integral I by

$$f(x) = \int_0^{x^2} \sin(\sqrt{t}) \, dt, \quad I = \int_0^1 f(t) \, dt$$

Estimate how many subdivisions are needed to calculate I to five decimal places of accuracy using the trapezoidal rule.

Note that if E_n is the error using n subintervals, then $|E_n| \leq \frac{M(b-a)^3}{12n^2}$, where M is the maximum absolute value of the second derivative of the function being integrated and a and b are the limits of integration.

Q[29]:

[hint](#) [answer](#) [solution](#)

Approximate the function $\log x$ with a rational function by approximating the integral

$\int_1^x \frac{1}{t} \, dt$ using Simpson's rule. Your rational function $f(x)$ should approximate $\log x$ with an error of not more than 0.1 for any x in the interval $[1, 3]$.

Q[30]:

[hint](#) [answer](#) [solution](#)

Using an approximation of the area under the curve $\frac{1}{x^2+1}$, show that the constant $\arctan 2$ is in the interval $\left[\frac{\pi}{4} + 0.321, \frac{\pi}{4} + 0.323\right]$.

You may assume use without proof that $\frac{d^4}{dx^4} \left\{ \frac{1}{1+x^2} \right\} = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$. You may use a calculator, but only to add, subtract, multiply, and divide.

1.13▲ Improper Integrals

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

For which values of b is the integral $\int_0^b \frac{1}{x^2 - 1} dx$ improper?

Q[2]:

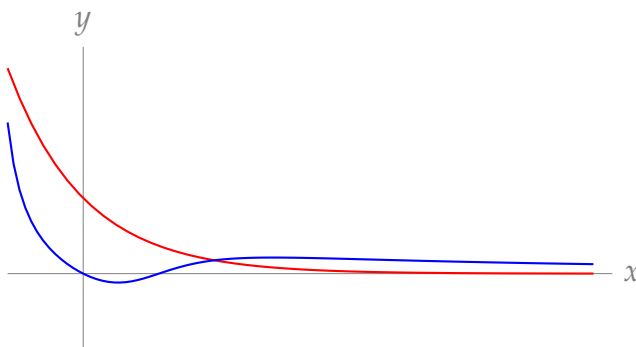
[hint](#) [answer](#) [solution](#)

For which values of b is the integral $\int_0^b \frac{1}{x^2 + 1} dx$ improper?

Q[3]:

[hint](#) [answer](#) [solution](#)

Below are the graphs $y = f(x)$ and $y = g(x)$. Suppose $\int_0^\infty f(x) dx$ converges, and $\int_0^\infty g(x) dx$ diverges. Assuming the graphs continue on as shown as $x \rightarrow \infty$, which graph is $f(x)$, and which is $g(x)$?



Q[4](*):

[hint](#) [answer](#) [solution](#)

Decide whether the following statement is true or false. If false, provide a counterexample. If true, provide a brief justification. (Assume that $f(x)$ and $g(x)$ are continuous functions.)

If $\int_1^\infty f(x) dx$ converges and $g(x) \geq f(x) \geq 0$ for all x , then $\int_1^\infty g(x) dx$ converges.

Q[5]:

[hint](#) [answer](#) [solution](#)

Let $f(x) = e^{-x}$ and $g(x) = \frac{1}{x+1}$. Note $\int_0^\infty f(x) dx$ converges while $\int_0^\infty g(x) dx$ diverges.

For each of the functions $h(x)$ described below, decide whether $\int_0^\infty h(x) dx$ converges or diverges, or whether there isn't enough information to decide. Justify your decision.

- (a) $h(x)$, continuous and defined for all $x \geq 0$, $h(x) \leq f(x)$.
- (b) $h(x)$, continuous and defined for all $x \geq 0$, $f(x) \leq h(x) \leq g(x)$.
- (c) $h(x)$, continuous and defined for all $x \geq 0$, $-2f(x) \leq h(x) \leq f(x)$.

►► Stage 2

Q[6](*):

[hint](#) [answer](#) [solution](#)

Evaluate the integral $\int_0^1 \frac{x^4}{x^5 - 1} dx$ or state that it diverges.

Q[7](*):

[hint](#) [answer](#) [solution](#)

Determine whether the integral $\int_{-2}^2 \frac{1}{(x+1)^{4/3}} dx$ is convergent or divergent. If it is convergent, find its value.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Does the improper integral $\int_1^{\infty} \frac{1}{\sqrt{4x^2 - x}} dx$ converge? Justify your answer.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Does the integral $\int_0^{\infty} \frac{dx}{x^2 + \sqrt{x}}$ converge or diverge? Justify your claim.

Q[10]:

[hint](#) [answer](#) [solution](#)

Does the integral $\int_{-\infty}^{\infty} \cos x dx$ converge or diverge? If it converges, evaluate it.

Q[11]:

[hint](#) [answer](#) [solution](#)

Does the integral $\int_{-\infty}^{\infty} \sin x dx$ converge or diverge? If it converges, evaluate it.

Q[12]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{10}^{\infty} \frac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8} dx$, or state that it diverges.

Q[13]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^{10} \frac{x-1}{x^2 - 11x + 10} dx$, or state that it diverges.

Q[14](*):

[hint](#) [answer](#) [solution](#)

Determine (with justification!) which of the following applies to the integral

$$\int_{-\infty}^{+\infty} \frac{x}{x^2 + 1} dx:$$

(i) $\int_{-\infty}^{+\infty} \frac{x}{x^2 + 1} dx$ diverges

(ii) $\int_{-\infty}^{+\infty} \frac{x}{x^2 + 1} dx$ converges but $\int_{-\infty}^{+\infty} \left| \frac{x}{x^2 + 1} \right| dx$ diverges

(iii) $\int_{-\infty}^{+\infty} \frac{x}{x^2 + 1} dx$ converges, as does $\int_{-\infty}^{+\infty} \left| \frac{x}{x^2 + 1} \right| dx$

Remark: these options, respectively, are that the integral diverges, converges conditionally, and converges absolutely. You'll see this terminology used for series in Section 3.4.1 of the text.

Q[15](*):

[hint](#) [answer](#) [solution](#)

Decide whether $I = \int_0^{\infty} \frac{|\sin x|}{x^{3/2} + x^{1/2}} dx$ converges or diverges. Justify.

Q[16](*):

[hint](#) [answer](#) [solution](#)

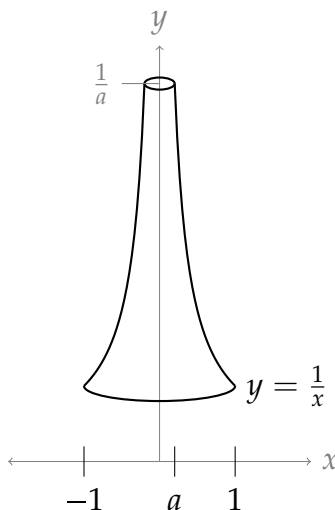
Does the integral $\int_0^{\infty} \frac{x+1}{x^{1/3}(x^2 + x + 1)} dx$ converge or diverge?

►► Stage 3

Q[17]:

[hint](#) [answer](#) [solution](#)

We craft a tall, vuvuzela-shaped solid by rotating the line $y = \frac{1}{x}$ from $x = a$ to $x = 1$ about the y -axis, where a is some constant between 0 and 1.



True or false: No matter how large a constant M is, there is some value of a that makes a solid with volume larger than M .

Q[18](*):

[hint](#) [answer](#) [solution](#)

What is the largest value of q for which the integral $\int_1^{\infty} \frac{1}{x^{5q}} dx$ diverges?

Q[19]:

[hint](#) [answer](#) [solution](#)

For which values of p does the integral $\int_0^{\infty} \frac{x}{(x^2 + 1)^p} dx$ converge?

Q[20]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_2^{\infty} \frac{1}{t^4 - 1} dt$, or state that it diverges.

Q[21]:

[hint](#) [answer](#) [solution](#)

Does the integral $\int_{-5}^5 \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x-1|}} + \frac{1}{\sqrt{|x-2|}} \right) dx$ converge or diverge?

Q[22](*):

[hint](#) [answer](#) [solution](#)

Is the integral $\int_0^{\infty} \frac{\sin^4 x}{x^2} dx$ convergent or divergent? Explain why.

Q[23]:

[hint](#) [answer](#) [solution](#)

Does the integral $\int_0^{\infty} \frac{x}{e^x + \sqrt{x}} dx$ converge or diverge?

Q[24]:

[hint](#) [answer](#) [solution](#)

Suppose $f(x)$ is continuous for all real numbers, and $\int_1^{\infty} f(x) dx$ converges.

(a) If $f(x)$ is odd, does $\int_{-\infty}^{-1} f(x) \, dx$ converge or diverge, or is there not enough information to decide?

(b) If $f(x)$ is even, does $\int_{-\infty}^{\infty} f(x) \, dx$ converge or diverge, or is there not enough information to decide?

Q[25]:

True or false: [There is some real number \$x\$, with \$x \geq 1\$, such that \$\int_0^x \frac{1}{e^t} \, dt = 1\$.](#) [hint](#) [answer](#) [solution](#)

1.14▴ More Integration Examples

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

Recall that we are using $\log x$ to denote the logarithm of x with base e . In other courses it is often denoted $\ln x$.

►► Stage 1

Q[1]:

Match the integration method to a common kind of integrand it's used to antidifferentiate. [hint](#) [answer](#) [solution](#)

(A) $u = f(x)$ substitution	(I) a function multiplied by its derivative
(B) trigonometric substitution	(II) a polynomial function times an exponential function
(C) integration by parts	(III) a rational function
(D) partial fractions	(IV) the square root of a quadratic function

►► Stage 2

Q[2]:

Evaluate $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$. [hint](#) [answer](#) [solution](#)

Q[3]:

Evaluate $\int \sqrt{3 - 5x^2} \, dx$. [hint](#) [answer](#) [solution](#)

Q[4]:

Evaluate $\int_0^{\infty} \frac{x-1}{e^x} \, dx$. [hint](#) [answer](#) [solution](#)

Q[5]:

Evaluate $\int \frac{-2}{3x^2 + 4x + 1} \, dx$. [hint](#) [answer](#) [solution](#)

Q[6]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_1^2 x^2 \log x \, dx$.

Q[7](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{x}{x^2 - 3} \, dx$.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals.

(a) $\int_0^4 \frac{x}{\sqrt{9 + x^2}} \, dx$

(b) $\int_0^{\pi/2} \cos^3 x \cdot \sin^2 x \, dx$

(c) $\int_1^e x^3 \log x \, dx$

Q[9](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals.

(a) $\int_0^{\pi/2} x \sin x \, dx$

(b) $\int_0^{\pi/2} \cos^5 x \, dx$

Q[10](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals.

(a) $\int_0^2 x e^x \, dx$

(b) $\int_0^1 \frac{1}{\sqrt{1 + x^2}} \, dx$

(c) $\int_3^5 \frac{4x}{(x^2 - 1)(x^2 + 1)} \, dx$

Q[11](*):

[hint](#) [answer](#) [solution](#)

Calculate the following integrals.

(a) $\int_0^3 \sqrt{9 - x^2} \, dx$

(b) $\int_0^1 \log(1 + x^2) \, dx$

(c) $\int_3^\infty \frac{x}{(x - 1)^2(x - 2)} \, dx$

Q[12]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{\sin^4 \theta - 5 \sin^3 \theta + 4 \sin^2 \theta + 10 \sin \theta}{\sin^2 \theta - 5 \sin \theta + 6} \cos \theta \, d\theta$.

Q[13](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals. *Show your work.*

(a) $\int_0^{\frac{\pi}{4}} \sin^2(2x) \cos^3(2x) \, dx$

(b) $\int (9 + x^2)^{-\frac{3}{2}} \, dx$

(c) $\int \frac{dx}{(x-1)(x^2+1)}$

(d) $\int x \arctan x \, dx$

Q[14](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals.

(a) $\int_0^{\pi/4} \sin^5(2x) \cos(2x) \, dx$

(b) $\int \sqrt{4-x^2} \, dx$

(c) $\int \frac{x+1}{x^2(x-1)} \, dx$

Q[15](*):

[hint](#) [answer](#) [solution](#)

Calculate the following integrals.

(a) $\int_0^{\sqrt{2}} \frac{1}{(2+x^2)^{3/2}} \, dx$

(b) $\int_0^1 x \log(1+x^2) \, dx$

(c) $\int_3^{\infty} \frac{1}{(x-1)^2(x-2)} \, dx$

Q[16](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals.

(a) $\int x \log x \, dx$

(b) $\int \frac{(x-1) \, dx}{x^2+4x+5}$

(c) $\int \frac{dx}{x^2-4x+3}$

(d) $\int \frac{x^2 \, dx}{1+x^6}$

Q[17](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals.

(a) $\int_0^1 \arctan x \, dx.$

(b) $\int \frac{2x-1}{x^2-2x+5} \, dx.$

Q[18](*):

[hint](#) [answer](#) [solution](#)

(a) Evaluate $\int \frac{x^2}{(x^3+1)^{101}} \, dx.$

(b) Evaluate $\int \cos^3 x \sin^4 x \, dx.$

Q[19]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{\pi/2}^{\pi} \frac{\cos x}{\sqrt{\sin x}} \, dx.$

Q[20](*):

[hint](#) [answer](#) [solution](#)

Evaluate the following integrals.

(a) $\int \frac{e^x}{(e^x+1)(e^x-3)} \, dx$

(b) $\int_2^4 \frac{x^2-4x+4}{\sqrt{12+4x-x^2}} \, dx$

Q[21](*):

[hint](#) [answer](#) [solution](#)

Evaluate these integrals.

(a) $\int \frac{\sin^3 x}{\cos^3 x} \, dx$

(b) $\int_{-2}^2 \frac{x^4}{x^{10}+16} \, dx$

Q[22]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int x\sqrt{x-1} \, dx.$

Q[23]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{3x^2+4x+6}{(x+1)^3} \, dx.$

Q[24]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{x^2+x+1} \, dx.$

Q[25]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sin x \cos x \tan x \, dx.$

Q[26]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{1}{x^3+1} \, dx.$

Q[27]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int (3x)^2 \arcsin x \, dx$.

►► Stage 3

Q[28]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^{\pi/2} \sqrt{\cos t + 1} \, dt$.

Q[29]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_1^e \frac{\log \sqrt{x}}{x} \, dx$.

Q[30]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_{0.1}^{0.2} \frac{\tan x}{\log(\cos x)} \, dx$.

Q[31](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^1 \frac{1}{x^2 - 5x + 6} \, dx$.

Q[32](*):

[hint](#) [answer](#) [solution](#)

Evaluate (with justification).

(a) $\int_0^3 (x+1)\sqrt{9-x^2} \, dx$

(b) $\int \frac{4x+8}{(x-2)(x^2+4)} \, dx$

(c) $\int_{-\infty}^{+\infty} \frac{1}{e^x + e^{-x}} \, dx$

Q[33]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \sqrt{\frac{x}{1-x}} \, dx$.

Q[34]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int_0^1 e^{2x} e^{e^x} \, dx$.

Q[35]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{xe^x}{(x+1)^2} \, dx$.

Q[36]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int \frac{x \sin x}{\cos^2 x} \, dx$.

You may use that $\int \sec x \, dx = \log |\sec x + \tan x| + C$.

Q[37]:

[hint](#) [answer](#) [solution](#)

Evaluate $\int x(x+a)^n dx$, where a and n are constants.

PROBABILITY

2.1▲ Introduction

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

►► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Let X be a random variable. Write, in our shorthand notation, “the probability that X takes the value 5 is 10%.”

►► Stage 2

Q[2]:

[hint](#) [answer](#) [solution](#)

Suppose X is a random variable and $Pr(X \leq 5) = \frac{99}{100}$. What is $Pr(X > 5)$?

Q[3]:

[hint](#) [answer](#) [solution](#)

Suppose X is a random variable and $Pr(X = x) = \frac{78}{93}$ for some number x . What is $Pr(X \neq x)$?

2.2▲ Probability density

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

►► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Draw a density diagram for each PDF below.

(a) $f(x) = 4x(x-1)(x-2), 0 \leq x \leq 1$

(b) $f(x) = \begin{cases} \frac{1}{2} & 0 < x < 1 \\ \frac{1}{3} & 1 < x < 2 \\ \frac{1}{6} & 2 < x < 3 \end{cases}$

(c) $f(x) = x^2, 1 \leq x \leq \sqrt[3]{2}$

►► Stage 2

Q[2]:

[hint](#) [answer](#) [solution](#)Let W be a continuous random variable with PDF

$$f(x) = \frac{10/\pi}{1 + 100x^2}.$$

Find $Pr(4 \leq W \leq 17)$.

Q[3]:

[hint](#) [answer](#) [solution](#)Let Q be a continuous random variable with PDF

$$f(x) = \begin{cases} \frac{1}{10} & 0 < x < 3 \\ \frac{1}{5} & 4 < x < 6 \\ \frac{3}{10} & 7 < x < 8 \end{cases}$$

Find $Pr(Q \geq 4.5)$.

Q[4]:

[hint](#) [answer](#) [solution](#)Let M be a continuous random variable with PDF

$$f(x) = \frac{x}{50}, \quad 0 \leq x < 10$$

What is the probability that M is in the interval $(0, 1)$ OR $(9, 10)$?

Q[5]:

[hint](#) [answer](#) [solution](#)

Suppose the function

$$f(x) = \begin{cases} e^x & -1 \leq x \leq b \\ 0 & \text{else} \end{cases}$$

is a PDF. What is b ?

Q[6]:

Suppose the function

[hint](#) [answer](#) [solution](#)

$$f(x) = \frac{A}{x^2 + 4}$$

is a PDF. What is A ?

►► Stage 3

Q[7]:

A continuous random variable X has PDF[hint](#) [answer](#) [solution](#)

$$f(x) = \sqrt{\frac{2}{\pi}} e^{-2x^2}$$

(There is a special name for this type of distribution: it's the normal distribution with expected value 0 and standard deviation $\frac{1}{2}$.)

The function $f(x)$ is a tough one to antidifferentiate, so it's hard to use $f(x)$ to evaluate probabilities exactly. Use Simpson's Rule with 4 intervals to approximate $Pr(0 \leq X \leq 1)$. Your final answer doesn't have to be simplified, but it should be calculator-ready.

2.3▲ Expected value

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

►► Stage 1

Q[1]:

Let X be a random variable. True or false: After one event, the most likely value of X to take is $\mathbb{E}(X)$.

[hint](#) [answer](#) [solution](#)

Q[2]:

Suppose Z is a continuous random variable whose sample space is the interval $[-1, 1]$, and $\mathbb{E}(Z) = 0$. True or false: Z must be uniformly distributed on $[-1, 1]$.

[hint](#) [answer](#) [solution](#)

►► Stage 2

Q[3]:

Let M be the continuous random variable with PDF[hint](#) [answer](#) [solution](#)

$$f(x) = \frac{x}{5000}, \quad 0 \leq x \leq 100$$

Compute $\mathbb{E}(M)$, and check your answer as in Section 2.3.3 of the text.

Q[4]:

[hint](#) [answer](#) [solution](#)Let N be the continuous random variable with PDF

$$f(x) = \frac{2/\pi}{x^2 + 1}, \quad -1 \leq x \leq 1$$

Compute $\mathbb{E}(N)$, and check your answer as in Section 2.3.3 of the text.

Q[5]:

[hint](#) [answer](#) [solution](#)Let P be the continuous random variable with PDF

$$f(x) = \begin{cases} \frac{2}{3}x(1-x) & 0 \leq x \leq 1 \\ \frac{2}{3}(x-2) & 2 \leq x \leq 3 \end{cases}$$

Compute $\mathbb{E}(P)$, and check your answer as in Section 2.3.3.

Q[6]:

[hint](#) [answer](#) [solution](#)Let Q be the continuous random variable with PDF

$$f(x) = \left(\frac{e}{e-2} \right) \cdot \frac{\ln x}{x^2}, \quad 1 \leq x \leq e$$

Compute $\mathbb{E}(Q)$, and check your answer as in Section 2.3.3.

Q[7]:

[hint](#) [answer](#) [solution](#)Suppose Y is a random variable that is uniformly distributed on the interval $[a, b]$. What is $\mathbb{E}(Y)$?**► Stage 3**

Q[8]:

[hint](#) [answer](#) [solution](#)Let X_p be the continuous random variable with PDF

$$f_p(x) = \frac{a_p}{x^p}, \quad 1 < x$$

where p is some positive number and a_p is some constant (possibly different constants for different values of p).

- (a) For which values of p does $f_p(x)$ actually have the characteristics of a PDF?
- (b) For those values of p from [a](#), what is a_p ?
- (c) Which values of p from [a](#) give finite expected values for $\mathbb{E}(X_p)$?

Q[9]:

[hint](#) [answer](#) [solution](#)Let A be the continuous random variable with PDF

$$f(x) = xe^x, \quad 0 \leq x \leq 1$$

Find $\mathbb{E}(A)$.

Q[10]:

[hint](#) [answer](#) [solution](#)

Let B be the continuous random variable with PDF

$$f(x) = \frac{2}{\ln^2 2} \left(\frac{\ln x}{x} \right), \quad 1 \leq x \leq 2$$

Find $\mathbb{E}(B)$.

Q[11]:

[hint](#) [answer](#) [solution](#)

Let C be the continuous random variable with PDF

$$f(x) = \frac{x}{x^2 - 1}, \quad 2 \leq x \leq b$$

where $b = \sqrt{3e^2 + 1}$. Find $\mathbb{E}(C)$. You may leave your answer in terms of b .

Q[12]:

[hint](#) [answer](#) [solution](#)

Let D be the continuous random variable with PDF

$$f(x) = \frac{4}{4 - \pi} \tan^2 x, \quad 0 \leq x \leq \frac{\pi}{4}$$

Find $\mathbb{E}(D)$.

Q[13]:

[hint](#) [answer](#) [solution](#)

A random variable X has PDF

$$f(x) = ce^{bx^2}, \quad -\infty < x < \infty$$

for a positive constant c and a negative constant b . (This is a type of “normal distribution.”)

Find the expected value of X .

2.4▲ Variance and standard deviation

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Match each term to its unofficial, intuitive description.

Terms:

- A. expected value
- B. variance
- C. standard deviation
- D. probability distribution function

Descriptions:

- 1. weights giving relative likelihood of X being in a particular region
- 2. long-term average
- 3. usual difference from the average
- 4. average squared difference from the average

Q[2]:

[hint](#) [answer](#) [solution](#)

Bundles of firewood sold at a particular campground have an average (mean) mass of 10 kg, with a standard deviation of 1.5 kg. What range of masses would be “typical” (within one standard deviation of the mean), and which would be “atypical” (more than one standard deviation from the mean)?

Q[3]:

[hint](#) [answer](#) [solution](#)

Two brands sell sunflower seeds in packages labelled 500 g. If you choose a bag at random from Brand A, the expected value of the actual mass of the bag is 510 g. If you choose a bag at random from Brand B, the expected value of the actual mass of the bag is 515 g. For both brands, the probability of a bag having mass less than 500 g is less than one percent.

For Brand A, you notice that all bags have nearly identical masses. For Brand B, there is greater diversity. Which brand has a higher standard deviation in the distribution of its masses?

Q[4]:

[hint](#) [answer](#) [solution](#)

Given an example of a random variable X that takes on values from $[-100, 100]$ (maybe not all of them) where $\sigma(X) = 100$, or explain why no such X exists.

Q[5]:

[hint](#) [answer](#) [solution](#)

Suppose X is a continuous random variable, uniformly distributed on its sample space, the interval $[a, b]$.

- (a) Find the variance and standard deviation of X .
- (b) On a number line from a to b , indicate which values are within one standard deviation of $\mathbb{E}(X)$.

Q[6]:

[hint](#) [answer](#) [solution](#)

Suppose X is a continuous random variable with sample space $[-a, a]$, and its PDF has even symmetry. What is $\mathbb{E}(X)$?

►► Stage 2

Q[7]:

[hint](#) [answer](#) [solution](#)A continuous random variable X has PDF

$$f(x) = \sin x, \quad 0 \leq x \leq \frac{\pi}{2}$$

Find the following.

(a) $\text{Var}(X)$ (b) $\sigma(X)$

Q[8]:

[hint](#) [answer](#) [solution](#)A continuous random variable W has PDF

$$f(x) = 1 - |x|, \quad -1 \leq x \leq 1$$

Find the expectation, variance, and standard deviation of W .**►► Open-Ended Questions**

Q[9]:

[hint](#) [answer](#) [solution](#)

When we motivated variance, we calculated “distance from the average” as

$$|E(X) - X|$$

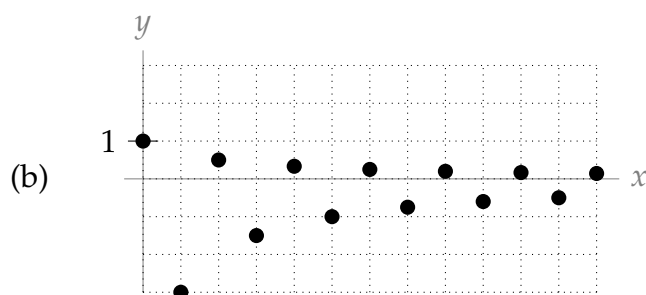
and then took the average of these values. What undesirable thing happens if we don't include the absolute value, and calculate “distance from the average” as $(E(X) - X)$?

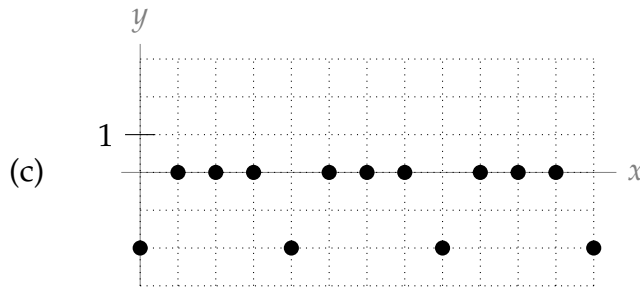
Exercises

►► Stage 1

hint answer solution

Assuming the sequences continue as shown, estimate the limit of each sequence from its graph.





Q[2]:

[hint](#) [answer](#) [solution](#)

Suppose a_n and b_n are sequences, and $a_n = b_n$ for all $n \geq 100$, but $a_n \neq b_n$ for $n < 100$.

True or false: $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$.

Q[3]:

[hint](#) [answer](#) [solution](#)

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences with $\lim_{n \rightarrow \infty} a_n = A$, $\lim_{n \rightarrow \infty} b_n = B$, and $\lim_{n \rightarrow \infty} c_n = C$. Assume A , B , and C are nonzero real numbers.

Evaluate the limits of the following sequences.

(a) $\frac{a_n - b_n}{c_n}$

(b) $\frac{c_n}{n}$

(c) $\frac{a_{2n+5}}{b_n}$

Q[4]:

[hint](#) [answer](#) [solution](#)

Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 1000$ for all $n \leq 1000$,
- $a_{n+1} < a_n$ for all n , and
- $\lim_{n \rightarrow \infty} a_n = -2$

Q[5]:

[hint](#) [answer](#) [solution](#)

Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 0$ for all even n ,
- $a_n < 0$ for all odd n ,
- $\lim_{n \rightarrow \infty} a_n$ does not exist.

Q[6]:

[hint](#) [answer](#) [solution](#)

Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 0$ for all even n ,
- $a_n < 0$ for all odd n ,
- $\lim_{n \rightarrow \infty} a_n$ exists.

Q[7]:

[hint](#) [answer](#) [solution](#)

Below is a list of sequences, and a list of functions.

- (a) Match each sequence $\{a_n\}_{n=1}^{\infty}$ to any and all functions $f(x)$ such that $f(n) = a_n$ for all positive whole numbers n .
- (b) Match each sequence $\{a_n\}_{n=1}^{\infty}$ to any and all functions $f(x)$ such that $\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x)$.

$$a_n = 1 + \frac{1}{n}$$

$$f(x) = \cos(\pi x)$$

$$b_n = 1 + \frac{1}{|n|}$$

$$g(x) = \frac{\cos(\pi x)}{x}$$

$$c_n = e^{-n}$$

$$h(x) = \begin{cases} \frac{x+1}{x} & x \text{ is a whole number} \\ 1 & \text{else} \end{cases}$$

$$d_n = (-1)^n$$

$$i(x) = \begin{cases} \frac{x+1}{x} & x \text{ is a whole number} \\ 0 & \text{else} \end{cases}$$

$$e_n = \frac{(-1)^n}{n}$$

$$j(x) = \frac{1}{e^x}$$

Q[8]:

[hint](#) [answer](#) [solution](#)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by $a_n = \cos n$.

- (a) Give three different whole numbers n that are within 0.1 of an odd integer multiple of π , and find the corresponding values of a_n .
- (b) Give three different whole numbers n such that a_n is close to 0. Justify your answers.
- (c) Give three different whole numbers n such that a_n is close to 1. Justify your answers.

Remark: this demonstrates intuitively, though not rigorously, why $\lim_{n \rightarrow \infty} \cos n$ is undefined.

We consistently find terms in the sequence that are close to -1 , and also consistently find terms in the sequence that are close to 0. Contrast this to a sequence like $\{\cos(2\pi n)\}$, whose terms are always 1, and whose limit therefore is 1. It is possible to turn the ideas of this question into a rigorous proof that $\lim_{n \rightarrow \infty} \cos n$ is undefined. See the solution.

► Stage 2

Q[9]:

[hint](#) [answer](#) [solution](#)

Determine the limits of the following sequences.

(a) $a_n = \frac{3n^2 - 2n + 5}{4n + 3}$

(b) $b_n = \frac{3n^2 - 2n + 5}{4n^2 + 3}$

(c) $c_n = \frac{3n^2 - 2n + 5}{4n^3 + 3}$

Q[10]:

[hint](#) [answer](#) [solution](#)

Determine the limit of the sequence $a_n = \frac{4n^3 - 21}{n^e + \frac{1}{n}}$.

Q[11]:

[hint](#) [answer](#) [solution](#)

Determine the limit of the sequence $b_n = \frac{\sqrt[4]{n} + 1}{\sqrt{9n + 3}}$.

Q[12]:

[hint](#) [answer](#) [solution](#)

Determine the limit of the sequence $d_n = e^{-1/n}$.

Q[13]:

[hint](#) [answer](#) [solution](#)

Determine the limit of the sequence $b_n = \frac{e^n}{2^n + n^2}$.

Q[14](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\lim_{n \rightarrow \infty} \left[\frac{6n^2 + 5n}{n^2 + 1} + 3 \cos(1/n^2) \right]$.

►► Stage 3

Q[15](*):

[hint](#) [answer](#) [solution](#)

Find the limit of the sequence $\left\{ \log \left(\sin \frac{1}{n} \right) + \log(2n) \right\}$.

Q[16]:

[hint](#) [answer](#) [solution](#)

Evaluate $\lim_{n \rightarrow \infty} \left[\sqrt{n^2 + 5n} - \sqrt{n^2 - 5n} \right]$.

Q[17]:

[hint](#) [answer](#) [solution](#)

Evaluate $\lim_{n \rightarrow \infty} \left[\sqrt{n^2 + 5n} - \sqrt{2n^2 - 5} \right]$.

Q[18]:

[hint](#) [answer](#) [solution](#)

Evaluate the limit of the sequence $\left\{ n \left[\left(2 + \frac{1}{n} \right)^{100} - 2^{100} \right] \right\}_{n=1}^{\infty}$.

Q[19]:

[hint](#) [answer](#) [solution](#)

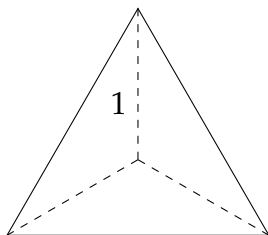
Write a sequence $\{a_n\}_{n=1}^{\infty}$ whose limit is $f'(a)$ for a function $f(x)$ that is differentiable at the point a .

Your answer will depend on f and a .

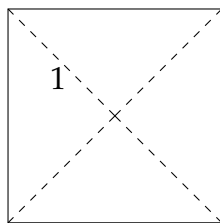
Q[20]:

[hint](#) [answer](#) [solution](#)

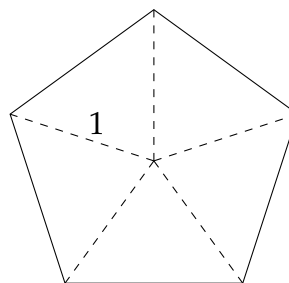
Let $\{A_n\}_{n=3}^{\infty}$ be the area of a regular polygon with n sides, with the distance from the centroid of the polygon to each corner equal to 1.



$$A(3) = \frac{3\sqrt{3}}{4}$$



$$A(4) = 2$$



$$A(5) = 2.5 \sin(0.4\pi)$$

(a) By dividing the polygon into n triangles, give a formula for A_n .

(b) What is $\lim_{n \rightarrow \infty} A_n$?

Q[21]:

[hint](#) [answer](#) [solution](#)

Suppose we define a sequence $\{f_n\}$, which depends on some constant x , as the following:

$$f_n(x) = \begin{cases} 1 & n \leq x < n+1 \\ 0 & \text{else} \end{cases}$$

For a fixed constant $x \geq 1$, $\{f_n\}$ is the sequence $\{0, 0, 0, \dots, 0, 1, 0, \dots, 0, 0, 0, \dots\}$. The sole nonzero element comes in position k , where k is what we get when we round x down to a whole number. If $x < 1$, then the sequence consists of all zeroes.

Since we can plug in different values of x , we can think of $f_n(x)$ as a function of sequences: a different x gives you a different sequence. On the other hand, if we imagine fixing n , then $f_n(x)$ is just a function, where $f_n(x)$ gives the n th term in the sequence corresponding to x .

(a) Sketch the curve $y = f_2(x)$.

(b) Sketch the curve $y = f_3(x)$.

(c) Define $A_n = \int_0^\infty f_n(x) dx$. Give a simple description of the sequence $\{A_n\}_{n=1}^\infty$.

(d) Evaluate $\lim_{n \rightarrow \infty} A_n$.

(e) Evaluate $\lim_{n \rightarrow \infty} f_n(x)$ for a constant x , and call the result $g(x)$.

(f) Evaluate $\int_0^\infty g(x) dx$.

Q[22]:

[hint](#) [answer](#) [solution](#)

Determine the limit of the sequence $b_n = \left(1 + \frac{3}{n} + \frac{5}{n^2}\right)^n$.

Q[23]:

[hint](#) [answer](#) [solution](#)

A sequence $\{a_n\}_{n=1}^\infty$ of real numbers satisfies the recursion relation $a_{n+1} = \frac{a_n + 8}{3}$ for $n \geq 1$.

(a) Suppose $a_1 = 4$. What is $\lim_{n \rightarrow \infty} a_n$?

(b) Find x if $x = \frac{x+8}{3}$.

(c) Suppose $a_1 = 1$. Show that $\lim_{n \rightarrow \infty} a_n = L$, where L is the solution to equation above.

Q[24]:

[hint](#) [answer](#) [solution](#)

Zipf's Law applied to word frequency can be phrased as follows:

The most-used word in a language is used n times as frequently as the n -th most word used in a language.

(a) Suppose the sequence $\{w_1, w_2, w_3, \dots\}$ is a list of all words in a language, where w_n is the word that is the n th most frequently used. Let f_n be the frequency of word w_n . Is $\{f_1, f_2, f_3, \dots\}$ an increasing sequence or a decreasing sequence?

(b) Give a general formula for f_n , treating f_1 as a constant.

(c) Suppose in a language, w_1 (the most frequently used word) has frequency 6%. If the language follows Zipf's Law, then what frequency does w_3 have?

(d) Suppose $f_6 = 0.3\%$ for a language following Zipf's law. What is f_{10} ?

(e) The word "the" is the most-used word in contemporary American English. In a collection of about 450 million words, "the" appeared 22,038,615 times. The second-most used word is "be," followed by "and." About how many usages of these words do you expect in the same collection of 450 million words?

Sources: *Zipf's word frequency law in natural language: A critical review and future directions*, Steven T. Piantadosi. *Psychon Bull Rev.* 2014 Oct; 21(5): 1112–1130. Accessed online 11 October 2017

Word Frequency Data, <https://www.wordfrequency.info/free.asp?s=y> Accessed online 11 October 2017

3.2▲ Series

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Write out the first five partial sums corresponding to the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

You don't need to simplify the terms.

Q[2]:

[hint](#) [answer](#) [solution](#)

Every student who comes to class brings their instructor cookies, and leaves them on the instructor's desk. Let C_k be the total number of cookies on the instructor's desk after the k th student comes.

If $C_{11} = 20$, and $C_{10} = 17$, how many cookies did the 11th student bring to class?

Q[3]:

[hint](#) [answer](#) [solution](#)

Suppose the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is $\{S_N\} = \left\{ \frac{N}{N+1} \right\}$.

(a) What is $\{a_n\}$?

(b) What is $\lim_{n \rightarrow \infty} a_n$?

(c) Evaluate $\sum_{n=1}^{\infty} a_n$.

Q[4]:

[hint](#) [answer](#) [solution](#)

Suppose the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is $\{S_N\} = \left\{ (-1)^N + \frac{1}{N} \right\}$.

What is $\{a_n\}$?

Q[5]:

[hint](#) [answer](#) [solution](#)

Let $f(N)$ be a formula for the N th partial sum of $\sum_{n=1}^{\infty} a_n$. (That is, $f(N) = S_N$.) If $f'(N) < 0$ for all $N > 1$, what does that say about a_n ?

Q[6]:

[hint](#) [answer](#) [solution](#)

Every student who comes to class brings their instructor cookies, and leaves them on the instructor's desk. Let C_k be the total number of cookies on the instructor's desk after the k th student comes.

If $C_{20} = 53$, and $C_{10} = 17$, what does $C_{20} - C_{10} = 36$ represent?

Q[7]:

[hint](#) [answer](#) [solution](#)

Suppose $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, and $\sum_{n=1}^{\infty} c_n = C$.

Evaluate $\sum_{n=1}^{\infty} (a_n + b_n + c_{n+1})$.

Q[8]:

[hint](#) [answer](#) [solution](#)

Suppose $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B \neq 0$, and $\sum_{n=1}^{\infty} c_n = C$.

True or false: $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n} + c_n \right) = \frac{A}{B} + C$.

Q[9]:

[hint](#) [answer](#) [solution](#)

Consider the spreadsheet set up as follows:

	A	B
1	0.1	=SUM(\$A\$1:A1)
2	=A2/10	↓
3	↓	↓

Answer the following questions *without* actually making the spreadsheet.

- Column A holds the terms of a sequence, with a_n in cell A_n . Give an explicit formula for a_n .
- What is held in cell B_n ? What is the role of the dollar signs?
- What number do you expect the values in column A to approach, as you go down the column?
- What number do you expect the values in column B to approach, as you go down the column?

Q[10]:

[hint](#) [answer](#) [solution](#)

Set up a spreadsheet that evaluates partial sums of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + n}.$$

Based on your spreadsheet, approximate $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Note: you may want to make use of the function `ROW()`, which gives the row number of its cell.

► Stage 2

Q[11](*):

[hint](#) [answer](#) [solution](#)

The n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is known to have the formula $s_n = \frac{1+3n}{5+4n}$.

- Find an expression for a_n , valid for $n \geq 2$.
- Show that the series $\sum_{n=1}^{\infty} a_n$ converges and find its value.

► Stage 3

Q[12]:

[hint](#) [answer](#) [solution](#)

Suppose a series $\sum_{n=1}^{\infty} a_n$ has sequence of partial sums $\{S_N\}$, and the series $\sum_{N=1}^{\infty} S_N$ has

sequence of partial sums $\{\mathcal{S}_M\} = \left\{ \sum_{N=1}^M S_N \right\}$.

If $\mathcal{S}_M = \frac{M+1}{M}$, what is a_n ?

3.3▲ Convergence Tests

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

►► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Select the series below that diverge by the divergence test.

(A) $\sum_{n=1}^{\infty} \frac{1}{n}$ (B) $\sum_{n=1}^{\infty} \frac{n^2}{n+1}$ (C) $\sum_{n=1}^{\infty} \sin n$ (D) $\sum_{n=1}^{\infty} \sin(\pi n)$

Q[2]:

[hint](#) [answer](#) [solution](#)

Select the series below whose terms satisfy the conditions to apply the integral test.

(A) $\sum_{n=1}^{\infty} \frac{1}{n}$ (B) $\sum_{n=1}^{\infty} \frac{n^2}{n+1}$ (C) $\sum_{n=1}^{\infty} \sin n$ (D) $\sum_{n=1}^{\infty} \frac{\sin n + 1}{n^2}$

Q[3]:

[hint](#) [answer](#) [solution](#)

Suppose there is some threshold after which a person is considered old, and before which they are young.

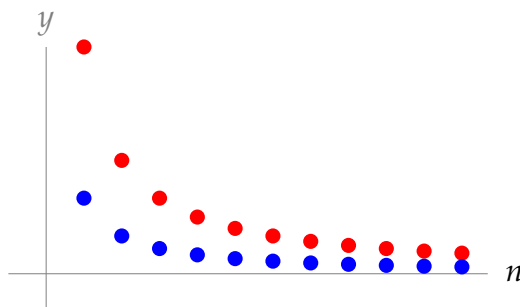
Let Olaf be an old person, and let Yuan be a young person.

- (a) Suppose I am older than Olaf. Am I old?
- (b) Suppose I am younger than Olaf. Am I old?
- (c) Suppose I am older than Yuan. Am I young?
- (d) Suppose I am younger than Yuan. Am I young?

Q[4]:

[hint](#) [answer](#) [solution](#)

Below are graphs of two sequences with positive terms. Assume the sequences continue as shown. Fill in the table with conclusions that can be made from the direct comparison test, if any.



	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$ _____	then $\sum b_n$ _____
and if $\{a_n\}$ is the blue series	then $\sum b_n$ _____	then $\sum b_n$ _____

Q[5]:

[hint](#) [answer](#) [solution](#)

For each pair of series below, decide whether the second series is a valid comparison series to determine the convergence of the first series, using the *direct* comparison test and/or the *limit* comparison test.

- (a) $\sum_{n=10}^{\infty} \frac{1}{n-1}$, compared to the divergent series $\sum_{n=10}^{\infty} \frac{1}{n}$.
- (b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2+1}$, compared to the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
- (c) $\sum_{n=5}^{\infty} \frac{n^3+5n+1}{n^6-2}$, compared to the convergent series $\sum_{n=5}^{\infty} \frac{1}{n^3}$.
- (d) $\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}}$, compared to the divergent series $\sum_{n=5}^{\infty} \frac{1}{\sqrt[4]{n}}$.

Q[6]:

[hint](#) [answer](#) [solution](#)

Suppose a_n is a sequence with $\lim_{n \rightarrow \infty} a_n = \frac{1}{2}$. Does $\sum_{n=7}^{\infty} a_n$ converge or diverge, or is it not possible to determine this from the information given? Why?

Q[7]:

[hint](#) [answer](#) [solution](#)

What flaw renders the following reasoning invalid?

Q: Determine whether $\sum_{n=1}^{\infty} (\sin(\pi n) + 2)$ converges or diverges.

A: We use the integral test. Let $f(x) = \sin(\pi x) + 2$. Note $f(x)$ is always positive, since $\sin(x) + 2 \geq -1 + 2 = 1$. Also, $f(x)$ is continuous.

$$\begin{aligned}
 \int_1^{\infty} [\sin(\pi x) + 2] dx &= \lim_{b \rightarrow \infty} \int_1^b [\sin(\pi x) + 2] dx \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{\pi} \cos(\pi x) + 2x \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{\pi} \cos(\pi b) + 2b + \frac{1}{\pi}(-1) - 2 \right] \\
 &= \infty
 \end{aligned}$$

By the integral test, since the integral diverges, also $\sum_{n=1}^{\infty} (\sin(\pi n) + 2)$ diverges.

Q[8]:

[hint](#) [answer](#) [solution](#)

What flaw renders the following reasoning invalid?

Q: Determine whether the series $\sum_{n=1}^{\infty} \frac{2^{n+1}n^2}{e^n + 2n}$ converges or diverges.

A: We want to compare this series to the series $\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n}$. Note both this series and the series in the question have positive terms.

First, we find that $\frac{2^{n+1}n^2}{e^n + 2n} > \frac{2^{n+1}}{e^n}$ when n is sufficiently large. The justification for this claim is as follows:

- We note that $e^n(n^2 - 1) > n^2 - 1 > 2n$ for n sufficiently large.
- Therefore, $e^n \cdot n^2 > e^n + 2n$
- Therefore, $2^{n+1} \cdot e^n \cdot n^2 > 2^{n+1}(e^n + 2n)$
- Since $e^n + 2n$ and e^n are both expressions that work out to be positive for the values of n under consideration, we can divide both sides of the inequality by these terms without having to flip the inequality. So,

$$\frac{2^{n+1}n^2}{e^n + 2n} > \frac{2^{n+1}}{e^n}.$$

Now, we claim $\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n}$ converges.

Note $\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n} = 2 \sum_{n=1}^{\infty} \frac{2^n}{e^n} = 2 \sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n$. This is a geometric series with $r = \frac{2}{e}$. Since $2/e < 1$, the series converges.

Now, by the Direct Comparison Test, we conclude that $\sum_{n=1}^{\infty} \frac{2^{n+1}n^2}{e^n + 2n}$ converges.

Q[9]:

[hint](#) [answer](#) [solution](#)

Which of the series below are alternating?

$$(A) \sum_{n=1}^{\infty} \sin n \quad (B) \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^3} \quad (C) \sum_{n=1}^{\infty} \frac{7}{(-n)^{2n}} \quad (D) \sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}}$$

Q[10]:

[hint](#) [answer](#) [solution](#)

Give an example of a convergent series for which the ratio test is inconclusive.

Q[11]:

[hint](#) [answer](#) [solution](#)

Imagine you're taking an exam, and you momentarily forget exactly how the inequality in the ratio test works. You remember there's a ratio, but you don't remember which

term goes on top; you remember there's something about the limit being greater than or less than one, but you don't remember which way implies convergence.

Explain why

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

or, equivalently,

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| < 1$$

should mean that the sum $\sum_{n=1}^{\infty} a_n$ *diverges* (rather than converging).

Q[12]:

[hint](#) [answer](#) [solution](#)

Give an example of a series $\sum_{n=a}^{\infty} a_n$, with a function $f(x)$ such that $f(n) = a_n$ for all whole numbers n , such that:

- $\int_a^{\infty} f(x) dx$ diverges, while
- $\sum_{n=a}^{\infty} a_n$ converges.

Q[13](*):

[hint](#) [answer](#) [solution](#)

Suppose that you want to use the Limit Comparison Test on the series $\sum_{n=0}^{\infty} a_n$ where

$a_n = \frac{2^n + n}{3^n + 1}$. Write down a sequence $\{b_n\}$ such that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists and is nonzero. (You don't have to carry out the Limit Comparison Test)

Q[14](*):

[hint](#) [answer](#) [solution](#)

Decide whether each of the following statements is true or false. If false, provide a counterexample. If true provide a brief justification.

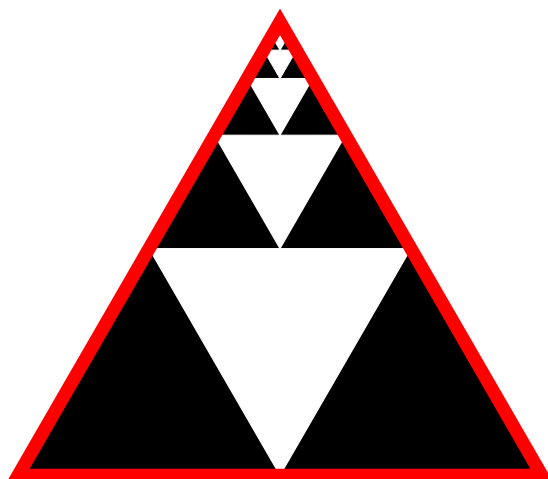
- (a) If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
- (b) If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.
- (c) If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Questions 15 through 17 invite you to explore geometric sums in a geometric way. This is complementary to than the algebraic method discussed in the text.

Q[15]:

[hint](#) [answer](#) [solution](#)

Suppose the triangle outlined in red in the picture below has area one.

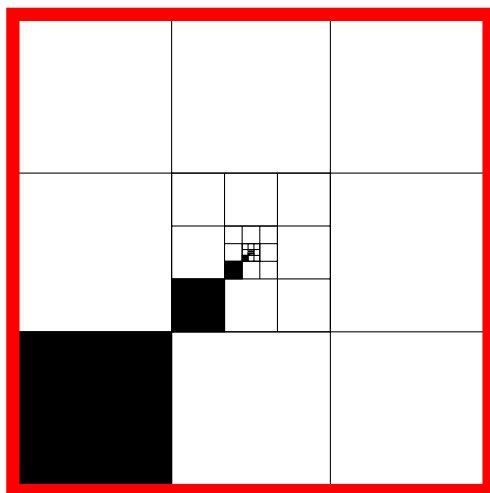


- (a) Express the combined area of the black triangles as a series, assuming the pattern continues forever.
- (b) Evaluate the series using the picture (*not* the formula from your book).

Q[16]:

[hint](#) [answer](#) [solution](#)

Suppose the square outlined in red in the picture below has area one.



- (a) Express the combined area of the black squares as a series, assuming the pattern continues forever.
- (b) Evaluate the series using the picture (*not* the formula from your book).

Q[17]:

[hint](#) [answer](#) [solution](#)

In the style of Questions [15](#) and [16](#), draw a picture that represents $\sum_{n=1}^{\infty} \frac{1}{3^n}$ as an area.

Q[18]:

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{n=0}^{100} \frac{1}{5^n}$.

Q[19]:

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{n=50}^{100} \frac{1}{5^n}$. (Note the starting index.)

► Stage 2

Q[20](*):

[hint](#) [answer](#) [solution](#)

Does the series $\sum_{n=2}^{\infty} \frac{n^2}{3n^2 + \sqrt{n}}$ converge?

Q[21](*):

[hint](#) [answer](#) [solution](#)

Determine, with explanation, whether the series $\sum_{n=1}^{\infty} \frac{5^k}{4^k + 3^k}$ converges or diverges.

Q[22](*):

[hint](#) [answer](#) [solution](#)

To what value does the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$ converge?

Q[23](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{k=7}^{\infty} \frac{1}{8^k}$

Q[24](*):

[hint](#) [answer](#) [solution](#)

Find the sum of the series $\sum_{n=2}^{\infty} \frac{3 \cdot 4^{n+1}}{8 \cdot 5^n}$. Simplify your answer completely.

Q[25](*):

[hint](#) [answer](#) [solution](#)

Relate the number $0.2\bar{3} = 0.23333\ldots$ to the sum of a geometric series, and use that to represent it as a rational number (a fraction or combination of fractions, with no decimals).

Q[26](*):

[hint](#) [answer](#) [solution](#)

Express $2.656565\ldots$ as a rational number, i.e. in the form p/q where p and q are integers.

Q[27](*):

[hint](#) [answer](#) [solution](#)

Express the decimal $0.\overline{321} = 0.321321321\ldots$ as a fraction.

Q[28](*):

[hint](#) [answer](#) [solution](#)

Evaluate

$$\sum_{n=1}^{\infty} \left[\left(\frac{1}{3} \right)^n + \left(-\frac{2}{5} \right)^{n-1} \right]$$

Q[29](*):

[hint](#) [answer](#) [solution](#)

Find the sum of the series $\sum_{n=0}^{\infty} \frac{1 + 3^{n+1}}{4^n}$.

Q[30](*):

[hint](#) [answer](#) [solution](#)

Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2}}$ is convergent or divergent. If it is convergent, find its value.

Q[31]:

[hint](#) [answer](#) [solution](#)

Does the following series converge or diverge? $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\sqrt{k+1}}$

Q[32]:

[hint](#) [answer](#) [solution](#)

Evaluate the following series, or show that it diverges: $\sum_{k=30}^{\infty} 3(1.001)^k$.

Q[33]:

[hint](#) [answer](#) [solution](#)

Evaluate the following series, or show that it diverges: $\sum_{n=3}^{\infty} \left(\frac{-1}{5}\right)^n$.

Q[34]:

[hint](#) [answer](#) [solution](#)

Does the following series converge or diverge? $\sum_{n=7}^{\infty} \sin(\pi n)$

Q[35]:

[hint](#) [answer](#) [solution](#)

Does the following series converge or diverge? $\sum_{n=7}^{\infty} \cos(\pi n)$

Q[36]:

[hint](#) [answer](#) [solution](#)

Does the following series converge or diverge? $\sum_{k=1}^{\infty} \frac{e^k}{k!}$.

Q[37]:

[hint](#) [answer](#) [solution](#)

Evaluate the following series, or show that it diverges: $\sum_{k=0}^{\infty} \frac{2^k}{3^{k+2}}$.

Q[38]:

[hint](#) [answer](#) [solution](#)

Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

Q[39]:

[hint](#) [answer](#) [solution](#)

Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{n^2 + 1}{2n^4 + n}$.

Q[40](*):

[hint](#) [answer](#) [solution](#)

Show that the series $\sum_{n=3}^{\infty} \frac{5}{n(\log n)^{3/2}}$ converges.

Q[41](*):

[hint](#) [answer](#) [solution](#)

Find the values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

Q[42](*):

[hint](#) [answer](#) [solution](#)

Does $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converge or diverge?

Q[43](*):

[hint](#) [answer](#) [solution](#)

Use the comparison test (not the limit comparison test) to show whether the series

$$\sum_{n=2}^{\infty} \frac{\sqrt{3n^2 - 7}}{n^3} \text{ converges or diverges.}$$

Q[44](*):

[hint](#) [answer](#) [solution](#)Determine whether the series $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}}$ converges.

Q[45](*):

[hint](#) [answer](#) [solution](#)Does $\sum_{n=1}^{\infty} \frac{n^4 2^{n/3}}{(2n+7)^4}$ converge or diverge?

Q[46](*):

[hint](#) [answer](#) [solution](#)

Determine, with explanation, whether each of the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

(b) $\sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2^n}$

Q[47](*):

[hint](#) [answer](#) [solution](#)

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k}$$

converges or diverges.

Q[48](*):

[hint](#) [answer](#) [solution](#)

Determine whether each of the following series converge or diverge.

(a) $\sum_{n=2}^{\infty} \frac{n^2 + n + 1}{n^5 - n}$

(b) $\sum_{m=1}^{\infty} \frac{3m + \sin \sqrt{m}}{m^2}$

Q[49]:

[hint](#) [answer](#) [solution](#)Evaluate the following series, or show that it diverges: $\sum_{n=5}^{\infty} \frac{1}{e^n}$.

Q[50](*):

[hint](#) [answer](#) [solution](#)Determine whether the series $\sum_{n=2}^{\infty} \frac{6}{7^n}$ is convergent or divergent. If it is convergent, find its value.

Q[51](*):

[hint](#) [answer](#) [solution](#)

Determine, with explanation, whether each of the following series converge or diverge.

(a) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$.

(b) $\sum_{n=1}^{\infty} \frac{2n+1}{2^{2n+1}}$

Q[52](*):

[hint](#) [answer](#) [solution](#)

Determine, with explanation, whether each of the following series converges or diverges.

(a) $\sum_{k=2}^{\infty} \frac{\sqrt[3]{k}}{k^2 - k}$.

(b) $\sum_{k=1}^{\infty} \frac{k^{10} 10^k (k!)^2}{(2k)!}$.

(c) $\sum_{k=3}^{\infty} \frac{1}{k(\log k)(\log \log k)}$.

Q[53](*):

[hint](#) [answer](#) [solution](#)

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^3 - 4}{2n^5 - 6n}$ is convergent or divergent.

Q[54](*):

[hint](#) [answer](#) [solution](#)

What is the smallest value of N such that the partial sum $\sum_{n=1}^N \frac{(-1)^n}{n \cdot 10^n}$ approximates $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 10^n}$ within an accuracy of 10^{-6} ?

Q[55](*):

[hint](#) [answer](#) [solution](#)

It is known that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ (you don't have to show this). Find N so that S_N , the N^{th} partial sum of the series, satisfies $|\frac{\pi^2}{12} - S_N| \leq 10^{-6}$. Be sure to say why your method can be applied to this particular series.

Q[56](*):

[hint](#) [answer](#) [solution](#)

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^2}$ converges to some number S (you don't have to prove this). According to the Alternating Series Estimation Theorem, what is the smallest value of N for which the N^{th} partial sum of the series is at most $\frac{1}{100}$ away from S ? For this value of N , write out the N^{th} partial sum of the series.

► Stage 3

Q[57]:

[hint](#) [answer](#) [solution](#)

Find the combined volume of an infinite collection of spheres, where for each whole number $n = 1, 2, 3, \dots$ there is exactly one sphere of radius $\frac{1}{\pi^n}$.

Q[58](*):

[hint](#) [answer](#) [solution](#)

Determine, with explanation, whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^n}{9^n n!}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{\log n}}$

Q[59](*):

[hint](#) [answer](#) [solution](#)

(a) Prove that $\int_2^{\infty} \frac{x + \sin x}{1 + x^2} dx$ diverges.

(b) Explain why you *cannot* conclude that $\sum_{n=1}^{\infty} \frac{n + \sin n}{1 + n^2}$ diverges from part (a) and the Integral Test.

(c) Determine, with explanation, whether $\sum_{n=1}^{\infty} \frac{n + \sin n}{1 + n^2}$ converges or diverges.

Q[60](*):

[hint](#) [answer](#) [solution](#)

Show that $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges and find an interval of length 0.05 or less that contains its exact value.

Q[61](*):

[hint](#) [answer](#) [solution](#)

Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges and that $1 > a_n \geq 0$ for all n . Prove that the series

$\sum_{n=1}^{\infty} \frac{a_n}{1 - a_n}$ also converges.

Q[62](*):

[hint](#) [answer](#) [solution](#)

Suppose that the series $\sum_{n=0}^{\infty} (1 - a_n)$ converges, where $a_n > 0$ for $n = 0, 1, 2, 3, \dots$. Determine whether the series $\sum_{n=0}^{\infty} 2^n a_n$ converges or diverges.

Q[63](*):

[hint](#) [answer](#) [solution](#)

Prove that if $a_n \geq 0$ for all n and if the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} a_n^2$ also converges.

A number of phenomena roughly follow a distribution called Zipf's law. We discuss some of these in Questions 64 and 65.

Q[64]:

[hint](#) [answer](#) [solution](#)

Suppose the frequency of word use in a language has the following pattern:

The n -th most frequently used word accounts for $\frac{\alpha}{n}$ percent of the total words used.

So, in a text of 100 words, we expect the most frequently used word to appear α times, while the second-most-frequently used word should appear about $\frac{\alpha}{2}$ times, and so on.

If books written in this language use 20,000 distinct words, then the most commonly used word accounts for roughly what percentage of total words used?

Q[65]:

[hint](#) [answer](#) [solution](#)

Suppose the sizes of cities in a country adhere to the following pattern: if the largest city has population α , then the n -th largest city has population $\frac{\alpha}{n}$.

If the largest city in this country has 2 million people and the smallest city has 1 person, then the population of the entire country is $\sum_{n=1}^{2 \times 10^6} \frac{2 \times 10^6}{n}$. (For many n 's in this sum $\frac{2 \times 10^6}{n}$ is not an integer. Ignore that.) Evaluate this sum approximately, with an error of no more than 1 million people.

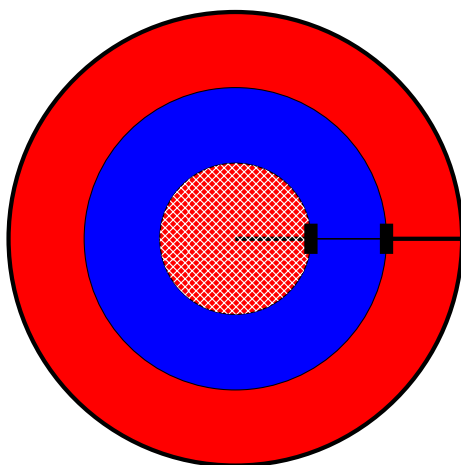
Q[66]:

[hint](#) [answer](#) [solution](#)

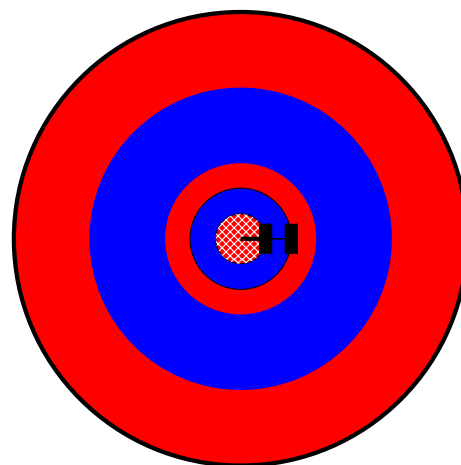
Create a bullseye using the following method:

Starting with a red circle of area 1, divide the radius into thirds, creating two rings and a circle. Colour the middle ring blue.

Continue the pattern with the inside circle: divide its radius into thirds, and colour the middle ring blue.

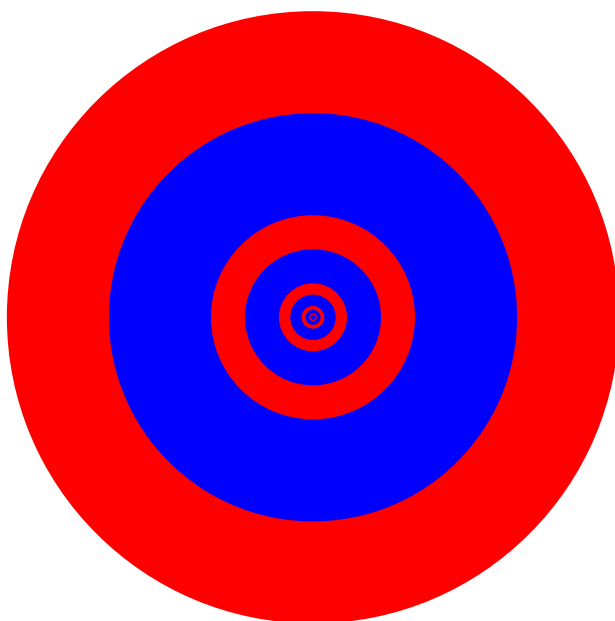


Step 1



Step 2

Continue in this way indefinitely: dividing the radius of the innermost circle into thirds, creating two rings and another circle, and colouring the middle ring blue.



What is the area of the red portion?

3.4▲ Absolute and Conditional Convergence

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

►► Stage 1

Q[1](*):

[hint](#) [answer](#) [solution](#)

Decide whether the following statement is true or false. If false, provide a counterexample. If true provide a brief justification.

If $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges.

Q[2]:

[hint](#) [answer](#) [solution](#)

Describe the series $\sum_{n=1}^{\infty} a_n$ based on whether $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge or diverge, using vocabulary from this section where possible.

	$\sum a_n$ converges	$\sum a_n$ diverges
$\sum a_n $ converges		
$\sum a_n $ diverges		

► Stage 2

Q[3](*):

[hint](#) [answer](#) [solution](#)

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{9n+5}$ is absolutely convergent, conditionally convergent, or divergent; justify your answer.

Q[4](*):

[hint](#) [answer](#) [solution](#)

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{1+n}$ is absolutely convergent, conditionally convergent, or divergent.

Q[5](*):

[hint](#) [answer](#) [solution](#)

The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1+4^n}{3+2^{2n}}$ either: converges absolutely; converges conditionally; diverges; or none of the above. Determine which is correct.

Q[6](*):

[hint](#) [answer](#) [solution](#)

Does the series $\sum_{n=5}^{\infty} \frac{\sqrt{n} \cos n}{n^2 - 1}$ converge conditionally, converge absolutely, or diverge?

Q[7](*):

[hint](#) [answer](#) [solution](#)

Determine (with justification!) whether the series $\sum_{n=1}^{\infty} \frac{n^2 - \sin n}{n^6 + n^2}$ converges absolutely, converges but not absolutely, or diverges.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Determine (with justification!) whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n^2 + 1)(n!)^2}$ converges absolutely, converges but not absolutely, or diverges.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Determine (with justification!) whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^{101}}$ converges absolutely, converges but not absolutely, or diverges.

Q[10]:

[hint](#) [answer](#) [solution](#)

Show that the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

Q[11]:

[hint](#) [answer](#) [solution](#)

Show that the series $\sum_{n=1}^{\infty} \left(\frac{\sin n}{4} - \frac{1}{8} \right)^n$ converges.

Q[12]:

[hint](#) [answer](#) [solution](#)

Show that the series $\sum_{n=1}^{\infty} \frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n}$ converges.

►► Stage 3

Q[13](*):

[hint](#) [answer](#) [solution](#)

Both parts of this question concern the series $S = \sum_{n=1}^{\infty} (-1)^{n-1} 24n^2 e^{-n^3}$.

- (a) Show that the series S converges absolutely.
- (b) Suppose that you approximate the series S by its fifth partial sum S_5 . Give an upper bound for the error resulting from this approximation.

Q[14]:

[hint](#) [answer](#) [solution](#)

You may assume without proof the following:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos(1)$$

Using this fact, approximate $\cos 1$ as a rational number, accurate to within $\frac{1}{1000}$.

Check your answer against a calculator's approximation of $\cos(1)$: what was your actual error?

Q[15]:

[hint](#) [answer](#) [solution](#)

Let a_n be defined as

$$a_n = \begin{cases} -e^{n/2} & \text{if } n \text{ is prime} \\ n^2 & \text{if } n \text{ is not prime} \end{cases}$$

Show that the series $\sum_{n=1}^{\infty} \frac{a_n}{e^n}$ converges.

3.5▲ Power Series

Exercises

Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

►► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Suppose $f(x) = \sum_{n=0}^{\infty} \left(\frac{3-x}{4}\right)^n$. What is $f(1)$?

Q[2]:

[hint](#) [answer](#) [solution](#)

Suppose $f(x) = \sum_{n=1}^{\infty} \frac{(x-5)^n}{n! + 2}$. Give a power series representation of $f'(x)$.

Q[3]:

[hint](#) [answer](#) [solution](#)

Let $f(x) = \sum_{n=a}^{\infty} A_n(x-c)^n$ for some positive constants a and c , and some sequence of constants $\{A_n\}$. For which values of x does $f(x)$ definitely converge?

Q[4]:

[hint](#) [answer](#) [solution](#)

Let $f(x)$ be a power series centred at $c = 5$. If $f(x)$ converges at $x = -1$, and diverges at $x = 11$, what is the radius of convergence of $f(x)$?

► Stage 2

Q[5](*):

[hint](#) [answer](#) [solution](#)

(a) Find the radius of convergence of the series

$$\sum_{k=0}^{\infty} (-1)^k 2^{k+1} x^k$$

(b) You are given the formula for the sum of a geometric series, namely:

$$1 + r + r^2 + \cdots = \frac{1}{1-r}, \quad |r| < 1$$

Use this fact to evaluate the series in part (a).

Q[6](*):

[hint](#) [answer](#) [solution](#)

Find the radius of convergence for the power series $\sum_{k=0}^{\infty} \frac{x^k}{10^{k+1}(k+1)!}$

Q[7](*):

[hint](#) [answer](#) [solution](#)

Find the radius of convergence for the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$.

Q[8](*):

[hint](#) [answer](#) [solution](#)

Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{\sqrt{n}}$, where x is a real number. Find the radius of convergence of this series.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{x+1}{3} \right)^n$$

Q[10](*):

[hint](#) [answer](#) [solution](#)

Find the *radius of convergence* for the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^{4/5}(5^n-4)}$$

Q[11](*):

[hint](#) [answer](#) [solution](#)

Find all values x for which the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$ converges.

Q[12](*):

[hint](#) [answer](#) [solution](#)

Find the radius of convergence for the following series.

(a)
$$\sum_{n=1}^{\infty} \frac{4^n}{n} (x-1)^n.$$

(b)
$$\sum_{n=1}^{\infty} \frac{4^n}{n} (2x+1)^n.$$

(c)
$$\sum_{n=1}^{\infty} \frac{4^n}{n} (2x+1)^{2n}.$$

Q[13](*):

[hint](#) [answer](#) [solution](#)

Find, with explanation, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2^n(n+2)}$$

Q[14](*):

[hint](#) [answer](#) [solution](#)Find the *radius of convergence* for the series $\sum_{n=1}^{\infty} (-1)^n n^2 (x-a)^{2n}$ where a is a constant.

Q[15](*):

[hint](#) [answer](#) [solution](#)

Find the radius of convergence of the following series:

(a)
$$\sum_{k=1}^{\infty} \frac{(x+1)^k}{k^2 9^k}.$$

(b)
$$\sum_{k=1}^{\infty} a_k (x-1)^k, \text{ where } a_k > 0 \text{ for } k = 1, 2, \dots \text{ and } \sum_{k=1}^{\infty} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right) = \frac{a_1}{a_2}.$$

Q[16](*):

[hint](#) [answer](#) [solution](#)Find a power series representation for $\frac{x^3}{1-x}$.

Q[17]:

[hint](#) [answer](#) [solution](#)Suppose $f'(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n+2}$, and $\int_5^x f(t) dt = 3x + \sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n(n+1)^2}$.Give a power series representation of $f(x)$.**► Stage 3**

Q[18](*):

[hint](#) [answer](#) [solution](#)Determine the values of x for which the series

$$\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \log n}$$

converges absolutely, converges conditionally, or diverges.

Q[19](*):

[hint](#) [answer](#) [solution](#)

(a) Find the power-series representation for $\int \frac{1}{1+x^3} dx$ centred at 0 (i.e. in powers of x).

(b) The power series above is used to approximate $\int_0^{1/4} \frac{1}{1+x^3} dx$. How many terms are required to guarantee that the resulting approximation is within 10^{-5} of the exact value? Justify your answer.

Q[20](*):

[hint](#) [answer](#) [solution](#)

(a) Show that $\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $-1 < x < 1$.

(b) Express $\sum_{n=0}^{\infty} n^2 x^n$ as a ratio of polynomials. For which x does this series converge?

Q[21](*):

[hint](#) [answer](#) [solution](#)

Suppose that you have a sequence $\{b_n\}$ such that the series $\sum_{n=0}^{\infty} (1-b_n)$ converges. Using the tests we've learned in class, prove that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} b_n x^n$$

is equal to 1.

Q[22](*):

[hint](#) [answer](#) [solution](#)

Assume $\{a_n\}$ is a sequence such that na_n decreases to C as $n \rightarrow \infty$ for some real number $C > 0$

Find the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$. Justify your answer carefully.

Q[23]:

[hint](#) [answer](#) [solution](#)

Let $f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n$, for some constant c and a sequence of constants $\{A_n\}$. Further, let $f(x)$ have a positive radius of convergence.

If $A_1 = 0$, show that $y = f(x)$ has a critical point at $x = c$. What is the relationship between the behaviour of the graph at that point and the value of A_2 ?

Q[24]:

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{n=3}^{\infty} \frac{n}{5^{n-1}}$.

Q[25]:

[hint](#) [answer](#) [solution](#)

Find a polynomial that approximates $f(x) = \log(1+x)$ to within an error of 10^{-5} for all values of x in $\left(0, \frac{1}{10}\right)$.

Then, use your polynomial to approximate $\log(1.05)$ as a rational number.

Q[26]:

[hint](#) [answer](#) [solution](#)

Find a polynomial that approximates $f(x) = \arctan x$ to within an error of 10^{-5} for all values of x in $\left(-\frac{1}{4}, \frac{1}{4}\right)$.

3.6▲ Taylor Series

Exercises

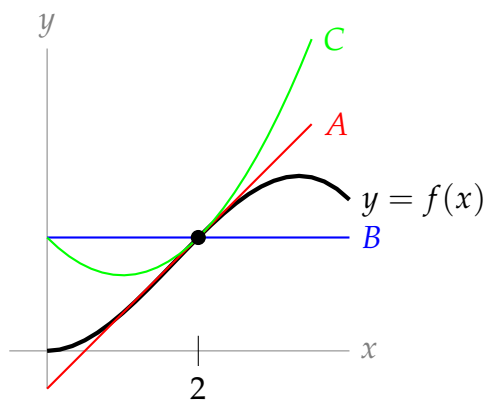
Jump to [HINTS](#), [ANSWERS](#), [SOLUTIONS](#) or [TABLE OF CONTENTS](#).

► Stage 1

Q[1]:

[hint](#) [answer](#) [solution](#)

Below is a graph of $y = f(x)$, along with the constant approximation, linear approximation, and quadratic approximation centred at $a = 2$. Which is which?



Q[2]:

[hint](#) [answer](#) [solution](#)

Suppose $T(x)$ is the Taylor series for $f(x) = \arctan^3(e^x + 7)$ centred at $a = 5$. What is $T(5)$?

Q[3]:

[hint](#) [answer](#) [solution](#)

Below are a list of common functions, and their Taylor series representations. Match the function to the Taylor series and give the radius of convergence of the series.

function	series
A. $\frac{1}{1-x}$	I. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$
B. $\log(1+x)$	II. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
C. $\arctan x$	III. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
D. e^x	IV. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
E. $\sin x$	V. $\sum_{n=0}^{\infty} x^n$
F. $\cos x$	VI. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Q[4]:

[hint](#) [answer](#) [solution](#)

(a) Suppose $f(x) = \sum_{n=0}^{\infty} \frac{n^2}{(n!+1)}(x-3)^n$ for all real x . What is $f^{(20)}(3)$ (the twentieth derivative of $f(x)$ at $x=3$)?

(b) Suppose $g(x) = \sum_{n=0}^{\infty} \frac{n^2}{(n!+1)}(x-3)^{2n}$ for all real x . What is $g^{(20)}(3)$?

(c) If $h(x) = \frac{\arctan(5x^2)}{x^2}$, what is $h^{(20)}(0)$? What is $h^{(22)}(0)$?

► Stage 2

In Questions 5 through 8, you will create Taylor series from scratch. In practice, it is often preferable to modify an existing series, rather than creating a new one, but you should understand both ways.

Q[5]:

[hint](#) [answer](#) [solution](#)

Using the definition of a Taylor series, find the Taylor series for $f(x) = \log(x)$ centred at $x=1$.

Q[6]:

[hint](#) [answer](#) [solution](#)

Find the Taylor series for $f(x) = \sin x$ centred at $a = \pi$.

Q[7]:

[hint](#) [answer](#) [solution](#)

Using the definition of a Taylor series, find the Taylor series for $g(x) = \frac{1}{x}$ centred at $x=10$. What is the radius of convergence of the resulting series?

Q[8]:

[hint](#) [answer](#) [solution](#)

Using the definition of a Taylor series, find the Taylor series for $h(x) = e^{3x}$ centred at $x = a$, where a is some constant. What is the radius of convergence of the resulting series?

In Questions 9 through 16, practice creating new Taylor series by modifying known Taylor series, rather than creating your series from scratch.

Q[9](*):

[hint](#) [answer](#) [solution](#)

Find the Maclaurin series for $f(x) = \frac{1}{2x-1}$.

Q[10](*):

[hint](#) [answer](#) [solution](#)

Let $\sum_{n=0}^{\infty} b_n x^n$ be the Maclaurin series for $f(x) = \frac{3}{x+1} - \frac{1}{2x-1}$,

i.e. $\sum_{n=0}^{\infty} b_n x^n = \frac{3}{x+1} - \frac{1}{2x-1}$. Find b_n .

Q[11](*):

[hint](#) [answer](#) [solution](#)

Find the coefficient c_5 of the fifth degree term in the Maclaurin series $\sum_{n=0}^{\infty} c_n x^n$ for e^{3x} .

Q[12](*):

[hint](#) [answer](#) [solution](#)

Express the Taylor series of the function

$$f(x) = \log(1+2x)$$

about $x = 0$ in summation notation.

Q[13](*):

[hint](#) [answer](#) [solution](#)

The first two terms in the Maclaurin series for $x^2 \sin(x^3)$ are $ax^5 + bx^{11}$, where a and b are constants. Find the values of a and b .

Q[14](*):

[hint](#) [answer](#) [solution](#)

Give the first two nonzero terms in the Maclaurin series for $\int \frac{e^{-x^2} - 1}{x} dx$.

Q[15](*):

[hint](#) [answer](#) [solution](#)

Find the Maclaurin series for $\int x^4 \arctan(2x) dx$.

Q[16](*):

[hint](#) [answer](#) [solution](#)

Suppose that $\frac{df}{dx} = \frac{x}{1+3x^3}$ and $f(0) = 1$. Find the Maclaurin series for $f(x)$.

In past chapters, we were only able to exactly evaluate very specific types of series: geometric. In Questions 17 through 25, we expand our range by relating given series to Taylor series.

Q[17](*):

[hint](#) [answer](#) [solution](#)

The Maclaurin series for $\arctan x$ is given by

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

which has radius of convergence equal to 1. Use this fact to compute the exact value of the series below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

Q[18](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

Q[19](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{k=0}^{\infty} \frac{1}{e^k k!}$.

Q[20](*):

[hint](#) [answer](#) [solution](#)

Evaluate the sum of the convergent series $\sum_{k=1}^{\infty} \frac{1}{\pi^k k!}$.

Q[21](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n}$.

Q[22](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n$.

Q[23]:

[hint](#) [answer](#) [solution](#)

Evaluate $\sum_{n=1}^{\infty} \frac{2^n}{n}$, or show that it diverges.

Q[24]:

[hint](#) [answer](#) [solution](#)

Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} (1 + 2^{2n+1})$$

or show that it diverges.

Q[25](*):

[hint](#) [answer](#) [solution](#)

(a) Show that the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ converges absolutely for all real numbers x .

(b) Evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n)!}$.

Q[26]:

[hint](#) [answer](#) [solution](#)

(a) Using the fact that $\arctan(1) = \frac{\pi}{4}$, how many terms of the Taylor series for arctangent would you have to add up to approximate π with an error of at most 4×10^{-5} ?

(b) Example 3.6.8 in the text mentions the formula

$$\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$$

Using the Taylor series for arctangent, how many terms would you have to add up to approximate π with an error of at most 4×10^{-5} ?

(c) Assume without proof the following:

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \arctan \left(\frac{3+2}{2 \cdot 3 - 1} \right)$$

Using the Taylor series for arctangent, how many terms would you have to add up to approximate π with an error of at most 4×10^{-5} ?

Q[27]:

[hint](#) [answer](#) [solution](#)

Suppose you wanted to approximate the number $\log(1.5)$ as a rational number using the Taylor expansion of $\log(1+x)$. How many terms would you need to add to get 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11} .)

Q[28]:

[hint](#) [answer](#) [solution](#)

Suppose you wanted to approximate the number e as a rational number using the Maclaurin expansion of e^x . How many terms would you need to add to get 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11} .)

You may assume without proof that $2 < e < 3$.

Q[29]:

[hint](#) [answer](#) [solution](#)

Suppose you wanted to approximate the number $\log(0.9)$ as a rational number using the Taylor expansion of $\log(1-x)$. Which partial sum should you use to get 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11} .)

Q[30]:

[hint](#) [answer](#) [solution](#)

Define the hyperbolic sine function as

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Suppose you wanted to approximate the number $\sinh(b)$ using the Maclaurin series of $\sinh x$, where b is some number in $(-2, 1)$. Which partial sum should you use to guarantee 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11} .)

You may assume without proof that $2 < e < 3$.

Q[31]:

[hint](#) [answer](#) [solution](#)

Let $f(x)$ be a function with

$$f^{(n)}(x) = \frac{(n-1)!}{2} \left[(1-x)^{-n} + (-1)^{n-1} (1+x)^{-n} \right]$$

for all $n \geq 1$.

Give reasonable bounds (both upper and lower) on the error involved in approximating $f\left(-\frac{1}{3}\right)$ using the partial sum S_6 of the Taylor series for $f(x)$ centred at $a = \frac{1}{2}$.

Remark: One function with this quality is the inverse hyperbolic tangent function.¹

►► Stage 3

Q[32](*):

[hint](#) [answer](#) [solution](#)

Use series to evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x}$.

Q[33](*):

[hint](#) [answer](#) [solution](#)

Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$.

Q[34]:

[hint](#) [answer](#) [solution](#)

Evaluate $\lim_{x \rightarrow 0} (1 + x + x^2)^{2/x}$ using a Taylor series for the natural logarithm.

Q[35]:

[hint](#) [answer](#) [solution](#)

Use series to evaluate

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{2x}\right)^x$$

Q[36]:

[hint](#) [answer](#) [solution](#)

Evaluate the series $\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{7^n}$ or show that it diverges.

Q[37]:

[hint](#) [answer](#) [solution](#)

Write the series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+1)(2n+2)}$ as a combination of familiar functions.

Q[38]:

[hint](#) [answer](#) [solution](#)

(a) Find the Maclaurin series for $f(x) = (1-x)^{-1/2}$. What is its radius of convergence?

(b) Manipulate the series you just found to find the Maclaurin series for $g(x) = \arcsin x$. What is its radius of convergence?

Q[39](*):

[hint](#) [answer](#) [solution](#)

Find the Taylor series for $f(x) = \log(x)$ centred at $a = 2$. Find the radius of convergence for this series.

Q[40](*):

[hint](#) [answer](#) [solution](#)

Let $I(x) = \int_0^x \frac{1}{1+t^4} dt$.

(a) Find the Maclaurin series for $I(x)$.

(b) Approximate $I(1/2)$ to within ± 0.0001 .

(c) Is your approximation in (b) larger or smaller than the true value of $I(1/2)$? Explain.

1 Of course it is! Actually, hyperbolic tangent is $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, and inverse hyperbolic tangent is its functional inverse.

Q[41](*):

[hint](#) [answer](#) [solution](#)

Using a Maclaurin series, the number $a = 1/5 - 1/7 + 1/18$ is found to be an approximation for $I = \int_0^1 x^4 e^{-x^2} dx$. Give the best upper bound you can for $|I - a|$.

Q[42](*):

[hint](#) [answer](#) [solution](#)

Find an interval of length 0.0002 or less that contains the number

$$I = \int_0^{\frac{1}{2}} x^2 e^{-x^2} dx$$

Q[43](*):

[hint](#) [answer](#) [solution](#)

Let $I(x) = \int_0^x \frac{e^{-t} - 1}{t} dt$.

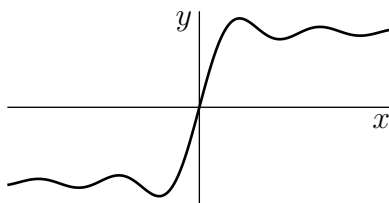
- Find the Maclaurin series for $I(x)$.
- Approximate $I(1)$ to within ± 0.01 .
- Explain why your answer to part (b) has the desired accuracy.

Q[44](*):

[hint](#) [answer](#) [solution](#)

The function $\Sigma(x)$ is defined by $\Sigma(x) = \int_0^x \frac{\sin t}{t} dt$.

- Find the Maclaurin series for $\Sigma(x)$.
- It can be shown that $\Sigma(x)$ has an absolute maximum which occurs at its smallest positive critical point (see the graph of $\Sigma(x)$ below). Find this critical point.
- Use the previous information to find the maximum value of $\Sigma(x)$ to within ± 0.01 .



Q[45](*):

[hint](#) [answer](#) [solution](#)

Let $I(x) = \int_0^x \frac{\cos t - 1}{t^2} dt$.

- Find the Maclaurin series for $I(x)$.
- Use this series to approximate $I(1)$ to within ± 0.01 .
- Is your estimate in (b) greater than $I(1)$? Explain.

Q[46](*):

[hint](#) [answer](#) [solution](#)

Let $I(x) = \int_0^x \frac{\cos t + t \sin t - 1}{t^2} dt$

- Find the Maclaurin series for $I(x)$.

(b) Use this series to approximate $I(1)$ to within ± 0.001

(c) Is your estimate in (b) greater than or less than $I(1)$?

Q[47](*):

[hint](#) [answer](#) [solution](#)

Define $f(x) = \int_0^x \frac{1 - e^{-t}}{t} dt$.

(a) Show that the Maclaurin series for $f(x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$.

(b) Use the ratio test to determine the values of x for which the Maclaurin series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n \text{ converges.}$$

Q[48](*):

[hint](#) [answer](#) [solution](#)

Show that $\int_0^1 \frac{x^3}{e^x - 1} dx \leq \frac{1}{3}$.

Q[49](*):

[hint](#) [answer](#) [solution](#)

Let $\cosh(x) = \frac{e^x + e^{-x}}{2}$.

(a) Find the power series expansion of $\cosh(x)$ about $x_0 = 0$ and determine its radius of convergence.

(b) Show that $3\frac{2}{3} \leq \cosh(2) \leq 3\frac{2}{3} + 0.1$.

(c) Show that $\cosh(t) \leq e^{\frac{1}{2}t^2}$ for all t .

Q[50]:

[hint](#) [answer](#) [solution](#)

The law of the instrument says “If you have a hammer then everything looks like a nail” — it is really a description of the “tendency of jobs to be adapted to tools rather than adapting tools to jobs.”² Anyway, this is a long way of saying that just because we know how to compute things using Taylor series doesn’t mean we should neglect other techniques.

(a) Using Newton’s method, approximate the constant $\sqrt[3]{2}$ as a root of the function $g(x) = x^3 - 2$. Using a calculator, make your estimation accurate to within 0.01.

(b) You may assume without proof that

$$\sqrt[3]{x} = 1 + \frac{1}{6}(x-1) + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2)(5)(8) \cdots (3n-4)}{3^n n!} (x-1)^n.$$

for all real numbers x . Using the fact that this is an alternating series, how many terms would you have to add for the partial sum to estimate $\sqrt[3]{2}$ with an error less than 0.01?

2 Quote from Silvan Tomkins’s *Computer Simulation of Personality: Frontier of Psychological Theory*. See also Birmingham screwdrivers.

Q[51]:

[hint](#) [answer](#) [solution](#)

Let $f(x) = \arctan(x^3)$. Write $f^{(10)}\left(\frac{1}{5}\right)$ as a sum of rational numbers with an error less than 10^{-6} using the Maclaurin series for arctangent.

Q[52]:

[hint](#) [answer](#) [solution](#)

Consider the following function:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- (a) Sketch $y = f(x)$.
- (b) Assume (without proof) that $f^{(n)}(0) = 0$ for all whole numbers n . Find the Maclaurin series for $f(x)$.
- (c) Where does the Maclaurin series for $f(x)$ converge?
- (d) For which values of x is $f(x)$ equal to its Maclaurin series?

Q[53]:

[hint](#) [answer](#) [solution](#)

Suppose $f(x)$ is an odd function, and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Simplify $\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n}$.

Part II

HINTS TO PROBLEMS

Hints for Exercises 1.1. — Jump to [TABLE OF CONTENTS](#).

H-1: Draw a rectangle that *encompasses* the entire shaded area, and one that is *encompassed by* the shaded area. The shaded area is no more than the area of the bigger rectangle, and no less than the area of the smaller rectangle.

H-2: We can improve on the method of Question 1 by using three rectangles that together encompass the shaded region, and three rectangles that together are encompassed by the shaded region.

H-3: Four rectangles suffice.

H-4: Try drawing a picture.

H-5: Try an oscillating function.

H-6: The ordering of the parts is intentional: each sum can be written by changing some small part of the sum before it.

H-7: If we raise -1 to an even power, we get $+1$, and if we raise it to an odd power, we get -1 .

H-8: Sometimes a little anti-simplification can make the pattern more clear.

(a) Re-write as $\frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + \frac{9}{243}$.

(b) Compare to the sum in the hint for (a).

(c) Re-write as $1 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + \frac{4}{1} + \frac{5}{10} + \frac{6}{100} + \frac{7}{1000}$.

H-9: (a), (b) These are geometric sums.

(c) You can write this as three separate sums.

(d) You can write this as two separate sums. Remember that e is a constant. Don't be thrown off by the index being n instead of i .

H-10:

(a) Write out the terms of the two sums.

(b) A change of index is an easier option than expanding the cubic.

(c) Which terms cancel?

(d) Remember $2n + 1$ is odd for every integer n . The index starts at $n = 2$, not $n = 1$.

H-11: Since the sum adds four pieces, there will be four rectangles. However, one might be extremely small.

H-12: Write out the general formula for the left Riemann sum from Definition 1.1.12 in the text and choose a , b and n to make it match the given sum.

H-13: Since the sum runs from 1 to 3, there are three intervals. Suppose $2 = \Delta x = \frac{b-a}{n}$. You may assume the sum given is a right Riemann sum (as opposed to left or midpoint).

H-14: Let $\Delta x = \frac{\pi}{20}$. Then what is $b - a$?

H-15: Notice that the index starts at $k = 0$, instead of $k = 1$. Write out the given sum explicitly without using summation notation, and sketch where the rectangles would fall on a graph of $y = f(x)$. Then try to identify $b - a$, and n , followed by “right”, “left”, or “midpoint”, and finally a .

H-16: The area is a triangle.

H-17: There is one triangle of positive area, and one of negative area.

H-19: Review Definition 1.1.12 in the text.

H-21: You’ll want the limit as n goes to infinity of a sum with n terms. If you’re having a hard time coming up with the sum in terms of n , try writing a sum with a finite number of terms of your choosing. Then, think about how that sum would change if it had n terms.

H-22: The main step is to express the given sum as the right Riemann sum,

$$\sum_{i=1}^n f(a + i\Delta x)\Delta x.$$

Don’t be afraid to guess Δx and $f(x)$ (review Definition 1.1.12 in the text). Then write out explicitly $\sum_{i=1}^n f(a + i\Delta x)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don’t match.

H-23: The main step is to express the given sum as the right Riemann sum

$\sum_{k=1}^n f(a + k\Delta x)\Delta x$. Don’t be afraid to guess Δx and $f(x)$ (review Definition 1.1.12 in the text). Then write out explicitly $\sum_{k=1}^n f(a + k\Delta x)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don’t match.

H-24: The main step is to express the given sum in the form $\sum_{i=1}^n f(x_i^*)\Delta x$. Don’t be afraid to guess Δx , x_i^* (for either a left or a right or a midpoint sum — review Definition 1.1.12 in the text) and $f(x)$. Then write out explicitly $\sum_{i=1}^n f(x_i^*)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don’t match.

H-25: The main step is to express the given sum in the form $\sum_{i=1}^n f(x_i^*)\Delta x$. Don’t be afraid to guess Δx , x_i^* (probably, based on the symbol R_n , assuming we have a right Riemann sum — review Definition 1.1.12 in the text) and $f(x)$. Then write out explicitly $\sum_{i=1}^n f(x_i^*)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don’t match.

H-26: Try several different choices of Δx and x_i^* .

H-27: Let $x = r^3$, and re-write the sum in terms of x .

H-28: Note the sum does not start at $r^0 = 1$.

H-29: Draw a picture. See Example 1.1.18 in the text.

H-30: Draw a picture. Remember $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$.

H-31: Draw a picture: the area we want is a trapezoid. If you don't remember a formula for the area of a trapezoid, think of it as the difference of two triangles.

H-32: You can draw a very similar picture to Question 31, but remember the areas are negative.

H-33: If $y = \sqrt{16 - x^2}$, then y is nonnegative, and $y^2 + x^2 = 16$.

H-34: Sketch the graph of $f(x)$.

H-35: At which time in the interval, for example, $0 \leq t \leq 0.5$, is the car moving the fastest?

H-36: What are the possible speeds the car could have reached at time $t = 0.25$?

H-37: You need to know the speed of the plane at the midpoints of your intervals, so (for example) noon to 1pm *is not* one of your intervals.

H-39: $\Delta x = \frac{2\pi}{75}$, which can be written in a spreadsheet using `2*PI()/75`.

H-40: Sure looks like a Riemann sum.

H-41: For part (b): don't panic! Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given identity. The third step is to evaluate the limit $n \rightarrow \infty$.

H-42: The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formulas. The third step is to evaluate the limit as $n \rightarrow \infty$.

H-43: The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formulas. The third step is to evaluate the limit $n \rightarrow \infty$.

H-44: You've probably seen this hint before. It is worth repeating. Don't panic! Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formula. The third step is to evaluate the limit $n \rightarrow \infty$.

H-45: Using the definition of a right Riemann sum, we can come up with an expression for $f(-5 + 10i)$. In order to find $f(x)$, set $x = -5 + 10i$.

H-46: Recall that for a positive constant a , $\frac{d}{dx} \{a^x\} = a^x \log a$, where $\log a$ is the natural logarithm (base e) of a .

H-47: Part (a) follows the same pattern as Question 46—there's just a little more algebra involved, since our lower limit of integration is not 0.

H-48: Your area can be divided into a section of a circle and a triangle. Then you can use geometry to find the area of each piece.

H-49:

(a) The difference between the upper and lower bounds is the area that is outside of the smaller rectangles but inside the larger rectangles. Drawing both sets of rectangles on one picture might make things clearer. Look for an easy way to compute the area you want.

(b) Use your answer from Part (a). Your answer will depend on f , a , and b .

H-50: Since $f(x)$ is linear, there exist real numbers m and c such that $f(x) = mx + c$. It's a little easier to first look at a single triangle from each sum, rather than the sums in their entirety.

Hints for Exercises 1.2. — Jump to [TABLE OF CONTENTS](#).

H-1:

(a) What is the length of this figure?

(b) Think about cutting the area into two pieces vertically.

(c) Think about cutting the area into two pieces another way.

H-2: Use the identity $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$.

H-4: Note that the limits of the integral given are in the opposite order from what we might expect: the smaller number is the top limit of integration.

Recall $\Delta x = \frac{b-a}{n}$.

H-5: Split the “target integral” up into pieces that can be evaluated using the given integrals.

H-6: Split the “target integral” up into pieces that can be evaluated using the given integrals.

H-7: Split the “target integral” up into pieces that can be evaluated using the given integrals.

H-8: For part (a), use the symmetry of the integrand. For part (b), the area $\int_0^1 \sqrt{1-x^2} \, dx$ is easy to find—how is this useful to you?

H-9: The evaluation of this integral was also the subject of Question 9 in Section 1.1. This time try using the method of Example 1.2.7 in the text.

H-10: Use symmetry.

H-11: Check Theorem 1.2.12 in the text.

H-12: Split the integral into a sum of two integrals. Interpret each geometrically.

H-13: Hmmmm. Looks like a complicated integral. It's probably a trick question. Check for symmetries.

H-14: Check for symmetries again.

H-15: What does the integrand look like to the left and right of $x = 3$?

H-16: In part (b), you'll have to factor a constant out through a square root. Remember the upper half of a circle looks like $\sqrt{r^2 - x^2}$.

H-17: For two functions $f(x)$ and $g(x)$, define $h(x) = f(x) \cdot g(x)$. If $h(-x) = h(x)$, then the product is even; if $h(-x) = -h(x)$, then the product is odd.

The table will *not* be the same as if we were multiplying even and odd *numbers*.

H-18: Note $f(0) = f(-0)$.

H-19: If $f(x)$ is even and odd, then $f(x) = -f(x)$ for every x .

H-20: Think about mirroring a function across an axis. What does this do to the slope?

Hints for Exercises 1.3. — Jump to [TABLE OF CONTENTS](#).

H-2: First find the general antiderivative by guessing and checking.

H-3: Be careful. Two of these make no sense at all.

H-4: Check by differentiating.

H-5: Check by differentiating.

H-6: Use the Fundamental Theorem of Calculus Part 1.

H-7: Use the Fundamental Theorem of Calculus, Part 1.

H-8: You already know that $F(x)$ is an antiderivative of $f(x)$.

H-9: (a) Recall $\frac{d}{dx}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$.

(b) All antiderivatives of $\sqrt{1-x^2}$ differ from one another by a constant. You already know one antiderivative.

H-10: In order to apply the Fundamental Theorem of Calculus Part 2, the integrand must be continuous over the interval of integration.

H-11: Use the definition of $F(x)$ as an area.

H-12: $F(x)$ represents net *signed* area.

H-13: Note $G(x) = -F(x)$, when $F(x)$ is defined as in Question 12.

H-14: Using the definition of the derivative, $F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$.

The area of a trapezoid with base b and heights h_1 and h_2 is $\frac{1}{2}b(h_1 + h_2)$.

H-15: There is only one!

H-16: If $\frac{d}{dx}\{F(x)\} = f(x)$, that tells us $\int f(x) dx = F(x) + C$.

H-17: When you're differentiating, you can leave the e^x factored out.

-
- H-18: After differentiation, you can simplify pretty far. Keep at it!
- H-19: This derivative also simplifies considerably. You might need to add fractions by finding a common denominator.
- H-20: Guess a function whose derivative is the integrand, then use the Fundamental Theorem of Calculus Part 2.
- H-21: Split the given integral up into two integrals.
- H-22: The integrand is similar to $\frac{1}{1+x^2}$, so something with arctangent seems in order.
- H-23: The integrand is similar to $\frac{1}{\sqrt{1-x^2}}$, so factoring out $\sqrt{2}$ from the denominator will make it look like some flavour of arcsine.
- H-24: We know how to antidifferentiate $\sec^2 x$, and there is an identity linking $\sec^2 x$ with $\tan^2 x$.
- H-25: Recall $2 \sin x \cos x = \sin(2x)$.
- H-26: $\cos^2 x = \frac{1 + \cos(2x)}{2}$
- H-28: There is a good way to test where a function is increasing, decreasing, or constant, that also has something to do with topic of this section.
- H-29: See Example 1.3.5 in the text.
- H-30: See Example 1.3.5 in the text.
- H-31: See Example 1.3.5 in the text.
- H-32: See Example 1.3.5 in the text.
- H-33: See Example 1.3.6 in the text.
- H-34: Apply $\frac{d}{dx}$ to both sides.
- H-35: What is the title of this section?
- H-36: See Example 1.3.6 in the text.
- H-37: See Example 1.3.6 in the text.
- H-38: See Example 1.3.6 in the text.
- H-39: See Example 1.3.6 in the text.
- H-40: Split up the domain of integration.
- H-41: It is possible to guess an antiderivative for $f'(x)f''(x)$ that is expressed in terms of $f'(x)$.
- H-42: When does the car stop? What is the relation between velocity and distance travelled?
-

H-43: See Example 1.3.5 in the text. For the absolute maximum part of the question, study the sign of $f'(x)$.

H-44: See Example 1.3.5 in the text. For the “minimum value” part of the question, study the sign of $f'(x)$.

H-45: See Example 1.3.5 in the text. For the “maximum” part of the question, study the sign of $F'(x)$.

H-46: Review the definition of the definite integral and in particular Definitions 1.1.10 and 1.1.12 in the text.

H-47: Review the definition of the definite integral and in particular Definitions 1.1.10 and 1.1.12 in the text.

H-48: Carefully check the Fundamental Theorem of Calculus: as written, it only applies directly to $F(x)$ when $x \geq 0$.

Is $F(x)$ even or odd?

H-49: In general, the equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is $y = f(a) + f'(a)(x - a)$.

H-50: Recall $\tan^2 x + 1 = \sec^2 x$.

H-51: Since the integration is with respect to t , the x^3 term can be moved outside the integral.

H-52: Remember that antiderivatives may have a constant term.

Hints for Exercises 1.4. — Jump to [TABLE OF CONTENTS](#).

H-1: Not everything needs an integral.

H-2: Once you can describe a function $f(x)$ as the rate of change of a function $F(x)$, then you know that $\int_a^b f(x) dx$ is $F(b) - F(a)$, the net change in F from a to b . Then, think about what “net change” means in the context of each model.

H-3: For total distance, start by finding all values of t for which $v(t)$ is positive, and all for which it is negative.

H-4: Mind your units.

H-5: At some point, you might want to write a function $f(t)$ as $f(t) = f(0) + f(t) - f(0)$, because $f(t) - f(0)$ looks like the Fundamental Theorem of Calculus, part 2.

H-6: Remember similar triangles from high school, and related rates from last semester.

H-7: It might be helpful to convert A and B into functions that give the position of the particles over time.

H-8: Review Example 1.4.4 in the text. Slice the city up into little pieces, and add the populations of the pieces together.

Hints for Exercises 1.5. — Jump to [TABLE OF CONTENTS](#).

H-1: One is true, the other false.

H-2: You can check whether the final answer is correct by differentiating.

H-3: Check the limits.

H-4: Check every step. Do they all make sense?

H-6: What is $\frac{d}{dx}\{f(g(x))\}$?

H-7: What is the derivative of the argument of the cosine?

H-8: What is the title of the current section?

H-9: What is the derivative of $x^3 + 1$?

H-10: What is the derivative of $\log x$?

H-11: What is the derivative of $1 + \sin x$?

H-12: $\cos x$ is the derivative of what?

H-13: What is the derivative of the exponent?

H-14: What is the derivative of the argument of the square root?

H-15: What is $\frac{d}{dx}\{\sqrt{\log x}\}$?

H-16: There is a short, slightly sneaky method — guess an antiderivative — and a really short, still-more-sneaky method.

H-17: Review the definition of the definite integral and in particular Definitions 1.1.10 and 1.1.12 in the text.

H-18: If $w = u^2 + 1$, then $u^2 = w - 1$.

H-19: Using a trigonometric identity, this is similar (though not identical) to $\int \tan \theta \cdot \sec^2 \theta \, d\theta$.

H-20: If you multiply the top and the bottom by e^x , what does this look like the antiderivative of?

H-21: You know methods other than substitution to evaluate definite integrals.

H-22: $\tan x = \frac{\sin x}{\cos x}$

H-23: Review the definition of the definite integral and in particular Definitions 1.1.10 and 1.1.12 in the text.

H-24: Review the definition of the definite integral and in particular Definitions 1.1.10 and 1.1.12 in the text.

H-25: Find the right Riemann sum for both definite integrals.

Hints for Exercises 1.6. — Jump to [TABLE OF CONTENTS](#).

H-1: When we say “area between,” we want positive area, *not* signed area.

H-2: We’re taking rectangles that reach from one function to the other.

H-3: Draw a sketch first.

H-4: Draw a sketch first.

H-5: You can probably find the intersections by inspection.

H-6: To find the intersection, plug $x = 4y^2$ into the equation $x + 12y + 5 = 0$.

H-7: If the bottom function is the x -axis, this is a familiar question.

H-8: Part of the job is to determine whether $y = x$ lies above or below $y = 3x - x^2$.

H-9: Guess the intersection points by trying small integers.

H-10: Draw a sketch first. You can also exploit a symmetry of the region to simplify your solution.

H-11: Figure out where the two curves cross. To determine which curve is above the other, try evaluating $f(x)$ and $g(x)$ for some simple value of x . Alternatively, consider x very close to zero.

H-12: Think about whether it will easier to use vertical strips or horizontal strips.

H-13: Writing an integral for this is nasty. How can you avoid it?

H-14: You are asked for the area, not the signed area. Be very careful about signs.

H-15: You are asked for the area, not the signed area. Draw a sketch of the region and be very careful about signs.

H-16: You have to determine whether

- the curve $y = f(x) = x\sqrt{25 - x^2}$ lies above the line $y = g(x) = 3x$ for all $0 \leq x \leq 4$ or
- the curve $y = f(x)$ lies below the line $y = g(x)$ for all $0 \leq x \leq 4$ or
- $y = f(x)$ and $y = g(x)$ cross somewhere between $x = 0$ and $x = 4$.

One way to do so is to study the sign of $f(x) - g(x) = x(\sqrt{25 - x^2} - 3)$.

H-17: Flex those geometry muscles.

H-18: These two functions have three points of intersection. This question is slightly messy, but uses the same concepts we’ve been practicing so far.

Hints for Exercises 1.7. — Jump to [TABLE OF CONTENTS](#).

H-1: Go ahead and try it!

H-2: Use the substitution $u = \sec x$.

H-3: Divide both sides of the second identity by $\cos^2 x$.

H-4: See Example 1.7.6 in the text. Note that the power of cosine is odd, and the power of sine is even (it's zero).

H-5: See Example 1.7.7 in the text. All you need is a helpful trig identity.

H-6: The power of cosine is odd, so we can reserve one cosine for du , and turn the rest into sines using the identity $\sin^2 x + \cos^2 x = 1$.

H-7: Since the power of sine is odd (and positive), we can reserve one sine for du , and turn the rest into cosines using the identity $\sin^2 + \cos^2 x = 1$.

H-8: When we have even powers of sine and cosine both, we use the identities in the last two lines of Equation 1.7.3 in the text.

H-9: Since the power of sine is odd, you can use the substitution $u = \cos x$.

H-10: Which substitution will work better: $u = \sin x$, or $u = \cos x$?

H-11: Try a substitution.

H-12: For practice, try doing this in two ways, with different substitutions.

H-13: Try the substitution $u = \sec x$.

H-14: Compare to Question 13.

H-15: What is the derivative of tangent?

H-16: Don't be scared off by the non-integer power of secant. You can still use the strategies in the notes for an odd power of tangent.

H-17: Since there are no secants in the problem, it's difficult to use the substitution $u = \sec x$ that we've enjoyed in the past. Example 1.7.13 in the text provides a template for antidifferentiating an odd power of tangent.

H-18: Write $\tan x = \frac{\sin x}{\cos x}$.

H-19: How have we handled integration in the past that involved an odd power of tangent?

H-20: $\frac{1}{\cos \theta} = \sec \theta$

H-21: $\cot x = \frac{\cos x}{\sin x}$

H-22: Try substituting.

H-23: Try the trig identity $\tan^2 x = \sec^2 x - 1$ and the substitution $y = \tan x$, $dy = \sec^2 x \, dx$ for (a). Then use (a) to answer (b).

Hints for Exercises 1.8. — Jump to [TABLE OF CONTENTS](#).

H-1: The beginning of this section has a template for choosing a substitution. Your goal is to use a trig identity to turn the argument of the square root into a perfect square, so you can cancel $\sqrt{(\text{something})^2} = |\text{something}|$.

H-2: You want to do the same thing you did in Question 1, but you'll have to complete the square first.

H-3: Since θ is acute, you can draw it as an angle of a right triangle. The given information will let you label two sides of the triangle, and the Pythagorean Theorem will lead you to the third.

H-4: You can draw a right triangle with angle θ , and use the given information to label two of the sides. The Pythagorean Theorem gives you the third side.

H-5: As in Question 1, choose an appropriate substitution. Your answer should be in terms of your original variable, x , which can be achieved using the methods of Question 3.

H-6: As in Question 1, choose an appropriate substitution. Your answer will be a number, so as long as you change your limits of integration when you substitute, you don't need to bother changing the antiderivative back into the original variable x . However, you might want to use the techniques of Question 4 to simplify your final answer.

H-7: Question 1 guides the way to finding the appropriate substitution. Since the integral is definite, your final answer will be a number. Your limits of integration should be common reference angles.

H-8: Question 1 guides the way to finding the appropriate substitution. Since you have an indefinite integral, make sure to get your answer back in terms of the original variable, x . Question 3 gives a reliable method for this.

H-9: A trig substitution is not the easiest path.

H-10: To antidifferentiate, change your trig functions into sines and cosines.

H-11: The integrand should simplify quite far after your substitution.

H-12: In part (a) you are asked to integrate an even power of $\cos x$. For part (b) you can use a trigonometric substitution to reduce the integral of part (b) almost to the integral of part (a).

H-13: What is the symmetry of the integrand?

H-14: See Example 1.8.3 in the text.

H-15: To integrate an even power of tangent, use the identity $\tan^2 x = \sec^2 x - 1$.

H-16: A trig substitution is not the easiest path.

H-17: Complete the square. Your final answer will have an inverse trig function in it.

H-18: To antidifferentiate even powers of cosine, use the formula $\cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta))$. Then, remember $\sin(2\theta) = 2 \sin \theta \cos \theta$.

H-19: After substituting, use the identity $\tan^2 x = \sec^2 x - 1$ more than once.

Remember $\int \sec x dx = \log |\sec x + \tan x| + C$.

H-20: There's no square root, but we can still make use of the substitution $x = \tan \theta$.

H-21: Complete the square — refer to Question 2 if you want a refresher. The constants aren't pretty, but don't let them scare you.

H-22: Make use of symmetry, and integrate with respect to y (rather than x).

H-23: Think of e^x as $(e^{x/2})^2$, and use a trig substitution. Then, use the identity $\sec^2 \theta = \tan^2 \theta + 1$.

H-24: Consider the ranges of the inverse trigonometric functions. For (c), also consider the domain of $\sqrt{x^2 - a^2}$.

Hints for Exercises 1.9. — Jump to [TABLE OF CONTENTS](#).

H-1: If a quadratic function can be factored as $(ax + b)(cx + d)$ for some constants a, b, c, d , then it has roots $-\frac{b}{a}$ and $-\frac{d}{c}$.

H-2: Review Example 1.9.1 in the text. Is the “Algebraic Method” or the “Sneaky Method” going to be easier?

H-3: For each part, use long division as in Example 1.9.4 of the text.

H-4: (a) Look for a pattern you can exploit to factor out a linear term.

(b) If you set $y = x^2$, this is quadratic. Remember $(x^2 - a) = (x + \sqrt{a})(x - \sqrt{a})$ as long as a is positive.

(c),(d) Look for integer roots, then use long division.

H-5: Why do we do partial fraction decomposition at all?

H-6: What is the title of this section?

H-7: You can save yourself some work in developing your partial fraction decomposition by renaming x^2 to y and comparing the result with Question 6.

H-8: Review Steps 3 (particularly the “Sneaky Method”) and 4 of Example 1.9.3 in the text.

H-9: Review Steps 3 (particularly the “Sneaky Method”) and 4 of Example 1.9.3 in the text. Remember $\frac{d}{dx} \{\arctan x\} = \frac{1}{1+x^2}$.

H-10: Fill in the blank: the integrand is a ____ function.

H-11: The integrand is yet another ____ function.

H-12: Since the degree of the numerator is the same as the degree of the denominator, we can't do our partial fraction decomposition before we simplify the integrand.

H-13: The degree of the numerator is not smaller than the degree of the denominator. Your final answer will have an arctangent in it.

H-14: Factor $(2x - 1)$ out of the denominator to get started. You don't need long division for this step.

H-15: When it comes time to integrate, look for a convenient substitution.

H-16: $\csc x = \frac{1}{\sin x} = \frac{\sin x}{\sin^2 x}$

H-17: Use the partial fraction decomposition from Question 16 to save yourself some time.

H-18: In the final integration, complete the square to make a piece of the integrand look more like the derivative of arctangent.

H-19: $\cos^2 \theta = 1 - \sin^2 \theta$

H-20: If you're having a hard time making the substitution, multiply the numerator and the denominator by e^x .

H-21: Try the substitution $u = \sqrt{1 + e^x}$. You'll need to do long division before you can use partial fraction decomposition.

H-22: You'll need to use two regions, because the curves cross.

H-23: For (b), use the Fundamental Theorem of Calculus Part 1.

Hints for Exercises 1.10. — Jump to [TABLE OF CONTENTS](#).

H-1: The horizontal cross-sections were discussed in Example 1.10.1 of the text.

H-2: What are the dimensions of the cross-sections?

H-3: There are two different kinds of washers.

H-4: Draw sketches.

H-5: Draw sketches.

H-6: Draw a sketch.

H-7: If you take horizontal slices (parallel to one face), they will all be equilateral triangles.

Be careful not to confuse the height of a *triangle* with the height of the tetrahedron.

H-8: Sketch the region.

H-9: Sketch the region first.

H-10: You can save yourself quite a bit of work by interpreting the integral as the area of a known geometric figure.

H-11: See Example 1.10.3 in the text.

H-12: See Example 1.10.5 in the text.

H-13: Sketch the region. To find where the curves intersect, look at where $\cos(\frac{x}{2})$ and $x^2 - \pi^2$ both have roots.

H-14: See Example 1.10.6 in the text.

H-15: See Example 1.10.6 in the text. Imagine cross-sections with shadow parallel to the y -axis, sticking straight out of the xy -plane.

H-16: See Example 1.10.1 in the text.

H-17: (a) Don't be put off by phrases like "rotating an ellipse about its minor axis." This is the same kind of volume you've been calculating all section.

(b) Hopefully, you sketched the ellipse in part (a). What was its smallest radius? Its largest? These correspond to the polar and equatorial radii, respectively.

(c) Combine your answers from (a) and (b).

(d) Remember that the absolute error is the absolute difference of your two results—that is, you subtract them and take the absolute value. The relative error is the absolute error divided by the actual value (which we're taking, for our purposes, to be your answer from (c)). When you take the relative error, lots of terms will cancel, so it's easiest to not use a calculator till the end.

H-18: To find the points of intersection, set $4 - (x - 1)^2 = x + 1$.

H-19: You can somewhat simplify your calculations in part (a) (but not part (b)) by using the fact that $\mathcal{R}$ is symmetric about the line $y = x$.

When you're solving an equation for x , be careful about your signs: $x - 1$ is negative.

H-21: Note that the curves cross. The area of this region was found in Problem 14 of Section 1.5. It would be useful to review that problem.

H-22: You can use ideas from this section to answer the question. If you take a very thin slice of the column, the density is almost constant, so you can find the mass. Then you can add up all your little slices. It's the same idea as volume, only applied to mass.

Do be careful about units: in the problem statement, some are given in metres, others in kilometres.

If you're having a hard time with the antiderivative, try writing the exponential function with base e . Remember $2 = e^{\log 2}$.

H-23: Use the symmetry of the function to re-write your integrals without an absolute value.

Hints for Exercises 1.11. — Jump to [TABLE OF CONTENTS](#).

H-1: Read back over Sections 1.5 and 1.11 of the text. When these methods are introduced, they are justified using the corresponding differentiation rules.

H-2: Remember our rule: $\int u dv = uv - \int v du$. So, we take u and use it to make du , and we take dv and use it to make v .

H-3: According to the quotient rule,

$$\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Antidifferentiate both sides of the equation, then solve for the expression in the question.

H-4: Remember all the antiderivatives differ only by a constant, so you can write them all as $v(x) + C$ for some C .

H-5: What integral do you have to evaluate, after you plug in your choices to the integration by parts formula?

H-6: You'll probably want to use integration by parts. (It's the title of the section, after all). You'll break the integrand into two parts, integrate one, and differentiate the other. Would you rather integrate $\log x$, or differentiate it?

H-7: This problem is similar to Question 6.

H-8: Example 1.11.5 in the text shows you how to find the antiderivative. Then the Fundamental Theorem of Calculus Part 2 gives you the definite integral.

H-9: Compare to Question 8. Try to do this one all the way through without peeking at another solution!

H-10: If at first you don't succeed, try using integration by parts a few times in a row. Eventually, one part will go away.

H-11: Similarly to Question 10, look for a way to use integration by parts a few times to simplify the integrand until it is antidifferentiable.

H-12: Use integration by parts twice to get an integral with only a trigonometric function in it.

H-13: If you let $u = \log t$ in the integration by parts, then du works quite nicely with the rest of the integrand.

H-14: Those square roots are a little disconcerting—get rid of them with a substitution.

H-15: This can be solved using the same ideas as Example 1.11.8 in the text.

H-16: Not every integral should be evaluated using integration by parts.

H-17: You know, or can easily look up, the derivative of arccosine. You can use a similar trick as the book did when antidifferentiating other inverse trigonometric functions in Example 1.11.9 of the text.

H-18: See Examples 1.11.9 and 1.10.5 in the text for refreshers on integrating arctangent, and using washers.

Remember $\tan^2 x + 1 = \sec^2 x$, and $\sec^2 x$ is easy to integrate.

H-19: Your integral can be broken into two integrals, which yield to two different integration methods.

H-20: After integrating by parts, do some algebraic manipulation to the integral until it's clear how to evaluate it.

H-21: After integration by parts, use a substitution.

H-22: Start by simplifying.

H-23: $\sin(2x) = 2 \sin x \cos x$

H-24: What is the derivative of xe^{-x} ?

H-25: You'll want to do an integration by parts for (a)—check the end result to get a guess as to what your parts should be. A trig identity and some amount of algebraic manipulation will be necessary to get the final form.

H-26: Think, first, about how to get rid of the square root in the argument of f'' , and, second, how to convert f'' into f' . Note that you are told that $f'(2) = 4$ and $f(0) = 1$, $f(2) = 3$.

H-27: Interpret the limit as a right Riemann sum.

H-28: Try an integration by parts.

H-29: To deal with the “inside function,” start with a substitution.

Hints for Exercises 1.12. — Jump to [TABLE OF CONTENTS](#).

H-1: The absolute error is the difference of the two values; the relative error is the absolute error divided by the exact value; the percent error is one hundred times the relative error.

H-3: Sketch the second derivative—it's quadratic.

H-4: You don't have to find the actual, exact maximum the second derivative achieves—you only have to give a reasonable “ceiling” that it never breaks through.

H-5: To compute the upper bound on the error, find an upper bound on the fourth derivative of cosine, then use Theorem 1.12.7 in the text.

To find the actual error, you need to find the actual value of A .

H-6: Find a function with $f''(x) = 3$ for all x in $[0, 1]$.

H-7: You're allowed to use common sense for this one.

H-8: Draw a sketch.

H-9: The error bound for the approximation is given in Theorem 1.12.7 in the text. You want this bound to be zero.

H-10: Review Subsection 1.12.4 in the text, but note the differences in the setups of the spreadsheets there and here.

H-11: See Section 1.12.1 in the text. To set up the volume integral, see Example 1.10.6 in the text. Note the dimensions given for the cross sections are diameters, not radii.

H-12: See Section 1.12.2 in the text, and compare to Question 11. Note the table gives diameters, not radii.

H-13: See §1.12.2 in the text. To set up the volume integral, see Example 1.10.6 in the text, or Question 12.

Note that the table gives the circumference, not radius, of the tree at a given height.

H-16: As usual, the biggest part of this problem is finding L . Don't be thrown off by the error bound being given slightly differently from Theorem 1.12.7 in the text: these expressions are equivalent, since $\Delta x = \frac{b-a}{n}$.

H-17: The function $e^{-2x} = \frac{1}{e^{2x}}$ is positive and decreasing, so its maximum occurs when x is as small as possible.

H-18:

H-19: The "best ... approximations that you can" means using the maximum number of intervals, given the information available.

The final sentence in part (b) is just a re-statement of the error bounds we're familiar with from Theorem 1.12.7 in the text. The information $|s^{(k)}(x)| \leq \frac{k}{1000}$ gives you values of M and L when you set $k = 2$ and $k = 4$, respectively.

H-20: Set the error bound to be less than 0.001, then solve for n .

H-21: Review Subsection 1.12.4 in the text.

H-22: Review Subsection 1.12.4 in the text.

H-23: Review Subsection 1.12.4 in the text. Remember the function `EXP ( )` works for the exponential function in most programs.

H-24: See Section 1.12.2 in the text. To set up the volume integral, see Example 1.10.2 in the text.

Since the cross-sections of the pool are semi-circular disks, a section that is d metres across will have area $\frac{1}{2}\pi \left(\frac{d}{2}\right)^2$ square feet. Based on the drawing, you may assume the very ends of the pool have distance 0 feet across.

H-25: See Example 1.12.8 in the text.

Don't get caught up in the interpretation of the integral. It's nice to see how integrals can be used, but for this problem, you're still just approximating the integral given, and bounding the error.

When you find the second derivative to bound your error, pay attention to the difference between the integrand and $g(r)$.

H-26: See Example 1.12.9 in the text. You'll want to use a calculator for the approximation in (a), and for finding the appropriate number of intervals in (b). Remember that Simpson's rule requires an even number of intervals.

H-27: See Example 1.12.9 in the text.

Rather than calculating the fourth derivative of the integrand, use the graph to find the largest absolute value it attains over our interval.

H-28: See Example 1.12.8 in the text.

You'll have to differentiate $f(x)$. To that end, you may also want to review the fundamental theorem of calculus and, in particular, Example 1.3.5 in the text.

You don't have to find the best possible value for M . A reasonable upper bound on $|f''(x)|$ will do.

To have five decimal places of accuracy, your error must be less than 0.000005. This ensures that, if you round your approximation to five decimal places, they will all be correct.

H-29: In using Simpson's rule to approximate $\int_1^x \frac{1}{t} dt$ with n intervals, $a = 1$, $b = x$, and $\Delta x = \frac{x-1}{n}$.

H-30:

- $\int_1^2 \frac{1}{1+x^2} dx = \arctan(2) - \frac{\pi}{4}$, so $\arctan(2) = \frac{\pi}{4} + \int_1^2 \frac{1}{1+x^2} dx$
- If an approximation A of the integral $\int_1^2 \frac{1}{1+x^2} dx$ has error at most ε , then $A - \varepsilon \leq \int_1^2 \frac{1}{1+x^2} dx \leq A + \varepsilon$.
- Looking at our target interval will tell you how small ε needs to be, which in turn will tell you how many intervals you need to use.
- You can show, by considering the numerator and denominator separately, that $|f^{(4)}(x)| \leq 30.75$ for every x in $[1, 2]$.
- If you use Simpson's rule to approximate $\int_1^2 \frac{1}{1+x^2} dx$, you won't need very many intervals to get the requisite accuracy.

Hints for Exercises 1.13. — Jump to [TABLE OF CONTENTS](#).

H-1: There are two kinds of impropriety in an integral: an infinite discontinuity in the integrand, and an infinite limit of integration.

H-2: The integrand is continuous for all x .

H-3: What matters is which function is bigger for large values of x , not near the origin.

H-4: Read both the question and Theorem 1.13.17 in the text *very* carefully.

H-5: (a) What if $h(x)$ is negative? What if it's not?

(b) What if $h(x)$ is very close to $f(x)$ or $g(x)$, rather than right in the middle?

(c) Note $|h(x)| \leq 2f(x)$.

H-6: First: is the integrand unbounded, and if so, where?

Second: when evaluating integrals, always check to see if you can use a simple substitution before trying a complicated procedure like partial fractions.

H-7: Is the integrand bounded?

H-8: See Example 1.13.21 in the text. Rather than antidifferentiating, you can find a nice comparison.

H-9: Which of the two terms in the denominator is more important when $x \approx 0$? Which one is more important when x is very large?

H-10: Remember to break the integral into two pieces.

H-11: Remember to break the integral into two pieces.

H-12: The easiest test in this case is limiting comparison, Theorem 1.13.22 in the text.

H-13: Not all discontinuities cause an integral to be improper—only infinite discontinuities.

H-14: Which of the two terms in the denominator is more important when x is very large?

H-15: Which of the two terms in the denominator is more important when $x \approx 0$? Which one is more important when x is very large?

H-16: What are the “problem x ’s” for this integral? Get a simple approximation to the integrand near each.

H-17: To find the volume of the solid, cut it into horizontal slices, which are thin circular disks.

The true/false statement is equivalent to saying that the improper integral giving the volume of the solid when $a = 0$ diverges to infinity.

H-18: Review Example 1.13.8 in the text. Remember the antiderivative of $\frac{1}{x}$ looks very different from the antiderivative of other powers of x .

H-19: Compare to Example 1.13.14 in the text. You can antidifferentiate with a u -substitution.

H-20: To evaluate the integral, you can factor the denominator.

Recall $\lim_{x \rightarrow \infty} \arctan x = \frac{\pi}{2}$. For the other limits, use logarithm rules, and beware of indeterminate forms.

H-21: Break up the integral. The absolute values give you a nice even function, so you can replace $|x - a|$ with $x - a$ if you’re careful about the limits of integration.

H-22: What is the limit of the integrand when $x \rightarrow 0$?

H-23: The only “source of impropriety” is the infinite domain of integration. Don’t be afraid to be a little creative to make a comparison work.

H-24: Look for a place to use Theorem 1.13.20 of the text.

Examples 1.2.10 and 1.2.11 in the text have nice results about the area under an even/odd curve.

H-25: x should be a real number

Hints for Exercises 1.14. — Jump to [TABLE OF CONTENTS](#).

H-1: Each option in each column should be used exactly once.

H-2: The integrand is the product of sines and cosines. See how this was handled with a substitution in Section 1.7.1 of the text.

After your substitution, you should have a polynomial expression in u —but it might take some simplification to get it into a form you can easily integrate.

H-3: We notice that the integrand has a quadratic polynomial under the square root. If that polynomial were a perfect square, we could get rid of the square root: try a trig substitution, as in Section 1.8 of the text.

The identity $\sin(2\theta) = 2 \sin \theta \cos \theta$ might come in handy.

H-4: Notice the integral is improper. When you compute the limit, l'Hôpital's rule might help.

If you're struggling to think of how to antidifferentiate, try writing $\frac{x-1}{e^x} = (x-1)e^{-x}$.

H-5: Which method usually works for rational functions (the quotient of two polynomials)?

H-6: It would be nice to replace logarithm with its derivative, $\frac{1}{x}$.

H-7: The integrand is a rational function, so it is possible to use partial fractions. But there is a much easier way!

H-8: You should prepare your own personal internal list of integration techniques ordered from easiest to hardest. You should have associated to each technique your own personal list of signals that you use to decide when the technique is likely to be useful.

H-9: Despite both containing a trig function, the two integrals are easiest to evaluate using different methods.

H-10: In (c), notice the denominator is not yet entirely factored.

H-11: Part (a) can be done by inspection – use a little highschool geometry! Part (b) is reminiscent of the antiderivative of logarithm—how did we find that one out? Part (c) is an improper integral.

H-12: Use the substitution $u = \sin \theta$.

H-13: For (c), try a little algebra to split the integral into pieces that are easy to antidifferentiate.

H-14: If you're stumped, review Sections 1.7, 1.8, and 1.9 in the text.

H-15: For part (c), see Example 1.9.4 in the text.

H-16: For part (b), first complete the square in the denominator. You can save some work by first comparing the derivative of the denominator with the numerator. For part (d) use a simple substitution.

H-17: For part (b), complete the square in the denominator. You can save some work by first comparing the derivative of the denominator with the numerator.

H-18: For part (a), the numerator is the derivative of a function that appears in the denominator.

H-19: The integral is improper.

H-20: For part (a), can you convert this into a partial fractions integral? For part (b), start by completing the square inside the square root.

H-21: For part (b), the numerator is the derivative of a function that is embedded in the denominator.

H-22: Try a substitution.

H-23: What's the usual trick for evaluating a rational function (quotient of polynomials)?

H-24: If the denominator were $x^2 + 1$, the antiderivative would be arctangent.

H-25: Simplify first.

H-26: $x^3 + 1 = (x + 1)(x^2 - x + 1)$

H-27: You have the product of two quite dissimilar functions in the integrand—try integration by parts.

H-28: Use the identity $\cos(2x) = 2\cos^2 x - 1$.

H-29: Using logarithm rules can make the integrand simpler.

H-30: What is the derivative of the function in the denominator? How could that be useful to you?

H-32: For part (a), split the integral in two. One part may be evaluated by interpreting it geometrically, without doing any integration at all. For part (c), multiply both the numerator and denominator by e^x and then make a substitution.

H-33: Let $u = \sqrt{1 - x}$.

H-34: Use the substitution $u = e^x$.

H-35: Use integration by parts. If you choose your parts well, the resulting integration will be very simple.

H-36: $\frac{\sin x}{\cos^2 x} = \tan x \sec x$

H-37: The cases $n = -1$ and $n = -2$ are different from all other values of n .

Hints for Exercises 2.1. — Jump to [TABLE OF CONTENTS](#).

H-1: Notation 2.1.4 in the text

H-2: Consider two disjoint outcomes.

H-3: Compare to Question 2

Hints for Exercises 2.2. — Jump to [TABLE OF CONTENTS](#).

H-1: First sketch the probability density function (PDF). Where the probability density function (PDF) is higher, the marks are densest – remember, it's the probability *density* function.

H-2: Try the substitution $u = 10x$

H-3: Use a sketch of $f(x)$ to find the area under the curve. Remember that on all non-specified intervals, $f(x) = 0$ or $f(x)$ DNE.

H-4: Because the two events are disjoint,

$$\Pr(0 < M < 1 \text{ OR } 9 < M < 10) = \Pr(0 < M < 1) + \Pr(9 < M < 10)$$

H-5: Properties of a PDF are given in Corollary 2.2.5 in the text.

H-6: Properties of a PDF are given in Corollary 2.2.5 in the text.

H-7: In the text, Simpson's Rule is Equation 1.12.4. It says

$$\int_a^b f(x) dx \approx \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \right. \\ \left. \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right] \frac{\Delta x}{3}$$

where $\Delta x = \frac{b-a}{2}$.

Hints for Exercises 2.3. — Jump to [TABLE OF CONTENTS](#).

H-1: Imagine X results from a coin flip, where $X = 0$ when the flip is tails and $X = 1$ when the flip is heads. What is $\mathbb{E}(X)$? Is this a likely outcome?

H-2: Can you make $\mathbb{E}(Z) = 1$ if $\mathcal{S} = \{-1, 1\}$?

H-3: $f(x)$ is increasing, so use both Theorems 2.3.5 and 2.3.6.

H-4: Make use of symmetry to avoid a long computation.

H-5: You'll compute two integrals separately.

H-6: Use a substitution.

H-7: Create the probability density function (PDF) of Y .

H-8: See Example 1.13.10 in the text. Corollary 2.2.5 has properties of probability density functions (PDFs).

H-9: Use integration by parts.

H-10: Use integration by parts to evaluate $\int \ln x dx$.

H-11: Use long division to get the integrand into the correct form for partial fractions.

H-12: Start with $\tan^2 x = \sec^2 x - 1$; then integration by parts.

H-13: Take advantage of symmetry, or use the substitution rule. Remember that b is negative.

Hints for Exercises 2.4. — Jump to [TABLE OF CONTENTS](#).

H-1: “Weight giving relative likelihood” isn’t necessarily a probability, but “how likely” is a probability.

H-2: The difference between two values, in this case, should be taken as an absolute value. So account for bundles being both too big and too small.

H-3: The first paragraph can be disregarded.

H-4: Let $S = \{-100, 100\}$

H-5: $f(x) = \frac{1}{b-a}, a \leq x \leq b$

H-6: If $f(x)$ has even symmetry, then $xf(x)$ has odd symmetry.

H-8: See Question 6 for the expectation;

$$f(x) = \begin{cases} 1+x & -1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \end{cases}$$

Hints for Exercises 3.1. — Jump to [TABLE OF CONTENTS](#).

H-1: Not every limit exists.

H-2: 100 isn’t all that big when you’re contemplating infinity. (Neither is any other number.)

H-3: $\lim_{n \rightarrow \infty} a_{2n+5} = \lim_{n \rightarrow \infty} a_n$

H-4: The sequence might be defined by different functions when n is large than when n is small.

H-5: Recall $(-1)^n$ is positive when n is even, and negative when n is odd.

H-6: Modify your answer from Question 5, but make the terms approach zero.

H-7: What might cause your answers in (a) and (b) to differ? Carefully read Theorem 3.1.6 in the text about convergent functions and their corresponding sequences.

H-8: You can use the fact that π is somewhat close to $\frac{22}{7}$, or you can use trial and error.

H-9: You can compare the leading terms, or factor a high power of n from the numerator and denominator.

H-10: This isn't a rational expression, but you can treat it in a similar way. Recall $e < 3$.

H-11: The techniques of evaluating limits of rational sequences are again useful here.

H-12: $e^{-1/n} = \frac{1}{e^{1/n}}$; what happens to $\frac{1}{n}$ as n grows?

H-13: L'Hôpital's rule might help you decide what happens if you are unsure.

H-14: $\cos 0 = 1$

H-15: This is trickier than it looks. Write $\frac{1}{n} = x$ and look at the limit as $x \rightarrow 0$.

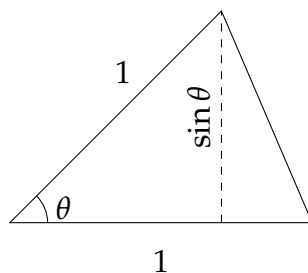
H-16: Multiply and divide by the conjugate.

H-17: Compared to Question 16, there's an easier path.

H-18: Consider $f'(x)$, when $f(x) = x^{100}$.

H-19: Look to Question 18 for inspiration.

H-20: The area of an isosceles triangle with two sides of length 1, meeting at an angle θ , is $\frac{1}{2} \sin \theta$.



H-21: Every term of A_n is the same, and $g(x)$ is a constant function.

H-22: You'll need to use a logarithm before you can apply l'Hôpital's rule.

H-23: (a) Write out the first few terms of the sequence.

(c) Consider how $a_{n+1} - L$ relates to $a_n - L$. What should happen to these numbers if a_n converges to L ?

H-24: Your answer from (b) will help you a lot with the subsequent parts.

Hints for Exercises 3.2. — Jump to [TABLE OF CONTENTS](#).

H-1: S_N is the sum of the terms corresponding to $n = 1$ through $n = N$.

H-2: Note C_k is the cumulative number of cookies.

H-3: How is (a) related to Question 2?

H-4: You'll have to calculate a_1 separately from the other terms.

H-5: When does adding a number decrease the total sum?

H-6: Note C_k is the cumulative number of cookies.

H-7: To find the difference between $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} c_{n+1}$, try writing out the first few terms.

H-8: You might want to first consider a simpler true or false: $\sum_{n=1}^{\infty} \frac{a_n}{b_n} \stackrel{?}{=} \frac{A}{B}$.

H-9: If you do really get stuck, you can try actually making the spreadsheet.

H-10: For maximum accuracy, keep computing partial sums until they appear to stop changing. This might take quite some time.

H-11: Review Definition 3.2.3 in the text.

H-12: Review Question 3 for using the sequence of partial sums.

Hints for Exercises 3.3. — Jump to [TABLE OF CONTENTS](#).

H-1: That is, which series have terms whose limit is not zero?

H-2: That is, if $f(x)$ is a function with $f(n) = a_n$ for all whole numbers n , is $f(x)$ nonnegative and decreasing?

H-3: This isn't a trick. It's meant to give you intuition to the direct comparison test.

H-4: The comparison test is Theorem 3.3.11 in the text. However, rather than trying to memorize which way the inequalities go in all cases, you can use the same reasoning as Question 3.

H-5: Think about Question 4 to remind yourself which way the inequalities have to go for direct comparison.

Note that all the comparison series have positive terms, so we don't need to worry about that part of the limit comparison test.

H-6: The divergence test is Theorem 3.3.2 in the text.

H-7: It is true that $f(x)$ is positive. What else has to be true of $f(x)$ for the integral test to apply?

H-8: Refer to Question 4.

H-9: The definition of an alternating series is given in the start of Section 3.3.5 in the text.

H-10: For the ratio test to be inconclusive, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ should be 1 or nonexistent.

H-11: By the divergence test, for a series $\sum a_n$ to *converge*, we need $\lim_{n \rightarrow \infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller.

H-12: If $f(x)$ is positive and decreasing, then the integral test tells you that the integral and the series either both increase or both decrease. So, in order to find an example with the properties required in the question, you need $f(x)$ to *not* be both positive and decreasing.

H-13: Review Theorem 3.3.14 and Example 3.3.15 in the text.

H-14: Don't jump to conclusions about properties of the a_n 's.

H-15: For (b), imagine cutting up the triangle into its black and white parts, then sharing it equally among a certain number of friends. What is the easiest number of friends to share with, making sure each has the same area in their pile?

H-16: Compare to Question 15.

H-17: Iteratively divide a shape into thirds.

H-18: Equation 3.3.1 in the text tells us $\sum_{n=0}^N ar^n = a \frac{1 - r^{N+1}}{1 - r}$, for $r \neq 1$.

H-19: To adjust the starting index, either factor out the first term in the series, or subtract two series. For the subtraction option, consider Question 6.

H-20: Always try the divergence test first (in your head).

H-21: Which test should you always try first (in your head)?

H-22: What kind of a series is this?

H-23: This is a special kind of series, that you should recognize.

H-24: This is a special case of a general series whose sum we know.

H-25: Review Example 3.3.10 in the text. To write the number as a geometric series, the first few terms might not fit the pattern of the rest of the terms.

H-26: Start by writing it as a geometric series.

H-27: Review Example 3.3.10 in the text. Since the pattern repeats every three decimals, your common ratio r will be $\frac{1}{10^3}$.

H-28: Split the series into two parts.

H-29: Split the series into two parts.

H-30: Review the integral test, which is Theorem 3.3.6 in the text.

H-31: A comparison might be helpful—try some algebraic manipulation to find a likely series to compare it to.

H-32: This is a geometric series.

H-33: Notice that the series is geometric, but it doesn't start at $n = 0$.

H-34: Note n only takes integer values: what's $\sin(\pi n)$ when n is an integer?

H-35: Note n only takes integer values: what's $\cos(\pi n)$ when n is an integer?

-
- H-36: What's the test that you should always think of when you see a factorial?
- H-37: This is a geometric series, but you'll need to do a little algebra to figure out r .
- H-38: Which test fits most often with factorials?
- H-39: Try finding a nice comparison.
- H-40: With the substitution $u = \log x$, the function $\frac{1}{x(\log x)^{3/2}}$ is easily integrable.
- H-41: Combine the integral test with the results about p -series, Example 3.3.7 in the text.
- H-42: Try the substitution $u = \sqrt{x}$.
- H-43: Review Example 3.3.12 in the text for developing intuition about comparisons, and Example 3.3.13 for an example where finding an appropriate comparison series calls for some creativity.
- H-44: What does the summand look like when k is very large?
- H-45: What does the summand look like when n is very large?
- H-46: $\cos(n\pi)$ is a sneaky way to write $(-1)^n$.
- H-47: What is the behaviour for large k ?
- H-48: When m is large, $3m + \sin \sqrt{m} \approx 3m$.
- H-49: This is a geometric series, but it doesn't start at $n = 0$.
- H-50: The series is geometric.

H-51: The first series can be written as $\sum_{n=1}^{\infty} \frac{1}{2n-1}$.

- H-53: What does the summand look like when n is very large?
- H-54: Review the alternating series test, which is given in Theorem 3.3.17 in the text.
- H-55: Review the alternating series test, which is given in Theorem 3.3.17 in the text.
- H-56: Review the alternating series test, which is given in Theorem 3.3.17 in the text.
- H-57: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.
- H-59: For part (a), see Example 1.13.23 in the text.

For part (b), review Theorem 3.3.6 in the text.

For part (c), see Example 3.3.15 in the text.

H-60: The truncation error arising from the approximation $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \approx \sum_{n=1}^N \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ is precisely $E_N = \sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$. You'll want to find a bound on this sum using the integral test.

A key observation is that, since $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$ is decreasing, we can show that

$$\frac{e^{-\sqrt{n}}}{\sqrt{n}} \leq \int_{n-1}^n \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$$

for every $n \geq 1$.

H-61: What does the fact that the series $\sum_{n=0}^{\infty} a_n$ converges guarantee about the behavior of a_n for large n ?

H-62: What does the fact that the series $\sum_{n=0}^{\infty} (1 - a_n)$ converges guarantee about the behavior of a_n for large n ?

H-63: What does the fact that the series $\sum_{n=1}^{\infty} a_n$ converges guarantee about the behavior of a_n for large n ? When is $x^2 \leq x$?

H-64: If we add together the frequencies of all the words, they should amount to 100%. We can approximate this sum using ideas from Example 3.3.5 in the text.

H-65: We are approximating a finite sum — not an infinite series. To get greater accuracy, use exact values for the first several terms in the sum, and use an integral to approximate the rest.

H-66: What is the ratio of areas between the outermost (red) ring and the next (blue) ring?

Hints for Exercises 3.4. — Jump to [TABLE OF CONTENTS](#).

H-1: What is conditional convergence?

H-2: If $\sum |a_n|$ converges, then $\sum a_n$ is guaranteed to converge as well. (That's Theorem 3.4.2 in the text.) So, one of the blank spaces describes an impossible sequence.

H-3: Apply the test you know to the series $\sum_{n=1}^{\infty} a_n$ as well as to the series $\sum_{n=1}^{\infty} |a_n|$

H-4: Be careful about the signs.

H-5: Does the alternating series test really apply?

H-6: What does the summand look like when n is very large?

H-7: What does the summand look like when n is very large?

H-8: This is a trick question. Be sure to verify *all* of the hypotheses of any convergence test you apply.

H-9: Try the substitution $u = \log x$.

H-10: Show that it converges absolutely.

H-11: Use a similar method to Question 10.

H-12: Show it converges absolutely using a direct comparison test.

H-13: For part (a), replace n by x in the absolute value of the summand. Can you integrate the resulting function?

H-14: You don't need to add up very many terms for this level of accuracy.

H-15: Use the direct comparison test to show that the series converges absolutely.

Hints for Exercises 3.5. — Jump to [TABLE OF CONTENTS](#).

H-1: $f(1)$ is the sum of a geometric series.

H-2: Calculate $\frac{d}{dx} \left\{ \frac{(x-5)^n}{n! + 2} \right\}$ when n is a constant.

H-3: There is only one.

H-4: Use Theorem 3.5.9 in the text.

H-5: Review the discussion immediately following Definition 3.5.1 in the text.

H-6: Review the discussion immediately following Definition 3.5.1 in the text.

H-7: Review the discussion immediately following Definition 3.5.1 in the text.

H-15: Start part (b) by computing the partial sums of $\sum_{k=1}^{\infty} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right)$

H-16: You should know a power series representation for $\frac{1}{1-x}$. Use it.

H-17: You can safely ignore one of the given equations, but not the other.

H-18: $n \geq \log n$ for all $n \geq 1$.

H-19: See Example 3.5.19 in the text. For part (b), review §3.3.5 in the text.

H-20: You know the geometric series expansion of $\frac{1}{1-x}$. What (calculus) operation(s) can you apply to that geometric series to convert it into the given series?

H-21: First show that the fact that the series $\sum_{n=0}^{\infty} (1 - b_n)$ converges guarantees that $\lim_{n \rightarrow \infty} b_n = 1$.

H-22: What does a_n look like for large n ?

H-23: Use the second derivative test.

H-24: What function has $\sum_{n=1}^{\infty} nx^{n-1}$ as its power series representation?

H-25: The power series representation in Example 3.5.18 is an alternating series when x is positive.

H-26: The power series representation in Example 3.5.19 is an alternating series when x is nonzero.

Hints for Exercises 3.6. — Jump to [TABLE OF CONTENTS](#).

H-1: Which of the functions are constant, linear, and quadratic?

H-2: You don't have to actually calculate the entire series $T(x)$ to answer the question.

H-3: If you don't have these memorized, it's good to be able to derive them. For instance, $\log(1+x)$ is the antiderivative of $\frac{1}{1+x}$, whose Taylor series can be found by modifying the geometric series $\sum x^n$.

H-4: See Example 3.6.11 in the text.

H-5: The series will bear some resemblance to the Maclaurin series for $\log(1+x)$.

H-6: The terms $f^{(n)}(\pi)$ are going to be similar to the terms $f^{(n)}(0)$ that we used in the Maclaurin series for sine.

H-7: The Taylor series will look similar to a geometric series.

H-8: Your answer will depend on a .

H-9: You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

H-10: You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

H-11: You should know the Maclaurin series for e^x . Use it.

H-12: Review Example 3.5.18 in the text.

H-13: You should know the Maclaurin series for $\sin x$. Use it.

H-14: You should know the Maclaurin series for e^x . Use it.

H-15: You should know the Maclaurin series for $\arctan(x)$. Use it.

H-16: You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

H-17: Set $(-1)^n \frac{x^{2n+1}}{2n+1} = C \frac{(-1)^n}{(2n+1)3^n}$, for some constant C . What are x and C ?

H-18: There is an important Taylor series, one of the series in Theorem 3.6.5 of the text, that looks a lot like the given series.

H-19: There is an important Taylor series, one of the series in Theorem 3.6.5 of the text, that looks a lot like the given series.

H-20: There is an important Taylor series, one of the series in Theorem 3.6.5 of the text, that looks a lot like the given series. Be careful about the limits of summation.

H-21: There is an important Taylor series, one of the series in Theorem 3.6.5 of the text, that looks a lot like the given series.

H-22: Split the series into a sum of two series. There is an important Taylor series, one of the series in Theorem 3.6.5 of the text, that looks a lot like each of the two series.

H-23: Try the ratio test.

H-24: Write it as the sum of two Taylor series.

H-25: Can you think of a way to eliminate the odd terms from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$?

H-26: The series you're adding up are alternating, so it's simple to bound the error using a partial sum.

H-27: The Taylor Series is alternating, so bounding the error in a partial-sum approximation is straightforward.

H-28: The Taylor Series is not alternating, so use Theorem 3.6.1-b in the text to bound the error in a partial-sum approximation.

H-29: The Taylor Series is not alternating, so use Theorem 3.6.1-b in the text to bound the error in a partial-sum approximation.

H-30: Use Theorem 3.6.1-b in the text to bound the error in a partial-sum approximation. This theorem requires you to consider values of c between x and $x = 0$; since x could be anything from -2 to 1 , you should think about values of c between -2 and 1 .

H-31: Use Theorem 3.6.1-b in the text to bound the error in a partial-sum approximation.

To bound the derivative over the appropriate range, remember how to find absolute extrema.

H-32: See Example 3.6.14 in the text

H-33: See Example 3.6.14 in the text

H-34: Set $f(x) = (1 + x + x^2)^{2/x}$, and find $\lim_{x \rightarrow 0} \log(f(x))$.

H-35: Use the substitution $y = \frac{1}{x}$, and compare to Question 34.

H-36: Start by differentiating $\sum_{n=0}^{\infty} x^n$.

H-37: The series bears a resemblance to the Taylor series for arctangent.

H-38: For simplification purposes, note $(1)(3)(5)(7) \cdots (2n-1) = \frac{(2n)!}{2^n n!}$.

H-39: You know the Maclaurin series for $\log(1 + y)$. Use it! Remember that you are asked for a series expansion in powers of $x - 2$. So you want y to be some constant times $x - 2$.

H-40: See Example 3.5.19 in the text. For parts (b) and (c), review §3.3.5 in the text.

H-41: Look at the signs of successive terms in the series.

H-42: The magic word is “series”.

H-43: See Example 3.6.9 in the text. For parts (b) and (c), review §3.3.5 in the text.

H-44: See Example 3.6.9 in the text. For part (b), review the fundamental theorem of calculus in §1.3 of the text. For part (c), review §3.3.5 in the text.

H-45: See Example 3.6.9 in the text. For parts (b) and (c), review §3.3.5 in the text.

H-46: See Example 3.6.9 in the text. For parts (b) and (c), review §3.3.5 in the text.

H-48: Use the Maclaurin series for e^x .

H-49: For part (c), compare two power series term-by-term.

H-50: For Newton’s method, recall we approximate a root of the function $g(x)$ in iterations: given an approximation x_n , our next approximation is $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$.

To gauge your error, note that from approximation to approximation, the first digits stabilize. Keep refining your approximation until the first two digits stop changing.

H-51: First, modify your known Maclaurin series for arctangent into a Maclaurin series for $f(x)$. This series is not hard to repeatedly differentiate, so use it to find a power series for $f^{(10)}(x)$.

H-52: Remember e^x is never negative for any real number x .

H-53: Since $f(x)$ is odd, $f(-x) = -f(x)$ for all x in its domain. Consider the even-indexed terms and odd-indexed terms of the Taylor series.

Part III

ANSWERS TO PROBLEMS

Answers to Exercises 1.1 — Jump to [TABLE OF CONTENTS](#)

A-1: The area is between 1.5 and 2.5 square units.

A-2: The shaded area is between 2.75 and 4.25 square units. (Other estimates are possible, but this is a reasonable estimate, using methods from this chapter.)

A-3: The area under the curve is a number in the interval $\left(\frac{3}{8} \left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right], \frac{3}{8} \left[1 + \frac{1}{\sqrt{2}}\right]\right)$.

A-4: left

A-5: Many answers are possible. One example is $f(x) = \sin x$, $[a, b] = [0, \pi]$, $n = 1$. Another example is $f(x) = \sin x$, $[a, b] = [0, 5\pi]$, $n = 5$.

A-6: Some of the possible answers are given, but more exist.

(a) $\sum_{i=3}^7 i$; $\sum_{i=1}^5 (i+2)$

(b) $\sum_{i=3}^7 2i$; $\sum_{i=1}^5 (2i+4)$

(c) $\sum_{i=3}^7 (2i+1)$; $\sum_{i=1}^5 (2i+5)$

(d) $\sum_{i=1}^8 (2i-1)$; $\sum_{i=0}^7 (2i+1)$

A-7: Some answers are below, but others are possible.

(a) $\sum_{i=1}^4 \frac{1}{3^i}$; $\sum_{i=1}^4 \left(\frac{1}{3}\right)^i$

(b) $\sum_{i=1}^4 \frac{2}{3^i}$; $\sum_{i=1}^4 2 \left(\frac{1}{3}\right)^i$

(c) $\sum_{i=1}^4 (-1)^i \frac{2}{3^i}$; $\sum_{i=1}^4 \frac{2}{(-3)^i}$

(d) $\sum_{i=1}^4 (-1)^{i+1} \frac{2}{3^i}$; $\sum_{i=1}^4 -\frac{2}{(-3)^i}$

A-8:

(a) $\sum_{i=1}^5 \frac{2i-1}{3^i}$

(b) $\sum_{i=1}^5 \frac{1}{3^i + 2}$

(c) $\sum_{i=1}^7 i \cdot 10^{4-i} \quad ; \quad \sum_{i=1}^7 \frac{i}{10^{i-4}}$

A-9:

(a) $\frac{5}{2} \left[1 - \left(\frac{3}{5} \right)^{101} \right]$

(b) $\frac{5}{2} \left(\frac{3}{5} \right)^{50} \left[1 - \left(\frac{3}{5} \right)^{51} \right]$

(c) 270

(d) $\frac{1 - \left(\frac{1}{e} \right)^b}{e - 1} + \frac{e}{4} [b(b+1)]^2$

A-10:

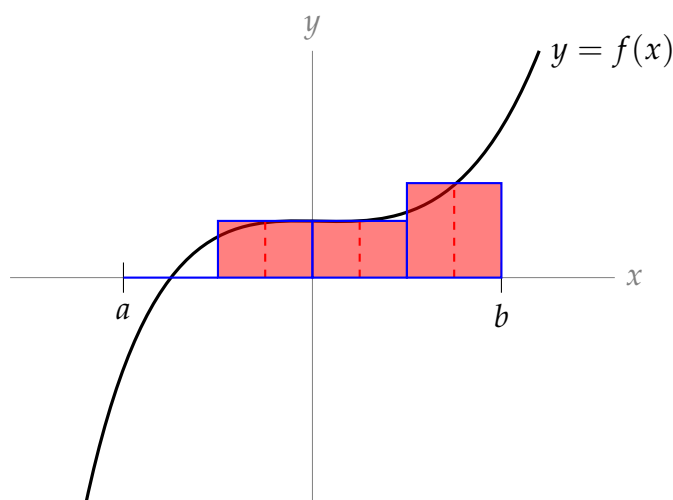
(a) $50 \cdot 51 = 2550$

(b) $\left[\frac{1}{2}(95)(96) \right]^2 - \left[\frac{1}{2}(4)(5) \right]^2 = 20,793,500$

(c) -1

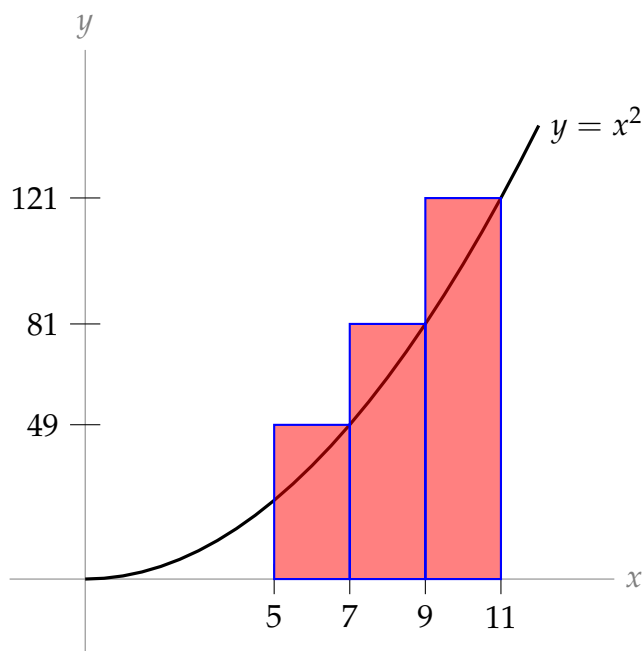
(d) -10

A-11:

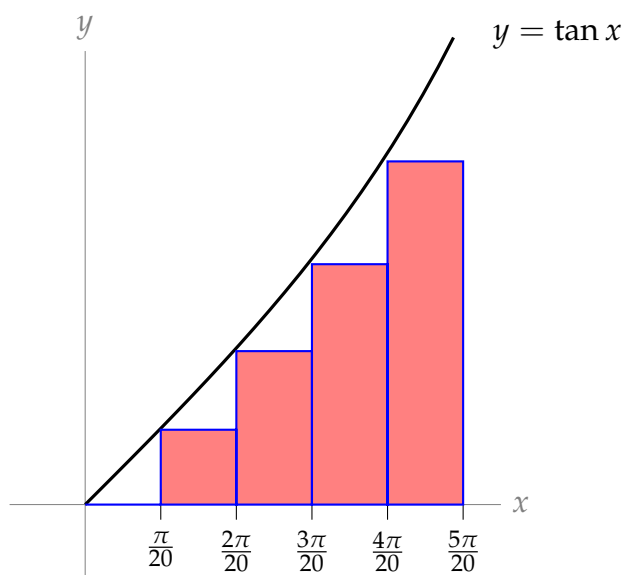


A-12: $n = 4$, $a = 2$, and $b = 6$

A-13: One answer is below, but other interpretations exist.



A-14: Many interpretations are possible—see the solution to Question 13 for a more thorough discussion—but the most obvious is given below.



A-15: Three answers are possible. It is a midpoint Riemann sum for f on the interval $[1, 5]$ with $n = 4$. It is also a left Riemann sum for f on the interval $[1.5, 5.5]$ with $n = 4$. It is also a right Riemann sum for f on the interval $[0.5, 4.5]$ with $n = 4$.

A-16: $\frac{25}{2}$

A-17: $\frac{21}{2}$

A-18: Assuming row i holds x_i , you should have the value $a + \frac{b-a}{n}$ in cell A1, and the value b in cell An.

A-19: $\sum_{i=1}^{50} \left(5 + (i - 1/2)\frac{1}{5}\right)^8 \frac{1}{5}$

A-20: 54

A-21: $\int_{-1}^7 f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(-1 + \frac{8i}{n}\right) \frac{8}{n}$

A-22: $f(x) = \sin^2(2 + x)$ and $b = 4$

A-23: $f(x) = x\sqrt{1 - x^2}$

A-24: $\int_0^3 e^{-x/3} \cos(x) \, dx$

A-25: $\int_0^1 xe^x \, dx$

A-26: Possible answers include:

$$\int_0^2 e^{-1-x} \, dx, \quad \int_1^3 e^{-x} \, dx, \quad 2 \int_{1/2}^{3/2} e^{-2x} \, dx, \quad \text{and} \quad 2 \int_0^1 e^{-1-2x} \, dx.$$

A-27: $\frac{r^{3n+3} - 1}{r^3 - 1}$

A-28: $r^5 \left(\frac{r^{96} - 1}{r - 1} \right)$

A-29: 5

A-30: 16

A-31: $\frac{b^2 - a^2}{2}$

A-32: $\frac{b^2 - a^2}{2}$

A-33: 4π

A-34: $\int_0^3 f(x) \, dx = 2.5$

A-35: 53 m

A-36: true

A-37: 3200 km

A-38: $\int_{-10}^{10} \frac{1}{1+x^2} \, dx \approx 2.942242279$

A-39: $\int_{-\pi}^{\pi} \sin^2 x \, dx \approx 3.141592654$

A-40: (a) There are many possible answers. Two are $\int_{-2}^0 \sqrt{4 - x^2} \, dx$ and $\int_0^2 \sqrt{4 - (-2 + x)^2} \, dx$. (b) π

A-41: (a) 30 (b) $41\frac{1}{4}$

A-42: $\frac{56}{3}$

A-43: 6

A-44: 12

A-45: $f(x) = \frac{3}{10} \left(\frac{x}{5} + 8 \right)^2 \sin \left(\frac{2x}{5} + 2 \right)$

A-46: $\frac{1}{\log 2}$

A-47: (a) $\frac{1}{\log 10} (10^b - 10^a)$

(b) $\frac{1}{\log c} (c^b - c^a)$; yes, it agrees.

A-48: $\frac{\pi}{4} - \frac{1}{2} \arccos(a) + \frac{1}{2} a \sqrt{1 - a^2}$

A-49:

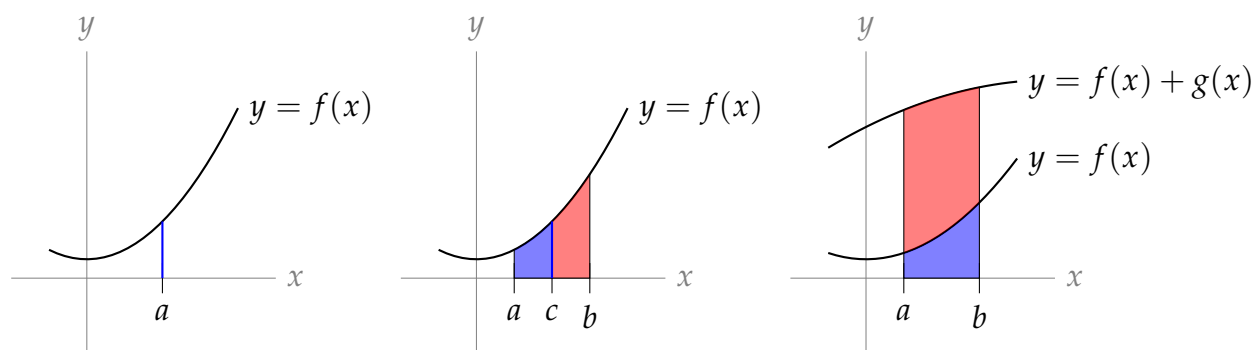
(a) $[f(b) - f(a)] \cdot \frac{b - a}{n}$

(b) Choose n to be an integer that is greater than or equal to $100 [f(b) - f(a)] (b - a)$.

A-50: true (but note, for a non-linear function, it is possible that the midpoint Riemann sum is *not* the average of the other two)

Answers to Exercises 1.2 — Jump to [TABLE OF CONTENTS](#)

A-1: Possible drawings:



A-2: $\sin b - \sin a$

A-3: (a) False. For example, the function

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

provides a counterexample.

(b) False. For example, the function $f(x) = x$ provides a counterexample.

(c) False. For example, the functions

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} \\ 1 & \text{for } x \geq \frac{1}{2} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{for } x \geq \frac{1}{2} \\ 1 & \text{for } x < \frac{1}{2} \end{cases}$$

provide a counterexample.

A-4: (a) $-\frac{1}{20}$ (b) positive (c) negative (d) positive

A-5: -21

A-6: -6

A-7: 20

A-8:

(a) $\frac{\pi}{4} - \frac{1}{2} \arccos(-a) - \frac{1}{2}a\sqrt{1-a^2} = -\frac{\pi}{4} + \frac{1}{2} \arccos(a) - \frac{1}{2}a\sqrt{1-a^2}$

(b) $\frac{1}{2} \arccos(a) - \frac{1}{2}a\sqrt{1-a^2}$

A-9: 5

A-10: 0

A-11: 5

A-12: $20 + 2\pi$

A-13: 0

A-14: 0

A-15: 0

A-16: (a) $y = \frac{1}{b} \sqrt{1 - (ax)^2}$ (b) $\frac{a}{b} \int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{\frac{1}{a^2} - x^2} \, dx$ (c) $\frac{\pi}{ab}$

A-17:

$\times$	even	odd
even	even	odd
odd	odd	even

A-18: $f(0) = 0$; $g(0)$ can be any real number

A-19: $f(x) = 0$ for every x

A-20: The derivative of an even function is odd, and the derivative of an odd function is even.

Answers to Exercises 1.3 — Jump to [TABLE OF CONTENTS](#)

A-1: $e^2 - e^{-2}$

A-2: $F(x) = \frac{x^4}{4} + \frac{1}{2} \cos 2x + \frac{1}{2}.$

A-3: (a) True (b) False (c) False, unless $\int_a^b f(x) dx = \int_a^b xf(x) dx = 0.$

A-4: false

A-5: false

A-6: $\sin(x^2)$

A-7: $\sqrt[3]{e}$

A-8: For any constant C , $F(x) + C$ is an antiderivative of $f(x)$. So, for example, $F(x)$ and $F(x) + 1$ are both antiderivatives of $f(x)$.

A-9:

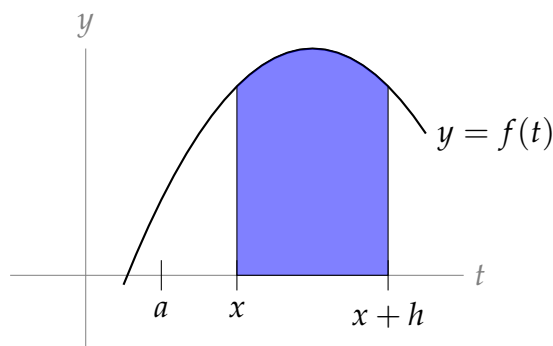
(a) We differentiate with respect to a . Recall $\frac{d}{dx}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$. To differentiate $\frac{1}{2}a\sqrt{1-a^2}$, we use the product and chain rules.

$$\begin{aligned} \frac{d}{da} \left\{ \frac{\pi}{4} - \frac{1}{2} \arccos(a) + \frac{1}{2} a \sqrt{1-a^2} \right\} &= 0 - \frac{1}{2} \cdot \frac{-1}{\sqrt{1-a^2}} + \left(\frac{1}{2} a \right) \cdot \frac{-2a}{2\sqrt{1-a^2}} + \frac{1}{2} \sqrt{1-a^2} \\ &= \frac{1}{2\sqrt{1-a^2}} - \frac{a^2}{2\sqrt{1-a^2}} + \frac{1-a^2}{2\sqrt{1-a^2}} \\ &= \frac{1-a^2+1-a^2}{2\sqrt{1-a^2}} \\ &= \frac{2(1-a^2)}{2\sqrt{1-a^2}} \\ &= \sqrt{1-a^2} \end{aligned}$$

(b) $F(x) = \frac{5\pi}{4} - \frac{1}{2} \arccos(x) + \frac{1}{2} x \sqrt{1-x^2}$

A-10: (a) 0 (b),(c) The FTC does not apply, because the integrand is not continuous over the interval of integration.

A-11:



A-12: (a) zero (b) increasing when $0 < x < 1$ and $3 < x < 4$; decreasing when $1 < x < 3$

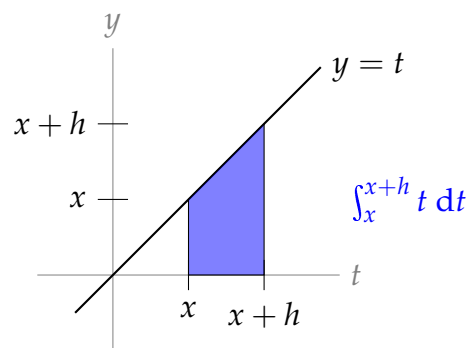
A-13: (a) zero (b) $G(x)$ is increasing when $1 < x < 3$, and it is decreasing when $0 < x < 1$ and when $3 < x < 4$.

A-14: Using the definition of the derivative,

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\int_a^{x+h} t \, dt - \int_a^x t \, dt}{h} \\ &= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} t \, dt}{h} \end{aligned}$$

The numerator describes the area of a trapezoid with base h and heights x and $x+h$.

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\frac{1}{2}h(x+x+h)}{h} \\ &= \lim_{h \rightarrow 0} \left(x + \frac{1}{2}h \right) \\ &= x \end{aligned}$$



So, $F'(x) = x$.

A-15: $f(t) = 0$

A-16: $\int \log(ax) \, dx = x \log(ax) - x + C$, where a is a given constant, and C is any constant.

A-17: $\int x^3 e^x \, dx = e^x (x^3 - 3x^2 + 6x - 6) + C$

A-18: $\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log \left| x + \sqrt{x^2 + a^2} \right| + C$ when a is a given constant. As usual, C is an arbitrary constant.

A-19: $\int \frac{x}{\sqrt{x(a+x)}} \, dx = \sqrt{x(a+x)} - a \log(\sqrt{x} + \sqrt{a+x}) + C$

A-20: $5 - \cos 2$

A-21: 2

A-22: $\frac{1}{5} \arctan(5x) + C$

A-23: $\arcsin\left(\frac{x}{\sqrt{2}}\right) + C$

A-24: $\tan x - x + C$

A-25: $-\frac{3}{4}\cos(2x) + C$, or equivalently, $\frac{3}{2}\sin^2 x + C$

A-26: $\frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

A-27: $F'\left(\frac{\pi}{2}\right) = \log(3) \quad G'\left(\frac{\pi}{2}\right) = -\log(3)$

A-28: $f(x)$ is increasing when $-\infty < x < 1$ and when $2 < x < \infty$.

A-29: $F'(x) = -\frac{\sin x}{\cos^3 x + 6}$

A-30: $4x^3 e^{(1+x^4)^2}$

A-31: $(\sin^6 x + 8)\cos x$

A-32: $F'(1) = 3e^{-1}$

A-33: $\frac{\sin u}{1 + \cos^3 u}$

A-34: $f(x) = 2x$

A-35: $f(4) = 4\pi$

A-36: (a) $(2x + 1)e^{-x^2}$ (b) $x = -1/2$

A-37: $e^{\sin x} - e^{\sin(x^4 - x^3)}(4x^3 - 3x^2)$

A-38: $-2x \cos(e^{-x^2}) - 5x^4 \cos(e^{x^5})$

A-39: $e^x \sqrt{\sin(e^x)} - \sqrt{\sin(x)}$

A-40: 14

A-41: $\frac{5}{2}$

A-42: 45 m

A-43: $f'(x) = (2 - 2x) \log(1 + e^{2x - x^2})$ and $f(x)$ achieves its absolute maximum at $x = 1$, because $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$.

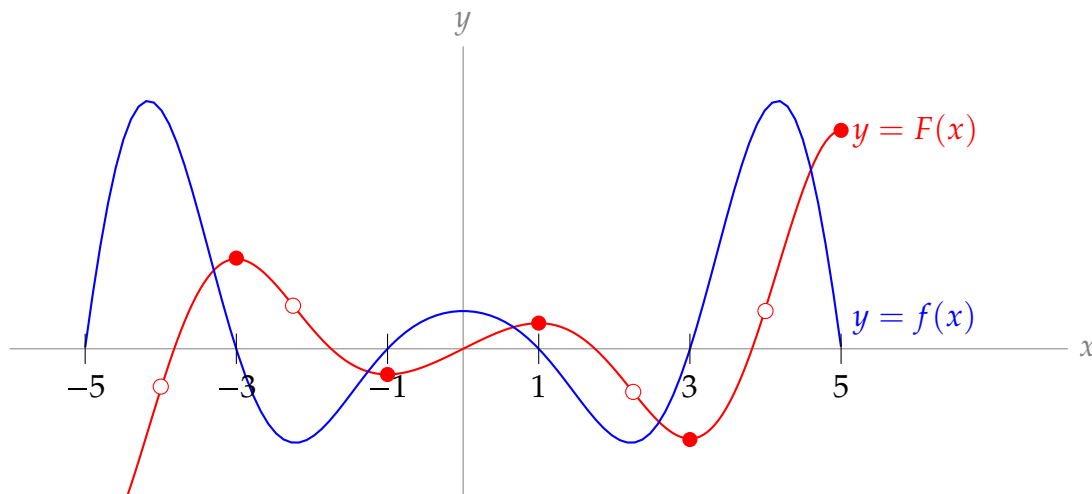
A-44: The minimum is $\int_0^{-1} \frac{dt}{1+t^4}$. As x runs from $-\infty$ to ∞ , the function $f(x) = \int_0^{x^2-2x} \frac{dt}{1+t^4}$ decreases until x reaches 1 and then increases all $x > 1$. So the minimum is achieved for $x = 1$. At $x = 1$, $x^2 - 2x = -1$.

A-45: F achieves its maximum value at $x = \pi$.

A-46: 2

A-47: $\log 2$

A-48: In the sketch below, open dots denote inflection points, and closed dots denote



extrema.

A-49: (a) $3x^2 \int_0^{x^3+1} e^{t^3} dt + 3x^5 e^{(x^3+1)^3}$ (b) $y = -3(x+1)$

A-50: Both students.

A-51: (a) $27(1 - \cos 3)$ (b) $x^3 \sin(x) + 3x^2[1 - \cos(x)]$

A-52: If $f(x) = 0$ for all x , then $F(x)$ is even and possibly also odd.

If $f(x) \neq 0$ for some x , then $F(x)$ is not even. It might be odd, and it might be neither even nor odd.

(Perhaps surprisingly, every antiderivative of an odd function is even.)

Answers to Exercises 1.4 — Jump to [TABLE OF CONTENTS](#)

A-1:

(a) $\int_0^1 s(t) dt$

(b) $\int_0^1 [b(t) + i(t) - d(t) - o(t)] dt$

(c) $L(2) - L(1)$

(d) $W'(1)$

A-2:

(a) The quantity described is the difference in the corn's height (so, how much it grew or shrank) from time 0 to time 1.

(b) The quantity described is the number of calls to the line in a 24-hour period (from midnight to midnight).

(c) The quantity describes is roughly the difference in average value between a home that is 750,000 sq cm (75 sq m) and one that is 1,000,000 sq cm (100 sq m).

A-3:

(a) $-\frac{1}{6}$

(b) $\frac{3}{4}$

A-4: $\frac{3}{50} \log 2$

A-5:

(a) Most quickly to the right at $t = 3$, most quickly to the left at $t = 1$

(b) Position at time t is $t^3 - 3t^2 + t + 2$

A-6: $\sqrt[3]{\frac{12}{25\pi}} \text{ m}$

A-7:

(a) A

(b) B

(c) $t = 0, \frac{10}{3}$

A-8: $100 \arctan 2 - 40$

Answers to Exercises 1.5 — Jump to [TABLE OF CONTENTS](#)

A-1: (a) true (b) false

A-2: The reasoning is not sound: when we do a substitution, we need to take care of the differential (dx). Remember the method of substitution comes from the chain rule: there should be a function *and its derivative*. Here's the way to do it:

Problem: Evaluate $\int (2x + 1)^2 dx$.

Work: We use the substitution $u = 2x + 1$. Then $du = 2dx$, so $dx = \frac{1}{2}du$:

$$\begin{aligned}\int (2x + 1)^2 dx &= \int u^2 \cdot \frac{1}{2} du \\ &= \frac{1}{6} u^3 + C \\ &= \frac{1}{6} (2x + 1)^3 + C\end{aligned}$$

A-3: The problem is with the limits of integration, as in Question 1. Here's how it ought to go:

Problem: Evaluate $\int_1^\pi \frac{\cos(\log t)}{t} dt$.

Work: We use the substitution $u = \log t$, so $du = \frac{1}{t}dt$. When $t = 1$, we have $u = \log 1 = 0$ and when $t = \pi$, we have $u = \log(\pi)$. Then:

$$\begin{aligned}\int_1^\pi \frac{\cos(\log t)}{t} dt &= \int_{\log 1}^{\log(\pi)} \cos(u) du \\ &= \int_0^{\log(\pi)} \cos(u) du \\ &= \sin(\log(\pi)) - \sin(0) = \sin(\log(\pi)).\end{aligned}$$

A-4: This one is OK.

A-5: $\int_0^1 \frac{f(u)}{\sqrt{1-u^2}} du$. Because the denominator $\sqrt{1-u^2}$ vanishes when $u = 1$, this is what is known as an improper integral. Improper integrals will be discussed in § 1.13 of the text.

A-6: some constant C

A-7: $\frac{1}{2}(\sin(e) - \sin(1))$

A-8: $\frac{1}{3}$

A-9: $-\frac{1}{300(x^3 + 1)^{100}} + C$

A-10: $\log 4$

A-11: $\log 2$

A-12: $\frac{4}{3}$

A-13: $e^6 - 1$

A-14: $\frac{1}{3}(4 - x^2)^{3/2} + C$

A-15: $e^{\sqrt{\log x}} + C$

A-16: 0

A-17: $\frac{1}{2}[\cos 1 - \cos 2] \approx 0.478$

A-18: $\frac{1}{2} - \frac{1}{2} \log 2$

A-19: $\frac{1}{2} \tan^2 \theta - \log |\sec \theta| + C$

A-20: $\arctan(e^x) + C$

A-21: $\frac{\pi}{4} - \frac{2}{3}$

A-22: $-\frac{1}{2}(\log(\cos x))^2 + C$

A-23: $\frac{1}{2} \sin(1)$

A-24: $\frac{1}{3}[2\sqrt{2} - 1] \approx 0.609$

A-25: Using the definition of a definite integral with right Riemann sums:

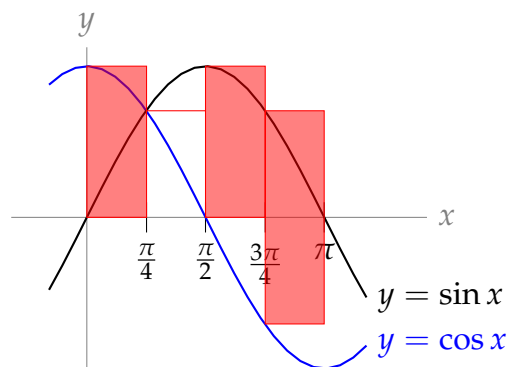
$$\begin{aligned}
 \int_a^b 2f(2x)dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \cdot 2f(2(a + i\Delta x)) & \Delta x &= \frac{b-a}{n} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{b-a}{n}\right) \cdot 2f\left(2\left(a + i\left(\frac{b-a}{n}\right)\right)\right) \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2b-2a}{n}\right) \cdot f\left(2a + i\left(\frac{2b-2a}{n}\right)\right) \\
 \int_{2a}^{2b} f(x)dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \cdot f(2a + i\Delta x) & \Delta x &= \frac{2b-2a}{n} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2b-2a}{n}\right) \cdot f\left(2a + i\left(\frac{2b-2a}{n}\right)\right)
 \end{aligned}$$

Since the Riemann sums are exactly the same,

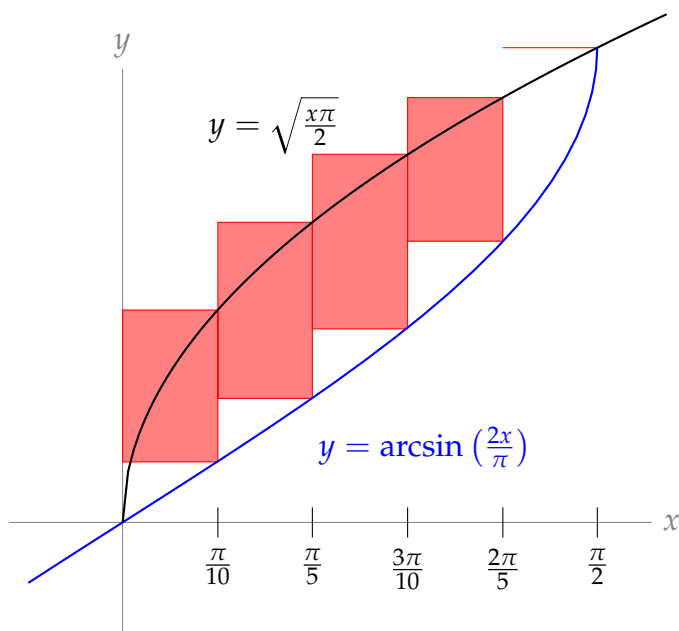
$$\int_a^b 2f(2x)dx = \int_{2a}^{2b} f(x)dx$$

Answers to Exercises 1.6 — Jump to [TABLE OF CONTENTS](#)

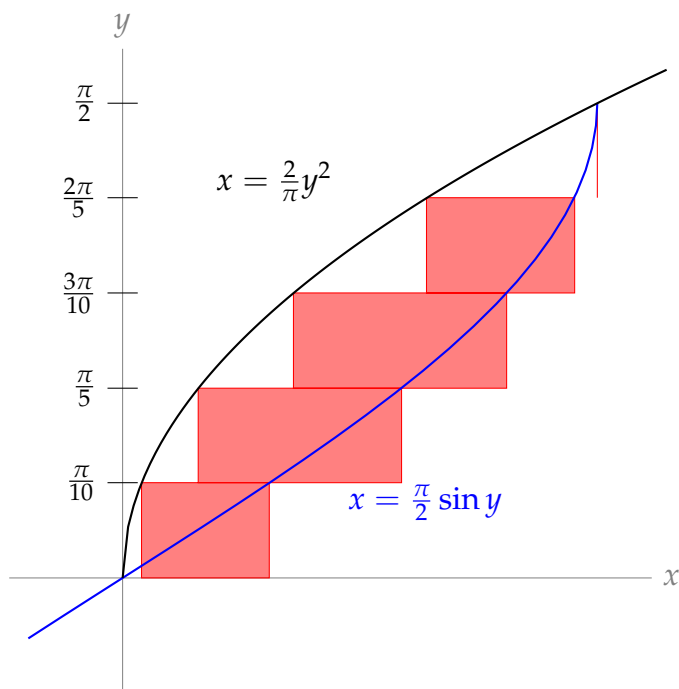
A-1: Area between curves $\approx \frac{\pi}{4} (2 + \sqrt{2})$



A-2: (a) Vertical rectangles:



(b) One possible answer:



A-3: $\int_0^{\sqrt{2}} [2x - x^3] \, dx$

A-4: $\int_{-3/2}^4 \left[\frac{4}{5}(6 - y^2) + 2y \right] \, dy$

A-5: $\int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] \, dx$

A-6: $\int_1^{25} \left[-\frac{1}{12}(x + 5) + \frac{1}{2}\sqrt{x} \right] \, dx$

A-7: $\frac{1}{8}$

A-8: $\frac{4}{3}$

A-9: $\frac{5}{3} - \frac{1}{\log 2}$

A-10: $\frac{8}{\pi} - 1$

A-11: $\frac{20}{9}$

A-12: $\frac{1}{6}$

A-13: 2π

A-14: $2\left[\pi - \frac{1}{4}\pi^2\right]$

A-15: $\frac{31}{6}$

A-16: $\frac{26}{3}$

A-17: $\frac{7\pi}{8} - \frac{1}{2}$

A-18: $12\sqrt{2} - \frac{13}{4}$

Answers to Exercises 1.7 — Jump to [TABLE OF CONTENTS](#)

A-1: (e)

A-2: $\frac{1}{n} \sec^n x + C$

A-3: We divide both sides by $\cos^2 x$, and simplify.

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \frac{\sin^2 x + \cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\ \frac{\sin^2 x}{\cos^2 x} + 1 &= \sec^2 x \\ \tan^2 x + 1 &= \sec^2 x\end{aligned}$$

A-4: $\sin x - \frac{\sin^3 x}{3} + C$

A-5: $\frac{\pi}{2}$

A-6: $\frac{\sin^{37} t}{37} - \frac{\sin^{39} t}{39} + C$

A-7: $\frac{1}{3 \cos^3 x} - \frac{1}{\cos x} + C$

A-8: $\frac{\pi}{8} - \frac{9\sqrt{3}}{64}$

A-9: $-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$

A-10: $\frac{1}{2.2} \sin^{2.2} x + C$

A-11: $\frac{1}{2} \tan^2 x + C$, or equivalently, $\frac{1}{2} \sec^2 x + C$

A-12: $\frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C$

A-13: $\frac{1}{3.5} \sec^{3.5} x - \frac{1}{1.5} \sec^{1.5} x + C$

A-14: $\frac{1}{4} \sec^4 x - \frac{1}{2} \sec^2 x + C$ or $\frac{1}{4} \tan^4 x + C$

A-15: $\frac{1}{5} \tan^5 x + C$

A-16: $\frac{1}{1.3} \sec^{1.3} x + \frac{1}{0.7} \cos^{0.7} x + C$

A-17: $= \frac{1}{4} \sec^4 x - \sec^2 x + \log |\sec x| + C$

A-18: $\frac{1}{2 \cos^2 x} + 2 \log |\cos x| - \frac{1}{2} \cos^2 x + C$

A-19: $2\sqrt{\sec x} + C$

A-20: $\tan \theta + C$

A-21: $\log |\sin x| + C$

A-22: $\frac{1}{2} \sin^2(e^x) + C$

A-23: (a) Using the trig identity $\tan^2 x = \sec^2 x - 1$ and the substitution $y = \tan x$, $dy = \sec^2 x \, dx$,

$$\begin{aligned} \int \tan^n x \, dx &= \int \tan^{n-2} x \tan^2 x \, dx = \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx \\ &= \int y^{n-2} \, dy - \int \tan^{n-2} x \, dx = \frac{y^{n-1}}{n-1} - \int \tan^{n-2} x \, dx \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \end{aligned}$$

(b) $\frac{13}{15} - \frac{\pi}{4} \approx 0.0813$

Answers to Exercises 1.8 — Jump to [TABLE OF CONTENTS](#)

A-1: (a) $x = \frac{4}{3} \sec \theta$ (b) $x = \frac{1}{2} \sin \theta$ (c) $x = 5 \tan \theta$

A-2: (a) $x - 2 = \sqrt{3} \sec u$ (b) $x - 1 = \sqrt{5} \sin u$ (c) $\left(2x + \frac{3}{2}\right) = \frac{\sqrt{31}}{2} \tan u$

(d) $x - \frac{1}{2} = \frac{1}{2} \sec u$

A-3: (a) $\frac{\sqrt{399}}{20}$ (b) $\frac{5\sqrt{2}}{7}$ (c) $\frac{\sqrt{x-5}}{2}$

A-4: (a) $\frac{\sqrt{4-x^2}}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{1-x}}$

A-5: $\frac{1}{4} \cdot \frac{x}{\sqrt{x^2+4}} + C$

A-6: $\frac{1}{2\sqrt{5}}$

A-7: $\frac{\pi}{6}$

A-8: $\log \left| \sqrt{1 + \frac{x^2}{25}} + \frac{x}{5} \right| + C$

A-9: $\frac{1}{2} \sqrt{2x^2 + 4x} + C$

A-10: $-\frac{1}{16} \frac{\sqrt{x^2+16}}{x} + C$

A-11: $\frac{\sqrt{x^2-9}}{9x} + C$

A-12: (a) We'll use the trig identity $\cos 2\theta = 2 \cos^2 \theta - 1$. It implies that

$$\begin{aligned} \cos^2 \theta = \frac{\cos 2\theta + 1}{2} &\implies \cos^4 \theta = \frac{1}{4} [\cos^2 2\theta + 2 \cos 2\theta + 1] = \frac{1}{4} \left[\frac{\cos 4\theta + 1}{2} + 2 \cos 2\theta + 1 \right] \\ &= \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8} \end{aligned}$$

So,

$$\begin{aligned}\int_0^{\pi/4} \cos^4 \theta \, d\theta &= \int_0^{\pi/4} \left(\frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8} \right) d\theta \\ &= \left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3}{8}\theta \right]_0^{\pi/4} \\ &= \frac{1}{4} + \frac{3}{8} \cdot \frac{\pi}{4} \\ &= \frac{8+3\pi}{32}\end{aligned}$$

as required.

(b) $\frac{8+3\pi}{16}$

A-13: 0

A-14: $2 \arcsin \frac{x}{2} + \frac{x}{2} \sqrt{4-x^2} + C$

A-15: $\sqrt{25x^2-4} - 2 \operatorname{arcsec} \frac{5x}{2} + C$

A-16: $\frac{40}{3}$

A-17: $\arcsin \frac{x+1}{2} + C$

A-18: $\frac{1}{4} \left(\arccos \left(\frac{1}{2x-3} \right) + \frac{\sqrt{4x^2-12x+8}}{(2x-3)^2} \right) + C$, or equivalently,

$\frac{1}{4} \left(\operatorname{arcsec} (2x-3) + \frac{\sqrt{4x^2-12x+8}}{(2x-3)^2} \right) + C$

A-19: $\log(1+\sqrt{2}) - \frac{1}{\sqrt{2}}$

A-20: $\frac{1}{2} \left(\arctan x + \frac{x}{x^2+1} \right) + C$

A-21: $\frac{1}{\sqrt{3}} \log \left| \left(\frac{6}{5}x + 1 \right) + \frac{2}{5} \sqrt{9x^2+15x} \right| + C$

A-22: $\frac{8\pi}{3} + 4\sqrt{3}$

A-23: $2\sqrt{1+e^x} + 2 \log |1 - \sqrt{1+e^x}| - x + C$

A-24: (a), (b): None. (c): $x < -a$

Answers to Exercises 1.9 — Jump to [TABLE OF CONTENTS](#)

A-1: (a) (iii) (b) (ii) (c) (ii) (d) (i)

A-2: 3

A-3: (a) $\frac{x^3 + 2x + 2}{x^2 + 1} = x + \frac{x + 2}{x^2 + 1}$

(b) $\frac{15x^4 + 6x^3 + 34x^2 + 4x + 20}{5x^2 + 2x + 8} = 3x^2 + 2 + \frac{4}{5x^2 + 2x + 8}$

(c) $\frac{2x^5 + 9x^3 + 12x^2 + 10x + 30}{2x^2 + 5} = x^3 + 2x + 6$

A-4: (a) $5x^3 - 3x^2 - 10x + 6 = (x + \sqrt{2})(x - \sqrt{2})(5x - 3)$

(b) $x^4 - 3x^2 - 5 = \left(x + \sqrt{\frac{3 + \sqrt{29}}{2}}\right) \left(x - \sqrt{\frac{3 + \sqrt{29}}{2}}\right) \left(x^2 + \frac{\sqrt{29} - 3}{2}\right)$

(c) $x^4 - 4x^3 - 10x^2 - 11x - 6 = (x + 1)(x - 6)(x^2 + x + 1)$

(d) $2x^4 + 12x^3 - x^2 - 52x + 15 = (x + 3)(x + 5) \left(x - \left(1 + \frac{\sqrt{2}}{2}\right)\right) \left(x - \left(1 - \frac{\sqrt{2}}{2}\right)\right)$

A-5: The goal of partial fraction decomposition is to write our integrand in a form that is easy to integrate. The antiderivative of (1) can be easily determined with the substitution $u = (ax + b)$. It's less clear how to find the antiderivative of (2).

A-6: $\log \frac{4}{3}$

A-7: $-\frac{1}{x} - \arctan x + C$

A-8: $4 \log |x - 3| - 2 \log(x^2 + 1) + C$

A-9: $F(x) = \log |x - 2| + \log(x^2 + 4) + 2 \arctan(x/2) + D$

A-10: $-2 \log |x - 3| + 3 \log |x + 2| + C$

A-11: $-9 \log |x + 2| + 14 \log |x + 3| + C$

A-12: $5x + \frac{1}{2} \log |x - 1| - \frac{7}{2} \log |x + 1| + C$

A-13: $x - \frac{2}{x} + \frac{5}{2} \arctan(2x) + C$

A-14: $-\frac{1}{2} \log |x - 2| + \frac{1}{2} \log |x + 2| + \frac{3}{2} \log |2x - 1| + C$

A-15: $\log \left(\frac{4 \cdot 6^3}{5^3} \right)$

A-16: $\frac{1}{2} \log \left| \frac{1 - \cos x}{1 + \cos x} \right| + C$

A-17: $\frac{-\cos x}{2 \sin^2 x} + \frac{1}{4} \log \left| \frac{1 - \cos x}{1 + \cos x} \right| + C$

A-18: $3\log 2 + \frac{1}{2} + \frac{2}{\sqrt{15}} \left(\arctan \left(\frac{7}{\sqrt{15}} \right) - \arctan \left(\frac{9}{\sqrt{15}} \right) \right)$

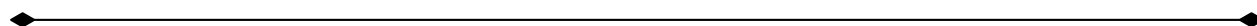
A-19: $\log \left| \frac{\sin \theta - 1}{\sin \theta - 2} \right| + C$

A-20: $t - \frac{1}{2} \log |e^{2t} + e^t + 1| - \frac{1}{\sqrt{3}} \arctan \left(\frac{2e^t + 1}{\sqrt{3}} \right) + C$

A-21: $2\sqrt{1+e^x} + \log \left| \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} \right| + C$

A-22: $2\log \frac{5}{3} + \frac{4}{\sqrt{3}} \arctan \frac{1}{4\sqrt{3}}$

A-23: (a) $\frac{1}{6} \left(\log \left| 2 \cdot \frac{x-3}{x+3} \right| \right)$ (b) $F'(x) = \frac{1}{x^2-9}$



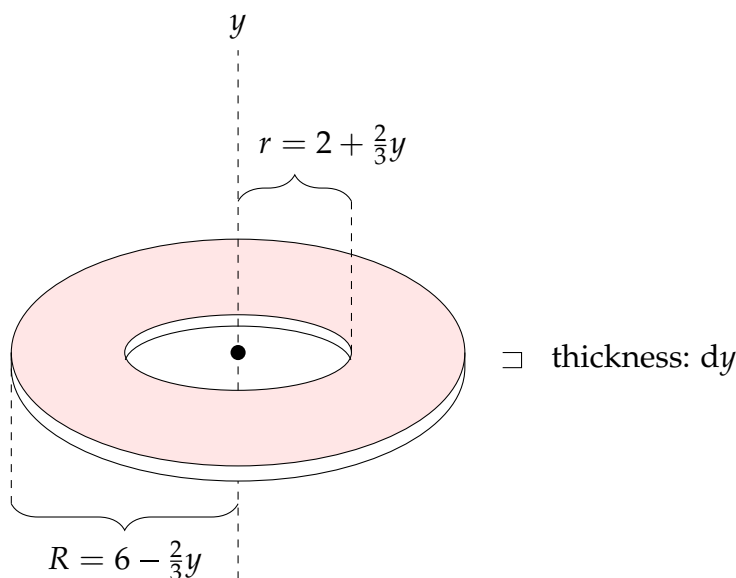
Answers to Exercises 1.10 — Jump to [TABLE OF CONTENTS](#)

A-1: The horizontal cross-sections are circles, but the vertical cross-sections are not.

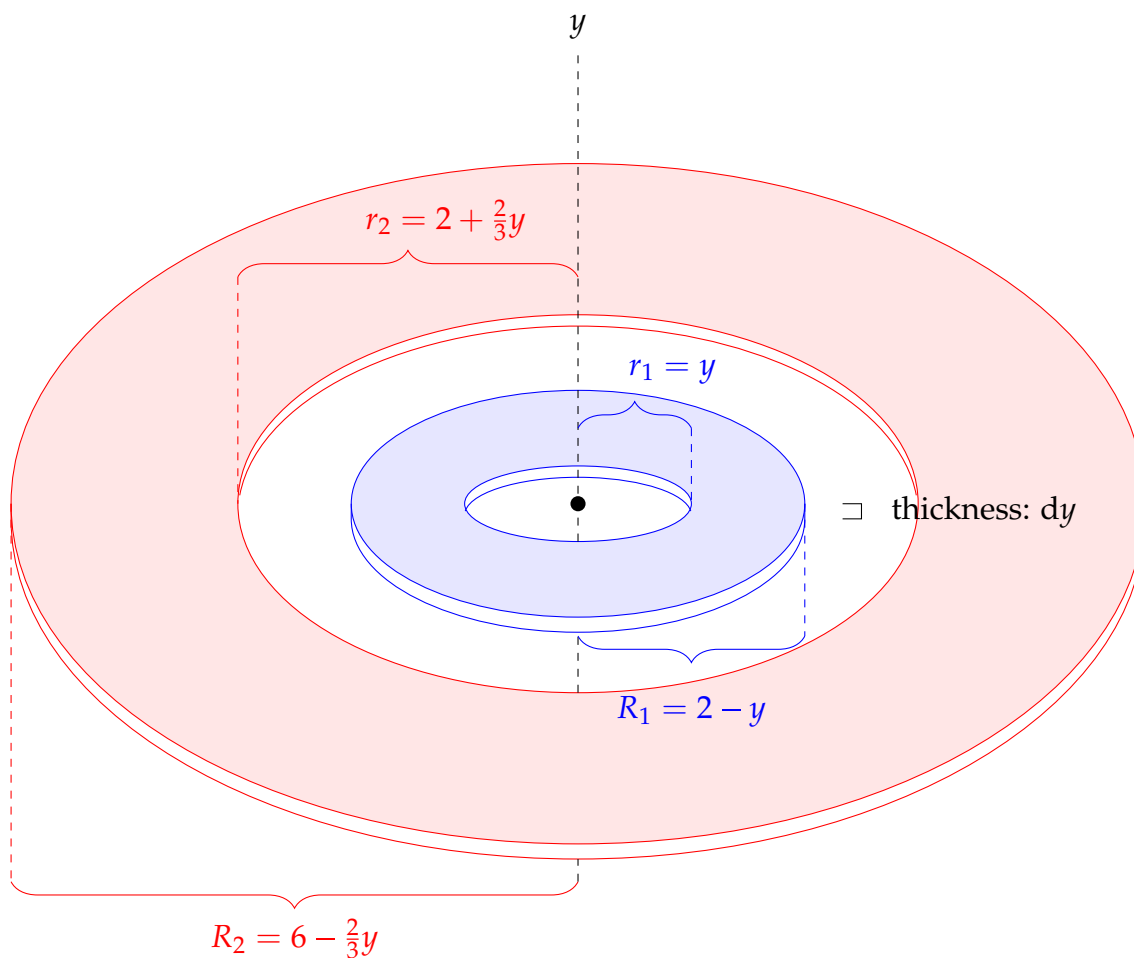
A-2: The columns have the same volume.

A-3:

Washers when $1 < y \leq 6$: If $y > 1$, then our washer has inner radius $2 + \frac{2}{3}y$, outer radius $6 - \frac{2}{3}y$, and height dy .



Washers when $0 \leq y < 1$: When $0 \leq y < 1$, we have a “double washer,” two concentric rings. The inner washer has inner radius $r_1 = y$ and outer radius $R_1 = 2 - y$. The outer washer has inner radius $r_2 = 2 + \frac{2}{3}y$ and outer radius $R_2 = 6 - \frac{2}{3}y$. The thickness of the washers is dy .



A-4: (a) $\pi \int_0^3 x e^{2x^2} dx$

(b) $\int_0^1 \pi [(3 + \sqrt{y})^2 - (3 - \sqrt{y})^2] dy + \int_1^4 \pi [(5 - y)^2 - (3 - \sqrt{y})^2] dy$

A-5: (a) $\int_{-1}^1 \pi [(5 - 4x^2)^2 - (2 - x^2)^2] dx$ (b)

$\int_{-1}^0 \pi [(5 + \sqrt{y+1})^2 - (5 - \sqrt{y+1})^2] dy$

A-6: $\pi \int_{-2}^2 [(9 - x^2)^2 - (x^2 + 1)^2] dx$

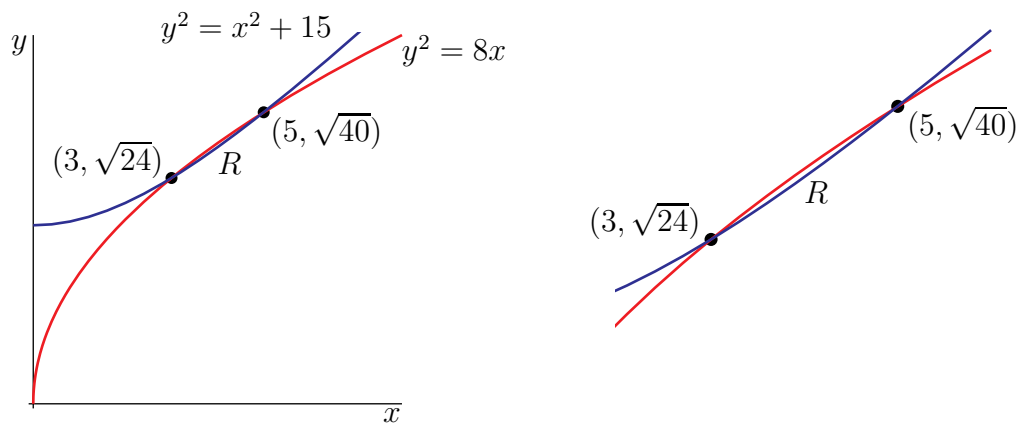
A-7: $\frac{\sqrt{2}}{12} \ell^3$

A-8: $\frac{\pi}{4} (e^{2a^2} - 1)$

A-9: $\pi \left[\frac{38}{3} - \frac{514}{3^4} \right] = \pi \frac{512}{81}$

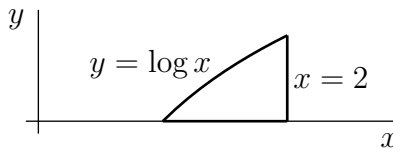
A-10: (a) $8\pi \int_{-1}^1 \sqrt{1 - x^2} dx$ (b) $4\pi^2$

A-11: (a) The region R is the region between the blue and red curves, with $3 \leq x \leq 5$, in the figures below.



(b) $\frac{4}{3}\pi \approx 4.19$

A-12: (a) The region R is sketched below.



(b) $\pi \left[4 \log 2 - \frac{3}{2} \right] \approx 3.998$

A-13: $\pi^2 + 8\pi^3 + \frac{8\pi^6}{5}$

A-14: $\frac{8}{3}$

A-15: $\frac{256 \times 8}{15} = 136.5\dot{3}$

A-16: $\frac{28}{3}\pi h$

A-17: (a) $\frac{4\pi}{3b^2a}$ cubic units (b) $a = \frac{1}{6356.752}$ and $b = \frac{1}{6378.137}$

(c) Approximately $1.08321 \times 10^{12} \text{ km}^3$, or $1.08321 \times 10^{21} \text{ m}^3$

(d) Absolute error is about $3.64 \times 10^9 \text{ km}^3$, and relative error is about 0.00336, or 0.336%.

A-18: (a) $\frac{9}{2}$ (b) $\pi \int_{-1}^2 [(4-x)^2 - (1+(x-1)^2)^2] dx$

A-19: (a) $\frac{\pi}{2} - 1$ (b) $\frac{\pi^2}{2} - \pi \approx 1.793$

A-20: (a) $V_1 = \frac{4}{3}\pi c^2$ (b) $V_2 = \frac{\pi c}{3} [4\sqrt{2} - 2]$ (c) $c = 0$ or $c = \sqrt{2} - \frac{1}{2}$

A-21:

$$\int_{\pi/2}^{\pi} \pi[(5 + \pi \sin x)^2 - (5 + 2\pi - 2x)^2] dx + \int_{\pi}^{3\pi/2} \pi[(5 + 2\pi - 2x)^2 - (5 + \pi \sin x)^2] dx$$

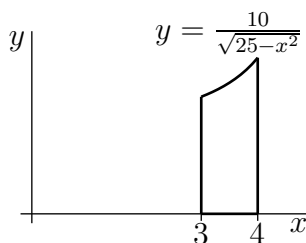
A-22: (a) $\frac{6000c\pi}{\log 2} \left(1 - \frac{1}{2^{10}}\right)$, which is close to $\frac{6000c\pi}{\log 2}$.

(b) 6km: that is, there is roughly the same mass of air in the lowest 6 km of the column as there is in the remaining 54 km.

A-23: Area: $\frac{4}{3} - \sqrt[4]{\frac{4}{3}}$ Volume: $\frac{\pi^2}{6} - \frac{\sqrt{3}\pi}{4}$

A-24: (a)

The region R is



(b) $10\pi \log \frac{9}{4} = 20\pi \log \frac{3}{2}$ (c) 20π

Answers to Exercises 1.11 — Jump to [TABLE OF CONTENTS](#)

A-1: chain; product

A-2: The part chosen as u will be differentiated. The part chosen as dv will be antiderivated.

A-3: $\int \frac{f'(x)}{g(x)} dx = \frac{f(x)}{g(x)} + \int \frac{f(x)g'(x)}{g^2(x)} dx$

A-4: All the antiderivatives differ only by a constant, so we can write them all as $v(x) + C$ for some C . Then, using the formula for integration by parts,

$$\begin{aligned} \int u(x) \cdot v'(x) dx &= \underbrace{u(x)}_u \underbrace{[v(x) + C]}_v - \int \underbrace{[v(x) + C]}_v \underbrace{u'(x) dx}_{du} \\ &= u(x)v(x) + Cu(x) - \int v(x)u'(x) dx - \int Cu'(x) dx \\ &= u(x)v(x) + Cu(x) - \int v(x)u'(x) dx - Cu(x) + D \\ &= u(x)v(x) - \int v(x)u'(x) dx + D \end{aligned}$$

where D is any constant.

Since the terms with C cancel out, it didn't matter what we chose for C —all choices end up the same.

A-5: Suppose we choose $dv = f(x)dx$, $u = 1$. Then $v = \int f(x)dx$, and $du = dx$. So, our integral becomes:

$$\int \underbrace{(1)}_u \underbrace{f(x)dx}_{dv} = \underbrace{(1)}_u \underbrace{\int f(x)dx}_v - \int \underbrace{\left(\int f(x)dx\right)}_v \underbrace{dx}_{du}$$

In order to figure out the first product (and the second integrand), you need to know the antiderivative of $f(x)$ —but that’s exactly what you’re trying to figure out!

A-6: $\frac{x^2 \log x}{2} - \frac{x^2}{4} + C$

A-7: $-\frac{\log x}{6x^6} - \frac{1}{36x^6} + C$

A-8: π

A-9: $\frac{\pi}{2} - 1$

A-10: $e^x (x^3 - 3x^2 + 6x - 6) + C$

A-11: $\frac{x^2}{2} \log^3 x - \frac{3x^2}{4} \log^2 x + \frac{3x^2}{4} \log x - \frac{3x^2}{8} + C$

A-12: $(2 - x^2) \cos x + 2x \sin x + C$

A-13: $(t^3 - \frac{5}{2}t^2 + 6t) \log t - \frac{1}{3}t^3 + \frac{5}{4}t^2 - 6t + C$

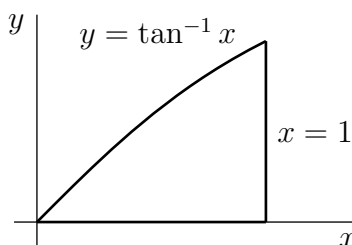
A-14: $e^{\sqrt{s}} (2s - 4\sqrt{s} + 4) + C$

A-15: $x \log^2 x - 2x \log x + 2x + C$

A-16: $e^{x^2+1} + C$

A-17: $y \arccos y - \sqrt{1 - y^2} + C$

A-18: (a) Area: $\frac{\pi}{4} - \frac{\log 2}{2}$



(b) Volume: $\frac{\pi^2}{2} - \pi$

A-19: $\pi \left(\frac{17e^{18} - 4373}{36} \right)$

A-20: $2y^2 \arctan(2y) - y + \frac{1}{2} \arctan(2y) + C$

A-21: $\frac{x^3}{3} \arctan x - \frac{1}{6}(1+x^2) + \frac{1}{6} \log(1+x^2) + C$

A-22: $\frac{2^x}{\log 2} \left(x - \frac{1}{\log 2} \right) + C$

A-23: $2e^{\cos x} [1 - \cos x] + C$

A-24: $\int \frac{xe^{-x}}{(1-x)^2} dx = \frac{xe^{-x}}{1-x} + e^{-x} + C = \frac{e^{-x}}{1-x} + C$

A-25: (a) We integrate by parts with $u = \sin^{n-1} x$ and $dv = \sin x dx$, so that $du = (n-1) \sin^{n-2} x \cos x$ and $v = -\cos x$.

$$\int \sin^n x dx = \underbrace{-\sin^{n-1} x \cos x}_{uv} + \underbrace{(n-1) \int \cos^2 x \sin^{n-2} x dx}_{-\int v du}$$

Using the identity $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned} &= -\sin^{n-1} x \cos x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx \end{aligned}$$

Moving the last term on the right hand side to the left hand side gives

$$n \int \sin^n x dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx$$

Dividing across by n gives the desired reduction formula.

(b) $\frac{35}{256} \pi \approx 0.4295$

A-26: 12

A-27: $\frac{2}{e}$

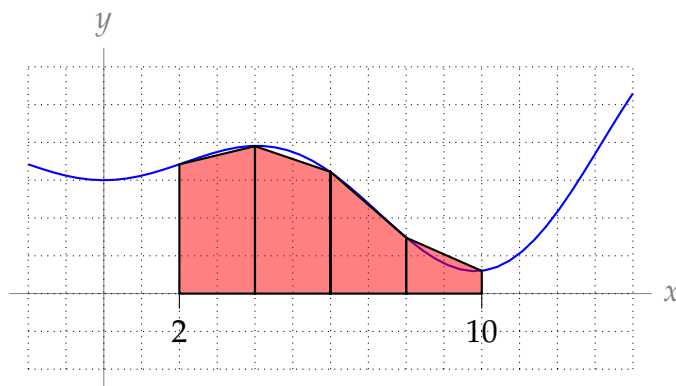
A-28: $\frac{x}{2} \sin^2 x - \frac{x}{4} + \frac{1}{4} \sin x \cos x + C$

A-29: $(\sin^2 x + 2) \cos(\cos x) + 2 \cos x \sin(\cos x) + C$

Answers to Exercises 1.12 — Jump to [TABLE OF CONTENTS](#)

A-1: Relative error: ≈ 0.08147 ; absolute error: 0.113; percent error: $\approx 8.147\%$.

A-2:



A-3: $M = 6.25, L = 2$

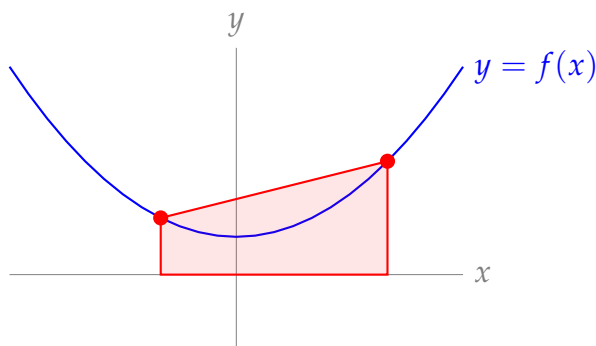
A-4: One reasonable answer is $M = 3$.

A-5: (a) $\frac{\pi^5}{180 \cdot 8}$ (b) 0 (c) 0

A-6: Possible answers: $f(x) = \frac{3}{2}x^2 + Cx + D$ for any constants C, D .

A-7: my mother

A-8: True. Because $f(x)$ is positive and concave up, the graph of $f(x)$ is always below the top edges of the trapezoids used in the trapezoidal rule.



A-9: Any polynomial of degree at most 3 will do. For example, $f(x) = 5x^3 - 27$, or $f(x) = x^2$.

A-10:

(a) $=B2 \star C2$

(b) $C2$ and $C(n+2)$

(c) $=3 + (-1) \wedge (\text{ROW}() + 1)$

A-11: $1720\pi \approx 5403.5 \text{ cm}^3$

A-12: $\frac{\pi}{12}(16.72) \approx 4.377 \text{ m}^3$

A-13: $\frac{12.94}{6\pi} \approx 0.6865 \text{ m}^3$

A-14: (a) 363,500 (b) 367,000

A-15: (a) $\frac{49}{2}$ (b) $\frac{77}{3}$

A-16: $\frac{8}{15}$

A-17: $\frac{1}{180 \times 3^4} = \frac{1}{14580}$

A-18: (a) $T_4 = \frac{1}{4} \left[\left(\frac{1}{2} \times 1 \right) + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \left(\frac{1}{2} \times \frac{1}{2} \right) \right],$

(b) $S_4 = \frac{1}{12} \left[1 + \left(4 \times \frac{4}{5} \right) + \left(2 \times \frac{2}{3} \right) + \left(4 \times \frac{4}{7} \right) + \frac{1}{2} \right]$

(c) $|I - S_4| \leq \frac{24}{180 \times 4^4} = \frac{1}{1920}$

A-19: (a) $T_4 = 8.03515$, $S_4 \approx 8.03509$

(b) $\left| \int_a^b f(x) \, dx - T_n \right| \leq \frac{2}{1000} \frac{8^3}{12(4)^2} \leq 0.00533,$

$\left| \int_a^b f(x) \, dx - S_n \right| \leq \frac{4}{1000} \frac{8^5}{180(4)^4} \leq 0.00284$

A-20: Any $n \geq 68$ works.

A-21: $\int_{\pi/2}^{2\pi/3} \sin x \, dx \approx 0.499293941$

A-22: $\int_{\frac{1}{2}}^1 \frac{1}{x} \, dx \approx 0.69315343$

A-23: $\int_0^1 e^{x^2} \, dx \approx 1.462651746$

A-24: $\frac{472}{3} \approx 494 \text{ ft}^3$

A-25: (a) 0.025635 (b) 1.8×10^{-5}

A-26: (a) ≈ 0.6931698 (b) $n \geq 12$ with n even

A-27: (a) 0.01345 (b) $n \geq 28$ with n even

A-28: $n \geq 259$

A-29: $\frac{x-1}{12} \left[1 + \frac{16}{x+3} + \frac{4}{x+1} + \frac{16}{3x+1} + \frac{1}{x} \right]$

A-30: *Note: for more detail, see the solutions.*

First, we use Simpson's rule with $n = 4$ to approximate $\int_1^2 \frac{1}{1+x^2} \, dx$. The choice of this method (what we're approximating, why $n = 4$, etc.) is explained in the solutions—here, we only show that it works.

$$\int_1^2 \frac{1}{1+x^2} \, dx \approx \frac{1}{12} \left[\frac{1}{2} + \frac{64}{41} + \frac{8}{13} + \frac{64}{65} + \frac{1}{5} \right] \approx 0.321748$$

For ease of notation, define $A = 0.321748$.

Now, we bound the error associated with this approximation. Define $N(x) = 24(5x^4 - 10x^2 + 1)$ and $D(x) = (x^2 + 1)^5$, so $N(x)/D(x)$ gives the fourth derivative of $\frac{1}{1+x^2}$. When $1 \leq x \leq 2$, $|N(x)| \leq N(2) = 984$ (because $N(x)$ is increasing over that interval) and $|D(x)| \geq D(1) = 2^5$ (because $D(x)$ is also increasing over that interval), so $\left| \frac{d^4}{dx^4} \left\{ \frac{1}{1+x^2} \right\} \right| = \left| \frac{N(x)}{D(x)} \right| \leq \frac{984}{2^5} = 30.75$. Now we find the error bound for Simpson's rule with $L = 30.75$, $b = 2$, $a = 1$, and $n = 4$.

$$\left| \int_1^2 \frac{1}{1+x^2} dx - A \right| = |\text{error}| \leq \frac{L(b-a)^5}{180 \cdot n^4} = \frac{30.75}{180 \cdot 4^4} < 0.00067$$

So,

$$\begin{array}{rclcl} -0.00067 & < & \int_1^2 \frac{1}{1+x^2} dx - A & < & 0.00067 \\ A - 0.00067 & < & \int_1^2 \frac{1}{1+x^2} dx & < & A + 0.00067 \\ A - 0.00067 & < & \arctan(2) - \arctan(1) & < & A + 0.00067 \\ A - 0.00067 & < & \arctan(2) - \frac{\pi}{4} & < & A + 0.00067 \\ \frac{\pi}{4} + A - 0.00067 & < & \arctan(2) & < & \frac{\pi}{4} + A + 0.00067 \\ \frac{\pi}{4} + 0.321748 - 0.00067 & < & \arctan(2) & < & \frac{\pi}{4} + 0.321748 + 0.00067 \\ \frac{\pi}{4} + 0.321078 & < & \arctan(2) & < & \frac{\pi}{4} + 0.322418 \\ \frac{\pi}{4} + 0.321 & < & \arctan(2) & < & \frac{\pi}{4} + 0.323 \end{array}$$

This was the desired bound.

Answers to Exercises 1.13 — Jump to [TABLE OF CONTENTS](#)

A-1: Any real number in $[1, \infty)$ or $(-\infty, -1]$, and $b = \pm\infty$.

A-2: $b = \pm\infty$

A-3: The red function is $f(x)$, and the blue function is $g(x)$.

A-4: False. For example, the functions $f(x) = e^{-x}$ and $g(x) = 1$ provide a counterexample.

A-5:

(a) Not enough information to decide. For example, consider $h(x) = 0$ versus $h(x) = -1$.

(b) Not enough information to decide. For example, consider $h(x) = f(x)$ versus $h(x) = g(x)$.

(c) $\int_0^\infty h(x) dx$ converges by the comparison test, since $|h(x)| \leq 2f(x)$ and $\int_0^\infty 2f(x) dx$ converges.

-
- A-6: The integral diverges.
A-7: The integral diverges.
A-8: The integral does not converge.
A-9: The integral converges.
A-10: The integral diverges.
A-11: The integral diverges.
A-12: The integral diverges.
A-13: The integral diverges.
A-14: The integral diverges.
A-15: The integral converges.
A-16: The integral converges.
A-17: false
A-18: $q = \frac{1}{5}$
A-19: $p > 1$
A-20: $\frac{\log 3 - \pi}{4} + \frac{1}{2} \arctan 2$
A-21: The integral converges.
A-22: The integral converges.
A-23: The integral converges.
A-24: (a) The integral converges. (b) The interval converges.
A-25: false
-

Answers to Exercises 1.14 — Jump to [TABLE OF CONTENTS](#)

- A-1: (A)–(I), (B)–(IV), (C)–(II), (D)–(III)
A-2: $\frac{1}{5} - \frac{2}{7} + \frac{1}{9} = \frac{8}{315}$
A-3: $\frac{3}{2\sqrt{5}} \arcsin \left(x\sqrt{\frac{5}{3}} \right) + \frac{x}{2} \sqrt{3 - 5x^2} + C$
A-4: 0
A-5: $\log \left| \frac{x+1}{3x+1} \right| + C$
A-6: $\frac{8}{3} \log 2 - \frac{7}{9}$
-

A-7: $\frac{1}{2} \log |x^2 - 3| + C$

A-8: (a) 2 (b) $\frac{2}{15}$ (c) $\frac{3e^4}{16} + \frac{1}{16}$

A-9: (a) 1 (b) $\frac{8}{15}$

A-10: (a) $e^2 + 1$ (b) $\log(\sqrt{2} + 1)$ (c) $\log \frac{15}{13} \approx 0.1431$

A-11: (a) $\frac{9}{4}\pi$ (b) $\log 2 - 2 + \frac{\pi}{2} \approx 0.264$ (b) $2 \log 2 - \frac{1}{2} \approx 0.886$

A-12: $\frac{1}{3} \sin^3 \theta - 2 \sin \theta + 12 \log \left| \frac{\sin \theta - 3}{\sin \theta - 2} \right| + C$

A-13: (a) $\frac{1}{15}$ (b) $\frac{1}{9} \cdot \frac{x}{\sqrt{x^2 + 9}} + C$ (c) $\frac{1}{2} \log |x - 1| - \frac{1}{4} \log(x^2 + 1) - \frac{1}{2} \arctan x + C$

(d) $\frac{1}{2} [x^2 \arctan x - x + \arctan x] + C$

A-14: (a) $\frac{1}{12}$ (b) $2 \sin^{-1} \frac{x}{2} + x \sqrt{1 - \frac{x^2}{4}} + C$

(c) $-2 \log |x| + \frac{1}{x} + 2 \log |x - 1| + C$

A-15: (a) $\frac{1}{2\sqrt{2}}$ (b) $\log 2 - \frac{1}{2} \approx 0.193$ (c) $\log 2 - \frac{1}{2} \approx 0.193$

A-16: (a) $\frac{1}{2} x^2 \log x - \frac{1}{4} x^2 + C$ (b) $\frac{1}{2} \log[x^2 + 4x + 5] - 3 \arctan(x + 2) + C$

(c) $\frac{1}{2} \log |x - 3| - \frac{1}{2} \log |x - 1| + C$ (d) $\frac{1}{3} \arctan x^3 + C$

A-17: (a) $\frac{\pi}{4} - \frac{1}{2} \log 2$ (b) $\log |x^2 - 2x + 5| + \frac{1}{2} \arctan \frac{x - 1}{2} + C$

A-18: (a) $-\frac{1}{300(x^3 + 1)^{100}} + C$ (b) $\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

A-19: -2

A-20: (a) $-\frac{1}{4} \log |e^x + 1| + \frac{1}{4} \log |e^x - 3| + C$ (b) $\frac{4\pi}{3} - 2\sqrt{3}$

A-21: (a) $\frac{1}{2} \sec^2 x + \log |\cos x| + C$ (b) $\frac{1}{10} \arctan 8 \approx 0.1446$

A-22: $\frac{2}{5} (x - 1)^{5/2} + \frac{2}{3} (x - 1)^{3/2} + C$

A-23: $3 \log |x + 1| + \frac{2}{x + 1} - \frac{5}{2(x + 1)^2} + C$

A-24: $\frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \right) + C$

A-25: $\frac{1}{2} (x - \sin x \cos x) + C$

A-26: $\frac{1}{3} \log |x + 1| - \frac{1}{6} \log |x^2 + x + 1| + \frac{1}{\sqrt{3}} \arctan \left(\frac{2x - 1}{\sqrt{3}} \right) + C$

A-27: $3x^3 \arcsin x + 3\sqrt{1 - x^2} - (1 - x^2)^{3/2} + C$

A-28: 2

A-29: $\frac{1}{4}$

A-30: $\log \left(\frac{\log(\cos(0.1))}{\log(\cos(0.2))} \right)$

A-31: $2 \log 2 - \log 3 = \log \frac{4}{3}$

A-32: (a) $\frac{9}{4}\pi + 9$ (b) $2 \log |x - 2| - \log(x^2 + 4) + C$ (c) $\frac{\pi}{2}$

A-33: $-\arcsin(\sqrt{1 - x}) - \sqrt{1 - x}\sqrt{x} + C$

A-34: $e^e(e - 1)$

A-35: $\frac{e^x}{x + 1} + C$

A-36: $x \sec x - \log |\sec x + \tan x| + C$

A-37: $\int x(x + a)^n dx = \begin{cases} \frac{(x+a)^{n+2}}{n+2} - a \frac{(x+a)^{n+1}}{n+1} + C & \text{if } n \neq -1, -2 \\ (x + a) - a \log |x + a| + C & \text{if } n = -1 \\ \log |x + a| + \frac{a}{x+a} + C & \text{if } n = -2 \end{cases}$

Answers to Exercises 2.1 — Jump to [TABLE OF CONTENTS](#)

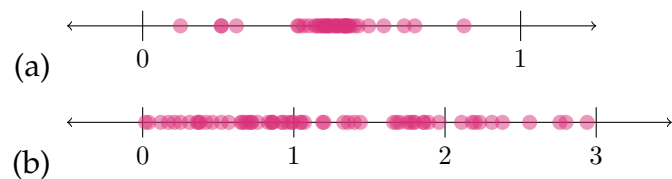
A-1: $Pr(X = 5) = 0.1$

A-2: $\frac{1}{100}$

A-3: $\frac{15}{93}$

Answers to Exercises 2.2 — Jump to [TABLE OF CONTENTS](#)

A-1:





(c)

A-2: $\frac{\arctan(170) - \arctan(40)}{\pi} \approx 0.006$

A-3: 0.6

A-4: $\frac{2}{10}$

A-5: $\ln\left(1 + \frac{1}{e}\right)$

A-6: $\frac{2}{\pi}$

A-7: $\frac{1}{12} \left[\sqrt{\frac{2}{\pi}} + 4 \left(\sqrt{\frac{2}{\pi}} e^{-1/8} \right) + 2 \left(\sqrt{\frac{2}{\pi}} e^{-1/2} \right) + 4 \left(\sqrt{\frac{2}{\pi}} e^{-9/8} \right) + \left(\sqrt{\frac{2}{\pi}} e^{-2} \right) \right]$

Answers to Exercises 2.3 — Jump to [TABLE OF CONTENTS](#)

A-1: false

A-2: false

A-3: $\mathbb{E}(M) = 66 + \frac{2}{3}$. From Section 2.3.3, we see $50 < \mathbb{E}(M) \leq 100$, and this is true of $66 + \frac{2}{3}$, so it passes the checks.

A-4: $\mathbb{E}(N) = 0$. From Section 2.3.3, we see $-1 \leq \mathbb{E}(M) \leq 1$, and this is true of 0, so it passes the check.

A-5: $\mathbb{E}(P) = \frac{17}{18}$. This accords with $0 \leq \mathbb{E}(P) \leq 3$ from Theorem 2.3.5

A-6: $\mathbb{E}(Q) = \frac{1}{2} \left(\frac{e}{e-2} \right)$. As a check, we expect $1 \leq \mathbb{E}(Q) \leq e$, and this is true.

A-7: $\frac{b+a}{2}$

A-8:

(a) $p > 1$

(b) $a_p = p - 1$

(c) $p > 2$

A-9: $e - 2$

A-10: $\frac{2}{\ln^2 2} (2 \ln 2 - 1)$

A-11: $(b - 2) + \frac{1}{2} \ln \left(4 \frac{b-1}{b+2} \right)$

A-12: $\frac{1}{4-\pi} \left(\pi - 2 \ln 2 - \frac{\pi^2}{8} \right)$

A-13: 0

Answers to Exercises 2.4 — Jump to [TABLE OF CONTENTS](#)

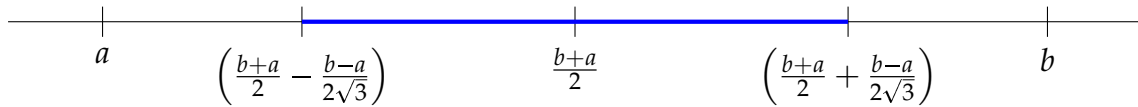
A-1: A -2 B -4 C -3 D -1

A-2: Bundles of 8.5 to 11.5 kg are typical; bundles less than 8.5 kg or more than 11.5 kg are atypical.

A-3: Brand B

A-4: X has sample space $S = \{-100, 100\}$ and is equally likely to take either value.

A-5: $\text{Var}(X) = \frac{(b-a)^2}{12}, \sigma(X) = \frac{b-a}{2\sqrt{3}}$



A-6: 0

A-7:

(a) $\text{Var}(X) = \pi - 3$

(b) $\sigma(X) = \sqrt{\pi - 3}$

A-8: $\mathbb{E}(X) = 0, \text{Var}(X) = \frac{1}{6}, \sigma(X) = \frac{1}{\sqrt{6}}$

Answers to Exercises 3.1 — Jump to [TABLE OF CONTENTS](#)

A-1: (a) -2 (b) 0 (c) the limit does not exist

A-2: true

A-3: (a) $\frac{A-B}{C}$ (b) 0 (c) $\frac{A}{B}$

A-4: Two possible answers, of many:

- $a_n = \begin{cases} 3000 - n & \text{if } n \leq 1000 \\ -2 + \frac{1}{n} & \text{if } n > 1000 \end{cases}$
- $a_n = \frac{1,002,001}{n} - 2$

A-5: One possible answer is $a_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, -1, \dots\}$.
Another is $a_n = n(-1)^n = \{-1, 2, -3, 4, -5, 6, -7, \dots\}$.

A-6: One sequence of many possible is $a_n = \frac{(-1)^n}{n} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots\right\}$.

A-7: (a) $a_n = b_n = h(n) = i(n), \quad c_n = j(n), \quad d_n = f(n), \quad e_n = g(n)$
(b) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \lim_{x \rightarrow \infty} h(x) = 1, \quad \lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} e_n = \lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} j(x) = 0,$
 $\lim_{n \rightarrow \infty} d_n, \lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow \infty} i(x)$ do not exist.

A-8: (a) Some possible answers: $a_{22} \approx -0.99996, a_{66} \approx -0.99965,$ and $a_{110} \approx -0.99902.$

(b) Some possible answers: $a_{11} \approx 0.0044, a_{33} \approx -0.0133,$ and $a_{55} \approx 0.0221.$

The integers 11, 33, and 55 were found by approximating π by $\frac{22}{7}$ and finding when an odd multiple of $\frac{11}{7}$ (which is the corresponding approximation of $\frac{\pi}{2}$) is an integer.

(c) Some possible answers: $a_{44} \approx 0.9998$, $a_{132} \approx 0.9986$ and $a_{220} \approx 0.09961$.
See the solution for how we found them.

A-9: (a) ∞ (b) $\frac{3}{4}$ (c) 0

A-10: ∞

A-11: 0

A-12: 1

A-13: ∞

A-14: 9

A-15: $\log 2$

A-16: 5

A-17: $-\infty$

A-18: $100 \cdot 2^{99}$.

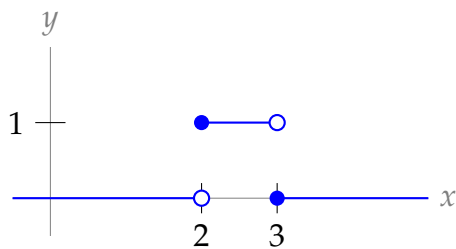
A-19: Possible answers are $\{a_n\} = \left\{ n \left[f \left(a + \frac{1}{n} \right) - f(a) \right] \right\}$

or $\{a_n\} = \left\{ n \left[f(a) - f \left(a - \frac{1}{n} \right) \right] \right\}$.

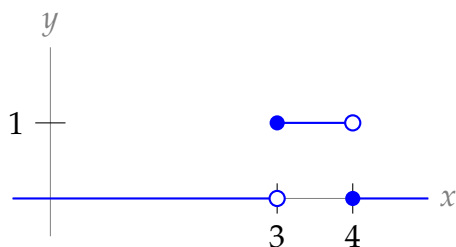
A-20: (a) $A_n = \frac{n}{2} \sin \left(\frac{2\pi}{n} \right)$ (b) π

A-21:

(a)



(b)



(c) $A_n = 1$ for all n

(d) $\lim_{n \rightarrow \infty} A_n = 1$.

(e) $g(x) = 0$

(f) $\int_0^\infty g(x) \, dx = 0$.

A-22: e^3

A-23: (a) 4 (b) $x = 4$ (c) see solution

A-24: (a) decreasing (b) $f_n = \frac{1}{n}f_1$ (c) 2% (d) 0.18%

(e) “be”: 11,019,308; “and”: 7,346,205

Answers to Exercises 3.2 — Jump to [TABLE OF CONTENTS](#)

A-1:

N	S_N
1	1
2	$1 + \frac{1}{2}$
3	$1 + \frac{1}{2} + \frac{1}{3}$
4	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
5	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

A-2: 3

A-3: (a) $a_n = \begin{cases} \frac{1}{2} & \text{if } n = 1 \\ \frac{1}{n(n+1)} & \text{else} \end{cases}$ (b) 0 (c) 1

A-4: $a_n = \begin{cases} 0 & \text{if } n = 1 \\ 2(-1)^n - \frac{1}{n(n-1)} & \text{else} \end{cases}$

A-5: $a_n < 0$ for all $n \geq 2$

A-6: All together, there were 36 cookies brought by Student 11 through Student 20.

A-7: $A + B + C - c_1$

A-8: in general, false

A-9:

(a) $a_n = \frac{1}{10^n}$

(b) B_n holds the n th partial sum, $\sum_{i=1}^n a_i$. The dollar signs fix the starting index at 1. Without them, it would just be $=\text{SUM}(A_n : A_n)$, i.e. the same as entry A_n .

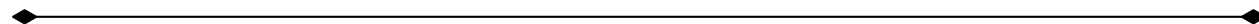
(c) 0

(d) $\frac{1}{9}$

A-10: $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ appears to be close to 0.5.

A-11: (a) $a_n = \frac{11}{16n^2 + 24n + 5}$ (b) $\frac{3}{4}$

A-12: $a_n = \begin{cases} \frac{2}{n(n-1)(n-2)} & \text{if } n \geq 3, \\ -\frac{5}{2} & \text{if } n = 2, \\ 2 & \text{if } n = 1 \end{cases}$



Answers to Exercises 3.3 — Jump to [TABLE OF CONTENTS](#)

A-1: (B), (C)

A-2: (A)

A-3: (a) I am old (b) not enough information to tell
(c) not enough information to tell (d) I am young

A-4:

	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$ CONVERGES	inconclusive
and if $\{a_n\}$ is the blue series	inconclusive	then $\sum b_n$ DIVERGES

A-5: (a) both direct comparison and limit comparison (b) direct comparison
(c) limit comparison (d) neither

A-6: It diverges by the divergence test, because $\lim_{n \rightarrow \infty} a_n \neq 0$.

A-7: The integral test does not apply because $f(x)$ is not decreasing.

A-8: The inequality goes the wrong way, so the direct comparison test (with this comparison series) is inconclusive.

A-9: (B), (D)

A-10: One possible answer: $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

A-11: By the divergence test, for a series $\sum a_n$ to converge, we need $\lim_{n \rightarrow \infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller. If $\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| < 1$ or (equivalently) $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $|a_{n+1}| > |a_n|$ for sufficiently large n , so the terms

are actually *growing* in magnitude. That means the series diverges, by the divergence test.

A-12: One possible answer: $f(x) = \sin(\pi x)$, $a_n = 0$ for every n .

By the integral test, any answer will use a function $f(x)$ that is *not* both positive and decreasing.

A-13: One possible answer: $b_n = \frac{2^n}{3^n}$

A-14: (a) In general false. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ provides a counterexample.

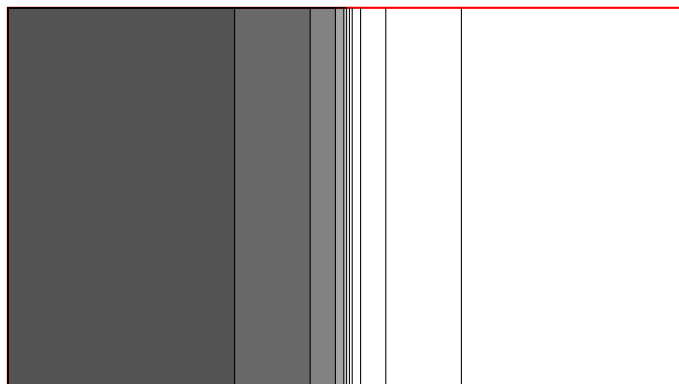
(b) In general false. If $a_n = (-1)^n \frac{1}{n}$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is again the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

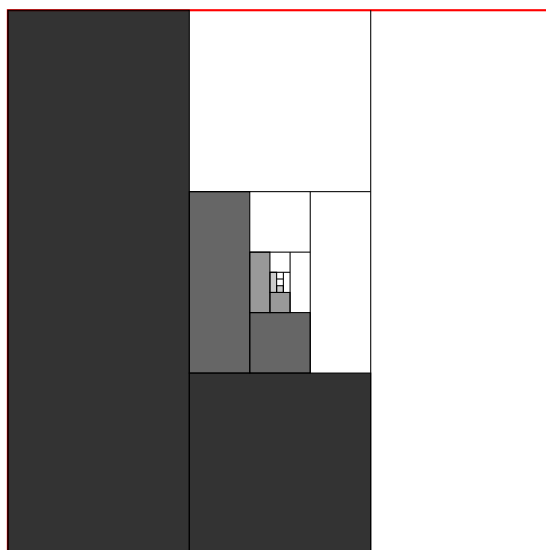
(c) In general false. Take, for example, $a_n = 0$ and $b_n = 1$.

A-15: (a) $\sum_{n=1}^{\infty} \frac{2}{4^n}$ (b) $\frac{2}{3}$

A-16: (a) $\sum_{n=1}^{\infty} \frac{1}{9^n}$ (b) $\frac{1}{8}$

A-17: Two possible pictures:





A-18: $\frac{5^{101} - 1}{4 \cdot 5^{100}}$

A-19: $\frac{5^{51} - 1}{4 \cdot 5^{100}}$

A-20: No. It diverges.

A-21: It diverges.

A-22: $\frac{3}{2}$

A-23: $\frac{1}{7 \times 8^6}$

A-24: $\frac{24}{5}$

A-25: $\frac{7}{30}$

A-26: $\frac{263}{99}$

A-27: $\frac{321}{999} = \frac{107}{333}$

A-28: $\frac{1}{2} + \frac{5}{7} = \frac{17}{14}$

A-29: $\frac{40}{3}$

A-30: The series diverges.

A-31: It diverges.

A-32: This is a geometric series with $r = 1.001$. Since $|r| > 1$, it is divergent.

A-33: The series converges to $-\frac{1}{150}$.

A-34: The series converges.

A-35: It diverges.

A-36: The series converges.

A-37: The series converges to $\frac{1}{3}$.

A-38: The series converges.

A-39: It converges.

A-40: Let $f(x) = \frac{5}{x(\log x)^{3/2}}$. Then $f(x)$ is positive and decreases as x increases. So the sum $\sum_{n=3}^{\infty} f(n)$ and the integral $\int_3^{\infty} f(x) dx$ either both converge or both diverge, by the integral test, which is Theorem 3.3.6 in the text. For the integral, we use the substitution $u = \log x$, $du = \frac{dx}{x}$ to get

$$\int_3^{\infty} \frac{5 dx}{x(\log x)^{3/2}} = \int_{\log 3}^{\infty} \frac{5 du}{u^{3/2}}$$

which converges by the p -test (which is Example 1.13.8 in the text) with $p = \frac{3}{2} > 1$.

A-41: $p > 1$

A-42: It converges.

A-43: The series $\sum_{n=2}^{\infty} \frac{\sqrt{3}}{n^2}$ converges by the p -test with $p = 2$.

Note that

$$0 < a_n = \frac{\sqrt{3n^2 - 7}}{n^3} < \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2}$$

for all $n \geq 2$. As the series $\sum_{n=2}^{\infty} \frac{\sqrt{3}}{n^2}$ converges, the comparison test says that $\sum_{n=2}^{\infty} \frac{\sqrt{3n^2 - 7}}{n^3}$ converges too.

A-44: The series converges.

A-45: It diverges.

A-46: (a) diverges (b) converges

A-47: The series diverges.

A-48: (a) converges (b) diverges

A-49: $\frac{1}{e^5 - e^4}$

A-50: $\frac{1}{7}$

A-51: (a) diverges by limit comparison with the harmonic series

(b) converges by the ratio test

A-52: (a) Converges by the limit comparison test with $b = \frac{1}{k^{5/3}}$.

(b) Diverges by the ratio test.

(c) Diverges by the integral test.

A-53: It converges.

A-54: $N = 5$

A-55: $N \geq 999$

A-56: We need $N = 4$ and then $S_4 = \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2}$

A-57: $\frac{4\pi}{3(\pi^3 - 1)}$

A-58: (a) converges (b) converges

A-59: (a) See the solution.

(b) $f(x) = \frac{x + \sin x}{1 + x^2}$ is not a decreasing function.

(c) See the solution.

A-60: The sum is between 0.9035 and 0.9535.

A-61: Since $\lim_{n \rightarrow \infty} a_n = 0$, there must be some integer N such that $\frac{1}{2} > a_n \geq 0$ for all $n > N$.
Then, for $n > N$,

$$\frac{a_n}{1 - a_n} \leq \frac{a_n}{1 - 1/2} = 2a_n$$

From the information in the problem statement, we know

$$\sum_{n=N+1}^{\infty} 2a_n = 2 \sum_{n=N+1}^{\infty} a_n \quad \text{converges.}$$

So, by the direct comparison test,

$$\sum_{n=N+1}^{\infty} \frac{a_n}{1 - a_n} \quad \text{converges as well.}$$

Since the convergence of a series is not affected by its first N terms, as long as N is finite, we conclude

$$\sum_{n=1}^{\infty} \frac{a_n}{1 - a_n} \quad \text{converges.}$$

A-62: It diverges.

A-63: See the solution.

A-64: About 9% to 10%

A-65: The total population is between 29,820,091 and 30,631,021 people.

A-66: $\frac{5}{8}$

Answers to Exercises 3.4 — Jump to [TABLE OF CONTENTS](#)

A-1: False. For example, $b_n = \frac{1}{n}$ provides a counterexample.

A-2:

	$\sum a_n$ converges	$\sum a_n$ diverges
$\sum a_n $ converges	converges absolutely	not possible
$\sum a_n $ diverges	converges conditionally	diverges

A-3: conditionally convergent

A-4: The series diverges.

A-5: It diverges.

A-6: It converges absolutely.

A-7: It converges absolutely.

A-8: It diverges.

A-9: It converges absolutely.

A-10: See solution.

A-11: See solution.

A-12: See solution.

A-13: (a) See the solution. (b) $|S - S_5| \leq 24 \times 36e^{-6^3}$

A-14: $\cos 1 \approx \frac{389}{720}$; the actual associated error (using a calculator) is about 0.000025.

A-15: See solution.

Answers to Exercises 3.5 — Jump to [TABLE OF CONTENTS](#)

A-1: 2

A-2: $f(x) = \sum_{n=1}^{\infty} \frac{n(x-5)^{n-1}}{n! + 2}$

A-3: only $x = c$

A-4: $R = 6$

A-5: (a) $R = \frac{1}{2}$ (b) $\frac{2}{1+2x}$ for all $|x| < \frac{1}{2}$

A-6: $R = \infty$

A-7: 1

A-8: $|x+2| < 1$

A-9: The radius of convergence is 3.

A-10: $R = 5$

A-11: The given series converges if and only if $-3 \leq x \leq -1$.

A-12: (a) $R = \frac{1}{4}$ (b) $R = \frac{1}{8}$ (c) $R = \frac{1}{4}$

A-13: The radius of convergence is 2.

A-14: The radius of convergence is 1.

A-15: (a) $R = 9$ (b) $R = 0$.

A-16: $\sum_{n=0}^{\infty} x^{n+3} = \sum_{n=3}^{\infty} x^n$

A-17: $f(x) = 3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)}$

A-18: The series converges absolutely for $|x| < 9$, converges conditionally for $x = -9$ and diverges otherwise.

A-19: (a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} + C$ (b) We need to keep two terms (the $n = 0$ and $n = 1$ terms).

A-20: (a) See the solution.

(b) $\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$. The series converges for $-1 < x < 1$.

A-21: See the solution.

A-22: 1

A-23: The point $x = c$ corresponds to a local maximum if $A_2 < 0$ and a local minimum if $A_2 > 0$.

A-24: $\frac{13}{80}$

A-25: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

A-26: $x - \frac{x^3}{3} + \frac{x^5}{5}$

Answers to Exercises 3.6 — Jump to [TABLE OF CONTENTS](#)

A-1: **A**: linear **B**: constant **C**: quadratic

A-2: $T(5) = \arctan^3(e^5 + 7)$

A-3:

A - V, radius = 1 B - I, radius = 1 C - IV, radius = 1
D - VI, radius = $+\infty$ E - II, radius = $+\infty$ F - III, radius = $+\infty$

A-4: (a) $f^{(20)}(3) = 20^2 \left(\frac{20!}{20! + 1} \right)$ (b) $g^{(20)}(3) = 10^2 \left(\frac{20!}{10! + 1} \right)$

(c) $h^{(20)}(0) = -\frac{20! \cdot 5^{11}}{11}$; $h^{(22)}(0) = 0$

A-5: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$

A-6: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1}$

A-7: $\frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{10-x}{10} \right)^n$ with radius of convergence 10.

A-8: $\sum_{n=0}^{\infty} \frac{3^n e^{3a}}{n!} (x-a)^n$, with infinite radius of convergence

A-9: $-\sum_{n=0}^{\infty} 2^n x^n$

A-10: $b_n = 3(-1)^n + 2^n$

A-11: $c_5 = \frac{3^5}{5!}$

A-12: $\sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1} x^{n+1}}{n+1}$ for all $|x| < \frac{1}{2}$

A-13: $a = 1, b = -\frac{1}{3!} = -\frac{1}{6}$.

A-14: $\int \frac{e^{-x^2} - 1}{x} dx = C - \frac{x^2}{2} + \frac{x^4}{8} + \dots$

It is not clear from the wording of the question whether or not the arbitrary constant C is to be counted as one of the “first two nonzero terms”.

A-15: $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+6}}{(2n+1)(2n+6)} + C = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+6}}{(2n+1)(n+3)} + C$

A-16: $f(x) = 1 + \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{3n+2} x^{3n+2}$

A-17: $\frac{\pi}{2\sqrt{3}}$

A-18: $\frac{1}{e}$

A-19: $e^{1/e}$

A-20: $e^{1/\pi} - 1$

A-21: $\log(3/2)$

A-22: $(e+2)e^e - 2$

A-23: The sum diverges—see the solution.

A-24: $\frac{1+\sqrt{2}}{\sqrt{2}}$

A-25: (a) See the solution. (b) $\frac{1}{2} \left(e + \frac{1}{e} \right)$

A-26: (a) 50,000 (b) three terms ($n = 0$ to $n = 2$) (c) six terms ($n = 0$ to $n = 5$)

A-27: 29

A-28: S_{13} or higher

A-29: S_9 or higher

A-30: S_{18} or higher

A-31: The error is in the interval $\left(\frac{-5^7}{14 \cdot 3^7} \left[1 + \frac{1}{3^7} \right], \frac{-5^7}{7 \cdot 6^7} \right) \approx (-0.199, -0.040)$

A-32: -1

A-33: $\frac{1}{5!} = \frac{1}{120}$

A-34: e^2

A-35: $\sqrt{e}$

A-36: $\frac{2}{(6/7)^3} = \frac{343}{108}$

A-37: $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+1)(2n+2)} = x^3 \arctan x - \frac{x^2}{2} \log(1+x^2)$

A-38: (a) the Maclaurin series for $f(x)$ is $\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} x^n$, and its radius of convergence is $R = 1$.

(b) the Maclaurin series for $\arcsin x$ is $\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2 (2n+1)} x^{2n+1}$, and its radius of convergence is $R = 1$.

A-39: $\log(x) = \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n} (x-2)^n$. Radius of convergence: 2

A-40: (a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1}$ (b) 0.493967

(c) The approximate value of part (b) is larger than the true value of $I(1/2)$

A-41: $\frac{1}{66}$

A-42: Any interval of length 0.0002 that contains 0.03592 and 0.03600 is fine.

A-43: (a) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n n!}$ (b) -0.80 (c) See the solution.

A-44: (a) $\Sigma(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$ (b) $x = \pi$ (c) 1.8525

A-45: (a) $I(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!(2n-1)}$ (b) $I(1) = -\frac{1}{2} + \frac{1}{4!3} \pm \frac{1}{6!5} = -0.486 \pm 0.001$

(c) $I(1) < -\frac{1}{2} + \frac{1}{4!3}$

A-46: (a) $I(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!} = \frac{1}{2!}x - \frac{1}{4!}x^3 + \frac{1}{6!}x^5 - \frac{1}{8!}x^8 + \dots$

(b) 0.460 (c) $I(1) < \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} < 0.460$

A-47: (a) See the solution. (b) The series converges for all x .

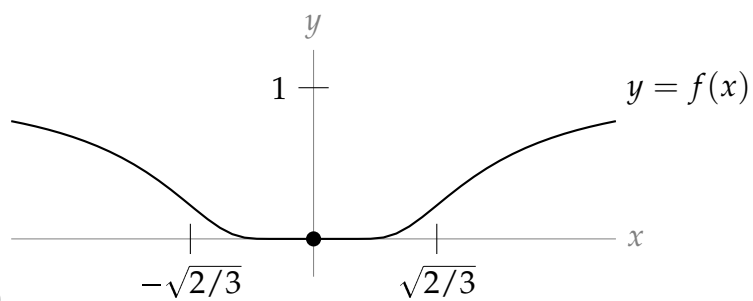
A-48: See the solution.

A-49: (a) $\cosh(x) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, with radius of convergence ∞ .

(b),(c): see solutions

A-50: (a) $\sqrt[3]{3} \approx 1.26$ (b) 12 terms (S_{11})

A-51: $\frac{15!}{5! \cdot 5^6} - \frac{21!}{7! \cdot 11! \cdot 5^{11}} + \frac{27!}{9! \cdot 17! \cdot 5^{17}} - \frac{33!}{11! \cdot 23! \cdot 5^{23}}$



A-52: (a)

(b) the constant function 0

(c) everywhere

(d) only at $x = 0$

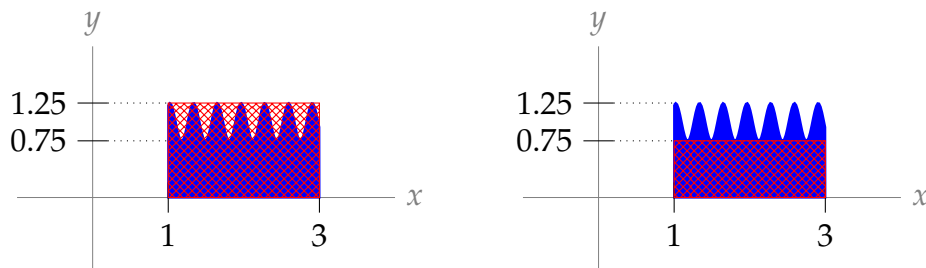
A-53: 0

Part IV

SOLUTIONS TO PROBLEMS

Solutions to Exercises 1.1 — Jump to [TABLE OF CONTENTS](#)

S-1:



The diagram on the left shows a rectangle with area $2 \times 1.25 = 2.5$ square units. Since the blue-shaded region is entirely inside this rectangle, the area of the blue-shaded region is no more than 2.5 square units.

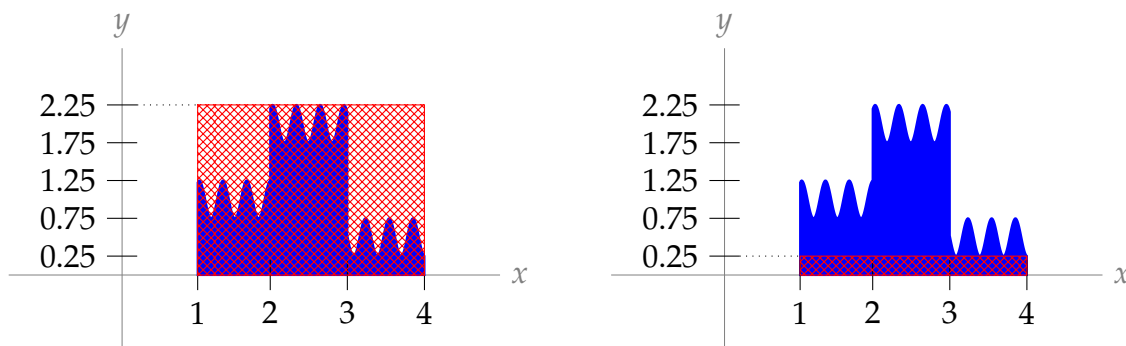
The diagram on the right shows a rectangle with area $2 \times 0.75 = 1.5$ square units. Since the blue-shaded region contains this entire rectangle, the area of the blue region is no less than 1.5 square units.

So, the area of the blue-shaded region is between 1.5 and 2.5 square units.

Remark: we could also give an obvious range, like “the shaded area is between zero and one million square units.” This would be true, but not very useful or interesting.

S-2:

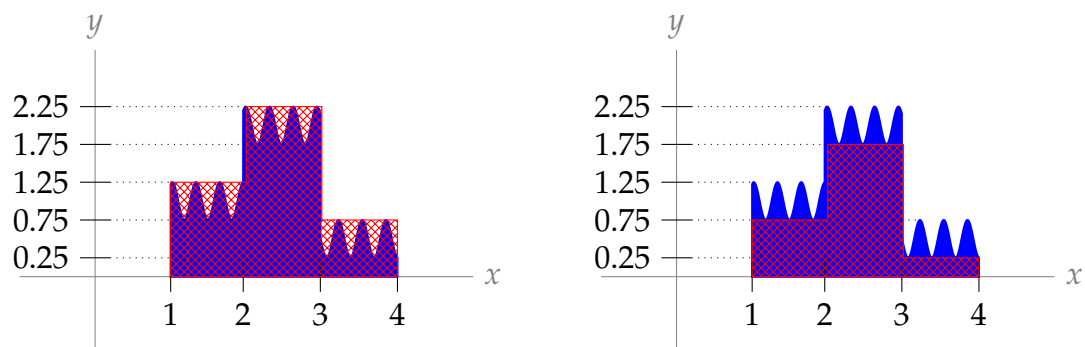
Solution 1: One naive way to solve this is to simply use the same method as Question 1.



The rectangle on the left has area $3 \times 2.25 = 6.75$ square units, and encompasses the entire shaded region. The rectangle on the right has area $3 \times 0.25 = 0.75$ square units, and is entirely contained inside the blue-shaded region. So, the area of the blue-shaded region is between 0.75 and 6.75 square units.

This is a legitimate approximation, but we can easily do much better. The shape of this graph suggests that using the areas of *three* rectangles would be a natural way to improve our estimate.

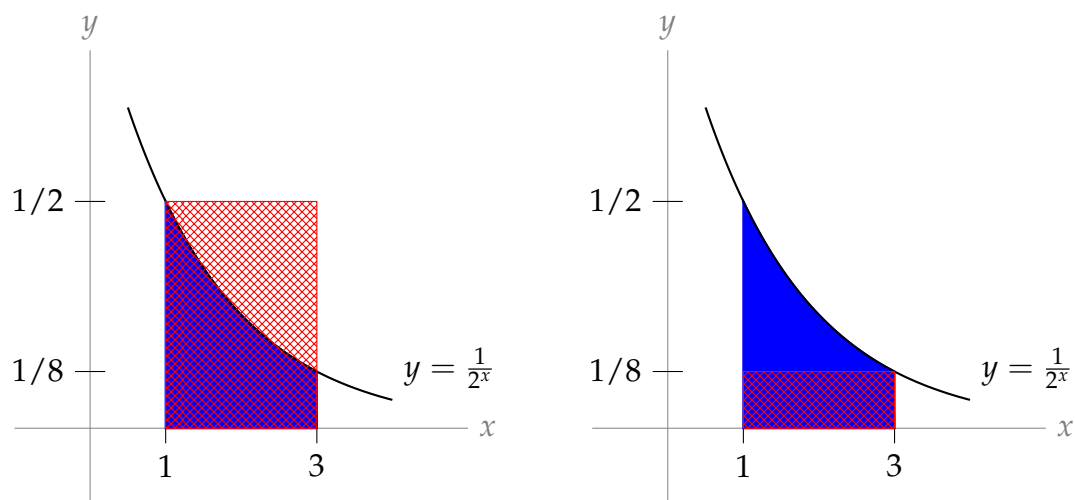
Solution 2: Let’s use these rectangles instead:



In the left picture, the red area is $(1 \times 1.25) + (1 \times 2.25) + (1 \times 0.75) = 4.25$ square units. In the right picture, the red area is $(1 \times 0.75) + (1 \times 1.75) + (1 \times 0.25) = 2.75$ square units. So, the blue shaded area is between 2.75 and 4.25 square units.

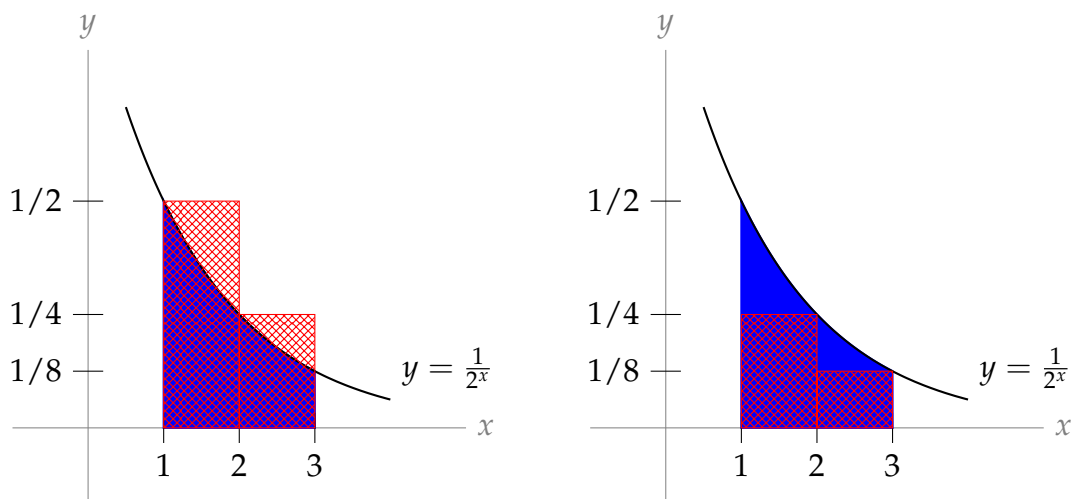
S-3: Remark: in the solution below, we find the appropriate approximation using trial and error. In Question 49, we take a more systematic approach.

Try 1: First, we can try by using a single rectangle as an overestimate, and a single rectangle as an underestimate.



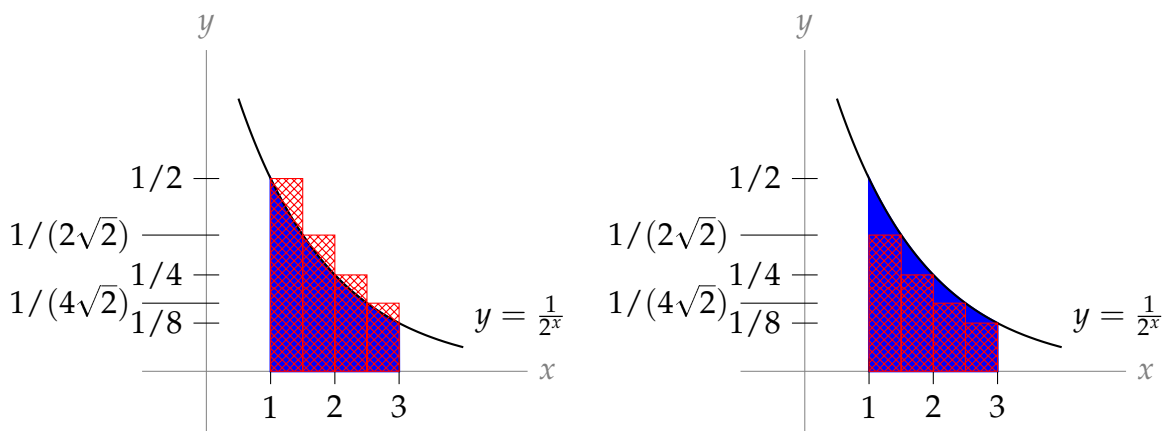
The area under the curve is less than the area of the rectangle on the left ($2 \times \frac{1}{2} = 1$) and greater than the area of the rectangle on the right ($2 \times \frac{1}{8} = \frac{1}{4}$). So, the area is in the range $(\frac{1}{4}, 1)$. Unfortunately, this range is too big—we need our range to have length at most 0.2. So, we refine our approximation by using more rectangles.

Try 2: Let's try using two rectangles each for the upper and lower bounds.



The rectangles in the left picture have area $\left(1 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) = \frac{3}{4}$, and the rectangles in the right picture have area $\left(1 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{8}\right) = \frac{3}{8}$. So, the area under the curve is in the interval $\left(\frac{3}{8}, \frac{3}{4}\right)$. The length of this interval is $\frac{3}{8}$, and $\frac{3}{8} > \frac{3}{15} = \frac{1}{5} = 0.2$. (Indeed, $\frac{3}{8} = 0.375 > 0.2$.) Since the length of our interval is still bigger than 0.2, we need even more rectangles.

Try 3: Let's go ahead and try four rectangles each for the upper and lower estimates.



The area of the rectangles on the left is:

$$\left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4\sqrt{2}}\right) = \frac{3}{8} \left[1 + \frac{1}{\sqrt{2}}\right],$$

and the area of the rectangles on the right is:

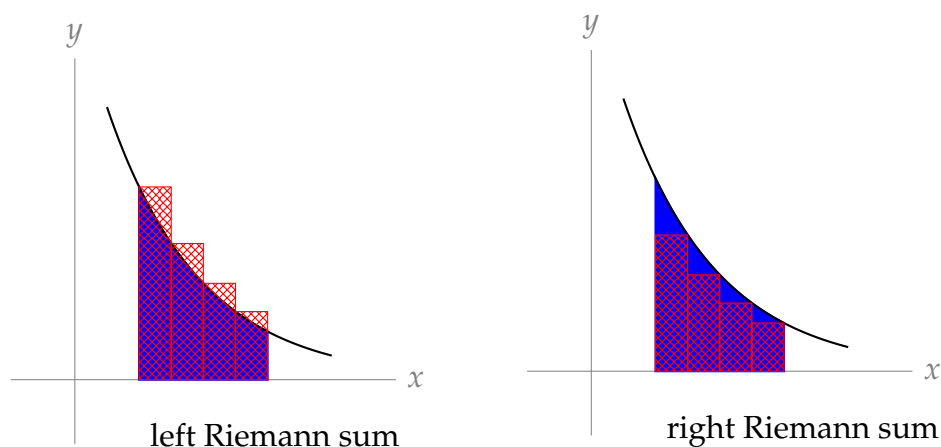
$$\left(\frac{1}{2} \times \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{8}\right) = \frac{3}{8} \left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right].$$

So, the area under the curve is in the interval $\left(\frac{3}{8} \left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right], \frac{3}{8} \left[1 + \frac{1}{\sqrt{2}}\right]\right)$. The length of this interval is $\frac{3}{16}$, and $\frac{3}{16} < \frac{3}{15} = \frac{1}{5} = 0.2$, as desired. (Indeed, $\frac{3}{16} = 0.1875 < 0.2$.)

Note, if we choose any value in the interval $\left(\frac{3}{8} \left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right], \frac{3}{8} \left[1 + \frac{1}{\sqrt{2}}\right]\right)$ as an approximation for the area under the curve, our error is no more than 0.2.

S-4: Since $f(x)$ is decreasing, it is larger on the left endpoint of an interval than on the right endpoint of an interval. So, a *left* Riemann sum gives a larger approximation. Notice this does not depend on n .

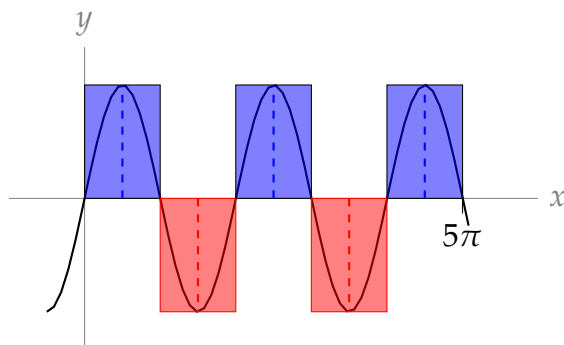
Furthermore, the actual area $\int_0^5 f(x)dx$ is larger than its right Riemann sum, and smaller than its left Riemann sum.



S-5: If $f(x)$ is always increasing or always decreasing, then the midpoint Riemann sum will be between the left and right Riemann sums. So, we need a function that goes up and down. Many examples are possible, but let's work with a familiar one: $\sin x$.

If our intervals have endpoints that are integer multiples of π , then the left and right Riemann sums will be 0, since $\sin(0) = \sin(\pi) = \sin(2\pi) = \dots = 0$. The midpoints of these intervals will give y -values of 1 and -1. So, for example, we can let $f(x) = \sin x$, $[a, b] = [0, \pi]$, and $n = 1$. Then the right and left Riemann sums are 0, while the midpoint Riemann sum is π .

We can extend the example of $f(x) = \sin x$ to have more intervals. As long as we have more positive terms than negative, the midpoint approximation will be a positive number, and so it will be larger than both the left and right Riemann sums. So, for example, we can let $f(x) = \sin x$, $[a, b] = [0, 5\pi]$, and $n = 5$. Then the midpoint Riemann sum is $\pi - \pi + \pi - \pi + \pi = \pi$, which is strictly larger than 0 and so it is larger than both the left and right Riemann sums.



S-6:

(a) Two possible answers are $\sum_{i=3}^7 i$ and $\sum_{i=1}^5 (i+2)$. The first has simpler terms (i versus $i+2$), while the second has simpler indices (we often like to start at $i=1$). Neither is objectively better than the other, but depending on your purposes you might find one more useful.

(b) The terms of this sum are each double the terms of the sum from part (a), so two possible answers are $\sum_{i=3}^7 2i$ and $\sum_{i=1}^5 (2i+4)$.

We often want to write a sum that involves even numbers: it will be useful for you to remember that the term $2i$ (with index i) generates evens.

(c) The terms of this sum are each one more than the terms of the sum from part (b), so two possible answers are $\sum_{i=3}^7 (2i+1)$ and $\sum_{i=1}^5 (2i+5)$.

In the last part, we used the expression $2i$ to generate even numbers; $2i+1$ will generate odds. So will the index $2i+5$, and indeed, $2i+k$ for any odd number k . The choice of what you add will depend on the limits of i .

(d) This sum adds up the odd numbers from 1 to 15. From Part (c), we know that the formula $2i+1$ is a simple way of generating odd numbers. Since our first term should be 1 and our last term should be 15, if we use $\sum (2i+1)$, then i should run from 0 to 7. So, one way of expressing our sum in sigma notation is $\sum_{i=0}^7 (2i+1)$.

Sometimes we like our sum to start at $i=1$ instead of $i=0$. If this is our desire, we can use $2i-1$ as our terms, and let i run from 1 to 8. This gives us another way of

expressing our sum: $\sum_{i=1}^8 (2i-1)$.

S-7:

(a) The denominators are successive powers of three, so one way of writing this is $\sum_{i=1}^4 \frac{1}{3^i}$.

Equivalently, the terms we're adding are powers of $1/3$, so we can also write

$$\sum_{i=1}^4 \left(\frac{1}{3}\right)^i.$$

(b) This sum is obtained from the sum in (a) by multiplying each term by two, so we can write $\sum_{i=1}^4 \frac{2}{3^i}$ or $\sum_{i=1}^4 2 \left(\frac{1}{3}\right)^i$.

- (c) The difference between this sum and the previous sum is its alternating sign, minus-plus-minus-plus. This behaviour appears when we raise a negative number to successive powers. We can multiply each term by $(-1)^i$, or we can slip a negative into the number that is already raised to the power i : $\sum_{i=1}^4 (-1)^i \frac{2}{3^i}$, or $\sum_{i=1}^4 \frac{2}{(-3)^i}$.
- (d) This sum is the negative of the sum in part (c), so we can simply multiply each term by negative one: $\sum_{i=1}^4 (-1)^{i+1} \frac{2}{3^i}$, or $\sum_{i=1}^4 -\frac{2}{(-3)^i}$.

Be careful with the second form: a common mistake is to think that $-\frac{2}{(-3)^i} = \frac{2}{3^i}$, but these are not the same.

S-8:

- (a) If we re-write the second term as $\frac{3}{9}$ instead of $\frac{1}{3}$, our sum becomes:

$$\frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + \frac{9}{243}$$

The numerators are the first five odd numbers, and the denominators are the first five positive powers of 3. We learned how to generate odd numbers in Question 6, and we learned how to generate powers of three in Question 7. Combining these, we can write our sum as $\sum_{i=1}^5 \frac{2i-1}{3^i}$.

- (b) The denominators of these terms differ from the denominators of part (a) by precisely two, while the numerators are simply 1. So, we can modify our previous answer: $\sum_{i=1}^5 \frac{1}{3^i + 2}$.

- (c) Let's re-write the sum to make the pattern clearer.

$$\begin{aligned} & 1000 + 200 + 30 + 4 + \frac{1}{2} + \frac{3}{50} + \frac{7}{1000} \\ = & 1 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + \frac{4}{1} + \frac{5}{10} + \frac{6}{100} + \frac{7}{1000} \\ = & 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2} + 7 \cdot 10^{-3} \\ = & \textcolor{red}{1} \cdot 10^{4-\textcolor{red}{1}} + \textcolor{red}{2} \cdot 10^{4-\textcolor{red}{2}} + \textcolor{red}{3} \cdot 10^{4-\textcolor{red}{3}} + \textcolor{red}{4} \cdot 10^{4-\textcolor{red}{4}} + \textcolor{red}{5} \cdot 10^{4-\textcolor{red}{5}} + \textcolor{red}{6} \cdot 10^{4-\textcolor{red}{6}} + \textcolor{red}{7} \cdot 10^{4-\textcolor{red}{7}} \end{aligned}$$

If we let the red numbers be our index i , this gives us the expression $\sum_{i=1}^7 i \cdot 10^{4-i}$.

Equivalently, we can write $\sum_{i=1}^7 \frac{i}{10^{i-4}}$.

S-9:

(a) Using Theorem 1.1.7.a in the text, with $a = 1$, $r = \frac{3}{5}$ and $n = 100$:

$$\sum_{i=0}^{100} \left(\frac{3}{5}\right)^i = \frac{1 - \left(\frac{3}{5}\right)^{101}}{1 - \frac{3}{5}} = \frac{5}{2} \left[1 - \left(\frac{3}{5}\right)^{101}\right]$$

(b) We want to use Theorem 1.1.7, part (a) again, but our sum doesn't start at $\left(\frac{3}{5}\right)^0 = 1$. We have two options: factor out the leading term, or use the difference of two sums that start where we want them to.

Solution 1: In this solution, we'll make our sum start at 1 by factoring out the leading term. We wrote our work out the long way (expanding the sigma into "dot-dot-dot" notation) for clarity, but it's faster to do the algebra in sigma notation all the way through.

$$\begin{aligned} \sum_{i=50}^{100} \left(\frac{3}{5}\right)^i &= \left(\frac{3}{5}\right)^{50} + \left(\frac{3}{5}\right)^{51} + \left(\frac{3}{5}\right)^{52} + \cdots + \left(\frac{3}{5}\right)^{100} \\ &= \left(\frac{3}{5}\right)^{50} \left[1 + \left(\frac{3}{5}\right) + \left(\frac{3}{5}\right)^2 + \cdots + \left(\frac{3}{5}\right)^{50}\right] \\ &= \left(\frac{3}{5}\right)^{50} \frac{1 - \left(\frac{3}{5}\right)^{51}}{1 - \frac{3}{5}} \\ &= \frac{5}{2} \left(\frac{3}{5}\right)^{50} \left[1 - \left(\frac{3}{5}\right)^{51}\right]. \end{aligned}$$

Solution 2: In this solution, we write our given expression as the difference of two sums, both starting at $i = 0$.

$$\begin{aligned} \sum_{i=50}^{100} \left(\frac{3}{5}\right)^i &= \sum_{i=0}^{100} \left(\frac{3}{5}\right)^i - \sum_{i=0}^{49} \left(\frac{3}{5}\right)^i \\ &= \frac{1 - \left(\frac{3}{5}\right)^{101}}{1 - \frac{3}{5}} - \frac{1 - \left(\frac{3}{5}\right)^{50}}{1 - \frac{3}{5}} \\ &= \frac{5}{2} \left[\left(\frac{3}{5}\right)^{50} - \left(\frac{3}{5}\right)^{101} \right] \\ &= \frac{5}{2} \left(\frac{3}{5}\right)^{50} \left[1 - \left(\frac{3}{5}\right)^{51}\right]. \end{aligned}$$

(c) Before we can use the equations in Theorem 1.1.7, we'll need to do a little

simplification.

$$\begin{aligned}
 \sum_{i=1}^{10} (i^2 - 3i + 5) &= \sum_{i=1}^{10} i^2 + \sum_{i=1}^{10} -3i + \sum_{i=1}^{10} 5 \\
 &= \sum_{i=1}^{10} i^2 - 3 \sum_{i=1}^{10} i + 5 \sum_{i=1}^{10} 1 \\
 &= \frac{1}{6}(10)(11)(21) - 3 \left(\frac{1}{2}(10 \cdot 11) \right) + 5 \cdot 10 \\
 &= 270
 \end{aligned}$$

- (d) As in part (c), we'll simplify first. The first part (shown here in red) is a geometric sum, but it does not start at 1 = $\left(\frac{1}{e}\right)^0$.

$$\begin{aligned}
 \sum_{n=1}^b \left[\left(\frac{1}{e}\right)^n + en^3 \right] &= \sum_{n=1}^b \left(\frac{1}{e}\right)^n + \sum_{n=1}^b en^3 \\
 &= \sum_{n=0}^b \left(\frac{1}{e}\right)^n - 1 + e \sum_{n=1}^b n^3 \\
 &= \frac{1 - \left(\frac{1}{e}\right)^{b+1}}{1 - \frac{1}{e}} - 1 + e \left[\frac{1}{2}b(b+1) \right]^2 \\
 &= \frac{\frac{1}{e} - \left(\frac{1}{e}\right)^{b+1}}{1 - \frac{1}{e}} + e \left[\frac{1}{2}b(b+1) \right]^2 \\
 &= \frac{1 - \left(\frac{1}{e}\right)^b}{e - 1} + \frac{e}{4} [b(b+1)]^2
 \end{aligned}$$

S-10:

- (a) The two pieces are very similar, which we can see by changing the index, or expanding them out:

$$\begin{aligned}
 \sum_{i=50}^{100} (i - 50) + \sum_{i=0}^{50} i &= (0 + 1 + 2 + \cdots + 50) + (0 + 1 + 2 + \cdots + 50) \\
 &= (1 + 2 + \cdots + 50) + (1 + 2 + \cdots + 50) \\
 &= 2(1 + 2 + \cdots + 50) \\
 &= 2 \sum_{i=1}^{50} i \\
 &= 2 \left(\frac{50 \cdot 51}{2} \right) = 50 \cdot 51 = 2550
 \end{aligned}$$

- (b) If we expand $(i - 5)^3 = i^3 - 15i^2 + 75i - 125$, we can break the sum into four parts, and evaluate each separately. However, it is much simpler to change the index and make the term $(i - 5)^3$ into i^3 .

$$\sum_{i=10}^{100} (i - 5)^3 = 5^3 + 6^3 + 7^3 + \cdots + 95^3$$

We have a formula to evaluate the sum of cubes if they start at 1, so we turn our expression into the difference of two sums starting at 1:

$$\begin{aligned} &= \left[1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + \cdots + 95^3 \right] - \left[1^3 + 2^3 + 3^3 + 4^3 \right] \\ &= \sum_{i=1}^{95} i^3 - \sum_{i=1}^4 i^3 \\ &= \left[\frac{1}{2}(95)(96) \right]^2 - \left[\frac{1}{2}(4)(5) \right]^2 \\ &= 20,793,500. \end{aligned}$$

- (c) Notice every two terms cancel with each other, since the sum is $(-1) + (+1)$, etc. Then the terms $n = 1$ through $n = 10$ cancel, and we're left only with the final term, $(-1)^{11} = -1$.

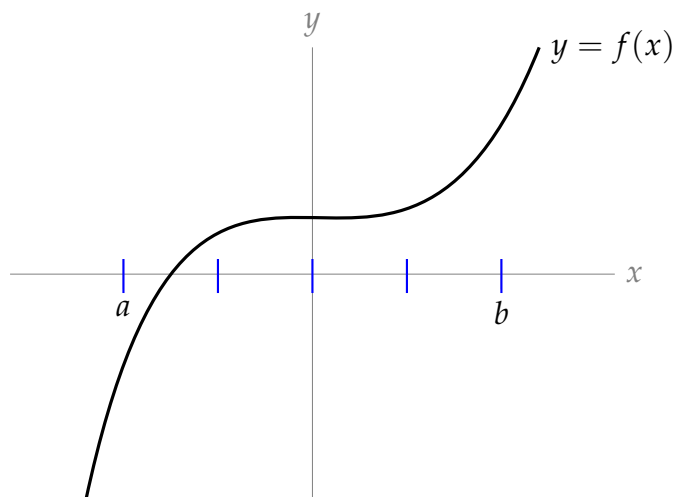
Written out more explicitly:

$$\begin{aligned} \sum_{n=1}^{11} (-1)^n &= -1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \\ &= [-1 + 1] + [-1 + 1] + [-1 + 1] + [-1 + 1] + [-1 + 1] - 1 \\ &= 0 + 0 + 0 + 0 + 0 - 1 = -1. \end{aligned}$$

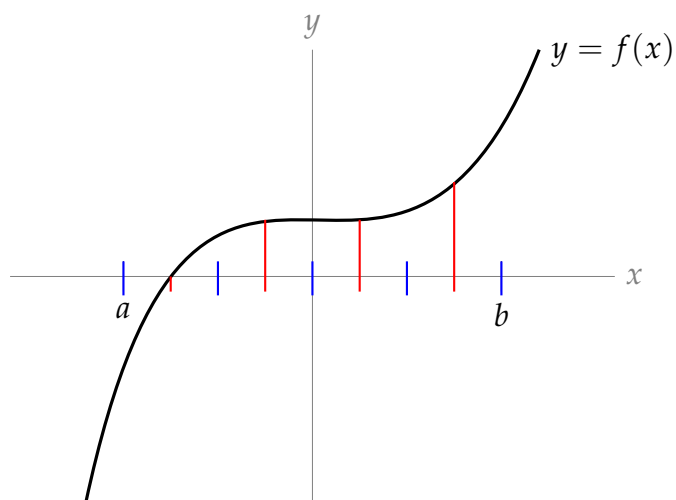
- (d) For every integer n , $2n + 1$ is odd, so $(-1)^{2n+1} = -1$. Then

$$\sum_{n=2}^{11} (-1)^{2n+1} = \sum_{n=2}^{11} -1 = -10.$$

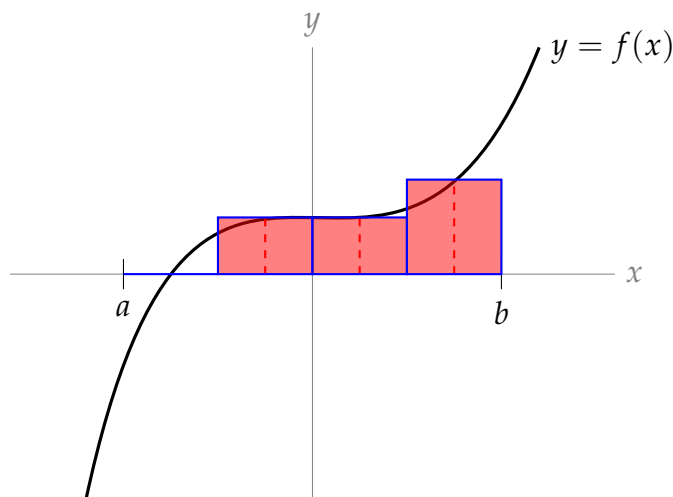
S-11: The index of the sum runs from 1 to 4: the first, second, third, and fourth rectangles. So, we have four rectangles in our Riemann sum. Let's start by drawing in the intervals along the x -axis taken up by these four rectangles. Note each has the same width: $\frac{b-a}{4}$.



Since this is a midpoint Riemann sum, the height of each rectangle is given by the y -value of the function in the midpoint of the interval. So, now let's find the height of the function at the midpoints of each of the four intervals.



The left-most interval has a height of about 0, so it gives a “trivial” rectangle with no height and no area. The middle two intervals have rectangles of about the same height, and the right-most interval has the highest rectangle.



S-12: In general, the left Riemann sum for the integral $\int_a^b f(x) \, dx$ is of the form

$$\sum_{k=1}^n f\left(a + (k-1)\frac{b-a}{n}\right) \frac{b-a}{n}$$

- To get the limits of summation to match the given sum, we need $n = 4$.
- Then to get the factor multiplying f to match that in the given sum, we need $\frac{b-a}{n} = 1$, so $b - a = 4$.
- Finally, to get the argument of f to match that in the given sum, we need

$$a + (k-1)\frac{b-a}{n} = a - \frac{b-a}{n} + k\frac{b-a}{n} = 1 + k$$

Subbing in $n = 4$ and $b - a = 4$ gives $a - 1 + k = 1 + k$, so $a = 2$ and $b = 6$.

S-13: The general form of a Riemann sum is $\sum_{i=1}^n \Delta x \cdot f(x_i^*)$, where $\Delta x = \frac{b-a}{n}$ is the width of each rectangle, and $f(x_i^*)$ is the height.

There are different ways to interpret the given sum as a Riemann sum. The most obvious is given in Solution 1. You may notice that we make some convenient assumptions in this solution about values for Δx and a , and we assume the sum is a right Riemann sum. Other visualizations of the sum arise from making more exotic choices. Some of these are explored in Solutions 2-4.

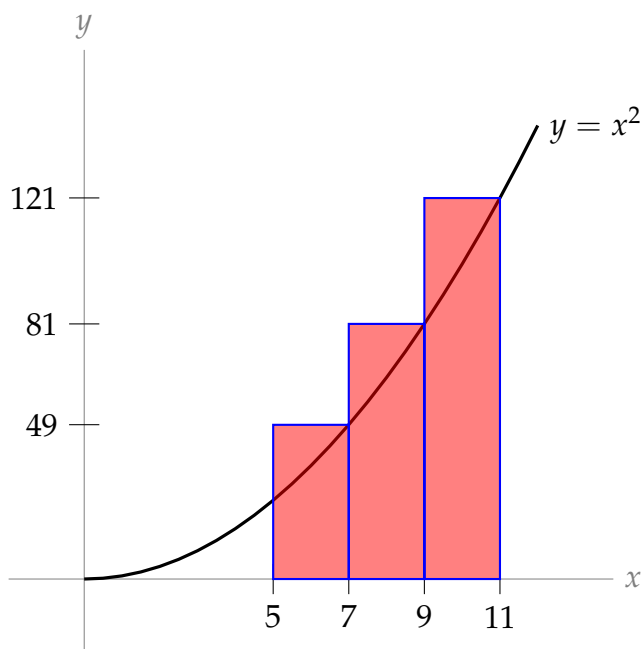
All cases have three rectangles, and the three rectangles will have the same areas: 98, 162, and 242 square units, respectively. This is because the terms of the given sum simplify to $98 + 162 + 242$.

Solution 1:

- Because the index runs from 1 to 3, there are three intervals: $n = 3$.

- Looking at our sum, it seems reasonable to interpret $\Delta x = 2$. Then, since $n = 3$, we conclude $\frac{b-a}{3} = 2$, hence $b - a = 6$.
- If $\Delta x = 2$, then $f(x_i^*) = (5 + 2i)^2$. Recall that x_i^* is the x -coordinate we use to decide the height of the i th rectangle. In a right Riemann sum, $x_i^* = a + i \cdot \Delta x$. So, using $2 = \Delta x$, we can let $f(x_i^*) = f(a + 2i) = (5 + 2i)^2$. This fits with the function $f(x) = x^2$, and $a = 5$.
- Since $b - a = 6$, and $a = 5$, this tells us $b = 11$

To sum up, we can interpret the Riemann sum as a right Riemann sum, with three intervals, of the function $f(x) = x^2$ from $x = 5$ to $x = 11$.



Solution 2: We could have chosen a different value for Δx .

- The index of the sum runs from 1 to 3, so we have $n = 3$.
- We didn't have to interpret Δx as 2—that was just the path of least resistance. We could have chosen it to be any other number—for the sake of argument, let's say $\Delta x = 10$. (Positive numbers are easiest to interpret, but negatives are technically allowed as well.)
- Then $10 = \frac{b-a}{n} = \frac{b-a}{3}$, so $b - a = 30$.
- Let's use the paradigm of a right Riemann sum, and match up the terms of the

sum given in the problem to the terms in the definition:

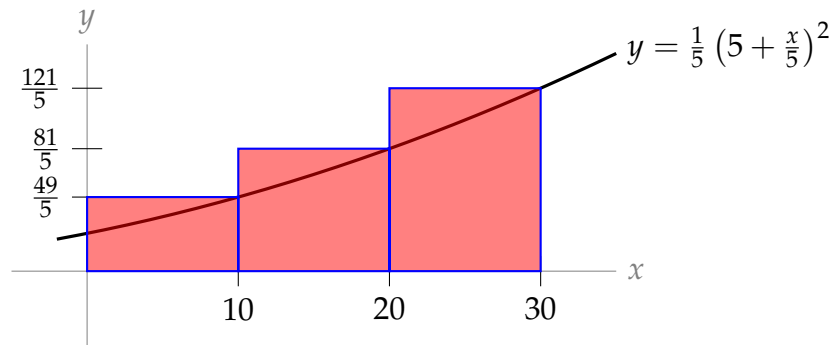
$$\Delta x \cdot f(a + i \cdot \Delta x) = 2 \cdot (5 + 2i)^2$$

$$10 \cdot f(a + 10i) = 2 \cdot (5 + 2i)^2$$

$$f(a + 10i) = \frac{1}{5} \cdot (5 + 2i)^2$$

$$f(a + 10i) = \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot 10i\right)^2$$

- The easiest value of a in this case is $a = 0$. Then $f(10i) = \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot 10i\right)^2$, so $f(x) = \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot x\right)^2$.
- If $a = 0$ and $b - a = 30$, then $b = 30$.
- To sum up: $n = 3$, $a = 0$, $b = 30$, $\Delta x = 10$, and $f(x) = \frac{1}{5} \cdot \left(5 + \frac{x}{5}\right)^2$.



By changing Δx , we changed the *widths* of the rectangles. The rectangles in this picture are wider and shorter than the rectangles in Solution 1. Their *areas* are the same: 98, 162, and 242.

Solution 3: We could have chosen a different value of a .

- Suppose $\Delta x = 2$, and we interpret our sum as a right Riemann sum, but we didn't assume $a = 5$. We could have chosen a to be any number—say, $a = 1$.
- Let's match up what we're given in the problem to what we're given as a definition:

$$\Delta x \cdot f(a + i \cdot \Delta x) = 2 \cdot (5 + 2i)^2$$

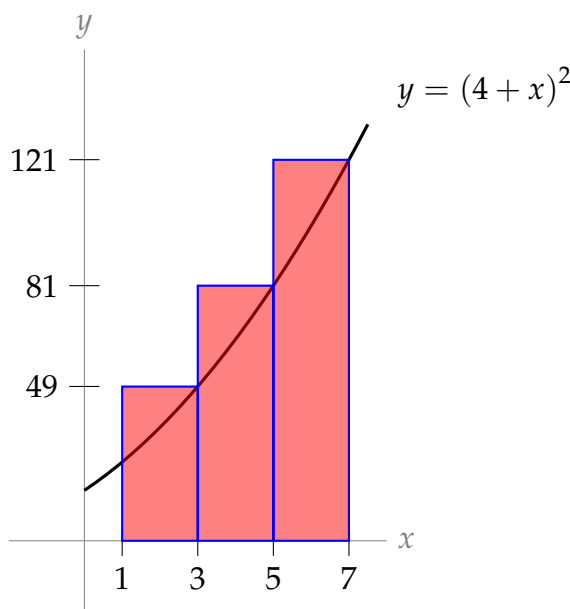
$$2 \cdot f(1 + 2i) = 2 \cdot (5 + 2i)^2$$

$$f(1 + 2i) = (5 + 2i)^2$$

$$f(1 + 2i) = (4 + 1 + 2i)^2$$

- Since $f(1 + 2i) = (4 + 1 + 2i)^2$, we have $f(x) = (4 + x)^2$

- Since $a = 1$ and $\frac{b-a}{3} = 2$, in this case $b = 7$.
- To sum up: $n = 3$, $a = 1$, $b = 7$, $\Delta x = 2$, and $f(x) = (4 + x)^2$.



This picture is a lot like the picture in Solution 1, but shifted to the left. By changing a , we changed the *left endpoint* of our region.

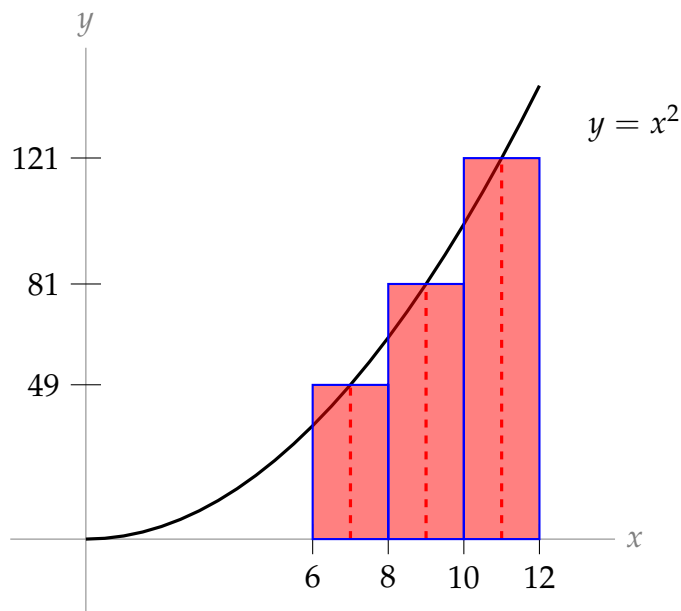
Solution 4: We could have chosen a different kind of Riemann sum.

- We didn't have to assume that we were dealing with a *right* Riemann sum. Suppose $\Delta x = 2$, and we have a *midpoint* Riemann sum.
- Let's match up what we're given in the problem with what we're given in the definition:

$$\begin{aligned}\Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right) \Delta x\right) &= 2 \cdot (5 + 2i)^2 \\ 2 \cdot f\left(a + \left(i - \frac{1}{2}\right) 2\right) &= 2 \cdot (5 + 2i)^2 \\ f\left(a + \left(i - \frac{1}{2}\right) 2\right) &= (5 + 2i)^2 \\ f(a + 2i - 1) &= (5 + 2i)^2 \\ f((a - 1) + 2i) &= (5 + 2i)^2\end{aligned}$$

- It is now convenient to set $a - 1 = 5$, hence $a = 6$.
- Then $f(5 + 2i) = (5 + 2i)^2$, so $f(x) = x^2$
- Since $2 = \frac{b-a}{3}$ and $a = 6$, we see $b = 12$.

- To sum up: $n = 3$, $a = 6$, $b = 12$, $\Delta x = 2$, and $f(x) = x^2$.



By choosing to interpret our sum as a midpoint Riemann sum instead of a right Riemann sum, we changed where our rectangles intersect the graph $y = f(x)$: instead of the graph hitting the right corner of the rectangle, it hits in the middle.

S-14: Many interpretations are possible—see the solution to Question 13 for a more thorough discussion—but the most obvious is given below. Recall the definition of a left Riemann sum:

$$\sum_{i=1}^n \Delta x \cdot f(a + (i-1)\Delta x)$$

We chose a left Riemann sum instead of right or midpoint because our given sum has $(i-1)$ in it, rather than $(i-\frac{1}{2})$ or simply i .

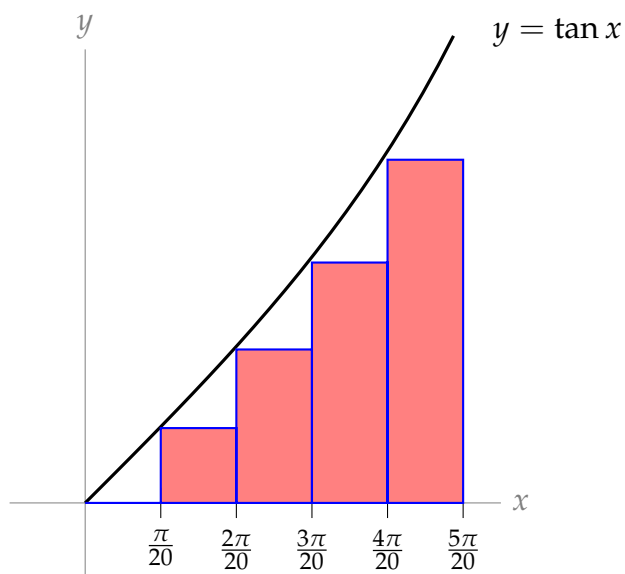
- Since the sum has five terms (i runs from 1 to 5), there are 5 rectangles. That is, $n = 5$.
- In the definition of the Riemann sum, note that the term Δx appears twice: once multiplied by the entire term, and once multiplied by $i-1$. So, a convenient choice for Δx is $\frac{\pi}{20}$, because this is the constant that is both multiplied at the start of the term, and multiplied by $i-1$.
- Since $\frac{\pi}{20} = \Delta x = \frac{b-a}{n} = \frac{b-a}{5}$, we see $b-a = \frac{5\pi}{20} = \frac{\pi}{4}$.
- We match the terms in the definition with the terms in the problem:

$$f(a + (i-1)\Delta x) = \tan\left(\frac{\pi(i-1)}{20}\right)$$

$$f\left(a + (i-1)\frac{\pi}{20}\right) = \tan\left((i-1)\frac{\pi}{20}\right)$$

So, we choose $a = 0$ and $f(x) = \tan x$.

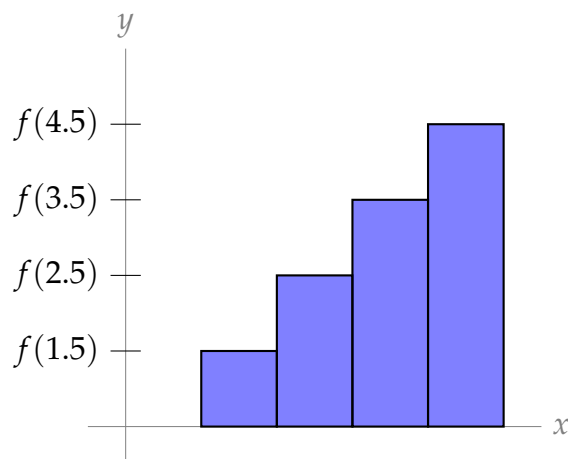
- Since $a = 0$ and $b - a = \frac{\pi}{4}$, we see $b = \frac{\pi}{4}$.



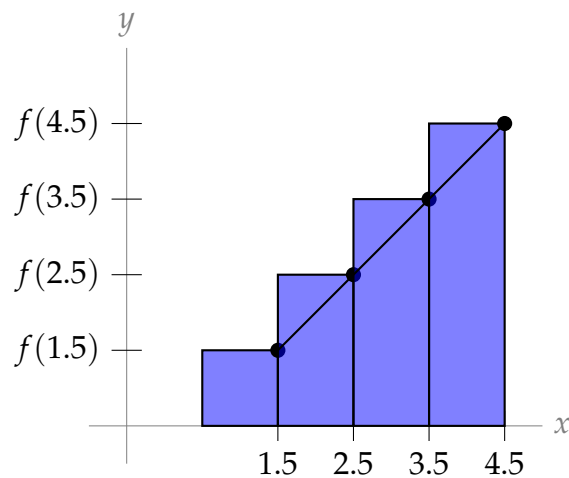
We note that the first rectangle of the five is a “trivial” rectangle, with height (and area) 0.

S-15: Since there are four terms in the sum, $n = 4$. (Note the sum starts at $k = 0$, instead of $k = 1$.) Since the function is multiplied by 1, $1 = \Delta x = \frac{b - a}{n} = \frac{b - a}{4}$, hence $b - a = 4$.

We can choose to view the given sum as a left, right, or midpoint Riemann sum. The choice we make determines the interval. Note that the heights of the rectangles are determined when $x = 1.5, 2.5, 3.5,$ and 4.5 .

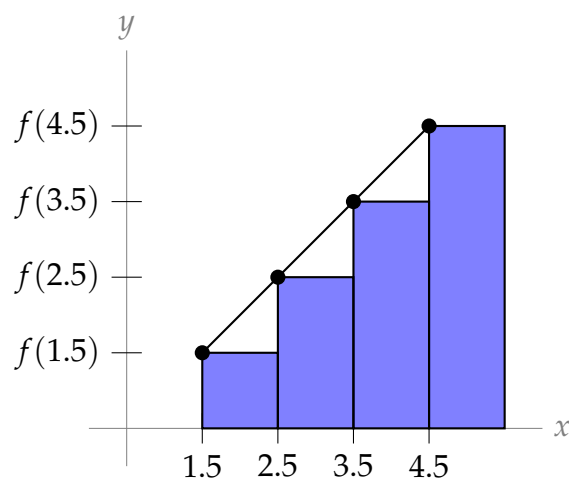


Option 1: right Riemann sum If our sum is a right Riemann sum, then we take the heights of the rectangles from the right endpoint of each interval.



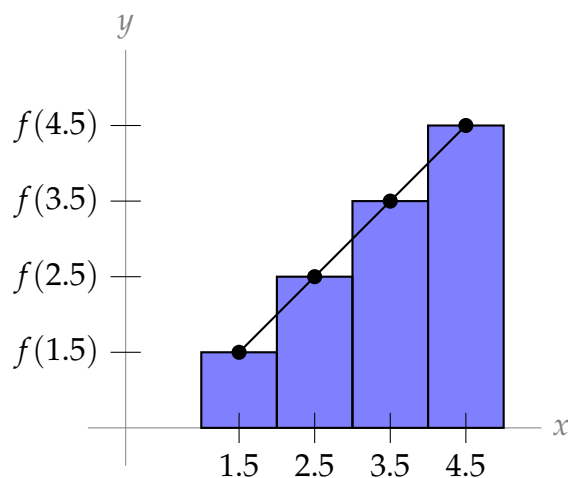
Then $a = 0.5$ and $b = 4.5$. Therefore: $\sum_{k=0}^3 f(1.5 + k) \cdot 1$ is a right Riemann sum on the interval $[0.5, 4.5]$ with $n = 4$.

Option 2: left Riemann sum If our sum is a left Riemann sum, then we take the heights of the rectangles from the left endpoint of each interval.



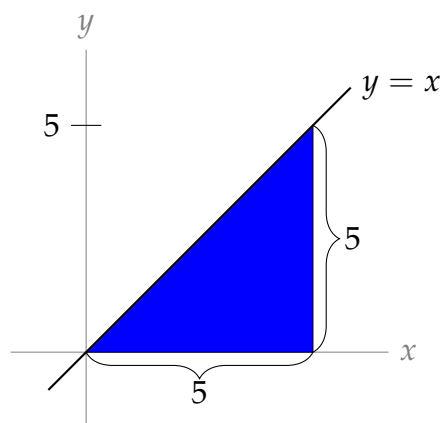
Then $a = 1.5$ and $b = 5.5$. Therefore: $\sum_{k=0}^3 f(1.5 + k) \cdot 1$ is a left Riemann sum on the interval $[1.5, 5.5]$ with $n = 4$.

Option 3: midpoint Riemann sum If our sum is a midpoint Riemann sum, then we take the heights of the rectangles from the midpoint of each interval.



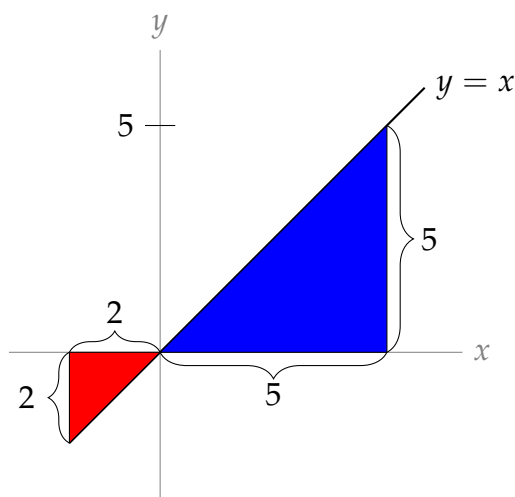
Then $a = 1$ and $b = 5$. Therefore: $\sum_{k=0}^3 f(1.5 + k) \cdot 1$ is a midpoint Riemann sum on the interval $[1, 5]$ with $n = 4$.

S-16: The area in question is a triangle with base 5 and height 5, so its area is $\frac{25}{2}$.



S-17:

There is a positive and a negative portion of this area. The positive area is a triangle with base 5 and height 5, so area $\frac{25}{2}$ square units. The negative area is a triangle with base 2 and height 2, so negative area $\frac{4}{2} = 2$ square units. So, the net area is $\frac{25}{2} - \frac{4}{2} = \frac{21}{2}$ square units.



S-18: Assuming row i holds x_i , you should have the value $a + \frac{b-a}{n}$ in cell A1, and the value b in cell An.

S-19: In general, the midpoint Riemann sum is given by

$$\sum_{i=1}^n f\left(a + (i - 1/2)\Delta x\right) \Delta x, \quad \text{where } \Delta x = \frac{b-a}{n}.$$

In this problem we are told that $f(x) = x^8$, $a = 5$, $b = 15$ and $n = 50$, so that $\Delta x = \frac{b-a}{n} = \frac{1}{5}$ and the desired Riemann sum is:

$$\sum_{i=1}^{50} \left(5 + (i - 1/2)\frac{1}{5}\right)^8 \frac{1}{5}$$

S-20: The given integral has interval of integration going from $a = -1$ to $b = 5$. So when we use three approximating rectangles, all of the same width, the common width is $\Delta x = \frac{b-a}{n} = 2$. The first rectangle has left endpoint $x_0 = a = -1$, the second has left hand endpoint $x_1 = a + \Delta x = 1$, and the third has left hand end point $x_2 = a + 2\Delta x = 3$. So

$$\int_{-1}^5 x^3 dx \approx [f(x_0) + f(x_1) + f(x_2)]\Delta x = [(-1)^3 + 1^3 + 3^3] \times 2 = 54$$

S-21: In the given integral, the domain of integration runs from $a = -1$ to $b = 7$. So, we have $\Delta x = \frac{(b-a)}{n} = \frac{(7-(-1))}{n} = \frac{8}{n}$. The left-hand end of the first subinterval is at $x_0 = a = -1$. So, the right-hand end of the i^{th} interval is at $x_i^* = -1 + \frac{8i}{n}$. So:

$$\int_{-1}^7 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(-1 + \frac{8i}{n}\right) \frac{8}{n}$$

S-22: We identify the given sum as the right Riemann sum $\sum_{i=1}^n f(a + i\Delta x)\Delta x$, with $a = 0$ (that's specified in the statement of the question). Since $\frac{4}{n}$ is multiplied in every term, and is also multiplied by i , we let $\Delta x = \frac{4}{n}$. Then $x_i^* = a + i\Delta x = \frac{4i}{n}$ and $f(x) = \sin^2(2 + x)$. So, $b = a + n\Delta x = 0 + n \cdot \frac{4}{n} = 4$.

S-23: The given sum is of the form

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2} \sqrt{1 - \frac{k^2}{n^2}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{n}\right) \frac{k}{n} \sqrt{1 - \left(\frac{k}{n}\right)^2} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta x f(x_k^*)$$

with $\Delta x = \frac{1}{n}$, $a = 0$, $x_k^* = \frac{k}{n} = a + k\Delta x$ and $f(x) = x\sqrt{1 - x^2}$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of $\int_0^1 f(x) \, dx$.

S-24: As i ranges from 1 to n , $3i/n$ range from $3/n$ to 3 with jumps of $\Delta x = 3/n$, so this is

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} e^{-i/n} \cos(3i/n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \int_a^b f(x) \, dx$$

where $x_i^* = 3i/n$, $f(x) = e^{-x/3} \cos(x)$, $a = x_0 = 0$ and $b = x_n = 3$. Thus

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} e^{-i/n} \cos(3i/n) = \int_0^3 e^{-x/3} \cos(x) \, dx$$

S-25: As i ranges from 1 to n , the exponent $\frac{i}{n}$ ranges from $\frac{1}{n}$ to 1 with jumps of $\Delta x = \frac{1}{n}$. So let's try $x_i^* = \frac{i}{n}$, $\Delta x = \frac{1}{n}$. Then:

$$R_n = \sum_{i=1}^n \frac{ie^{i/n}}{n^2} = \sum_{i=1}^n \frac{i}{n} e^{i/n} \frac{1}{n} = \sum_{i=1}^n x_i^* e^{x_i^*} \Delta x = \sum_{i=1}^n f(x_i^*) \Delta x$$

with $f(x) = xe^x$, and the limit

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \int_a^b f(x) \, dx$$

Since we chose $x_i^* = \frac{i}{n} = 0 + i\Delta x$, we let $a = 0$. Then $\frac{1}{n} = \Delta x = \frac{b-a}{n} = \frac{b}{n}$ tells us $b = 1$. Thus,

$$\lim_{n \rightarrow \infty} R_n = \int_0^1 xe^x \, dx.$$

S-26:

Choice #1: If we set $\Delta x = \frac{2}{n}$ and $x_i^* = \frac{2i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 0$, then

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-1-2i/n} \cdot \frac{2}{n} \right) &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-1-x_i^*} \Delta x \right) \\
&= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n f(x_i^*) \Delta x \right) \quad \text{with } f(x) = e^{-1-x} \\
&= \int_a^b f(x) \, dx \quad \text{with } a = x_0 = 0 \text{ and } b = x_n = 2 \\
&= \int_0^2 e^{-1-x} \, dx
\end{aligned}$$

Choice #2: If we set $\Delta x = \frac{2}{n}$ and $x_i^* = 1 + \frac{2i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 1$, then

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-1-2i/n} \cdot \frac{2}{n} \right) &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-x_i^*} \Delta x \right) \\
&= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n f(x_i^*) \Delta x \right) \quad \text{with } f(x) = e^{-x} \\
&= \int_a^b f(x) \, dx \quad \text{with } a = x_0 = 1 \text{ and } b = x_n = 3 \\
&= \int_1^3 e^{-x} \, dx
\end{aligned}$$

Choice #3: If we set $\Delta x = \frac{1}{n}$ and $x_i^* = \frac{i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 0$, then

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-1-2i/n} \cdot \frac{2}{n} \right) &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-1-2x_i^*} 2\Delta x \right) \\
&= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n f(x_i^*) \Delta x \right) \quad \text{with } f(x) = 2e^{-1-2x} \\
&= \int_a^b f(x) \, dx \quad \text{with } a = x_0 = 0 \text{ and } b = x_n = 1 \\
&= 2 \int_0^1 e^{-1-2x} \, dx
\end{aligned}$$

Choice #4: If we set $\Delta x = \frac{1}{n}$ and $x_i^* = \frac{1}{2} + \frac{i}{n}$, i.e. $x_i = a + i\Delta x$ with $a = \frac{1}{2}$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-1-2i/n} \cdot \frac{2}{n} \right) &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e^{-2x_i} 2\Delta x \right) \\ &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n f(x_i^*) \Delta x \right) \quad \text{with } f(x) = 2e^{-2x} \\ &= \int_a^b f(x) \, dx \quad \text{with } a = x_0 = \frac{1}{2} \text{ and } b = x_n = \frac{3}{2} \\ &= 2 \int_{1/2}^{3/2} e^{-2x} \, dx \end{aligned}$$

S-27: This is similar to the familiar form of a geometric sum, but the powers go up by threes. So, we make a substitution. If $x = r^3$, then:

$$1 + r^3 + r^6 + r^9 + \cdots + r^{3n} = 1 + x + x^2 + x^3 + \cdots + x^n$$

Now, using Equation 1.1.3 in the text,

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

Substituting back in $x = r^3$, we find our sum is equal to $\frac{(r^3)^{n+1} - 1}{r^3 - 1}$, or $\frac{r^{3n+3} - 1}{r^3 - 1}$.

S-28: The sum does not start at 1, so we need to do some algebra. We can either factor out the first term, or subtract off the initial terms that are missing.

Solution 1: If we factor out r^5 , then what's left fits the form of Equation 1.1.3 in the text:

$$r^5 + r^6 + r^7 + \cdots + r^{100} = r^5 \left[1 + r + r^2 + \cdots + r^{95} \right] = r^5 \left(\frac{r^{96} - 1}{r - 1} \right).$$

Solution 2: We know how to evaluate sums of this form if they start at 1, so we re-write our sum as follows:

$$\begin{aligned} r^5 + r^6 + r^7 + \cdots + r^{100} &= \left(1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^{100} \right) - \left(1 + r + r^2 + r^3 + r^4 \right) \\ &= \frac{r^{101} - 1}{r - 1} - \frac{r^5 - 1}{r - 1} \\ &= \frac{r^{101} - 1 - r^5 + 1}{r - 1} = \frac{r^{101} - r^5}{r - 1} = r^5 \left(\frac{r^{96} - 1}{r - 1} \right). \end{aligned}$$

S-29: Recall that

$$|x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$$

so that

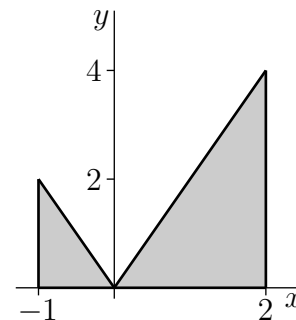
$$|2x| = \begin{cases} -2x & \text{if } x \leq 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

To picture the geometric figure whose area the integral represents observe that

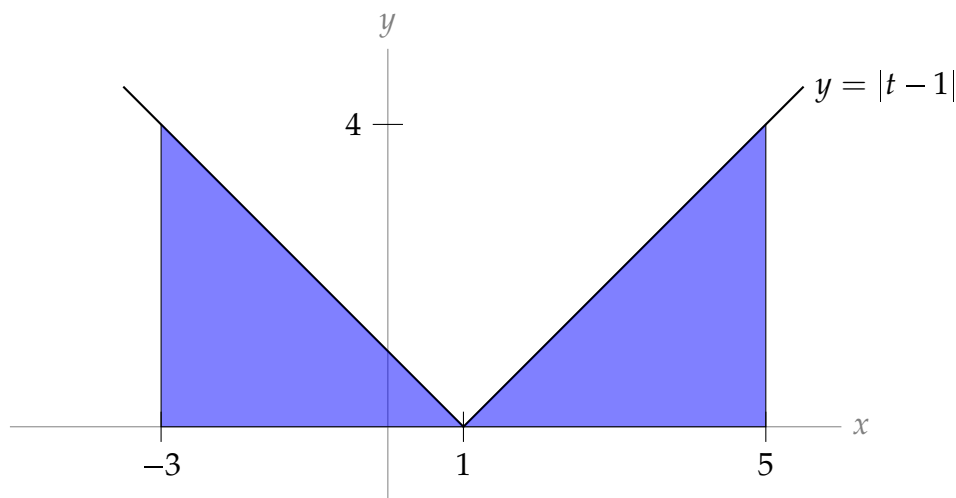
- at the left hand end of the domain of integration $x = -1$ and the integrand $|2x| = |-2| = 2$ and
- as x increases from -1 towards 0 , the integrand $|2x| = -2x$ decreases linearly, until
- when x hits 0 the integrand hits $|2x| = |0| = 0$ and then
- as x increases from 0 , the integrand $|2x| = 2x$ increases linearly, until
- when x hits $+2$, the right hand end of the domain of integration, the integrand hits $|2x| = |4| = 4$.

So the integral $\int_{-1}^2 |2x| \, dx$ is the area of the union of the two shaded triangles (one of base 1 and of height 2 and the other of base 2 and height 4) in the figure on the right below and

$$\int_{-1}^2 |2x| \, dx = \frac{1}{2} \times 1 \times 2 + \frac{1}{2} \times 2 \times 4 = 5$$



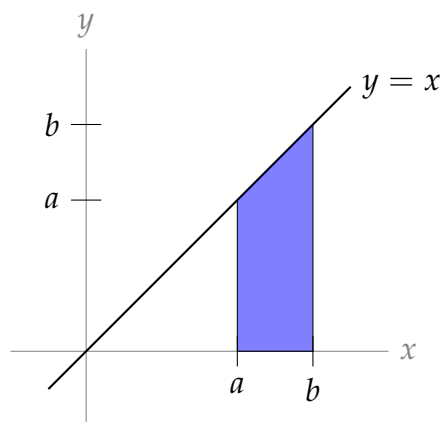
S-30: The area we want is two triangles, both above the x -axis. Each triangle has base 4 and height 4, so the total area is $2 \cdot \left(\frac{4 \cdot 4}{2}\right) = 16$.



If you had a hard time sketching the function, recall that the absolute value of a number leaves it unchanged if it is positive or zero, and flips the sign if it is negative. So, when

$t - 1 \geq 0$ (that is, when $t \geq 1$), our function is simply $f(t) = |t - 1| = t - 1$. On the other hand, when $t - 1$ is negative (that is, when $t < 1$), the absolute value changes the sign, so $f(t) = |t - 1| = -(t - 1) = -t + 1$.

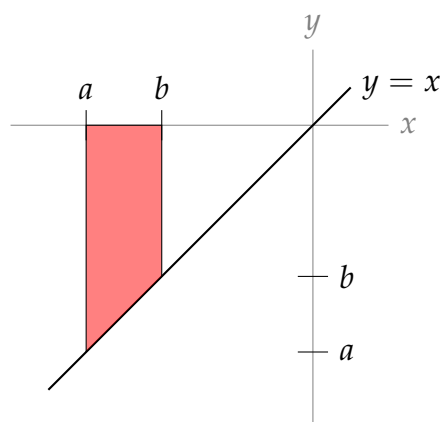
S-31: The area we want is a trapezoid with base $(b - a)$ and heights a and b , so its area is $\frac{(b - a)(b + a)}{2} = \frac{b^2 - a^2}{2}$.



Instead of using a formula for the area of a trapezoid, you can find the blue area as the area of a triangle with base and height b , minus the area of a triangle with base and height a .

S-32: The area is negative. The shape is a trapezoid with base length $(b - a)$ and heights $0 - a = -a$ and $0 - b = -b$ (note: those are nonnegative numbers), so its area is $\frac{(b - a)(-b - a)}{2} = \frac{-b^2 + a^2}{2}$. Since the shape is below the x -axis, we change its sign.

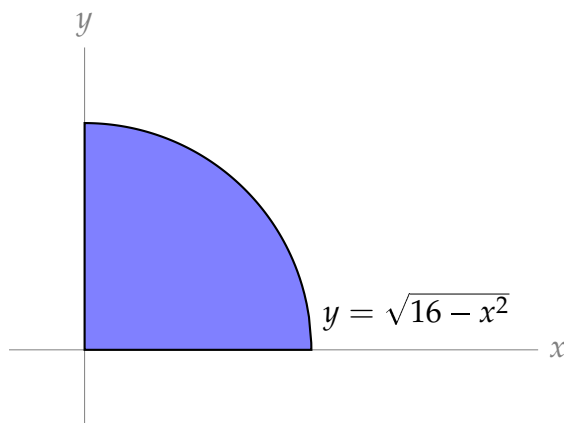
Thus, the integral evaluates to $\frac{b^2 - a^2}{2}$.



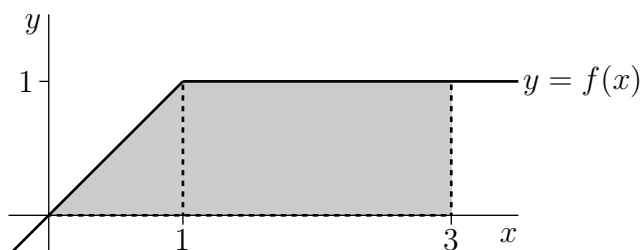
The signs can be a little hard to keep track of. The base of our trapezoid is $|a - b|$; since $b > a$, this is $b - a$. The heights of the trapezoid are $|a|$ and $|b|$; since these are both negative, $|a| = -a$ and $|b| = -b$.

We note that this is the same result as in Question 31.

S-33: If $y = \sqrt{16 - x^2}$, then y is nonnegative, and $y^2 + x^2 = 16$. So, the graph $y = \sqrt{16 - x^2}$ is the upper half of a circle of radius 4. Since x only runs from 0 to 4, we have a quarter of a circle of radius 4. Then the area under the curve is $\frac{1}{4} [\pi \cdot 4^2] = 4\pi$.



S-34: Here is a sketch the graph of $f(x)$.



There is a linear increase from $x = 0$ to $x = 1$, followed by a constant. Using the interpretation of $\int_0^3 f(x) \, dx$ as the area between $y = f(x)$ and the x -axis with x between 0 and 3, we can break this area into:

- $\int_0^1 f(x) \, dx$: a right-angled triangle of height 1 and base 1 and hence area 0.5.
- $\int_1^3 f(x) \, dx$: a rectangle of height 1 and base 2 and hence area 2.

Summing up: $\int_0^3 f(x) \, dx = 2.5$.

S-35: The car's speed increases with time. So its highest speed on any time interval occurs at the right hand end of the interval and the best possible upper estimate for the distance traveled is given by the *right* Riemann sum with $\Delta x = 0.5$, which is

$$[v(0.5) + v(1.0) + v(1.5) + v(2.0)] \times 0.5 = [14 + 22 + 30 + 40] \times 0.5 = 53 \text{ m}$$

S-36: There is a key detail in the statement of Question 35: namely, that the car is continuously accelerating. So, although we don't know exactly what's going on in

between our brief snippets of information, we know that the car is not going any faster during an interval than at the end of that interval. Therefore, the car certainly travelled no farther than our estimation.

We ask this question in order to point out an important detail. If we did not have the information that the car was continuously accelerating, we would not be able to give a certain upper bound on its distance travelled. It would be possible that, when the car is not being observed (for example, when $t = 0.25$), it is going much faster than when it is being observed.

S-37: First, note that the distance travelled by the plane is equal to the area under the graph of its speed.

We need to know the speed of the plane at the midpoints of our intervals. So (for example) noon to 1pm is *not* one of your intervals—we don't know the speed at 12:30. (A common idea is to average the two end values, 700 and 800. This is a fine approximation, but it is not a Riemann sum.) So, we use the two intervals 12:00 to 2:00, and 2:00 to 4:00. Then our intervals have length 2 hours, and at the midpoints of the intervals the speed of the plane is 700 kph and 900 kph, respectively. So, our midpoint Riemann sum gives us:

$$700(2) + 900(2) = 3200$$

an approximation of 3200 km travelled by the plane from noon to 4:00 pm.

Remark: if we had been asked to approximate the distance travelled from 11:30 am to 4:30 pm, then we could have used the midpoint rule with five intervals and made use of every entry in the data table. With the question as stated, however, we ignore three out of five entries in the table because they are not the midpoints of our intervals.

S-38: We can set up a spreadsheet like the one below:

	A	B
1	<code>= -10 + ROW() / 5</code>	<code>= 1 / (1 + (A1) ^ 2)</code>
2	↓	↓

and copy the entries down to row 100. Column A holds the x -values $a + i\Delta x = -10 + i/5$, where i is the row. Column B holds the values of the integrand at those points, $f(a + i\Delta x)$. To find the Riemann sum

$$\sum_{i=1}^{100} \Delta x \cdot f(a + i\Delta x) = \frac{1}{5} \sum_{i=1}^{100} f(a + i\Delta x),$$

we write `= .2 * SUM(B1 : B100)` into a blank cell, and find

$$\int_{-10}^{10} \frac{1}{1 + x^2} dx \approx 2.942242279.$$

S-39: We can set up a spreadsheet like the one below:

	A	B
1	<code>=-PI () +ROW () *2*PI () /75</code>	<code>=(SIN (A1)) ^2</code>
2	↓	↓

and copy the entries down to row 75. Column A holds the x -values $a + i\Delta x = -\pi + \frac{2\pi}{75}i$, where i is the row. Column B holds the values of the integrand at those points, $f(a + i\Delta x)$. To find the Riemann sum

$$\sum_{i=1}^{75} \Delta x \cdot f(a + i\Delta x) = \frac{2\pi}{75} \sum_{i=1}^{100} f(a + i\Delta x) ,$$

we write `=2*PI () /75*SUM (B1 :B75)` into a blank cell, and find

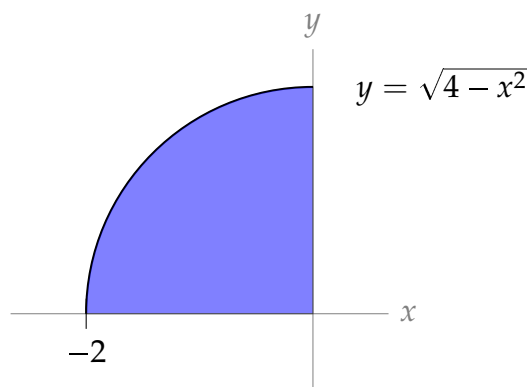
$$\int_{-\pi}^{\pi} \sin^2 x \, dx \approx 3.141592654 .$$

S-40:

Solution #1: Set $x_i^* = -2 + \frac{2i}{n}$. Then $a = x_0 = -2$ and $b = x_n = 0$ and $\Delta x = \frac{2}{n}$. So

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \sqrt{4 - \left(-2 + \frac{2i}{n}\right)^2} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \quad \text{with } f(x) = \sqrt{4 - x^2} \text{ and } \Delta x = \frac{2}{n} \\ &= \int_{-2}^0 \sqrt{4 - x^2} \, dx \end{aligned}$$

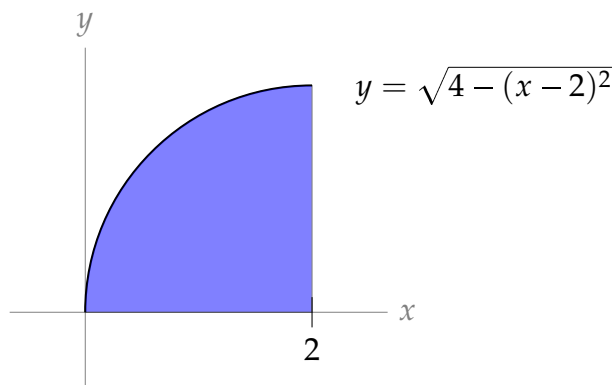
For the integral $\int_{-2}^0 \sqrt{4 - x^2} \, dx$, $y = \sqrt{4 - x^2}$ is equivalent to $x^2 + y^2 = 4$, $y \geq 0$. So the integral represents the area between the upper half of the circle $x^2 + y^2 = 4$ (which has radius 2) and the x -axis with $-2 \leq x \leq 0$, which is a quarter circle with area $\frac{1}{4} \cdot \pi 2^2 = \pi$.



Solution #2: Set $x_i^* = \frac{2i}{n}$. Then $a = x_0 = 0$ and $b = x_n = 2$ and $\Delta x = \frac{2}{n}$. So

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \sqrt{4 - \left(-2 + \frac{2i}{n}\right)^2} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \quad \text{with } f(x) = \sqrt{4 - (-2 + x)^2}, \Delta x = \frac{2}{n} \\ &= \int_0^2 \sqrt{4 - (-2 + x)^2} \, dx \end{aligned}$$

For the integral $\int_0^2 \sqrt{4 - (-2 + x)^2} \, dx$, $y = \sqrt{4 - (x - 2)^2}$ is equivalent to $(x - 2)^2 + y^2 = 4$, $y \geq 0$. So the integral represents the area between the upper half of the circle $(x - 2)^2 + y^2 = 4$ (which is centered at $(2, 0)$ and has radius 2) and the x -axis with $0 \leq x \leq 2$, which is a quarter circle with area $\frac{1}{4} \cdot \pi 2^2 = \pi$.



S-41: (a) The left Riemann sum is defined as

$$L_n = \sum_{i=1}^n f(x_{i-1}) \Delta x \quad \text{with } x_i = a + i \Delta x$$

We subdivide into $n = 3$ intervals, so that $\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$, $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. The function $f(x) = 7 + x^3$ has the values $f(x_0) = 7 + 0^3 = 7$, $f(x_1) = 7 + 1^3 = 8$, and $f(x_2) = 7 + 2^3 = 15$, from which we evaluate

$$L_3 = [f(x_0) + f(x_1) + f(x_2)] \Delta x = [7 + 8 + 15] \times 1 = 30$$

(b) We divide into n intervals so that $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = a + i \Delta x = \frac{3i}{n}$. The right Riemann sum is therefore:

$$R_n = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left[7 + \frac{(3i)^3}{n^3} \right] \frac{3}{n} = \sum_{i=1}^n \left[\frac{21}{n} + \frac{81 i^3}{n^4} \right]$$

To calculate the sum:

$$\begin{aligned} R_n &= \left(\frac{21}{n} \sum_{i=1}^n 1 \right) + \left(\frac{81}{n^4} \sum_{i=1}^n i^3 \right) \\ &= \left(\frac{21}{n} \times n \right) + \left(\frac{81}{n^4} \times \frac{n^4 + 2n^3 + n^2}{4} \right) \\ &= 21 + \frac{81}{4} (1 + 2/n + 1/n^2) \end{aligned}$$

To evaluate the limit exactly, we take $n \rightarrow \infty$. The expressions involving $1/n$ vanish leaving:

$$\int_0^3 (7 + x^3) \, dx = \lim_{n \rightarrow \infty} R_n = 21 + \frac{81}{4} = 41 \frac{1}{4}$$

S-42: In general, the right-endpoint Riemann sum approximation to the integral $\int_a^b f(x) \, dx$ using n rectangles is

$$\sum_{i=1}^n f(a + i\Delta x) \Delta x$$

where $\Delta x = \frac{b-a}{n}$. In this problem, $a = 2$, $b = 4$, and $f(x) = x^2$, so that $\Delta x = \frac{2}{n}$ and the right-endpoint Riemann sum approximation becomes

$$\begin{aligned} \sum_{i=1}^n f\left(2 + \frac{2i}{n}\right) \frac{2}{n} &= \sum_{i=1}^n \left(2 + \frac{2i}{n}\right)^2 \frac{2}{n} \\ &= \sum_{i=1}^n \left(4 + \frac{8i}{n} + \frac{4i^2}{n^2}\right) \frac{2}{n} \\ &= \sum_{i=1}^n \left(\frac{8}{n} + \frac{16i}{n^2} + \frac{8i^2}{n^3}\right) \\ &= \sum_{i=1}^n \frac{8}{n} + \sum_{i=1}^n \frac{16i}{n^2} + \sum_{i=1}^n \frac{8i^2}{n^3} \\ &= \frac{8}{n} \sum_{i=1}^n 1 + \frac{16}{n^2} \sum_{i=1}^n i + \frac{8}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{8}{n} n + \frac{16}{n^2} \cdot \frac{n(n+1)}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= 8 + 8\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \end{aligned}$$

So

$$\int_2^4 x^2 \, dx = \lim_{n \rightarrow \infty} \left[8 + 8\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \right] = 8 + 8 + \frac{4}{3} \times 2 = \frac{56}{3}$$

S-43: We'll use right Riemann sums with $a = 0$ and $b = 2$. When there are n rectangles,

$\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and $x_i = a + i\Delta x = 2i/n$. So we need to evaluate

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left((x_i)^3 + x_i \right) \Delta x \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\left(\frac{2i}{n} \right)^3 + \frac{2i}{n} \right) \frac{2}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left(\frac{8i^3}{n^3} + \frac{2i}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{16}{n^4} \sum_{i=1}^n i^3 + \frac{4}{n^2} \sum_{i=1}^n i \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{16(n^4 + 2n^3 + n^2)}{n^4 \cdot 4} + \frac{4(n^2 + n)}{n^2 \cdot 2} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{16}{4} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) + \frac{4}{2} \left(1 + \frac{1}{n} \right) \right) \\
 &= \frac{16}{4} + \frac{4}{2} = 6.
 \end{aligned}$$

S-44: We'll use right Riemann sums with $a = 1$, $b = 4$ and $f(x) = 2x - 1$. When there are n rectangles, $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = a + i\Delta x = 1 + 3i/n$. So we need to evaluate

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (2x_i - 1) \Delta x \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 + \frac{6i}{n} - 1 \right) \frac{3}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left(\frac{6i}{n} + 1 \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{18}{n^2} \sum_{i=1}^n i + \frac{3}{n} \sum_{i=1}^n 1 \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{18 \cdot n(n+1)}{n^2 \cdot 2} + \frac{3}{n} n \right) \\
 &= \lim_{n \rightarrow \infty} \left(9 \left(1 + \frac{1}{n} \right) + 3 \right) \\
 &= 9 + 3 = 12.
 \end{aligned}$$

S-45: Using the definition of a right Riemann sum,

$$\sum_{i=1}^{10} 3(7 + 2i)^2 \sin(4i) = \sum_{i=1}^{10} \Delta x f(a + i\Delta x)$$

Since $\Delta x = 10$ and $a = -5$,

$$\sum_{i=1}^{10} 3(7+2i)^2 \sin(4i) = \sum_{i=1}^{10} 10f(-5+10i)$$

Dividing both expressions by 10,

$$\sum_{i=1}^{10} \frac{3}{10}(7+2i)^2 \sin(4i) = \sum_{i=1}^{10} f(-5+10i)$$

So, we have an expression for $f(-5+10i)$:

$$f(-5+10i) = \frac{3}{10}(7+2i)^2 \sin(4i)$$

In order to find $f(x)$, let $x = -5+10i$. Then $i = \frac{x}{10} + \frac{1}{2}$.

$$\begin{aligned} f(x) &= \frac{3}{10} \left(7 + 2 \left(\frac{x}{10} + \frac{1}{2} \right) \right)^2 \sin \left(4 \left(\frac{x}{10} + \frac{1}{2} \right) \right) \\ &= \frac{3}{10} \left(\frac{x}{5} + 8 \right)^2 \sin \left(\frac{2x}{5} + 2 \right). \end{aligned}$$

S-46: As in the text, we'll set up a Riemann sum for the given integral. Right Riemann sums have the simplest form, so we use a right Riemann sum, but we could equally well use left or midpoint.

$$\begin{aligned} \int_0^1 2^x dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x f(a + i\Delta x) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} f\left(\frac{i}{n}\right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \cdot 2^{i/n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(2^{1/n} + 2^{2/n} + 2^{3/n} + \cdots + 2^{n/n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{2^{1/n}}{n} \left(1 + 2^{1/n} + 2^{2/n} + \cdots + 2^{\frac{n-1}{n}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{2^{1/n}}{n} \left(1 + 2^{1/n} + \left(2^{1/n} \right)^2 + \cdots + \left(2^{1/n} \right)^{n-1} \right) \end{aligned}$$

The sum in parenthesis has the form of a geometric sum, with $r = 2^{1/n}$:

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{2^{1/n}}{n} \left(\frac{(2^{1/n})^n - 1}{2^{1/n} - 1} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{2^{1/n}}{n} \left(\frac{2 - 1}{2^{1/n} - 1} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{2^{1/n}}{n(2^{1/n} - 1)}
 \end{aligned}$$

Note as $n \rightarrow \infty$, $1/n \rightarrow 0$, so the numerator has limit 1, while the denominator has indeterminate form $\infty \cdot 0$. So, we'll do a little algebra to get this into a l'Hôpital-style indeterminate form:

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \cdot 2^{1/n}}{2^{1/n} - 1} \\
 &= \lim_{n \rightarrow \infty} \underbrace{\frac{\frac{1}{n}}{1 - 2^{-1/n}}}_{\substack{\text{num} \rightarrow 0 \\ \text{den} \rightarrow 0}}
 \end{aligned}$$

Now we can use l'Hôpital's rule. Recall $\frac{d}{dx} \{2^x\} = 2^x \log 2$, where $\log x$ is the natural logarithm of x , also sometimes written $\ln x$. We'll need to use the chain rule when we differentiate the denominator.

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{\frac{-1}{n^2}}{-2^{-1/n} \log 2 \cdot \frac{1}{n^2}} \\
 &= \lim_{n \rightarrow \infty} \frac{2^{1/n}}{\log 2} \\
 &= \frac{1}{\log 2}
 \end{aligned}$$

Using a calculator, we see this is about 1.44 square units.

S-47: As in the text, we'll set up a Riemann sum for the given integral. Right Riemann

sums have the simplest form:

$$\begin{aligned}
\int_a^b 10^x dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x f(a + i \Delta x) \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} f\left(a + i \frac{b-a}{n}\right) \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} \cdot 10^{a + i \frac{b-a}{n}} \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} \cdot 10^a \cdot \left(10^{\frac{b-a}{n}}\right)^i \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \cdot 10^a \left(\left(10^{\frac{b-a}{n}}\right)^1 + \left(10^{\frac{b-a}{n}}\right)^2 + \left(10^{\frac{b-a}{n}}\right)^3 + \dots + \left(10^{\frac{b-a}{n}}\right)^n \right) \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \cdot 10^a \cdot 10^{\frac{b-a}{n}} \left(1 + \left(10^{\frac{b-a}{n}}\right) + \left(10^{\frac{b-a}{n}}\right)^2 + \dots + \left(10^{\frac{b-a}{n}}\right)^{n-1} \right)
\end{aligned}$$

Now the sum in parentheses has the form of a geometric sum, with $r = 10^{\frac{b-a}{n}}$:

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \cdot 10^a \cdot 10^{\frac{b-a}{n}} \left(\frac{\left(10^{\frac{b-a}{n}}\right)^n - 1}{10^{\frac{b-a}{n}} - 1} \right) \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \cdot 10^a \cdot 10^{\frac{b-a}{n}} \left(\frac{10^{b-a} - 1}{10^{\frac{b-a}{n}} - 1} \right)
\end{aligned}$$

The coloured parts do not depend on n , so for simplicity we can move them outside the limit.

$$\begin{aligned}
&= (b-a) \cdot 10^a \left(10^{b-a} - 1 \right) \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \left(\frac{10^{\frac{b-a}{n}}}{10^{\frac{b-a}{n}} - 1} \right) \\
&= (b-a) \cdot \left(10^b - 10^a \right) \lim_{n \rightarrow \infty} \underbrace{\left(\frac{1/n}{1 - 10^{-\frac{b-a}{n}}} \right)}_{\substack{\text{num} \rightarrow 0 \\ \text{den} \rightarrow 0}}
\end{aligned}$$

Now we can use l'Hôpital's rule. Recall $\frac{d}{dx} \{10^x\} = 10^x \log 10$, where $\log x$ is the natural logarithm of x , also sometimes written $\ln x$. For the denominator, we will have to use the chain rule.

$$\begin{aligned}
 &= (b-a) \cdot (10^b - 10^a) \lim_{n \rightarrow \infty} \left(\frac{-1/n^2}{-10^{-\frac{b-a}{n}} \cdot \log 10 \cdot \frac{b-a}{n^2}} \right) \\
 &= (b-a) \cdot (10^b - 10^a) \lim_{n \rightarrow \infty} \left(\frac{1}{10^{-\frac{b-a}{n}} \cdot \log 10 \cdot (b-a)} \right) \\
 &= (b-a) \cdot (10^b - 10^a) \left(\frac{1}{\log 10 \cdot (b-a)} \right) \\
 &= \frac{1}{\log 10} (10^b - 10^a)
 \end{aligned}$$

For part (b), we can guess that if 10 were changed to c , our answer would be

$$\int_a^b c^x dx = \frac{1}{\log c} (c^b - c^a)$$

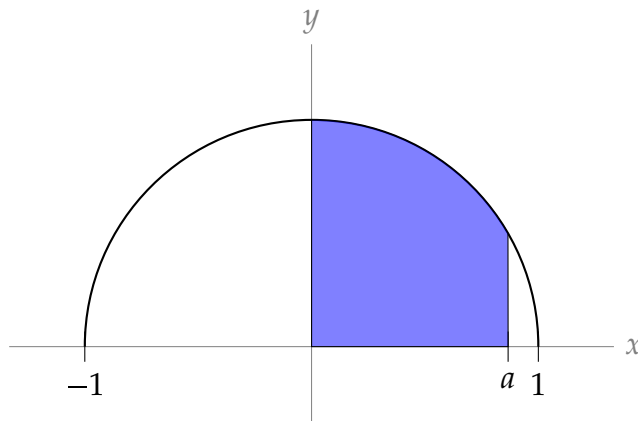
In Question 46, we had $a = 0$, $b = 1$, and $c = 2$. In this case, the formula we guessed above gives

$$\int_0^1 2^x dx = \frac{1}{\log 2} (2^1 - 2^0) = \frac{1}{\log 2}$$

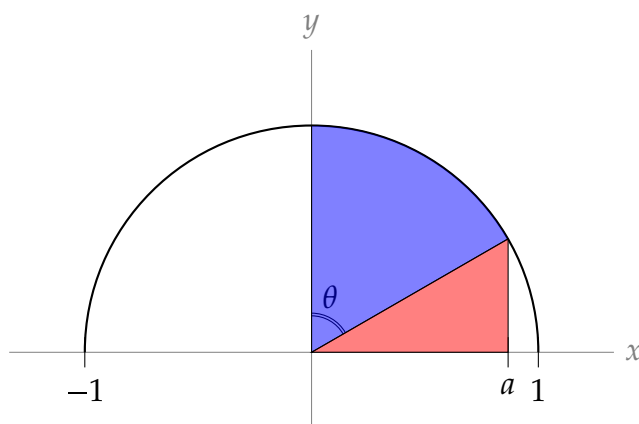
This does indeed match the answer we calculated.

(In fact, we can directly show $\int_a^b c^x dx = \frac{1}{\log c} (c^b - c^a)$ using the method of this problem.)

S-48: First, we note $y = \sqrt{1-x^2}$ is the upper half of a circle of radius 1, centred at the origin. We're taking the area under the curve from 0 to a , so the area in question is as shown in the picture below.



In order to use geometry to find this area, we break it up into two pieces: a sector of a circle, and a triangle, shown below.



Area of sector: The sector is a portion of a circle with radius 1, with inner angle θ . So, its area is $\frac{\theta}{2\pi}$ (area of circle) $= \frac{\theta}{2\pi} (\pi) = \frac{\theta}{2}$.

Our job now is to find θ in terms of a . Note $\frac{\pi}{2} - \theta$ is the inner angle of the red triangle, which lies in the unit circle. So, $\cos\left(\frac{\pi}{2} - \theta\right) = a$. Then $\frac{\pi}{2} - \theta = \arccos(a)$, and so $\theta = \frac{\pi}{2} - \arccos(a)$.

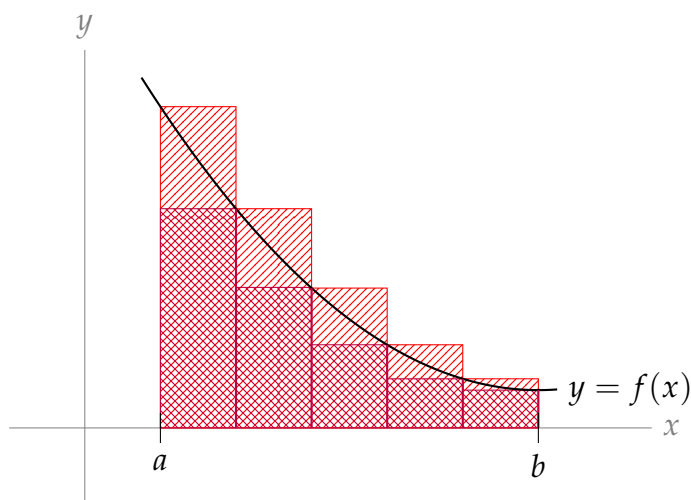
Then the area of the sector is $\frac{\pi}{4} - \frac{1}{2} \arccos(a)$ square units.

Area of triangle: The triangle has base a . Its height is the y -value of the function when $x = a$, so its height is $\sqrt{1 - a^2}$. Then the area of the triangle is $\frac{1}{2}a\sqrt{1 - a^2}$.

We conclude $\int_0^a \sqrt{1 - x^2} \, dx = \frac{\pi}{4} - \frac{1}{2} \arccos(a) + \frac{1}{2}a\sqrt{1 - a^2}$.

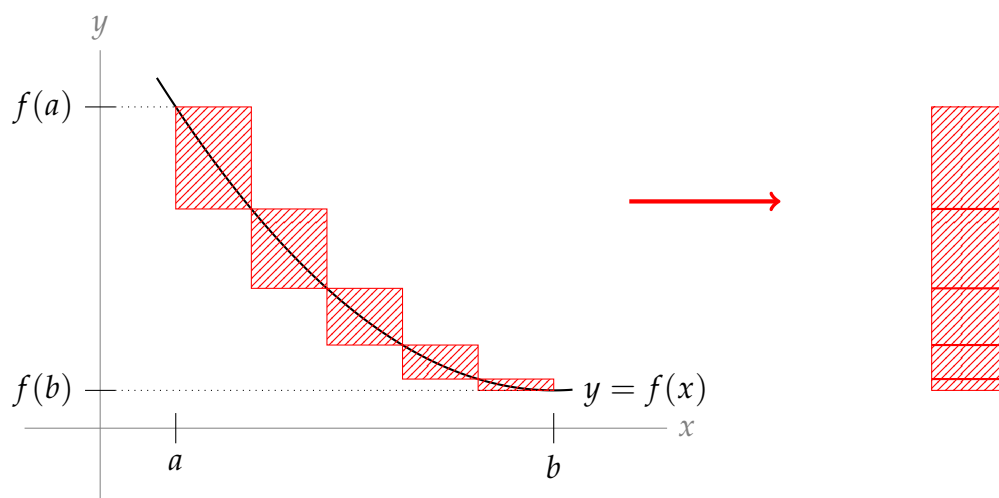
S-49:

- (a) The difference between our upper and lower bounds is the difference in areas between the larger set of rectangles and the smaller set of rectangles. Drawing them on a single picture makes this a little clearer.



Each of the rectangles has width $\frac{b-a}{n}$, since we took a segment of the x -axis with length $b - a$ and chopped it into n pieces. We could calculate the height of each rectangle, but it would be a little complicated, since it differs for each of them. An

easier method is to notice that the area we want to calculate can be imagined as a single rectangle:



The rectangle has base $\frac{b-a}{n}$. Its highest coordinate is $f(a)$, and its lowest is $f(b)$, so its height is $f(b) - f(a)$. Therefore, the difference in area between our lower bound and our upper bound is:

$$[f(b) - f(a)] \cdot \frac{b-a}{n}$$

- (b) We want to give a range with length at most 0.01, and guarantee that the area under the curve $y = f(x)$ is inside that range. In the previous part, we figured out that when we use n rectangles, the length of our range is $[f(b) - f(a)] \cdot \frac{b-a}{n}$. So, all we have to do is set this to be less than or equal to 0.01, and solve for n :

$$\begin{aligned} [f(b) - f(a)] \cdot \frac{b-a}{n} &\leq 0.01 \\ 100 [f(b) - f(a)] \cdot (b-a) &\leq n \end{aligned}$$

We can choose n to be an integer that is greater than or equal to $100 [f(b) - f(a)] \cdot (b-a)$. Using that many rectangles, we find an upper and lower bound for the area under the curve. If we choose any number between our upper and lower bound as an approximation for the area under the curve, our error is no more than 0.01.

Remark: this question depends on the fact that f is decreasing and positive from a to b . In general, bounding errors on approximations like this is not so straightforward.

S-50: Since $f(x)$ is linear, there exist real numbers m and c such that $f(x) = mx + c$. Now we can do some calculations. Suppose we have a rectangle in our Riemann sum that takes up the interval $[x, x + w]$.

- If we are using a left Riemann sum, our rectangle has height $f(x) = mx + c$. Then it has area $w(mx + c)$.
- If we are using a right Riemann sum, our rectangle has height $f(x + w) = m(x + w) + c = mx + c + mw$. Then it has area $w(mx + c + mw)$.

- If we are using a midpoint Riemann sum, our rectangle has height $f(x + \frac{1}{2}w) = m(x + \frac{1}{2}w) + c = mx + c + \frac{1}{2}mw$. Then it has area $w \left(mx + c + \frac{1}{2}w \right)$.

So, for each rectangle in our sums, the midpoint rectangle has the same area as the average of the left and right rectangles:

$$w \left(mx + c + \frac{1}{2}mw \right) = \frac{w(mx + c) + w(mx + c + mw)}{2}$$

It follows that the midpoint Riemann sum has a value equal to the average of the values of the left and right Riemann sums. To see this, let the rectangles in the midpoint Riemann sum have areas $M_1, M_2, \dots, M_n$, let the rectangles in the left Riemann sum have areas $L_1, L_2, \dots, L_n$, and let the rectangles in the right Riemann sum have areas $R_1, R_2, \dots, R_n$. Then the midpoint Riemann sum evaluates to $M_1 + M_2 + \dots + M_n$, and:

$$\begin{aligned} \frac{[L_1 + L_2 + \dots + L_n] + [R_1 + R_2 + \dots + R_n]}{2} &= \frac{L_1 + R_1}{2} + \frac{L_2 + R_2}{2} + \dots + \frac{L_n + R_n}{2} \\ &= M_1 + M_2 + \dots + M_n \end{aligned}$$

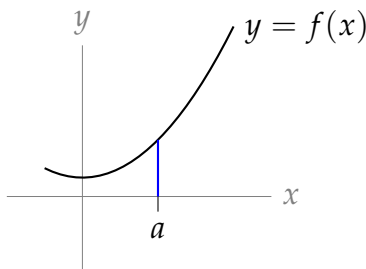
So, the statement is true.

(Note, however, it is false for many non-linear functions $f(x)$.)

Solutions to Exercises 1.2 — Jump to [TABLE OF CONTENTS](#)

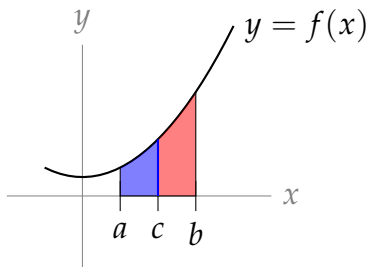
S-1:

(a) $\int_a^a f(x) dx = 0$



The area under the curve is zero, because it's a region with no width.

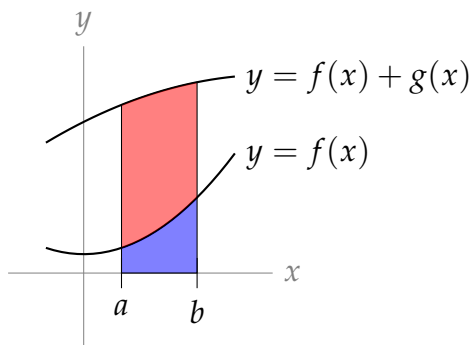
(b) $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$



If we assume $a \leq c \leq b$, then this identity simply tells us that if we add up the area under the curve from a to c , and from c to b , then we get the whole area under the curve from a to b .

(The situation is slightly more complicated when c is not between a and b , but it still works out.)

$$(c) \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$$



The blue-shaded area in the picture above is $\int_a^b f(x) \, dx$. The area under the curve $f(x) + g(x)$ but above the curve $f(x)$ (shown in red) is $\int_a^b g(x) \, dx$.

S-2: Using the identity

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx,$$

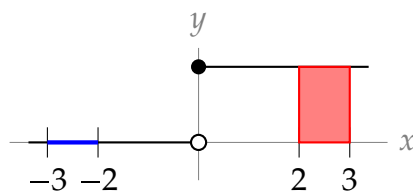
we see

$$\begin{aligned} \int_a^b \cos x \, dx &= \int_a^0 \cos x \, dx + \int_0^b \cos x \, dx \\ &= -\int_0^a \cos x \, dx + \int_0^b \cos x \, dx \\ &= -\sin a + \sin b \\ &= \sin b - \sin a \end{aligned}$$

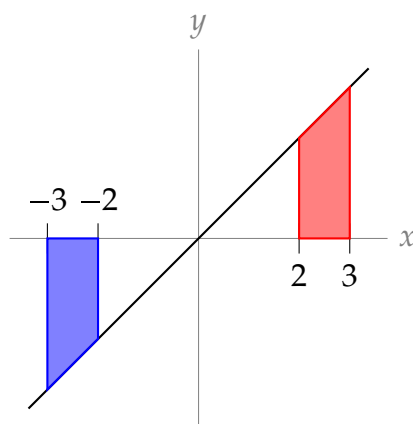
S-3: (a) False. For example if

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

then $\int_{-3}^2 f(x) \, dx = 0$ and $-\int_3^2 f(x) \, dx = \int_2^3 f(x) \, dx = 1$.



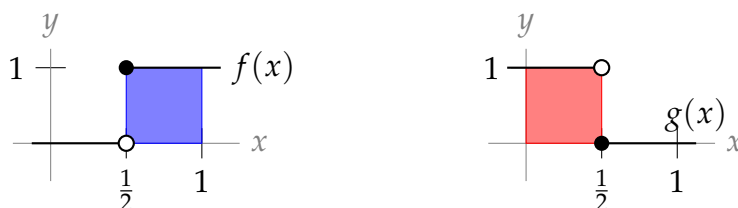
(b) False. For example, if $f(x) = x$, then $\int_{-3}^{-2} f(x) dx$ is negative while $\int_2^3 f(x) dx$ is positive, so they cannot be the same.



(c) False. For example, consider the functions

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} \\ 1 & \text{for } x \geq \frac{1}{2} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{for } x \geq \frac{1}{2} \\ 1 & \text{for } x < \frac{1}{2} \end{cases}$$

Then $f(x) \cdot g(x) = 0$ for all x , so $\int_0^1 f(x) \cdot g(x) dx = 0$. However, $\int_0^1 f(x) dx = \frac{1}{2}$ and $\int_0^1 g(x) dx = \frac{1}{2}$, so $\int_0^1 f(x) dx \cdot \int_0^1 g(x) dx = \frac{1}{4}$.



S-4:

$$(a) \Delta x = \frac{b-a}{n} = \frac{0-5}{100} = -\frac{1}{20}$$

Note: if we were to use the Riemann-sum definition of a definite integral, this is how we would justify the identity $\int_a^b f(x) dx = -\int_b^a f(x) dx$.

(b) The heights of the rectangles are given by $f(x_i)$, where $x_i = a + i\Delta x = 5 - \frac{i}{20}$. Since $f(x)$ only gives positive values, $f(x_i) > 0$, so the heights of the rectangles are positive.

(c) Our Riemann sum is the sum of the signed areas of individual rectangles. Each rectangle has a negative base (Δx) and a positive height ($f(x_i)$). So, each term of our sum is negative. If we add up negative numbers, the sum is negative. So, the Riemann sum is negative.

(d) Since $f(x)$ is always above the x -axis, $\int_0^5 f(x) dx$ is positive.

S-5: The operation of integration is linear (that's part (d) of the "arithmetic of integration" Theorem 1.2.1 in the text), so that:

$$\begin{aligned}\int_2^3 [6f(x) - 3g(x)] dx &= \int_2^3 6f(x) dx - \int_2^3 3g(x) dx \\ &= 6 \int_2^3 f(x) dx - 3 \int_2^3 g(x) dx = (6 \times (-1)) - (3 \times 5) = -21\end{aligned}$$

S-6: The operation of integration is linear (that's part (d) of the "arithmetic of integration" Theorem 1.2.1 in the text), so that:

$$\begin{aligned}\int_0^2 [2f(x) + 3g(x)] dx &= \int_0^2 2f(x) dx + \int_0^2 3g(x) dx \\ &= 2 \int_0^2 f(x) dx + 3 \int_0^2 g(x) dx = (2 \times 3) + (3 \times (-4)) = -6\end{aligned}$$

S-7: Using part (d) of the "arithmetic of integration" Theorem 1.2.1, followed by parts (c) and (b) of the "arithmetic for the domain of integration" Theorem 1.2.3 in the in the text,

$$\begin{aligned}\int_{-1}^2 [3g(x) - f(x)] dx &= 3 \int_{-1}^2 g(x) dx - \int_{-1}^2 f(x) dx \\ &= 3 \int_{-1}^0 g(x) dx + 3 \int_0^2 g(x) dx - \int_{-1}^0 f(x) dx - \int_0^2 f(x) dx \\ &= 3 \int_{-1}^0 g(x) dx + 3 \int_0^2 g(x) dx + \int_0^{-1} f(x) dx - \int_0^2 f(x) dx \\ &= 3 \times 3 + 3 \times 4 + 1 - 2 = 20\end{aligned}$$

S-8:

(a) Since $\sqrt{1-x^2}$ is an even function,

$$\begin{aligned}\int_a^0 \sqrt{1-x^2} dx &= \int_0^{|a|} \sqrt{1-x^2} dx = \frac{\pi}{4} - \frac{1}{2} \arccos(|a|) + \frac{1}{2}|a|\sqrt{1-|a|^2} \\ &= \frac{\pi}{4} - \frac{1}{2} \arccos(-a) - \frac{1}{2}a\sqrt{1-a^2}\end{aligned}$$

Alternatively, since $\arccos(-a) = \pi - \arccos(a)$ we also have

$$\int_a^0 \sqrt{1-x^2} \, dx = -\frac{\pi}{4} + \frac{1}{2} \arccos(a) - \frac{1}{2} a \sqrt{1-a^2}$$

(b) Note $\int_0^1 \sqrt{1-x^2} \, dx = \frac{\pi}{4}$, since the area under the curve represents one-quarter of the unit circle. Then,

$$\begin{aligned} \int_a^1 \sqrt{1-x^2} \, dx &= \int_0^1 \sqrt{1-x^2} \, dx - \int_0^a \sqrt{1-x^2} \, dx \\ &= \frac{\pi}{4} - \left(\frac{\pi}{4} - \frac{1}{2} \arccos(a) + \frac{1}{2} a \sqrt{1-a^2} \right) \\ &= \frac{1}{2} \arccos(a) - \frac{1}{2} a \sqrt{1-a^2} \end{aligned}$$

S-9: Recall that

$$|x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$$

so that

$$|2x| = \begin{cases} -2x & \text{if } x \leq 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

Also recall, from Example 1.2.6 in the text that

$$\int_a^b x \, dx = \frac{b^2 - a^2}{2}$$

So

$$\begin{aligned} \int_{-1}^2 |2x| \, dx &= \int_{-1}^0 |2x| \, dx + \int_0^2 |2x| \, dx = \int_{-1}^0 (-2x) \, dx + \int_0^2 2x \, dx \\ &= -2 \int_{-1}^0 x \, dx + 2 \int_0^2 x \, dx = -2 \cdot \frac{0^2 - (-1)^2}{2} + 2 \cdot \frac{2^2 - 0^2}{2} \\ &= 1 + 4 = 5 \end{aligned}$$

S-10: We note that the integrand $f(x) = x|x|$ is an odd function, because $f(-x) = -x|-x| = -x|x| = -f(x)$. Then, by Theorem 1.2.12.b in the text,

$$\int_{-5}^5 x|x| \, dx = 0.$$

S-11: Using Theorem 1.2.12.a in the text,

$$10 = \int_{-2}^2 f(x) dx = 2 \int_0^2 f(x) dx$$
$$5 = \int_0^2 f(x) dx$$

Also,

$$\int_{-2}^2 f(x) dx = \int_{-2}^0 f(x) dx + \int_0^2 f(x) dx$$

So,

$$\begin{aligned} \int_{-2}^0 f(x) dx &= \int_{-2}^2 f(x) dx - \int_0^2 f(x) dx \\ &= 10 - 5 = 5 \end{aligned}$$

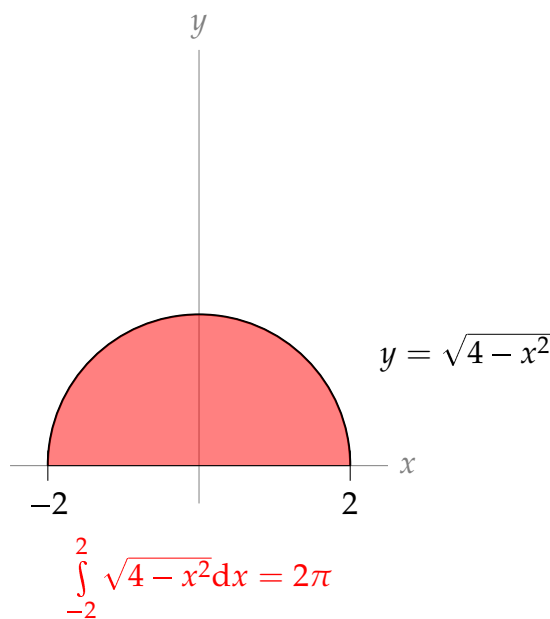
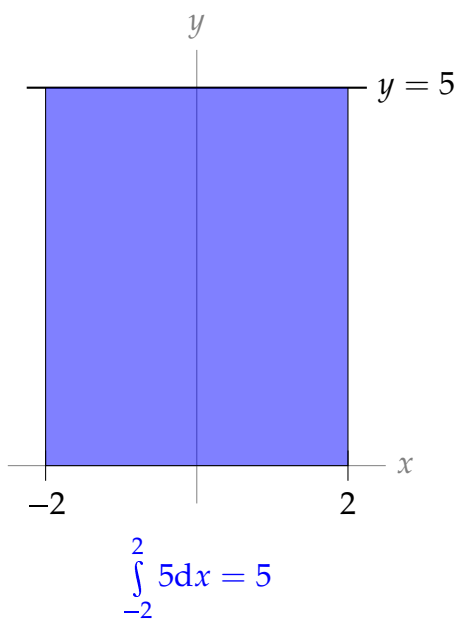
Indeed, for any even function $f(x)$, $\int_{-a}^0 f(x) dx = \int_0^a f(x) dx$.

S-12: We first use additivity:

$$\int_{-2}^2 (5 + \sqrt{4 - x^2}) dx = \int_{-2}^2 5 dx + \int_{-2}^2 \sqrt{4 - x^2} dx$$

The first integral represents the area of a rectangle of height 5 and width 4 and so equals 20. The second integral represents the area above the x -axis and below the curve $y = \sqrt{4 - x^2}$ or $x^2 + y^2 = 4$. That is a semicircle of radius 2, which has area $\frac{1}{2}\pi 2^2$. So

$$\int_{-2}^2 (5 + \sqrt{4 - x^2}) dx = 20 + 2\pi$$



S-13: Note that the integrand $f(x) = \frac{\sin x}{\log(3+x^2)}$ is an odd function, because:

$$f(-x) = \frac{\sin(-x)}{\log(3+(-x)^2)} = \frac{-\sin x}{\log(3+x^2)} = -f(x)$$

The domain of integration $-2012 \leq x \leq 2012$ is symmetric about $x = 0$. So, by Theorem 1.2.12 of the text,

$$\int_{-2012}^{+2012} \frac{\sin x}{\log(3+x^2)} dx = 0$$

S-14: Note that the integrand $f(x) = x^{1/3} \cos x$ is an odd function, because:

$$f(-x) = (-x)^{1/3} \cos(-x) = -x^{1/3} \cos x = -f(x)$$

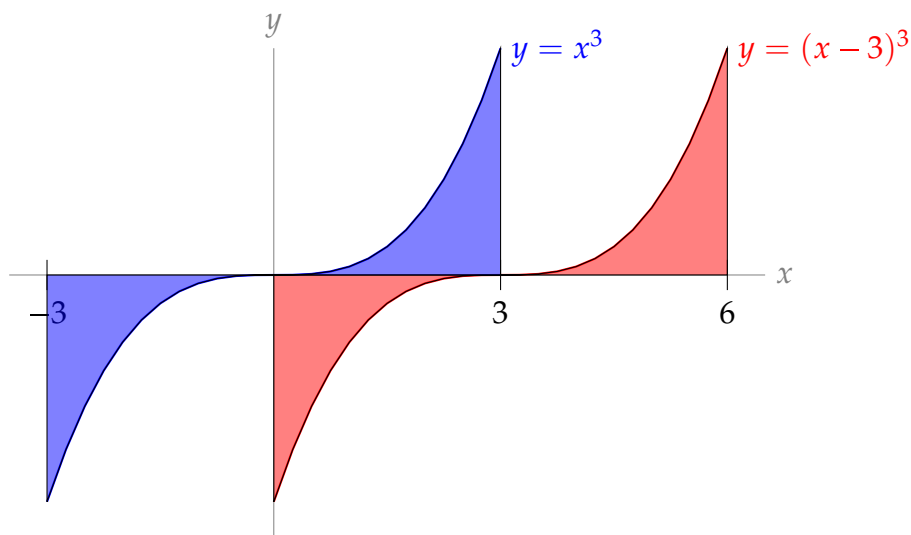
The domain of integration $-2012 \leq x \leq 2012$ is symmetric about $x = 0$. So, by Theorem 1.2.12 of the text,

$$\int_{-2012}^{+2012} x^{1/3} \cos x dx = 0$$

S-15: Our integrand $f(x) = (x-3)^3$ is neither even nor odd. However, it does have a similar symmetry. Namely, $f(3+x) = -f(3-x)$. So, f is “negatively symmetric” across the line $x = 3$. This suggests that the integral should be 0: the positive area to the right of $x = 3$ will be the same as the negative area to the left of $x = 3$.

Another way to see this is to notice that the graph of $f(x) = (x-3)^3$ is equivalent to the graph of $g(x) = x^3$ shifted three units to the right, and $g(x)$ is an odd function. So,

$$\int_0^6 (x-3)^3 dx = \int_{-3}^3 x^3 dx = 0$$



S-16:

(a)

$$\begin{aligned}(ax)^2 + (by)^2 &= 1 \\ by &= \sqrt{1 - (ax)^2} \\ y &= \frac{1}{b} \sqrt{1 - (ax)^2}\end{aligned}$$

(b) The values of x in the domain of the function above are those that satisfy $1 - (ax)^2 \geq 0$. That is, $-\frac{1}{a} \leq x \leq \frac{1}{a}$. Therefore, the upper half of the ellipse has area

$$\frac{1}{b} \int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{1 - (ax)^2} \, dx$$

The upper half of a circle has equation $y = \sqrt{r^2 - x^2}$.

$$\begin{aligned}&= \frac{1}{b} \int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{a^2 \left(\frac{1}{a^2} - x^2 \right)} \, dx \\&= \frac{1}{b} \int_{-\frac{1}{a}}^{\frac{1}{a}} a \sqrt{\frac{1}{a^2} - x^2} \, dx \\&= \frac{a}{b} \int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{\frac{1}{a^2} - x^2} \, dx\end{aligned}$$

(c) The function $y = \sqrt{\frac{1}{a^2} - x^2}$ is the upper-half of the circle centred at the origin with radius $\frac{1}{a}$. So, the expression from (b) evaluates to $\left(\frac{a}{b}\right) \frac{\pi}{2a^2} = \frac{\pi}{2ab}$.

The expression from (b) was half of the ellipse, so the area of the ellipse is $\frac{\pi}{ab}$.

Remark: this was a slightly long-winded way of getting the result. The reasoning is basically this:

- The area of the unit circle $x^2 + y^2 = 1$ is π .
- The ellipse $(ax)^2 + y^2 = 1$ is obtained by shrinking the unit circle horizontally by a factor of a . So, its area is $\frac{\pi}{a}$.
- Further, the ellipse $(ax)^2 + (by)^2 = 1$ is obtained from the previous ellipse by shrinking it vertically by a factor of b . So, its area is $\frac{\pi}{ab}$.

S-17: Let's recall the definitions of even and odd functions: $f(x)$ is *even* if $f(-x) = f(x)$ for every x in its domain, and $f(x)$ is *odd* if $f(-x) = -f(x)$ for every x in its domain.

Let $h(x) = f(x) \cdot g(x)$.

even $\times$ even: If f and g are both even, then

$$h(-x) = f(-x) \cdot g(-x) = f(x) \cdot g(x) = h(x), \text{ so their product is even.}$$

odd $\times$ odd: If f and g are both odd, then

$$h(-x) = f(-x) \cdot g(-x) = [-f(x)] \cdot [-g(x)] = f(x) \cdot g(x) = h(x), \text{ so their product is even.}$$

even $\times$ odd: If f is even and g is odd, then

$$h(-x) = f(-x) \cdot g(-x) = f(x) \cdot [-g(x)] = -[f(x) \cdot g(x)] = -h(x), \text{ so their product is odd. Because multiplication is commutative, the order we multiply the functions in doesn't matter.}$$

We note that the table would be the same as if we were *adding* (not multiplying) even and odd *numbers* (not functions).

S-18: Since $f(x)$ is odd, $f(0) = -f(-0) = -f(0)$. So, $f(0) = 0$.

However, this restriction does not apply to $g(x)$. For example, for any constant c , let $g(x) = c$. Then $g(x)$ is even and $g(0) = c$. So, $g(0)$ can be any real number.

S-19: Let x be any real number.

- $f(x) = f(-x)$ (since $f(x)$ is even), and
- $f(x) = -f(-x)$ (since $f(x)$ is odd).
- So, $f(x) = -f(x)$.
- Then (adding $f(x)$ to both sides) we see $2f(x) = 0$, so $f(x) = 0$.

So, $f(x) = 0$ for every x .

S-20:

Solution 1: Suppose $f(x)$ is an odd function. We investigate $f'(x)$ using the chain rule:

$$\begin{aligned} f(-x) &= -f(x) && \text{(odd function)} \\ \frac{d}{dx}\{f(-x)\} &= \frac{d}{dx}\{-f(x)\} \\ -f'(-x) &= -f'(x) && \text{(chain rule)} \\ f'(-x) &= f'(x) \end{aligned}$$

So, when $f(x)$ is odd, $f'(x)$ is even.

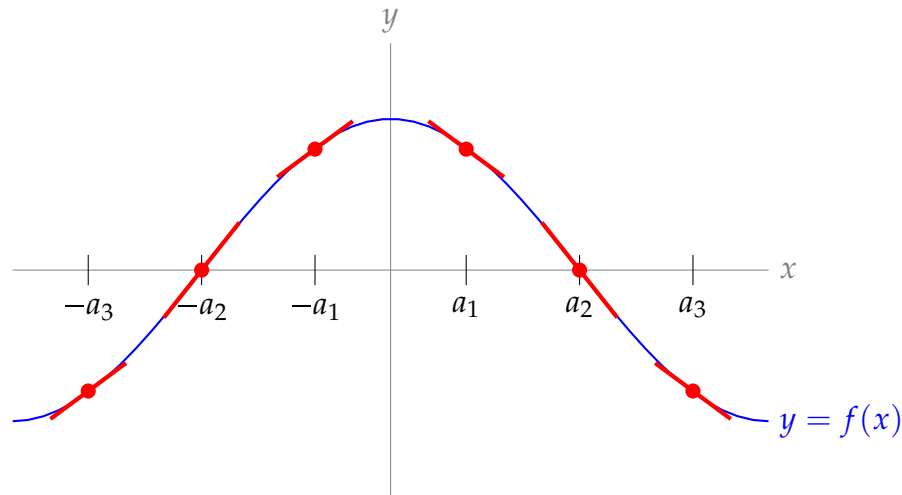
Similarly, suppose $f(x)$ is even.

$$\begin{aligned}
 f(-x) &= f(x) && \text{(even function)} \\
 \frac{d}{dx}\{f(-x)\} &= \frac{d}{dx}\{f(x)\} \\
 -f'(-x) &= f'(x) && \text{(chain rule)} \\
 f'(-x) &= -f'(x)
 \end{aligned}$$

So, when $f(x)$ is even, $f'(x)$ is odd.

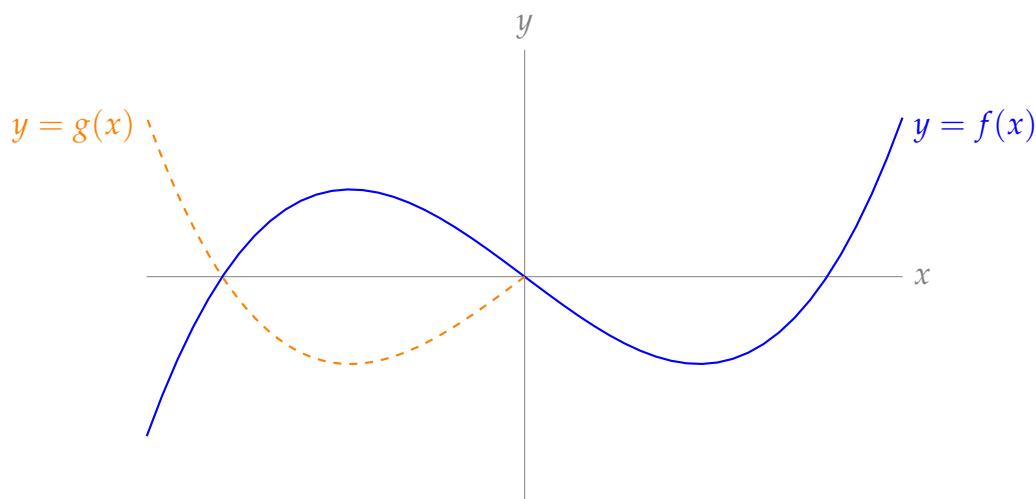
Solution 2: Another way to think about this problem is to notice that “mirroring” a function changes the sign of its derivative. Then since an even function is “mirrored once” (across the y -axis), it should have $f'(x) = -f'(-x)$, and so the derivative of an even function should be an odd function. Since an odd function is “mirrored twice” (across the y -axis and across the x -axis), it should have $f'(x) = -(-f'(-x)) = f'(-x)$. So the derivative of an odd function should be even. These ideas are presented in more detail below.

First, we consider the case where $f(x)$ is even, and investigate $f'(x)$.

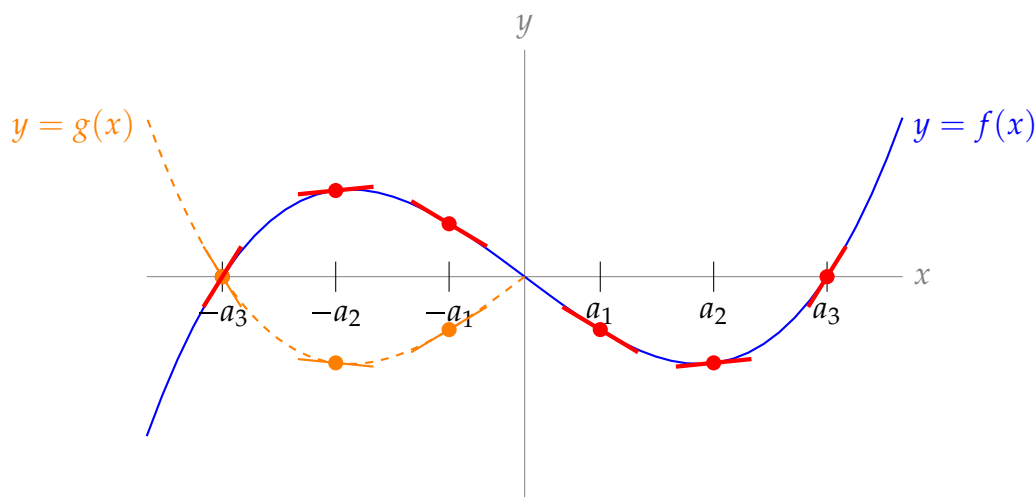


The whole function has a mirror-like symmetry across the y -axis. So, at x and $-x$, the function will have the same “steepness,” but if one is increasing then the other is decreasing. That is, $f'(-x) = -f'(x)$. (In the picture above, compare the slope at some point a_i with its corresponding point $-a_i$.) So, $f'(x)$ is odd when $f(x)$ is even.

Second, let’s consider the case where $f(x)$ is odd, and investigate $f'(x)$. Suppose the blue graph below is $y = f(x)$. If $f(x)$ were *even*, then to the left of the y -axis, it would look like the orange graph, which we’ll call $y = g(x)$.



From our work above, we know that, for every $x > 0$, $-f'(x) = g'(-x)$. When $x < 0$, $f(x) = -g(x)$. So, if $x > 0$, then $-f'(x) = g'(-x) = -f'(-x)$. In other words, $f'(x) = f'(-x)$. Similarly, if $x < 0$, then $f'(x) = -g'(x) = f'(-x)$. Therefore $f'(x)$ is even. (In the graph below, you can anecdotally verify that $f'(a_i) = f'(-a_i)$.)



Solutions to Exercises 1.3 — [Jump to TABLE OF CONTENTS](#)

S-1: The Fundamental Theorem of Calculus Part 2 (Theorem 1.3.1 in the text) tells us that

$$\begin{aligned} \int_1^{\sqrt{5}} f(x) \, dx &= F(\sqrt{5}) - F(1) \\ &= (e^{(\sqrt{5}^2-3)} + 1) - (e^{(1^2-3)} + 1) \\ &= e^{5-3} - e^{1-3} = e^2 - e^{-2} \end{aligned}$$

S-2: First, let's find a general antiderivative of $x^3 - \sin(2x)$.

- One function with derivative x^3 is $\frac{x^4}{4}$.
- To find an antiderivative of $\sin(2x)$, we might first guess $\cos(2x)$; checking, we see $\frac{d}{dx}\{\cos(2x)\} = -2\sin(2x)$. So, we only need to multiply by $-\frac{1}{2}$:

$$\frac{d}{dx}\left\{-\frac{1}{2}\cos 2x\right\} = \sin(2x).$$

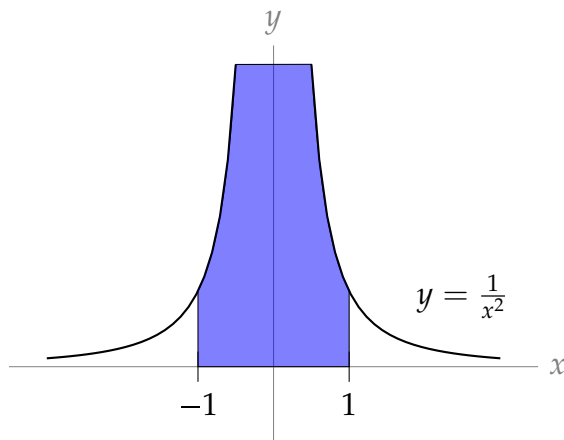
So, the general antiderivative of $f(x)$ is $\frac{x^4}{4} + \frac{1}{2}\cos 2x + C$. To satisfy $F(0) = 1$, we need³

$$\left[\frac{x^4}{4} + \frac{1}{2}\cos 2x + C\right]_{x=0} = 1 \iff \frac{1}{2} + C = 1 \iff C = \frac{1}{2}$$

So $F(x) = \frac{x^4}{4} + \frac{1}{2}\cos 2x + \frac{1}{2}$.

S-3: (a) This is true, by part 2 of the Fundamental Theorem of Calculus, Theorem 1.3.1 in the text with $G(x) = f(x)$ and $f(x)$ replaced by $f'(x)$.

(b) This is not only false, but it makes no sense at all. The integrand is strictly positive so the integral has to be strictly positive. In fact it's $+\infty$. The Fundamental Theorem of Calculus does not apply because the integrand has an infinite discontinuity at $x = 0$.



(c) This is not only false, but it makes no sense at all, unless $\int_a^b f(x) dx = \int_a^b x f(x) dx = 0$. The left hand side is a number. The right hand side is a number times x .

$$\underbrace{\int_a^b x f(x) dx}_{\text{area}} \quad \text{vs} \quad \underbrace{x}_{\text{variable}} \cdot \underbrace{\int_a^b f(x) dx}_{\text{area}}$$

For example, if $a = 0$, $b = 1$ and $f(x) = 1$, then the left hand side is $\int_0^1 x dx = \frac{1}{2}$ and the

3 The symbol $\iff$ is read "if and only if". This is used in mathematics to express the logical equivalence of two statements. To be more precise, the statement $P \iff Q$ tells us that P is true whenever Q is true and Q is true whenever P is true.

right hand side is $x \int_0^1 dx = x$.

S-4: This is a tempting thought:

$$\int \frac{1}{x} dx = \log |x| + C$$

so perhaps similarly

$$\int \frac{1}{x^2} dx \stackrel{?}{=} \log |x^2| + C = \log(x^2) + C$$

We check by differentiating:

$$\frac{d}{dx} \{\log(x^2)\} = \frac{d}{dx} \{2 \log x\} = \frac{2}{x} \neq \frac{1}{x^2}$$

So, it wasn't so easy: false.

When we're guessing antiderivatives, we often need to adjust our original guesses a little. Changing constants works well; changing functions usually does not.

S-5: This is tempting:

$$\frac{d}{dx} \{\sin(e^x)\} = e^x \cos(e^x)$$

so perhaps

$$\frac{d}{dx} \left\{ \frac{\sin(e^x)}{e^x} \right\} \stackrel{?}{=} \cos(e^x)$$

We check by differentiating:

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{\sin(e^x)}{e^x} \right\} &= \frac{e^x (\cos(e^x) \cdot e^x) - \sin(e^x) e^x}{e^{2x}} && \text{(quotient rule)} \\ &= \cos(e^x) - \frac{\sin(e^x)}{e^x} \\ &\neq \cos(e^x) \end{aligned}$$

So, the statement is false.

When we're guessing antiderivatives, we often need to adjust our original guesses a little. Dividing by constants works well; dividing by functions usually does not.

S-6: "The instantaneous rate of change of $F(x)$ with respect to x " is another way of saying " $F'(x)$ ". From the Fundamental Theorem of Calculus Part 1, we know this is $\sin(x^2)$.

S-7: The slope of the tangent line to $y = F(x)$ when $x = 3$ is exactly $F'(3)$. By the Fundamental Theorem of Calculus Part 1, $F'(x) = e^{1/x}$. Then $F'(3) = e^{1/3} = \sqrt[3]{e}$.

S-8: For any constant C , $F(x) + C$ is an antiderivative of $f(x)$, because $\frac{d}{dx}\{F(x) + C\} = \frac{d}{dx}\{F(x)\} = f(x)$. So, for example, $F(x)$ and $F(x) + 1$ are both antiderivatives of $f(x)$.

S-9:

(a) We differentiate with respect to a . Recall $\frac{d}{dx}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$. To differentiate $\frac{1}{2}a\sqrt{1-a^2}$, we use the product and chain rules.

$$\begin{aligned}\frac{d}{da} \left\{ \frac{\pi}{4} - \frac{1}{2} \arccos(a) + \frac{1}{2} a \sqrt{1-a^2} \right\} &= 0 - \frac{1}{2} \cdot \frac{-1}{\sqrt{1-a^2}} + \left(\frac{1}{2} a \right) \cdot \frac{-2a}{2\sqrt{1-a^2}} + \frac{1}{2} \sqrt{1-a^2} \\ &= \frac{1}{2\sqrt{1-a^2}} - \frac{a^2}{2\sqrt{1-a^2}} + \frac{1-a^2}{2\sqrt{1-a^2}} \\ &= \frac{1-a^2+1-a^2}{2\sqrt{1-a^2}} \\ &= \frac{2(1-a^2)}{2\sqrt{1-a^2}} \\ &= \sqrt{1-a^2}\end{aligned}$$

(b) Let $G(x) = \frac{\pi}{4} - \frac{1}{2} \arccos(x) + \frac{1}{2} x \sqrt{1-x^2}$. We showed in part (a) that $G(x)$ is an antiderivative of $\sqrt{1-x^2}$. Since $F(x)$ is also an antiderivative of $\sqrt{1-x^2}$, $F(x) = G(x) + C$ for some constant C (this is Lemma 1.3.8 in the text).

Note $G(0) = \int_0^0 \sqrt{1-x^2} dx = 0$, so if $F(0) = \pi$, then $F(x) = G(x) + \pi$. That is,

$$F(x) = \frac{5\pi}{4} - \frac{1}{2} \arccos(x) + \frac{1}{2} x \sqrt{1-x^2}.$$

S-10:

(a) The antiderivative of $\cos x$ is $\sin x$, and $\cos x$ is continuous everywhere, so

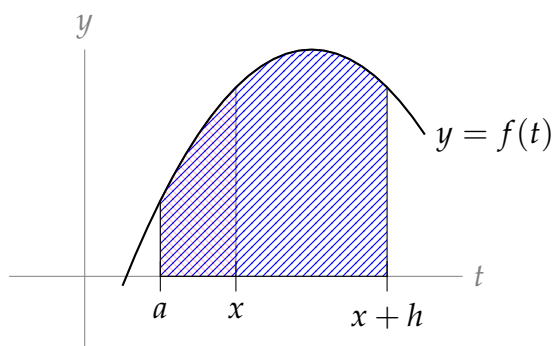
$$\int_{-\pi}^{\pi} \cos x dx = \sin(\pi) - \sin(-\pi) = 0.$$

(b) Since $\sec^2 x$ is discontinuous at $x = \pm \frac{\pi}{2}$, the Fundamental Theorem of Calculus Part 2 does not apply to $\int_{-\pi}^{\pi} \sec^2 x dx$.

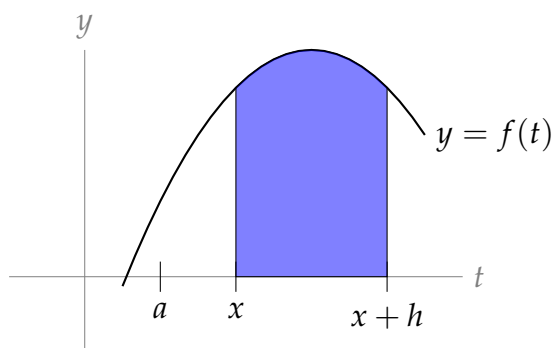
(c) Since $\frac{1}{x+1}$ is discontinuous at $x = -1$, the Fundamental Theorem of Calculus Part 2 does not apply to $\int_{-2}^0 \frac{1}{x+1} dx$.

S-11:

Using the definition of F , $F(x)$ is the area under the curve from a to x , and $F(x+h)$ is the area under the curve from a to $x+h$. These are shown on the same diagram, below.



Then the area represented by $F(x+h) - F(x)$ is the area that is outside the red, but inside the blue. Equivalently, it is $\int_x^{x+h} f(t) dt$.



S-12: We evaluate $F(0)$ using the definition: $F(0) = \int_0^0 f(t) dt = 0$. Although $f(0) > 0$, the area from $t = 0$ to $t = 0$ is zero.

As x moves along, $F(x)$ adds bits of signed area. If it's adding positive area, it's increasing, and if it's adding negative area, it's decreasing. So, $F(x)$ is increasing when $0 < x < 1$ and $3 < x < 4$, and $F(x)$ is decreasing when $1 < x < 3$.

S-13: This question is nearly identical to Question 12, with

$$G(x) = \int_x^0 f(t) dt = - \int_0^x f(t) dt = -F(x).$$

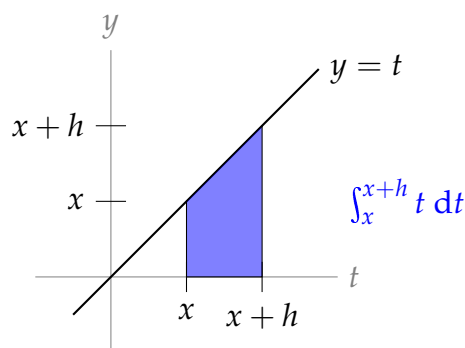
So, $G(x)$ increases when $F(x)$ decreases, and vice-versa. Therefore: $G(0) = 0$, $G(x)$ is increasing when $1 < x < 3$, and $G(x)$ is decreasing when $0 < x < 1$ and when $3 < x < 4$.

S-14: Using the definition of the derivative,

$$\begin{aligned}
 F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\int_a^{x+h} t \, dt - \int_a^x t \, dt}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} t \, dt}{h}
 \end{aligned}$$

The numerator describes the area of a trapezoid with base h and heights x and $x+h$.

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{2}h(x + x+h)}{h} \\
 &= \lim_{h \rightarrow 0} \left(x + \frac{1}{2}h \right) \\
 &= x
 \end{aligned}$$



So, $F'(x) = x$.

S-15: If $F(x)$ is constant, then $F'(x) = 0$. By the Fundamental Theorem of Calculus Part 1, $F'(x) = f(x)$. So, the only possible continuous function fitting the question is $f(x) = 0$.

This makes intuitive sense: if moving x doesn't add or subtract area under the curve, then there must not be any area under the curve—the curve should be the same as the x -axis.

As an aside, we mention that there are other, *non-continuous* functions $f(t)$ such that $\int_0^x f(t) \, dt = 0$ for all x . For example, $f(t) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$. These kinds of removable discontinuities will not factor heavily in our discussion of integrals.

S-16:

$$\begin{aligned}
 \frac{d}{dx} \{x \log(ax) - x\} &= x \left(\frac{a}{ax} \right) + \log(ax) - 1 && \text{(product rule, chain rule)} \\
 &= \log(ax)
 \end{aligned}$$

So, we know

$$\int \log(ax) \, dx = x \log(ax) - x + C \quad \text{where } a \text{ is a given constant, and } C \text{ is any constant.}$$

Remark: $\int \log(ax) \, dx$ can be calculated using the method of Integration by Parts, which you will learn in Section 1.11 of the text.

S-17:

$$\begin{aligned} \frac{d}{dx} \left\{ e^x (x^3 - 3x^2 + 6x - 6) \right\} &= e^x (3x^2 - 6x + 6) + e^x (x^3 - 3x^2 + 6x - 6) \quad (\text{product rule}) \\ &= e^x (3x^2 - 6x + 6 + x^3 - 3x^2 + 6x - 6) \\ &= x^3 e^x \end{aligned}$$

So,

$$\int x^3 e^x \, dx = e^x (x^3 - 3x^2 + 6x - 6) + C$$

Remark: $\int x^3 e^x \, dx$ can be calculated using the method of Integration by Parts, which you will learn in Section 1.11 of the text.

S-18:

$$\begin{aligned} \frac{d}{dx} \left\{ \log \left| x + \sqrt{x^2 + a^2} \right| \right\} &= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + a^2}} \cdot 2x \right) \quad (\text{chain rule}) \\ &= \frac{1 + \frac{x}{\sqrt{x^2 + a^2}}}{x + \sqrt{x^2 + a^2}} = \frac{\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}}{x + \sqrt{x^2 + a^2}} \\ &= \frac{1}{\sqrt{x^2 + a^2}} \end{aligned}$$

So,

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

Remark: $\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$ can be calculated using the method of Trigonometric Substitution, which you will learn in Section 1.8 of the text.

S-19: Using the chain rule:

$$\begin{aligned} & \frac{d}{dx} \left\{ \sqrt{x(a+x)} - a \log(\sqrt{x} + \sqrt{a+x}) \right\} \\ &= \frac{x + (a+x)}{2\sqrt{x(a+x)}} - a \left(\frac{1}{\sqrt{x} + \sqrt{a+x}} \cdot \left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{a+x}} \right) \right) \\ &= \frac{2x+a}{2\sqrt{x(a+x)}} - a \left(\frac{1}{\sqrt{x} + \sqrt{a+x}} \cdot \left(\frac{\sqrt{a+x} + \sqrt{x}}{2\sqrt{x(a+x)}} \right) \right) \\ &= \frac{2x+a}{2\sqrt{x(a+x)}} - a \left(\frac{1}{2\sqrt{x(a+x)}} \right) \\ &= \frac{2x}{2\sqrt{x(a+x)}} = \frac{x}{\sqrt{x(a+x)}} \end{aligned}$$

So,

$$\int \frac{x}{\sqrt{x(a+x)}} dx = \sqrt{x(a+x)} - a \log(\sqrt{x} + \sqrt{a+x}) + C$$

Remark: $\int \frac{x}{\sqrt{x(a+x)}} dx$ can be calculated using the method of Trigonometric Substitution, which you will learn in Section 1.8 of the text.

S-20: By the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_0^2 (x^3 + \sin x) dx &= \left[\frac{x^4}{4} - \cos x \right]_0^2 \\ &= \left(\frac{2^4}{4} - \cos 2 \right) - (0 - \cos 0) \\ &= 4 - \cos 2 + 1 = 5 - \cos 2. \end{aligned}$$

S-21: By part (d) of our “Arithmetic of Integration” theorem, Theorem 1.2.1 in the text,

$$\int_1^2 \frac{x^2+2}{x^2} dx = \int_1^2 \left[1 + \frac{2}{x^2} \right] dx = \int_1^2 dx + 2 \int_1^2 \frac{1}{x^2} dx$$

Then by the Fundamental Theorem of Calculus Part 2,

$$\int_1^2 dx + 2 \int_1^2 \frac{1}{x^2} dx = \left[x \right]_1^2 + 2 \left[-\frac{1}{x} \right]_1^2 = [2-1] + 2 \left[-\frac{1}{2} + 1 \right] = 2$$

S-22: The integrand is similar to $\frac{1}{1+x^2}$, which is the derivative of arctangent. Indeed, we have

$$\int \frac{1}{1+25x^2} dx = \int \frac{1}{1+(5x)^2} dx.$$

So, a reasonable first guess for the antiderivative might be

$$F(x) \stackrel{?}{=} \arctan(5x).$$

However, because of the chain rule,

$$F'(x) = \frac{5}{1 + (5x)^2}.$$

In order to “fix” the numerator, we make a second guess:

$$F(x) = \frac{1}{5} \arctan(5x)$$

$$F'(x) = \frac{1}{5} \left(\frac{5}{1 + (5x)^2} \right) = \frac{1}{1 + 25x^2}$$

$$\text{So, } \int \frac{1}{1 + 25x^2} dx = \frac{1}{5} \arctan(5x) + C.$$

S-23: The integrand is similar to $\frac{1}{\sqrt{1-x^2}}$. In order to formulate a guess for the antiderivative, let's factor out $\sqrt{2}$ from the denominator:

$$\begin{aligned} \int \frac{1}{\sqrt{2-x^2}} dx &= \int \frac{1}{\sqrt{2\left(1-\frac{x^2}{2}\right)}} dx \\ &= \int \frac{1}{\sqrt{2}\sqrt{1-\frac{x^2}{2}}} dx \\ &= \int \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} dx \end{aligned}$$

At this point, we might guess that our antiderivative is something like

$F(x) = \arcsin\left(\frac{x}{\sqrt{2}}\right)$. To explore this possibility, we can differentiate, and see what we get.

$$\frac{d}{dx} \left\{ \arcsin\left(\frac{x}{\sqrt{2}}\right) \right\} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}$$

This is exactly what we want! So,

$$\int \frac{1}{\sqrt{2-x^2}} dx = \arcsin\left(\frac{x}{\sqrt{2}}\right) + C$$

S-24: We know that $\int \sec^2 x \, dx = \tan x + C$, and $\sec^2 x = \tan^2 x + 1$, so

$$\begin{aligned}\int \tan^2 x \, dx &= \int \sec^2 x - 1 \, dx \\ &= \int \sec^2 x \, dx - \int 1 \, dx \\ &= \tan x - x + C\end{aligned}$$

S-25:

Solution 1: This might not obviously look like the derivative of anything familiar, but it does look like half of a familiar trig identity: $2 \sin x \cos x = \sin(2x)$.

$$\begin{aligned}\int 3 \sin x \cos x \, dx &= \int \frac{3}{2} \cdot 2 \sin x \cos x \, dx \\ &= \int \frac{3}{2} \sin(2x) \, dx\end{aligned}$$

So, we might guess that the antiderivative is something like $-\cos(2x)$. We only need to figure out the constants.

$$\begin{aligned}\frac{d}{dx}\{-\cos(2x)\} &= 2 \sin(2x) \\ \text{So, } \frac{d}{dx}\left\{-\frac{3}{4}\cos(2x)\right\} &= \frac{3}{2}\sin(2x) \\ \text{Therefore, } \int 3 \sin x \cos x \, dx &= -\frac{3}{4}\cos(2x) + C\end{aligned}$$

Solution 2: You might notice that the integrand looks like it came from the chain rule, since $\cos x$ is the derivative of $\sin x$. Using this observation, we can work out the antiderivative:

$$\begin{aligned}\frac{d}{dx}\{\sin^2 x\} &= 2 \sin x \cos x \\ \frac{d}{dx}\left\{\frac{3}{2}\sin^2 x\right\} &= 3 \sin x \cos x \\ \text{So, } \int 3 \sin x \cos x \, dx &= \frac{3}{2}\sin^2 x + C\end{aligned}$$

These two answers look different. Using the identity $\cos(2x) = 1 - 2\sin^2(x)$, we reconcile them:

$$\begin{aligned}-\frac{3}{4}\cos(2x) + C &= -\frac{3}{4}(1 - 2\sin^2 x) + C \\ &= \frac{3}{2}\sin^2 x + \left(C - \frac{3}{4}\right)\end{aligned}$$

The $\frac{3}{4}$ here is not significant. Remember that C is used to designate a constant that can take any value between $-\infty$ and $+\infty$. So $C - \frac{3}{4}$ is also just a constant that can take any value between $-\infty$ and $+\infty$. As the two answers we found differ by a constant, they are equivalent.

S-26: It's not immediately obvious which function has $\cos^2 x$ as its derivative, but we can make the situation a little clearer by using the identity $\cos^2 x = \frac{1 + \cos(2x)}{2}$:

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{1}{2} \cdot (1 + \cos(2x)) \, dx \\ &= \int \frac{1}{2} \, dx + \int \frac{1}{2} \cos(2x) \, dx \\ &= \frac{1}{2}x + C + \int \frac{1}{2} \cos(2x) \, dx\end{aligned}$$

For the remaining integral, we might guess something like $F(x) = \sin(2x)$. Let's figure out the appropriate constant:

$$\begin{aligned}\frac{d}{dx} \{\sin(2x)\} &= 2 \cos(2x) \\ \frac{d}{dx} \left\{ \frac{1}{4} \sin(2x) \right\} &= \frac{1}{2} \cos(2x) \\ \text{So, } \int \frac{1}{2} \cos(2x) \, dx &= \frac{1}{4} \sin(2x) + C \\ \text{Therefore, } \int \cos^2 x \, dx &= \frac{1}{2}x + \frac{1}{4} \sin(2x) + C\end{aligned}$$

S-27: By the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned}F'(x) &= \frac{d}{dx} \int_0^x \log(2 + \sin t) \, dt = \log(2 + \sin x) \\ G'(y) &= \frac{d}{dy} \left[- \int_0^y \log(2 + \sin t) \, dt \right] = -\log(2 + \sin y)\end{aligned}$$

So,

$$F'\left(\frac{\pi}{2}\right) = \log 3 \quad G'\left(\frac{\pi}{2}\right) = -\log(3)$$

S-28: By the Fundamental Theorem of Calculus Part 1,

$$f'(x) = 100(x^2 - 3x + 2)e^{-x^2} = 100(x - 1)(x - 2)e^{-x^2}$$

As $f(x)$ is increasing whenever $f'(x) > 0$ and $100e^{-x^2}$ is always strictly bigger than 0, we have $f(x)$ increasing if and only if $(x - 1)(x - 2) > 0$, which is the case if and only if

$(x - 1)$ and $(x - 2)$ are of the same sign. Both are positive when $x > 2$ and both are negative when $x < 1$. So $f(x)$ is increasing when $-\infty < x < 1$ and when $2 < x < \infty$.

Remark: even without the Fundamental Theorem of Calculus, since $f(x)$ is the area under a curve from 1 to x , $f(x)$ is increasing when the curve is above the x -axis (because we're adding positive area), and it's decreasing when the curve is below the x -axis (because we're adding negative area).

S-29: Write $G(x) = \int_0^x \frac{1}{t^3 + 6} dt$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = \frac{1}{x^3 + 6}$. Since $F(x) = G(\cos x)$, the chain rule gives us

$$F'(x) = G'(\cos x) \cdot (-\sin x) = -\frac{\sin x}{\cos^3 x + 6}$$

S-30: Define $g(x) = \int_0^x e^{t^2} dt$. By the Fundamental Theorem of Calculus Part 1, $g'(x) = e^{x^2}$. As $f(x) = g(1 + x^4)$ the chain rule gives us

$$f'(x) = 4x^3 g'(1 + x^4) = 4x^3 e^{(1+x^4)^2}$$

S-31: Define $g(x) = \int_0^x (t^6 + 8) dt$. By the fundamental theorem of calculus, $g'(x) = x^6 + 8$. We are to compute the derivative of $f(x) = g(\sin x)$. The chain rule gives

$$\frac{d}{dx} \left\{ \int_0^{\sin x} (t^6 + 8) dt \right\} = g'(\sin x) \cdot \cos x = (\sin^6 x + 8) \cos x$$

S-32: Let $G(x) = \int_0^x e^{-t} \sin\left(\frac{\pi t}{2}\right) dt$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = e^{-x} \sin\left(\frac{\pi x}{2}\right)$ and, since $F(x) = G(x^3)$, $F'(x) = 3x^2 G'(x^3) = 3x^2 e^{-x^3} \sin\left(\frac{\pi x^3}{2}\right)$. Then $F'(1) = 3e^{-1} \sin\left(\frac{\pi}{2}\right) = 3e^{-1}$.

S-33: Define $G(x) = \int_x^0 \frac{dt}{1+t^3} = -\int_0^x \frac{1}{1+t^3} dt$, so that $G'(x) = -\frac{1}{1+x^3}$ by the Fundamental Theorem of Calculus Part 1. Then by the chain rule,

$$\frac{d}{du} \left\{ \int_{\cos u}^0 \frac{dt}{1+t^3} \right\} = \frac{d}{du} G(\cos u) = G'(\cos u) \cdot \frac{d}{du} \cos u = -\frac{1}{1+\cos^3 u} \cdot (-\sin u).$$

S-34: Applying $\frac{d}{dx}$ to both sides of $x^2 = 1 + \int_1^x f(t) dt$ gives, by the Fundamental Theorem of Calculus Part 1, $2x = f(x)$.

S-35: Apply $\frac{d}{dx}$ to both sides of $x \sin(\pi x) = \int_0^x f(t) dt$. Then, by the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned} f(x) &= \frac{d}{dx} \int_0^x f(t) dt = \frac{d}{dx} \{x \sin(\pi x)\} \\ \implies f(x) &= \frac{d}{dx} \{x \sin(\pi x)\} = \sin(\pi x) + \pi x \cos(\pi x) \\ \implies f(4) &= \sin(4\pi) + 4\pi \cos(4\pi) = 4\pi \end{aligned}$$

S-36: (a) Write

$$F(x) = G(x^2) - H(-x) \quad \text{with} \quad G(y) = \int_0^y e^{-t} dt, \quad H(y) = \int_0^y e^{-t^2} dt$$

By the Fundamental Theorem of Calculus Part 1,

$$G'(y) = e^{-y}, \quad H'(y) = e^{-y^2}$$

Hence, by the chain rule,

$$F'(x) = 2xG'(x^2) - (-1)H'(-x) = 2xe^{-(x^2)} + e^{-(-x)^2} = (2x + 1)e^{-x^2}$$

(b) Observe that $F'(x) < 0$ for $x < -1/2$ and $F'(x) > 0$ for $x > -1/2$. Hence $F(x)$ is decreasing for $x < -1/2$ and increasing for $x > -1/2$, and $F(x)$ must take its minimum value when $x = -1/2$.

S-37: Define $G(y) = \int_0^y e^{\sin t} dt$. Then:

$$\begin{aligned} F(x) &= \int_0^x e^{\sin t} dt + \int_{x^4-x^3}^0 e^{\sin t} dt = \int_0^x e^{\sin t} dt - \int_0^{x^4-x^3} e^{\sin t} dt \\ &= G(x) - G(x^4 - x^3) \end{aligned}$$

By the Fundamental Theorem of Calculus Part 1,

$$G'(y) = e^{\sin y}$$

Hence, by the chain rule,

$$\begin{aligned} F'(x) &= G'(x) - G'(x^4 - x^3) \frac{d}{dx} \{x^4 - x^3\} \\ &= G'(x) - G'(x^4 - x^3) (4x^3 - 3x^2) \\ &= e^{\sin x} - e^{\sin(x^4-x^3)} (4x^3 - 3x^2) \end{aligned}$$

S-38: Define with $G(y) = \int_0^y \cos(e^t) dt$. Then:

$$\begin{aligned} F(x) &= \int_{x^5}^{-x^2} \cos(e^t) dt = \int_0^{-x^2} \cos(e^t) dt + \int_{x^5}^0 \cos(e^t) dt \\ &= \int_0^{-x^2} \cos(e^t) dt - \int_0^{x^5} \cos(e^t) dt \\ &= G(-x^2) - G(x^5) \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$G'(y) = \cos(e^y)$$

Hence, by the chain rule,

$$\begin{aligned} F'(x) &= G'(-x^2) \frac{d}{dx}\{-x^2\} - G'(x^5) \frac{d}{dx}\{x^5\} \\ &= G'(-x^2) (-2x) - G'(x^5) (5x^4) \\ &= -2x \cos(e^{-x^2}) - 5x^4 \cos(e^{x^5}) \end{aligned}$$

S-39: Define with $G(y) = \int_0^y \sqrt{\sin t} dt$. Then:

$$\begin{aligned} F(x) &= \int_x^{e^x} \sqrt{\sin t} dt \\ &= \int_0^{e^x} \sqrt{\sin t} dt + \int_x^0 \sqrt{\sin t} dt = \int_0^{e^x} \sqrt{\sin t} dt - \int_0^x \sqrt{\sin t} dt \\ &= G(e^x) - G(x) \end{aligned}$$

By the Fundamental Theorem of Calculus Part 1,

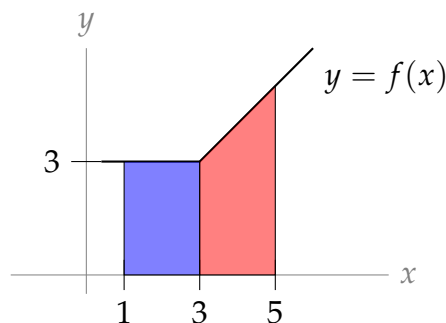
$$G'(y) = \sqrt{\sin y}$$

Hence, by the chain rule,

$$\begin{aligned} F'(x) &= G'(e^x) \frac{d}{dx}\{e^x\} - G'(x) \\ &= e^x G'(e^x) - G'(x) \\ &= e^x \sqrt{\sin(e^x)} - \sqrt{\sin(x)} \end{aligned}$$

S-40: Splitting up the domain of integration,

$$\begin{aligned}\int_1^5 f(x) \, dx &= \int_1^3 f(x) \, dx + \int_3^5 f(x) \, dx \\ &= \int_1^3 3 \, dx + \int_3^5 x \, dx \\ &= 3x \Big|_{x=1}^{x=3} + \frac{x^2}{2} \Big|_{x=3}^{x=5} \\ &= 14\end{aligned}$$



S-41: By the chain rule,

$$\frac{d}{dx} \{ (f'(x))^2 \} = 2f'(x) f''(x)$$

so $\frac{1}{2}f'(x)^2$ is an antiderivative for $f'(x) f''(x)$ and, by the Fundamental Theorem of Calculus Part 2,

$$\int_1^2 f'(x) f''(x) \, dx = \left[\frac{1}{2} (f'(x))^2 \right]_{x=1}^{x=2} = \frac{1}{2} f'(2)^2 - \frac{1}{2} f'(1)^2 = \frac{5}{2}$$

Remark: evaluating antiderivatives of this type will occupy the next section, Section 1.5 of the text.

S-42: The car stops when $v(t) = 30 - 10t = 0$, which occurs at time $t = 3$. The distance covered up to that time is

$$\int_0^3 v(t) \, dt = (30t - 5t^2) \Big|_0^3 = (90 - 45) - 0 = 45 \text{ m.}$$

S-43: Define $g(x) = \int_0^x \log(1 + e^t) \, dt$. By the Fundamental Theorem of Calculus Part 1, $g'(x) = \log(1 + e^x)$. But $f(x) = g(2x - x^2)$, so by the chain rule,

$$f'(x) = g'(2x - x^2) \cdot \frac{d}{dx} \{2x - x^2\} = (2 - 2x) \cdot \log(1 + e^{2x-x^2})$$

Observe that $e^{2x-x^2} > 0$ for all x so that $1 + e^{2x-x^2} > 1$ for all x and $\log(1 + e^{2x-x^2}) > 0$ for all x . Since $2 - 2x$ is positive for $x < 1$ and negative for $x > 1$, $f'(x)$ is also positive for $x < 1$ and negative for $x > 1$. That is, $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$. So $f(x)$ achieves its absolute maximum at $x = 1$.

S-44: Let $f(x) = \int_0^{x^2-2x} \frac{dt}{1+t^4}$ and $g(x) = \int_0^x \frac{dt}{1+t^4}$. Then $g'(x) = \frac{1}{1+x^4}$ and, since $f(x) = g(x^2 - 2x)$, $f'(x) = (2x - 2)g'(x^2 - 2x) = 2\frac{x-1}{1+(x^2-2x)^4}$. This is zero for $x = 1$, negative for $x < 1$ and positive for $x > 1$. Thus as x runs from $-\infty$ to ∞ , $f(x)$ decreases until x reaches 1 and then increases all $x > 1$. So the minimum of $f(x)$ is achieved for $x = 1$. At $x = 1$, $x^2 - 2x = -1$ and $f(1) = \int_0^{-1} \frac{dt}{1+t^4}$.

S-45: Define $G(x) = \int_0^x \sin(\sqrt{t}) dt$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = \sin(\sqrt{x})$. Since $F(x) = G(x^2)$, and since $x > 0$, we have

$$F'(x) = 2xG'(x^2) = 2x \sin|x| = 2x \sin x.$$

Thus F increases as x runs from 0 to π (since $F'(x) > 0$ there) and decreases as x runs from π to 4 (since $F'(x) < 0$ there). Thus F achieves its maximum value at $x = \pi$.

S-46: The given sum is of the form

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{\pi}{n} \sin\left(\frac{j\pi}{n}\right) = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j^*) \Delta x$$

with $\Delta x = \frac{\pi}{n}$, $x_j^* = \frac{j\pi}{n}$ and $f(x) = \sin(x)$. Since $x_0^* = 0$ and $x_n^* = \pi$, the right hand side is the definition (using the right Riemann sum) of

$$\int_0^\pi f(x) dx = \int_0^\pi \sin(x) dx = [-\cos(x)]_0^\pi = 2$$

where we evaluate the definite integral using the Fundamental Theorem of Calculus Part 2.

S-47: The given sum is of the form

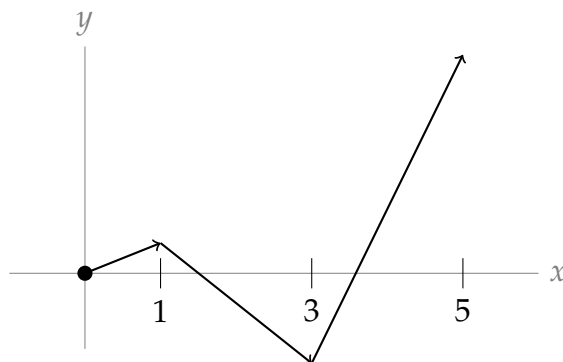
$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{1}{1 + \frac{j}{n}} = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j) \Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j = \frac{j}{n}$ and $f(x) = \frac{1}{1+x}$. The right hand side is the definition (using the right Riemann sum) of

$$\int_0^1 f(x) dx = \int_0^1 \frac{1}{1+x} dx = \log|1+x| \Big|_0^1 = \log 2$$

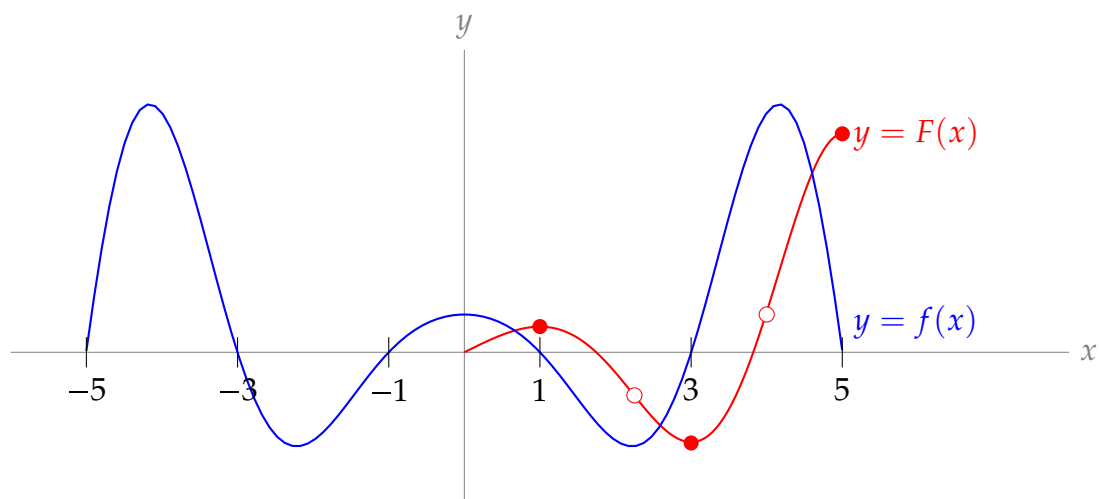
$F(x)$, $x \geq 0$ We learned quite a lot last semester about curve sketching. We can use those techniques here. We have to be quite careful about the sign of x , though. We can only directly apply the Fundamental Theorem of Calculus Part 1 (as it's written in your text) when $x \geq 0$. So first, let's graph the right-hand portion. Notice $f(x)$ has even symmetry—so, if we know one half of $F(x)$, we should be able to figure out the other half with relative ease.

- $F(0) = \int_0^0 f(t) dt = 0$ (so, $F(x)$ passes through the origin)
- Using the Fundamental Theorem of Calculus Part 1, $F'(x) > 0$ when $0 < x < 1$ and when $3 < x < 5$; $F'(x) < 0$ when $1 < x < 3$. So, $F(x)$ is decreasing from 1 to 3, and increasing from 0 to 1 and also from 3 to 5. That gives us a skeleton to work with.



We get the relative sizes of the maxes and mins by eyeballing the area under $y = f(t)$. The first lobe (from $x = 0$ to $x = 1$) has a small positive area, so $F(1)$ is a small positive number. The next lobe (from $x = 1$ to $x = 3$) has a larger absolute area than the first, so $F(3)$ is negative. Indeed, the second lobe seems to have more than twice the area of the first, so $|F(3)|$ should be larger than $F(1)$. The third lobe is larger still, and even after subtracting the area of the second lobe it looks much larger than the first or second lobe, so $|F(3)| < F(5)$.

- We can use $F''(x)$ to get the concavity of $F(x)$. Note $F''(x) = f'(x)$. We observe $f(x)$ is decreasing on (roughly) $(0, 2.5)$ and $(4, 5)$, so $F(x)$ is concave down on those intervals. Further, $f(x)$ is increasing on (roughly) $(2.5, 4)$, so $F(x)$ is concave up there, and has inflection points at about $x = 2.5$ and $x = 4$.



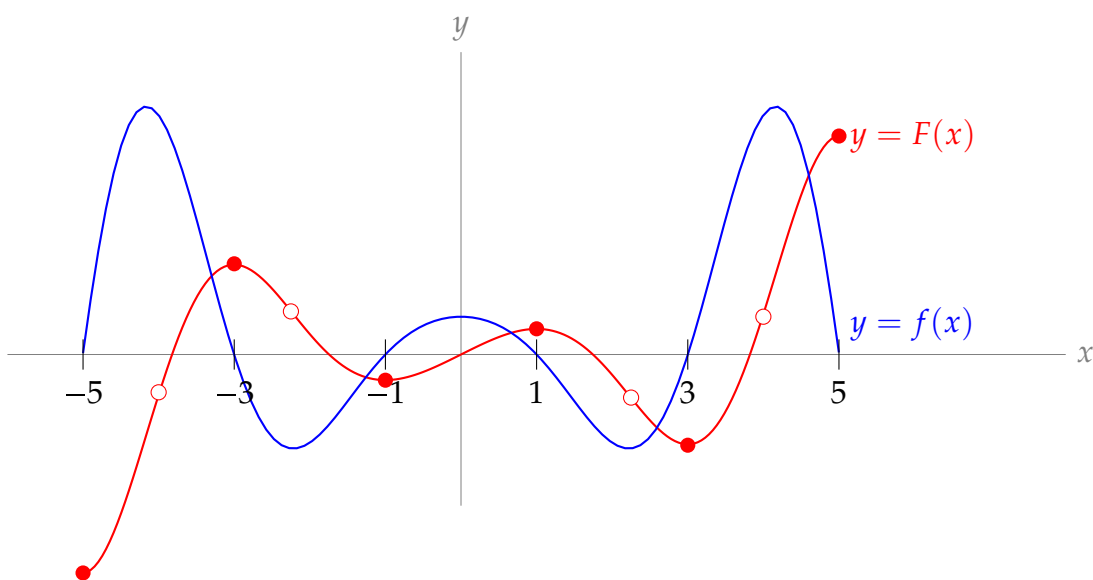
In the sketch above, closed dots are extrema, and open dots are inflection points.

F(x), $x < 0$ Now we can consider the left half of the graph. If you stare at it long enough, you might convince yourself that $F(x)$ is an odd function. We can also show this with the following calculation:

$$\begin{aligned} F(-x) &= \int_0^{-x} f(t) \, dt \\ &= \int_x^0 f(t) \, dt = - \int_0^x f(t) \, dt \\ &= -F(x) \end{aligned}$$

As in Example 1.2.10 of the text, since $f(t)$ is even,

Knowing that $F(x)$ is odd allows us to finish our sketch.



S-49: (a) Using the product rule, followed by the chain rule, followed by the

Fundamental Theorem of Calculus Part 1,

$$\begin{aligned}f'(x) &= 3x^2 \int_0^{x^3+1} e^{t^3} dt + x^3 \frac{d}{dx} \int_0^{x^3+1} e^{t^3} dt \\&= 3x^2 \int_0^{x^3+1} e^{t^3} dt + x^3 [3x^2] \left[\frac{d}{dy} \int_0^y e^{t^3} dt \right]_{y=x^3+1} \\&= 3x^2 \int_0^{x^3+1} e^{t^3} dt + x^3 [3x^2] [e^{y^3}]_{y=x^3+1} \\&= 3x^2 \int_0^{x^3+1} e^{t^3} dt + x^3 [3x^2] e^{(x^3+1)^3} \\&= 3x^2 \int_0^{x^3+1} e^{t^3} dt + 3x^5 e^{(x^3+1)^3}\end{aligned}$$

(b) In general, the equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x - a)$$

Substituting in the given $f(x)$ and $a = -1$:

$$\begin{aligned}f(a) &= f(-1) = (-1)^3 \int_0^0 e^{t^3} dt = 0 \\f'(a) &= f'(-1) = 3(-1)^2 \int_0^0 e^{t^3} dt + 3(-1)^5 e^0 \\&= 0 - 3 = -3 \\(x - a) &= x - (-1) = x + 1\end{aligned}$$

So, the equation of the tangent line is

$$y = -3(x + 1).$$

S-50: Recall that “+C” means that we can add any constant to the function. Since $\tan^2 x = \sec^2 x - 1$, Students A and B have equivalent answers: they only differ by a constant.

So, if one is right, both are right; if one is wrong, both are wrong. We check Student A's work:

$$\frac{d}{dx} \{\tan^2 x + x + C\} = \frac{d}{dx} \{\tan^2 x\} + 1 + 0 = f(x) - 1 + 1 = f(x)$$

So, Student A's answer is indeed an antiderivative of $f(x)$. Therefore, *both students* ended up with the correct answer.

Remark: it is a frequent occurrence that equivalent answers might look quite different. As you are comparing your work to others', this is a good thing to keep in mind!

S-51:

(a) When $x = 3$,

$$F(3) = \int_0^3 3^3 \sin(t) \, dt = 27 \int_0^3 \sin t \, dt$$

Using the Fundamental Theorem of Calculus Part 2,

$$\begin{aligned} &= 27 [-\cos t]_{t=0}^{t=3} = 27 [-\cos 3 - (-\cos 0)] \\ &= 27(1 - \cos 3) \end{aligned}$$

(b) Since the integration is with respect to t , the x^3 term can be moved outside the integral. That is: for the purposes of the integral, x^3 is a constant (although for the purposes of the derivative, it certainly is not).

$$F(x) = \int_0^x x^3 \sin(t) \, dt = x^3 \int_0^x \sin(t) \, dt$$

Using the product rule and the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned} F'(x) &= x^3 \cdot \sin(x) + 3x^2 \int_0^x \sin(t) \, dt \\ &= x^3 \sin(x) + 3x^2 [-\cos(t)]_{t=0}^{t=x} \\ &= x^3 \sin(x) + 3x^2 [-\cos(x) - (-\cos(0))] \\ &= x^3 \sin(x) + 3x^2 [1 - \cos(x)] \end{aligned}$$

Remark: Since x and t play different roles in our problem, it's crucial that they have different names. This is one reason why we should avoid the common mistake of writing $\int_a^x f(x) dx$ when we mean $\int_a^x f(t) dt$.

S-52: If $F(x)$ is even, then $f(x)$ is odd (by the result of Question 20 in Section 1.2). So, $F(x)$ can only be even if $f(x)$ is both even and odd. By the result in Question 19, Section 1.2, this means $F(x)$ is only even if $f(x) = 0$ for all x . Note if $f(x) = 0$, then $F(x)$ is a constant function. So, it is certainly even, and it might be odd as well if $F(x) = f(x) = 0$.

Therefore, if $f(x) \neq 0$ for some x , then $F(x)$ is not even. It could be odd, or it could be neither even nor odd. We can come up with examples of both types: if $f(x) = 1$, then $F(x) = x$ is an odd antiderivative, and $F(x) = x + 1$ is an antiderivative that is neither even nor odd.

Interestingly, the antiderivative of an odd function is always even. The proof is a little beyond what we might ask you, but is given below for completeness. The proof goes like this: First, we'll show that if $g(x)$ is odd, then there is *some* antiderivative of $g(x)$ that is even. Then, we'll show that *every* antiderivative of $g(x)$ is even.

So, suppose $g(x)$ is odd and define $G(x) = \int_0^x g(t) dt$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = g(x)$, so $G(x)$ is an antiderivative of $g(x)$. Since $g(x)$ is odd, for

any $x \geq 0$, the net signed area under the curve along $[0, x]$ is the *negative* of the net signed area under the curve along $[-x, 0]$. So,

$$\begin{aligned}\int_0^x g(t) \, dt &= -\int_{-x}^0 g(t) \, dt \quad (\text{See Example 1.2.11 in the text}) \\ &= \int_0^{-x} g(t) \, dt\end{aligned}$$

By the definition of $G(x)$,

$$G(x) = G(-x)$$

That is, $G(x)$ is even. We've shown that there exists *some* antiderivative of $g(x)$ that is even; it remains to show that *all of them* are even.

Recall that every antiderivative of $g(x)$ differs from $G(x)$ by some constant. So, any antiderivative of $g(x)$ can be written as $G(x) + C$, and $G(-x) + C = G(x) + C$. So, every antiderivative of an odd function is even.

Solutions to Exercises 1.4 — Jump to [TABLE OF CONTENTS](#)

S-1:

- (a) The function $s(t)$ gives the rate of change of the amount of data uploaded over time. So, the net change is given by its integral: $\int_0^1 s(t) \, dt$.
- (b) The population is changing at a rate of (rate of individuals added)-(rate of individuals lost), which is $N'(t) = b(t) + i(t) - d(t) - o(t)$. So, the net change is $\int_0^1 [b(t) + i(t) - d(t) - o(t)] \, dt$.
- (c) We're given a function for the amount of light emitted, and we're asked a question about the amount of light emitted. The difference between the two scenarios is $L(2) - L(1)$.
- (d) Think back to last semester: the rate of change of $W(x)$ with respect to x when $x = 1$ is $W'(1)$.

S-2:

- (a) The integrand is the rate of change of height, so the definite integral is net change in height. That is: the quantity described is the difference in the corn's height (so, how much it grew or shrank) from time 0 to time 1.
- (b) Let $f(t)$ be the number of calls that the line has fielded since time $t = 0$. Then $f'(t)$ is the rate at which new calls are coming in, i.e. $c(t)$. So, the integral of $c(t)$ gives net change in $f(t)$. That is: the quantity described is the number of calls to the line in a 24-hour period (from midnight to midnight).

(c) Let $F(a)$ be the average price of a home of a square centimetres. Then

$$f(a) = \frac{F(a+1) - F(a)}{1} \approx \lim_{h \rightarrow 0} \frac{F(a+h) - F(a)}{h} = F'(a).$$

(Note for the interval given, where $750\,000 \leq a \leq 1\,000\,000$, $h = 1$ is indeed a pretty small number.) So, the integral of $f(a)$ gives (roughly) the net change in $F(a)$.

That is: the quantity describes is roughly the difference in average value between a home that is 750,000 sq cm (75 sq m) and one that is 1,000,000 sq cm (100 sq m).

S-3:

(a) Velocity is rate of change of position, and the integral of rate of change is net change. So, the net change in position is

$$\int_1^2 (3t - 2t^2) dt = \left[\frac{3}{2}t^2 - \frac{2}{3}t^3 \right]_1^2 = -\frac{1}{6}$$

(b) Note $v(t) = 2t \left(\frac{3}{2} - t \right)$. So, in the time period in question, $v(t) > 0$ when $1 < t < 1.5$, and $v(t) < 0$ when $1.5 < t < 2$.

- When $1 < t < 1.5$, the particle is travelling to the right, so its distance travelled is:

$$\int_1^{3/2} (3t - 2t^2) dt = \left[\frac{3}{2}t^2 - \frac{2}{3}t^3 \right]_1^{3/2} = \frac{9}{8} - \frac{5}{6} = \frac{7}{24}.$$

- When $1.5 < t < 2$, the particle is travelling to the left, so its distance travelled is:

$$\int_{3/2}^2 (3t - 2t^2) dt = \left[\frac{3}{2}t^2 - \frac{2}{3}t^3 \right]_{3/2}^2 = \frac{9}{8} - \frac{2}{3} = \frac{11}{24}.$$

So, its total distance travelled is $\frac{7+11}{24} = \frac{3}{4}$.

S-4: We're asked about litres per hour, so let's start by converting. The rate at which water is being lost is $\frac{3}{50} \cdot \frac{1}{1+x}$ litres per hour. So, the net change from $x = 0$ hours to $x = 1$ hours is

$$\frac{3}{50} \int_0^1 \frac{1}{1+x} dx = \frac{3}{50} [\log(1+x)]_0^1 = \frac{3}{50} \log 2.$$

S-5:

(a) Let $v(t)$ be the velocity of the particle at time t . We're given $v(0) = 1$. Recall

$$v'(t) = a(t).$$

$$\begin{aligned} v(t) &= v(0) + \int_0^t v'(x) \, dx \\ &= 1 + \int_0^t [6x - 6] \, dx \\ &= 1 + \left[3x^2 - 6x \right]_0^t \\ &= 1 + 3t^2 - 6t \end{aligned}$$

Now, we find the velocity at the desired times.

- $v(0) = 1$
- $v(1) = -2$
- $v(2) = 1$
- $v(3) = 10$

So, of these four times, the particle is moving most quickly to the right at $t = 3$, and most quickly to the left at $t = 1$.

- (b) Let $s(t)$ be the position of the particle at time t . We're given $s(0) = 2$, and recall $s'(t) = v(t)$.

$$\begin{aligned} s(t) &= s(0) + \int_0^t s'(x) \, dx \\ &= 2 + \int_0^t [1 + 3x^2 - 6x] \, dx \\ &= 2 + \left[x + x^3 - 3x^2 \right]_0^t \\ &= 2 + t + t^3 - 3t^2 \end{aligned}$$

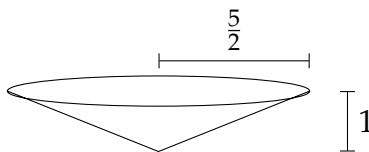
S-6: To find the quantity we want, we'll first find the *amount* of water in the basin at $t = 1$, then we'll convert that to the *height* of water in the basin. (Since we started with no water and height 0, the height at $t = 1$ is the same as the amount the height rose from $t = 0$ to $t = 1$.)

We know the rate of change of the volume of water in the basin. To find the net change, we integrate. Recall from last semester that $\frac{d}{dt}[\sqrt{t}] = \frac{1}{2\sqrt{t}}$. So, the antiderivative of $\frac{1}{2\sqrt{t}}$ is $\sqrt{t}$.

$$\int_0^1 \frac{1}{2\sqrt{t}} \, dt = [\sqrt{t}]_0^1 = 1$$

So, there is 1 cubic m of water in the basin at time $t = 1$.

To convert this to height, recall that the volume of a cone with base radius r and height h is $\frac{\pi}{3}r^2h$. We know at $t = 1$, the volume is 1, so to find h , we should do something about r . This is where similar triangles come in.



In the entire basin, the base radius is $\frac{5}{2}$ m and the height is 1 m. As it fills, it creates a smaller cone with the same ratio, so $\frac{r}{h} = \frac{5/2}{1} = \frac{5}{2}$. That is, $r = \frac{5}{2}h$.

$$\begin{aligned} 1 &= \frac{\pi}{3}r^2h = \frac{\pi}{3}\left(\frac{5}{2}h\right)^2h \\ &= \frac{25\pi}{12}h^3 \\ \implies \frac{12}{25\pi} &= h^3 \\ \implies h &= \sqrt[3]{\frac{12}{25\pi}} \end{aligned}$$

So, the height in the basin to rise $\sqrt[3]{\frac{12}{25\pi}}$ m.

S-7: *Remark: there are lots of different ways to solve this problem. We present one. Our solution avoids the need to think too hard about difficult numbers like the cube root of $\frac{1500}{7}$, but of course there's nothing wrong with thinking hard about difficult numbers.*

When $t \approx 0$, we see $A(t) \approx 10t$ and $B(t) = 7t^2$. We recall from last semester that lower powers dominate near the origin, so $A(t)$ is faster when the race starts. That is, particle A is leading at the start of the race.

For both (b) and (c), it would be helpful to have a function that gives the position of each particle as a function of time. Let's call these $s_A(t)$ and $s_B(t)$, in the obvious way. Remember that velocity is the rate of change of position, so the definite integral of velocity gives net change in position. (So, the functions below are actually the change in position from the starting position. If we define the starting position to be 0, they're also the actual position.)

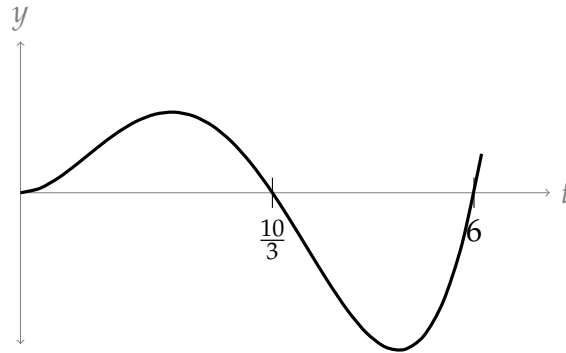
$$\begin{aligned} s_A(t) &= \int_0^t [x^3 + 10x] dx = \frac{1}{4}t^4 + 5t^2 \\ s_B(t) &= \int_0^t [7x^2] dx = \frac{7}{3}t^3 \end{aligned}$$

To figure out which particle wins, we could figure out which position function has the smallest positive value of t that makes it equal to 500. This leads to some numbers that might be awkward to deal with without a calculator. So, instead of that (perfectly fine) method, we'll skip ahead to (c), which gives us a better overview of the race.

Let $D(t) = s_A(t) - s_B(t)$. When $D(t) = 0$, the two particles are in the same position; when $D(t) > 0$, particle A is in the lead; and when $D(t) < 0$, particle B is in the lead. (From part (a), we know $D(t)$ will be positive when t is slightly larger than 0.)

$$\begin{aligned} D(t) &= s_A(t) - s_B(t) = \frac{1}{4}t^4 - \frac{7}{3}t^3 + 5t^2 \\ &= t^2 \left(\frac{1}{4}t^2 - \frac{7}{3}t + 5 \right) \end{aligned}$$

With the help of the quadratic formula, we find the roots of this function are $t = 0$, $t = \frac{10}{3}$, and $t = 6$. However, we're not yet sure of our model domain. The sketch of $D(t)$ below shows where the function is positive, negative, and 0. If the race ends in a region where $D(t) > 0$, then A wins; if it ends in a region where $D(t) < 0$, then B wins.



- $s_B(6) = \frac{7 \cdot 6^3}{3} = 504 > 500$, so we know the race ends *before* $t = 6$. (Note if *at least one* particle has crossed the 500-metre mark, then the race is effectively over.)

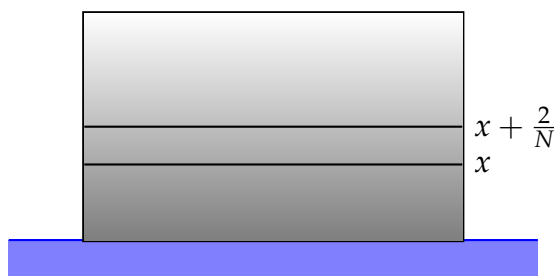
Now we need to decide whether the race ends before or after $t = \frac{10}{3}$. We'll investigate $t = 4$, since this is easier to work with than $t = \frac{10}{3}$.

- $s_A(4) = \frac{4^4}{4} + 5(4^2) = 144 < 500$. So, at $t = 4$, particle A has not yet crossed the finish line.
- $s_B(4) = \frac{7}{3}(4^3) = \frac{7}{3} \cdot 64 < 500$, so particle B has also not yet crossed the finish line at $t = 4$.
- We conclude that the race ends sometime between $t = 4$ and $t = 6$. In this region, $D(t) < 0$, so B is ahead. That is, B wins the race.

Furthermore, we now know that the model domain contains only two points where the two particles are in a tie: $t = 0$ and $t = \frac{10}{3}$.

S-8: First, let's find how far from the river the city stretches. We're told it ends where $d(x) = 0$, so it ends at $x = 2$.

We'll cut up the city into N thin slices, each of width $\frac{2}{N}$. Consider a slice from distance x to distance $x + \frac{2}{N}$ from the river.



In this thin slice, the density is approximately $d(x)$ people per square km, and the area is $\frac{2}{N} \cdot 5$ km², so the number of people in the slice is approximately $\frac{2}{N} \cdot 5d(x)$. So, the number of people in the entire city is approximately

$$\sum_{i=1}^N \frac{2}{N} \cdot 5d\left(\frac{2}{N}i\right).$$

This is the Riemann sum for

$$\int_0^2 5d(x) dx,$$

so the number of people in the city is

$$\begin{aligned} 5 \int_0^2 d(x) dx &= 5 \int_0^2 \left[\frac{20}{1+x^2} - 4 \right] dx \\ &= 100 \int_0^2 \frac{1}{1+x^2} dx - 5 \int_0^2 4 dx \\ &= 100 \arctan(2) - 40 \end{aligned}$$

Solutions to Exercises 1.5 — [Jump to TABLE OF CONTENTS](#)

S-1: (a) This is true: it is an application of Theorem 1.5.2 in the text with $f(x) = \sin x$ and $u(x) = e^x$.

(b) This is false: the upper limit of integration is incorrect. Using Theorem 1.5.6 in the text, the correct form is

$$\int_0^1 \sin(e^x) \cdot e^x dx = \int_1^e \sin(u) du = -\cos(e) + \cos(1) = \cos(1) - \cos(e).$$

Alternately, we can use the Fundamental Theorem of Calculus Part 2, and our answer from (a):

$$\int_0^1 \sin(e^x) \cdot e^x dx = [-\cos(e^x) + C]_0^1 = \cos(1) - \cos(e).$$

S-2: The reasoning is not sound: when we do a substitution, we need to take care of the differential (dx). Remember the method of substitution comes from the chain rule: there should be a function *and its derivative*. Here's the way to do it:

Problem: Evaluate $\int (2x + 1)^2 dx$.

Work: We use the substitution $u = 2x + 1$. Then $du = 2dx$, so $dx = \frac{1}{2}du$:

$$\begin{aligned}\int (2x + 1)^2 dx &= \int u^2 \cdot \frac{1}{2} du \\ &= \frac{1}{6} u^3 + C \\ &= \frac{1}{6} (2x + 1)^3 + C\end{aligned}$$

S-3: The problem is with the limits of integration, as in Question 1. Here's how it ought to go:

Problem: Evaluate $\int_1^\pi \frac{\cos(\log t)}{t} dt$.

Work: We use the substitution $u = \log t$, so $du = \frac{1}{t} dt$. When $t = 1$, we have $u = \log 1 = 0$ and when $t = \pi$, we have $u = \log(\pi)$. Then:

$$\begin{aligned}\int_1^\pi \frac{\cos(\log t)}{t} dt &= \int_{\log 1}^{\log(\pi)} \cos(u) du \\ &= \int_0^{\log(\pi)} \cos(u) du \\ &= \sin(\log(\pi)) - \sin(0) = \sin(\log(\pi)).\end{aligned}$$

S-4: Perhaps shorter ways exist, but the reasoning here is valid.

Problem: Evaluate $\int_0^{\pi/4} x \tan(x^2) dx$.

Work: We begin with the substitution $u = x^2$, $du = 2x dx$:

If $u = x^2$, then $\frac{du}{dx} = 2x$, so indeed $du = 2x dx$.

$$\begin{aligned}\int_0^{\pi/4} x \tan(x^2) dx &= \int_0^{\pi/4} \frac{1}{2} \tan(x^2) \cdot 2x dx && \text{algebra} \\ &= \int_0^{\pi^2/16} \frac{1}{2} \tan u du\end{aligned}$$

Every piece is changed from x to u : integrand, differential, limits.

$$= \frac{1}{2} \int_0^{\pi^2/16} \frac{\sin u}{\cos u} du \qquad \tan u = \frac{\sin u}{\cos u}$$

Now we use the substitution $v = \cos u$, $dv = -\sin u du$:

$$= \frac{1}{2} \int_{\cos 0}^{\cos(\pi^2/16)} -\frac{1}{v} dv$$

Every piece is changed from u to v : integrand, differential, limits.

$$\begin{aligned} &= -\frac{1}{2} \int_1^{\cos(\pi^2/16)} \frac{1}{v} dv && \cos(0) = 1 \\ &= -\frac{1}{2} \left[\log |v| \right]_1^{\cos(\pi^2/16)} && \text{FTC Part 2} \\ &= -\frac{1}{2} \left(\log \left(\cos(\pi^2/16) \right) - \log(1) \right) \\ &= -\frac{1}{2} \log \left(\cos(\pi^2/16) \right) && \log(1) = 0 \end{aligned}$$

S-5: We substitute:

$$\begin{aligned} u &= \sin x, \\ du &= \cos x \, dx, \\ \cos x &= \sqrt{1 - \sin^2 x} = \sqrt{1 - u^2}, \\ dx &= \frac{du}{\cos x} = \frac{du}{\sqrt{1 - u^2}} \\ u(0) &= \sin 0 = 0 \\ u\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) = 1 \end{aligned}$$

So,

$$\int_{x=0}^{x=\pi/2} f(\sin x) \, dx = \int_{u=0}^{u=1} f(u) \frac{du}{\sqrt{1 - u^2}}$$

Because the denominator $\sqrt{1 - u^2}$ vanishes when $u = 1$, this is what is known as an improper integral. Improper integrals will be discussed in § 1.13 of the text.

S-6: Using the chain rule, we see that

$$\frac{d}{dx} \{f(g(x))\} = f'(g(x))g'(x)$$

So, $f(g(x))$ is an antiderivative of $f'(g(x))g'(x)$. All antiderivatives of $f'(g(x))g'(x)$ differ by only a constant, so:

$$\begin{aligned} \int f'(g(x))g'(x) \, dx - f(g(x)) &= f(g(x)) + C - f(g(x)) \\ &= C \end{aligned}$$

That is, our expression simplifies to some constant C .

Remark: since

$$\int f'(g(x))g'(x) \, dt - f(g(x)) = C$$

we conclude

$$\int f'(g(x))g'(x) \, dt = f(g(x)) + C$$

which is precisely how we perform substitution on integrals.

S-7: We write $u(x) = e^{x^2}$ and find $du = u'(x) \, dx = 2xe^{x^2} \, dx$. Note that $u(1) = e^{1^2} = e$ when $x = 1$, and $u(0) = e^{0^2} = 1$ when $x = 0$. Therefore:

$$\begin{aligned} \int_0^1 xe^{x^2} \cos(e^{x^2}) \, dx &= \frac{1}{2} \int_{x=0}^{x=1} \cos(u(x)) u'(x) \, dx \\ &= \frac{1}{2} \int_{u=1}^{u=e} \cos(u) \, du \\ &= \frac{1}{2} \left[\sin(u) \right]_1^e = \frac{1}{2} (\sin(e) - \sin(1)). \end{aligned}$$

S-8: Substituting $y = x^3$, $dy = 3x^2 \, dx$:

$$\int_1^2 x^2 f(x^3) \, dx = \frac{1}{3} \int_1^8 f(y) \, dy = \frac{1}{3}$$

S-9: Setting $u = x^3 + 1$, we have $du = 3x^2 \, dx$ and so

$$\begin{aligned} \int \frac{x^2 \, dx}{(x^3 + 1)^{101}} &= \int \frac{du/3}{u^{101}} \\ &= \frac{1}{3} \int u^{-101} \, du \\ &= \frac{1}{3} \cdot \frac{u^{-100}}{-100} \\ &= -\frac{1}{3 \times 100 u^{100}} + C \\ &= -\frac{1}{300(x^3 + 1)^{100}} + C \end{aligned}$$

S-10: Setting $u = \log x$, we have $du = \frac{1}{x} \, dx$ and so

$$\int_e^{e^4} \frac{dx}{x \cdot \log x} = \int_{x=e}^{x=e^4} \frac{1}{\log x} \cdot \frac{1}{x} \, dx = \int_{u=1}^{u=4} \frac{1}{u} \, du,$$

since $u = \log(e) = 1$ when $x = e$ and $u = \log(e^4) = 4$ when $x = e^4$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_1^4 \frac{1}{u} \, du = \left[\log |u| \right]_1^4 = \log 4 - \log 1 = \log 4.$$

S-11: Setting $u = 1 + \sin x$, we have $du = \cos x \, dx$ and so

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} dx = \int_{x=0}^{x=\pi/2} \frac{1}{1 + \sin x} \cos x \, dx = \int_{u=1}^{u=2} \frac{du}{u}$$

since $u = 1 + \sin 0 = 1$ when $x = 0$ and $u = 1 + \sin(\pi/2) = 2$ when $x = \pi/2$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_{u=1}^{u=2} \frac{du}{u} = \left[\log |u| \right]_1^2 = \log 2$$

S-12: Setting $u = \sin x$, we have $du = \cos x \, dx$ and so

$$\int_0^{\pi/2} \cos x \cdot (1 + \sin^2 x) dx = \int_{x=0}^{x=\pi/2} (1 + \sin^2 x) \cdot \cos x \, dx = \int_{u=0}^{u=1} (1 + u^2) du,$$

since $u = \sin 0 = 0$ when $x = 0$ and $u = \sin(\pi/2) = 1$ when $x = \pi/2$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_0^1 (1 + u^2) du = \left[u + \frac{u^3}{3} \right]_0^1 = \left(1 + \frac{1}{3} \right) - 0 = \frac{4}{3}.$$

S-13: Substituting $t = x^2 - x$, $dt = (2x - 1) \, dx$ and noting that $t = 0$ when $x = 1$ and $t = 6$ when $x = 3$,

$$\int_1^3 (2x - 1)e^{x^2 - x} dx = \int_0^6 e^t dt = [e^t]_0^6 = e^6 - 1$$

S-14: We use the substitution $u = 4 - x^2$, for which $du = -2x \, dx$:

$$\begin{aligned} \int \frac{x^2 - 4}{\sqrt{4 - x^2}} x \, dx &= \int \frac{1}{2} \cdot \frac{4 - x^2}{\sqrt{4 - x^2}} (-2x) \, dx \\ &= \frac{1}{2} \int \frac{u}{\sqrt{u}} du \\ &= \frac{1}{2} \int \sqrt{u} \, du \\ &= \frac{1}{2} \frac{u^{3/2}}{3/2} + C \\ &= \frac{1}{3} (4 - x^2)^{3/2} + C \end{aligned}$$

S-15:

Solution 1: If we let $u = \sqrt{\log x}$, then $du = \frac{1}{2x\sqrt{\log x}} dx$, and:

$$\int \frac{e^{\sqrt{\log x}}}{2x\sqrt{\log x}} dx = \int e^u du = e^u + C = e^{\sqrt{\log x}} + C$$

Solution 2: In Solution 1, we made a pretty slick choice. We might have tried to work with something a little less convenient. For example, it's not unnatural to think that $u = \log x$, $du = \frac{1}{x} dx$ would be a good choice. In that case:

$$\int \frac{e^{\sqrt{\log x}}}{2x\sqrt{\log x}} dx = \int \frac{e^{\sqrt{u}}}{2\sqrt{u}} du$$

Now, we should be able to see that $w = \sqrt{u}$, $dw = \frac{1}{2\sqrt{u}} du$ is a good choice:

$$\begin{aligned} \int \frac{e^{\sqrt{u}}}{2\sqrt{u}} du &= \int e^w dw \\ &= e^w + C \\ &= e^{\sqrt{u}} + C \\ &= e^{\sqrt{\log x}} + C \end{aligned}$$

S-16:

The straightforward method: We use the substitution $u = x^2$, for which $du = 2x dx$, and note that $u = 4$ for both $x = 2$ and $x = -2$:

$$\int_{-2}^2 x e^{x^2} dx = \int_{-2}^2 \frac{1}{2} e^{x^2} 2x dx = \int_4^4 \frac{1}{2} e^u du = 0$$

The slightly sneaky method: We note that $\frac{d}{dx} \{e^{x^2}\} = 2x e^{x^2}$, so that $\frac{1}{2}e^{x^2}$ is an antiderivative for the integrand $x e^{x^2}$. So

$$\int_{-2}^2 x e^{x^2} dx = \left[\frac{1}{2} e^{x^2} \right]_{-2}^2 = \frac{1}{2} e^4 - \frac{1}{2} e^4 = 0$$

The really sneaky method: The integrand $f(x) = x e^{x^2}$ is an odd function (meaning that $f(-x) = -f(x)$). So by Theorem 1.2.12 in the text every integral of the form $\int_{-a}^a x e^{x^2} dx$ is zero.

S-17: The given sum is of the form

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{j}{n^2} \sin \left(1 + \frac{j^2}{n^2} \right) = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j^*) \Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x \sin(1 + x^2)$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned} \int_0^1 f(x) \, dx &= \int_0^1 x \sin(1 + x^2) \, dx \\ &= \frac{1}{2} \int_1^2 \sin(y) \, dy \quad \text{with } y = 1 + x^2, \, dy = 2x \, dx \\ &= \frac{1}{2} \left[-\cos(y) \right]_{y=1}^{y=2} \\ &= \frac{1}{2} [\cos 1 - \cos 2] \end{aligned}$$

Using a calculator, we see this is close to 0.478.

S-18: Often, the denominator of a function is a good guess for the substitution. So, let's try setting $w = u^2 + 1$. Then $dw = 2u \, du$:

$$\int_0^1 \frac{u^3}{u^2 + 1} \, du = \frac{1}{2} \int_0^1 \frac{u^2}{u^2 + 1} \, 2u \, du$$

The numerator now is u^2 , and looking at our substitution, we see $u^2 = w - 1$:

$$\begin{aligned} &= \frac{1}{2} \int_1^2 \frac{w - 1}{w} \, dw \\ &= \frac{1}{2} \int_1^2 \left(1 - \frac{1}{w} \right) \, dw \\ &= \frac{1}{2} [w - \log |w|]_{w=1}^{w=2} \\ &= \frac{1}{2} (2 - \log 2 - 1) = \frac{1}{2} - \frac{1}{2} \log 2 \end{aligned}$$

S-19: The only thing we really have to work with is a tangent, so it's worth considering what would happen if we substituted $u = \tan \theta$. Then $du = \sec^2 \theta \, d\theta$. This doesn't show up in the integrand as it's written, but we can try and bring it out by using the identity $\tan^2 \theta = \sec^2 \theta - 1$:

$$\begin{aligned} \int \tan^3 \theta \, d\theta &= \int \tan \theta \cdot \tan^2 \theta \, d\theta \\ &= \int \tan \theta \cdot (\sec^2 \theta - 1) \, d\theta \\ &= \int \tan \theta \cdot \sec^2 \theta \, d\theta - \int \tan \theta \, d\theta \end{aligned}$$

In Example 1.5.17 of the text, we learned $\int \tan \theta \, d\theta = \log |\sec \theta| + C$

$$\begin{aligned} &= \int u \, du - \log |\sec \theta| + C \\ &= \frac{1}{2} u^2 - \log |\sec \theta| + C \\ &= \frac{1}{2} \tan^2 \theta - \log |\sec \theta| + C \end{aligned}$$

S-20: At first glance, it's not clear what substitution to use. If we try the denominator, $u = e^x + e^{-x}$, then $du = (e^x - e^{-x}) \, dx$, but it's not clear how to make this work with our integral. So, we can try something else.

If we want to tidy things up, we might think to take $u = e^x$ as a substitution. Then $du = e^x \, dx$, so we need an e^x in the numerator. That can be arranged.

$$\begin{aligned} \int \frac{1}{e^x + e^{-x}} \cdot \left(\frac{e^x}{e^x} \right) dx &= \int \frac{e^x}{(e^x)^2 + 1} dx \\ &= \int \frac{1}{u^2 + 1} du \\ &= \arctan(u) + C \\ &= \arctan(e^x) + C \end{aligned}$$

S-21: We often like to take the “inside” function as our substitution, in this case $u = 1 - x^2$, so $du = -2x \, dx$. This takes care of *part* of the integral:

$$\int_0^1 (1 - 2x) \sqrt{1 - x^2} \, dx = \int_0^1 \sqrt{1 - x^2} \, dx + \int_0^1 (-2x) \sqrt{1 - x^2} \, dx$$

The left integral is tough to solve with substitution, but luckily we don't have to—it's the area of a quarter of a circle of radius 1.

$$\begin{aligned} &= \frac{\pi}{4} + \int_1^0 \sqrt{u} \, du \\ &= \frac{\pi}{4} + \left[\frac{2}{3} u^{3/2} \right]_{u=1}^{u=0} \\ &= \frac{\pi}{4} + 0 - \frac{2}{3} = \frac{\pi}{4} - \frac{2}{3} \end{aligned}$$

S-22:

Solution 1: We often find it useful to take “inside” functions as our substitutions, so let’s try $u = \cos x$, $du = -\sin x \, dx$. In order to dig up a sine, we use the identity $\tan x = \frac{\sin x}{\cos x}$:

$$\begin{aligned}\int \tan x \cdot \log(\cos x) \, dx &= - \int \frac{-\sin x}{\cos x} \cdot \log(\cos x) \, dx \\ &= - \int \frac{1}{u} \log(u) \, du\end{aligned}$$

Now, it is convenient to let $w = \log u$, $dw = \frac{1}{u} \, du$:

$$\begin{aligned}- \int \frac{1}{u} \log(u) \, du &= - \int w \, dw \\ &= -\frac{1}{2} w^2 + C \\ &= -\frac{1}{2} (\log u)^2 + C \\ &= -\frac{1}{2} (\log(\cos x))^2 + C\end{aligned}$$

Solution 2: We might guess that it’s useful to have $u = \log(\cos x)$,

$$du = \frac{-\sin x}{\cos x} \, dx = -\tan x \, dx:$$

$$\begin{aligned}\int \tan x \cdot \log(\cos x) \, dx &= - \int -\tan x \cdot \log(\cos x) \, dx \\ &= - \int u \, du \\ &= -\frac{1}{2} u^2 + C \\ &= -\frac{1}{2} (\log(\cos x))^2 + C\end{aligned}$$

S-23: The given sum is of the form

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{j}{n^2} \cos\left(\frac{j^2}{n^2}\right) = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j^*) \Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x \cos(x^2)$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}\int_0^1 f(x) \, dx &= \int_0^1 x \cos(x^2) \, dx \\ &= \frac{1}{2} \int_0^1 \cos(y) \, dy \quad \text{with } y = x^2, \, dy = 2x \, dx \\ &= \frac{1}{2} \left[\sin(y) \right]_0^1 \\ &= \frac{1}{2} \sin 1\end{aligned}$$

S-24: The given sum is of the form

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{j}{n^2} \sqrt{1 + \frac{j^2}{n^2}} = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j^*) \Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x\sqrt{1+x^2}$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned} \int_0^1 f(x) \, dx &= \int_0^1 x\sqrt{1+x^2} \, dx \\ &= \frac{1}{2} \int_1^2 \sqrt{y} \, dy \quad \text{with } y = 1+x^2, \, dy = 2x \, dx \\ &= \frac{1}{2} \left[\frac{2}{3} y^{3/2} \right]_{y=1}^{y=2} \\ &= \frac{1}{3} [2\sqrt{2} - 1] \end{aligned}$$

Using a calculator, we see this is approximately 0.609.

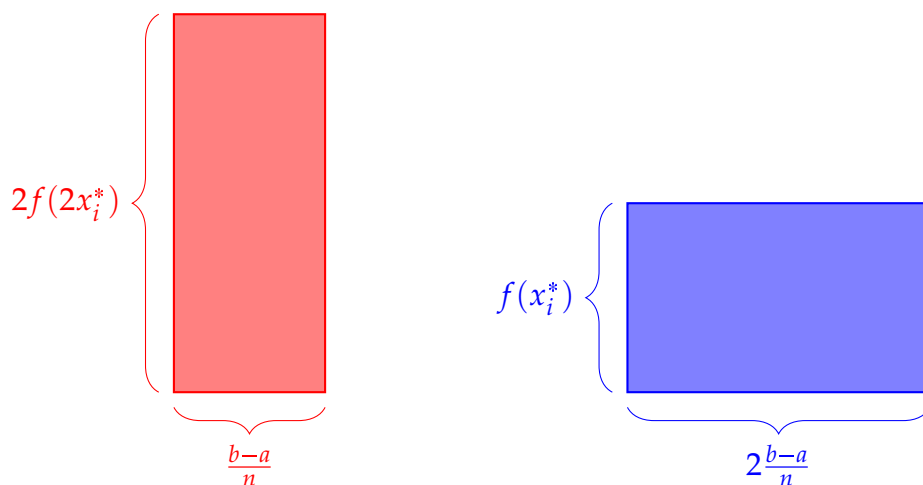
S-25: Using the definition of a definite integral with right Riemann sums:

$$\begin{aligned} \int_a^b 2f(2x) \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \cdot 2f(2(a + i\Delta x)) & \Delta x &= \frac{b-a}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{b-a}{n} \right) \cdot 2f \left(2 \left(a + i \left(\frac{b-a}{n} \right) \right) \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2b-2a}{n} \right) \cdot f \left(2a + i \left(\frac{2b-2a}{n} \right) \right) \\ \int_{2a}^{2b} f(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \cdot f(2a + i\Delta x) & \Delta x &= \frac{2b-2a}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2b-2a}{n} \right) \cdot f \left(2a + i \left(\frac{2b-2a}{n} \right) \right) \end{aligned}$$

Since the Riemann sums are exactly the same,

$$\int_a^b 2f(2x) \, dx = \int_{2a}^{2b} f(x) \, dx$$

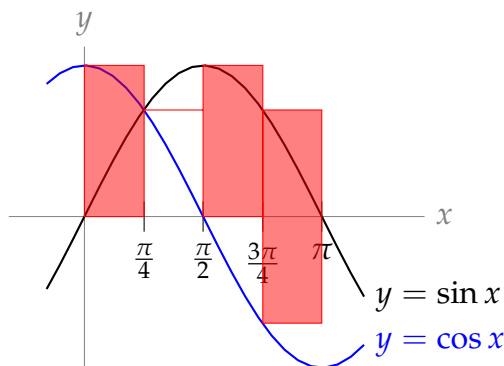
Looking at the Riemann sum in this way is instructive, because it is very clear why the two integrals should be equal (without using substitution). The rectangles in the first Riemann sum are half as wide, but twice as tall, as the rectangles in the second Riemann sum. So, the two Riemann sums have rectangles of the same area.



In the integral on the left, the variable is red x and in the integral on the right, the variable is blue x . Red x and blue x are not the same. In fact $2x_i^* = x_i^*$. (Not every substitution corresponds to such a simple picture.)

Solutions to Exercises 1.6 — [Jump to TABLE OF CONTENTS](#)

S-1:



The intervals of our rectangles are $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, and $[\frac{3\pi}{4}, \pi]$. Since we're taking a left Riemann sum, we find the height of the rectangles at the left endpoints of the intervals.

$x = 0$: The distance from $\cos 0$ to $\sin 0$ is 1, so our first rectangle has height 1.

$x = \frac{\pi}{4}$: The distance from $\cos \frac{\pi}{4}$ to $\sin \frac{\pi}{4}$ is 0, so our second rectangle has height 0.

$x = \frac{\pi}{2}$: The distance from $\cos \frac{\pi}{2}$ to $\sin \frac{\pi}{2}$ is 1, so our third rectangle has height 1.

$x = \frac{3\pi}{4}$: The distance from $\cos \frac{3\pi}{4}$ to $\sin \frac{3\pi}{4}$ is $\sin(3\pi/4) - \cos(3\pi/4) = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$, so our fourth rectangle has height $\sqrt{2}$.

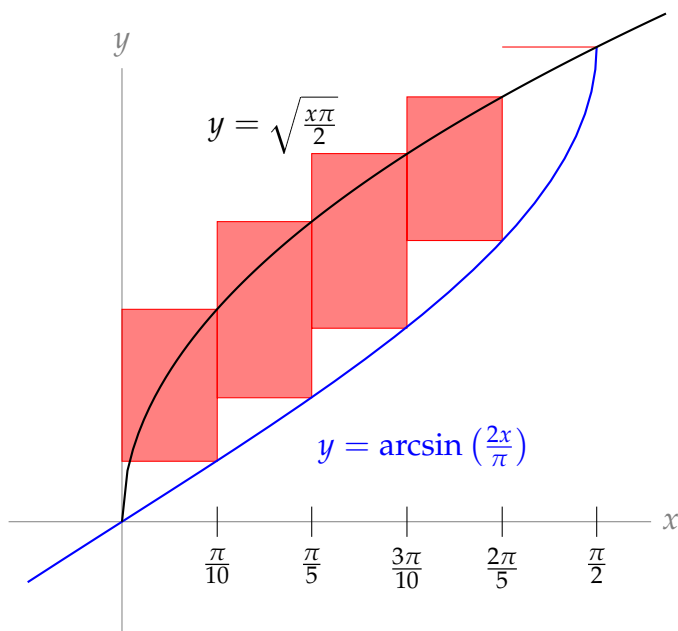
So, our approximation for the area between the two curves is

$$\frac{\pi}{4} (1 + 0 + 1 + \sqrt{2}) = \frac{\pi}{4} (2 + \sqrt{2})$$

S-2:

- (a) We are finding the area in the interval from $x = 0$ to $x = \frac{\pi}{2}$. Since we're taking $n = 5$ rectangles, our rectangles cover the following intervals:

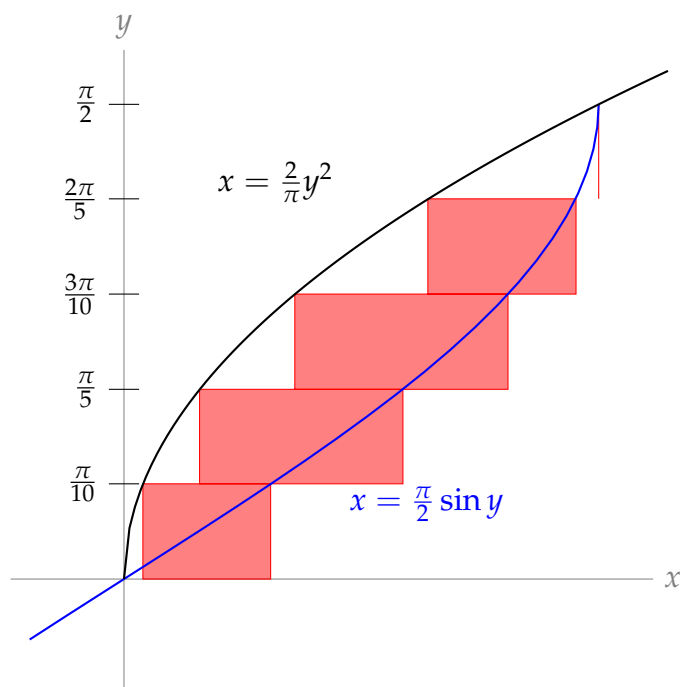
$$\left[0, \frac{\pi}{10}\right], \quad \left[\frac{\pi}{10}, \frac{\pi}{5}\right], \quad \left[\frac{\pi}{5}, \frac{3\pi}{10}\right], \quad \left[\frac{3\pi}{10}, \frac{2\pi}{5}\right], \quad \left[\frac{2\pi}{5}, \frac{\pi}{2}\right].$$



- (b) We are finding the area in the interval from $y = 0$ to $y = \frac{\pi}{2}$. (In general, when we switch from horizontal rectangles to vertical, the limits of integration will change—it's only coincidence that they are the same in this example.) Since we're taking $n = 5$ rectangles, these rectangles cover the following intervals of the y -axis:

$$\left[0, \frac{\pi}{10}\right], \quad \left[\frac{\pi}{10}, \frac{\pi}{5}\right], \quad \left[\frac{\pi}{5}, \frac{3\pi}{10}\right], \quad \left[\frac{3\pi}{10}, \frac{2\pi}{5}\right], \quad \left[\frac{2\pi}{5}, \frac{\pi}{2}\right].$$

The question doesn't specify which endpoints we're using. Let's use upper endpoints, to match part (a).



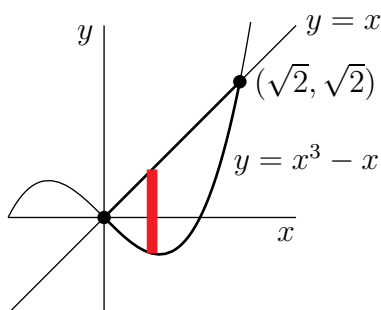
S-3: The curves intersect when $y = x$ and $y = x^3 - x$. To find these points, we set:

$$\begin{aligned} x &= x^3 - x \\ 0 &= x^3 - 2x \\ 0 &= x(x^2 - 2) \\ 0 &= x \quad \text{or} \quad 0 = x^2 - 2 \end{aligned}$$

For $x \geq 0$, the curves intersect at $(0, 0)$ and $(\sqrt{2}, \sqrt{2})$.

A handy observation is that, since both curves are continuous and they do not meet each other between $x = 0$ and $x = \sqrt{2}$, we don't have to worry about dividing our area into two regions: one of the functions is always on the top, and the other is always on the bottom.

Using vertical strips:



The top and bottom boundaries of the specified region are $y = T(x) = x$ and

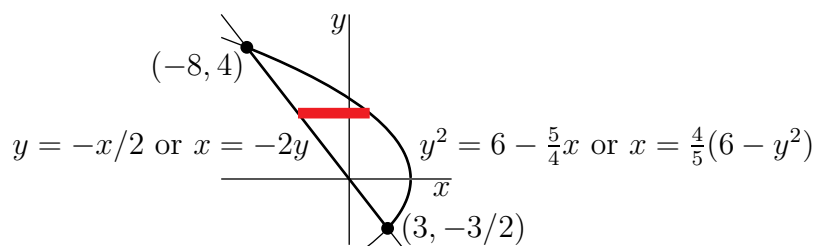
$y = B(x) = x^3 - x$, respectively. So,

$$\text{Area} = \int_0^{\sqrt{2}} [T(x) - B(x)] \, dx = \int_0^{\sqrt{2}} [x - (x^3 - x)] \, dx = \int_0^{\sqrt{2}} 2x - x^3 \, dx$$

S-4: We need to find where the curves intersect.

$$\begin{aligned} \frac{x^2}{4} &= y^2 = 6 - \frac{5x}{4} \\ \frac{1}{4}x^2 + \frac{5}{4}x - 6 &= 0 \\ x^2 + 5x - 24 &= 0 \\ (x + 8)(x - 3) &= 0 \\ x = -8, \quad x = 3 \end{aligned}$$

The curves intersect at $(-8, 4)$ and $(3, -\frac{3}{2})$. Using horizontal strips:



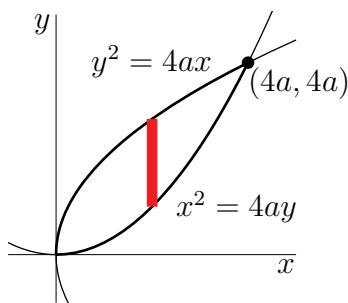
we have

$$\text{Area} = \int_{-3/2}^4 \left[\frac{4}{5}(6 - y^2) + 2y \right] \, dy$$

S-5: If the curves intersect at (x, y) , then

$$\begin{aligned} (x^2)^2 &= (4a)^2 y^2 = (4a)^2 4ax \\ x^4 &= (4a)^3 x \\ x^4 - (4a)^3 x &= 0 \\ x(x^3 - (4a)^3) &= 0 \\ x = 0 \quad \text{or} \quad x^3 &= (4a)^3 \end{aligned}$$

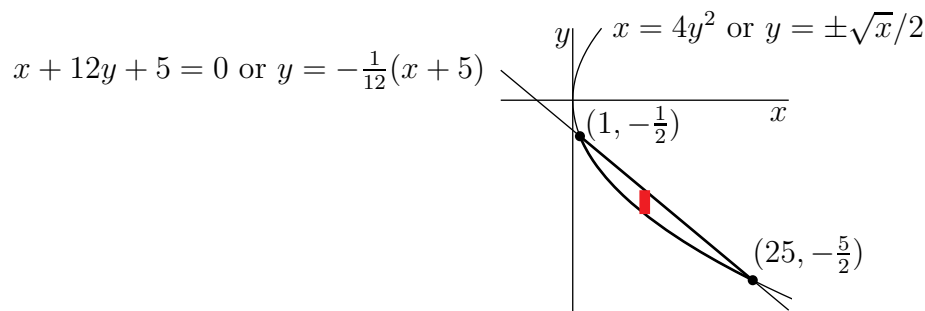
The curves intersect at $(0, 0)$ and $(4a, 4a)$. (It is also possible to find these points by inspection.) Using vertical strips:



We want the y -values of the functions. We write the top function as $y = \sqrt{4ax}$ (we care about the positive square root, not the negative one) and we write the bottom function as $y = \frac{x^2}{4a}$. Then we have

$$\text{Area} = \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx$$

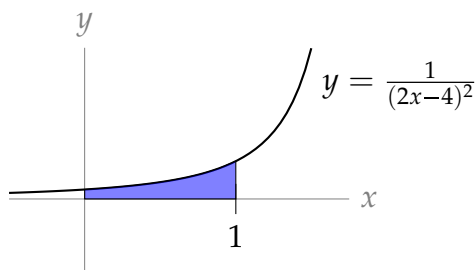
S-6: The curves intersect when $x = 4y^2$ and $0 = 4y^2 + 12y + 5 = (2y + 5)(2y + 1)$. So, the curves intersect at $(1, -\frac{1}{2})$ and $(25, -\frac{5}{2})$. Using vertical strips:



we have

$$\text{Area} = \int_1^{25} \left[-\frac{1}{12}(x + 5) + \frac{1}{2}\sqrt{x} \right] dx$$

S-7:



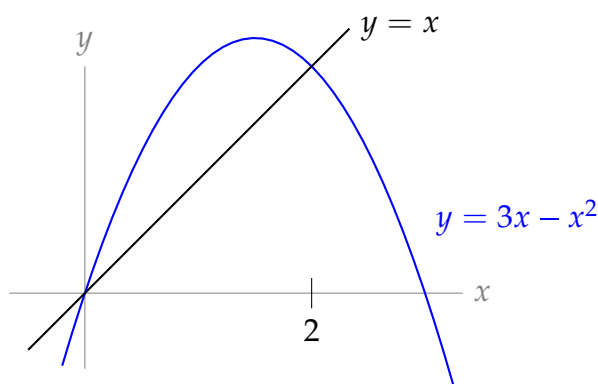
The area between the curve $y = \frac{1}{(2x-4)^2}$ and the x -axis, with x running from $a = 0$ to $b = 1$, is exactly the definite integral of $\frac{1}{(2x-4)^2}$ with limits 0 and 1.

$$\begin{aligned}
 \text{Area} &= \int_0^1 \frac{dx}{(2x-4)^2} & u &= 2x-4, \quad du = 2 \, dx \\
 &= \frac{1}{2} \int_{-4}^{-2} \frac{1}{u^2} du = \frac{1}{2} \left[\frac{-1}{u} \right]_{u=-4}^{u=-2} \\
 &= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{4} \right] = \frac{1}{8}
 \end{aligned}$$

S-8: If the curves $y = f(x) = x$ and $y = g(x) = 3x - x^2$ intersect at (x, y) , then

$$\begin{aligned}
 3x - x^2 &= y = x \\
 x^2 - 2x &= 0 \\
 x(x-2) &= 0 \\
 x = 0 &\quad \text{or} \quad x = 2
 \end{aligned}$$

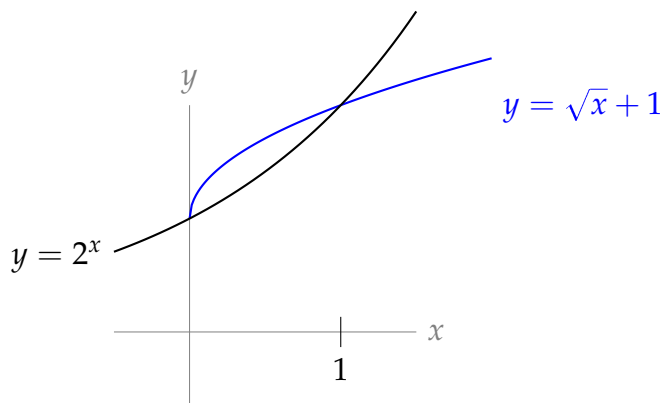
Furthermore, $g(x) - f(x) = 2x - x^2 = x(2 - x)$ is positive for all $0 \leq x \leq 2$. That is, the curve $y = 3x - x^2$ lies above the line $y = x$ for all $0 \leq x \leq 2$.



We therefore evaluate the integral:

$$\int_0^2 [(3x - x^2) - x] \, dx = \int_0^2 [2x - x^2] \, dx = \left[x^2 - \frac{x^3}{3} \right]_0^2 = \left[4 - \frac{8}{3} \right] - 0 = \frac{4}{3}$$

S-9: The following sketch contains the graphs of $y = 2^x$ and $y = \sqrt{x} + 1$.

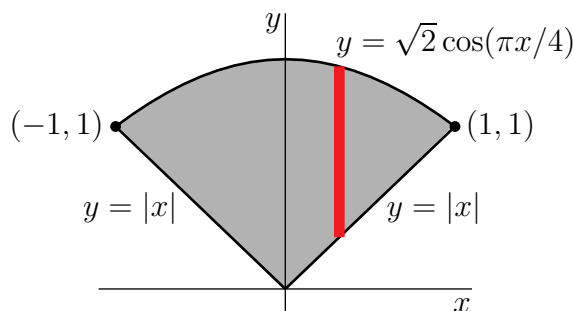


From the sketch, it looks like the two curves cross when $x = 0$ and when $x = 1$ and nowhere else. Indeed, when $x = 0$ we have $2^x = \sqrt{x} + 1 = 1$ and when $x = 1$ we have $2^x = \sqrt{x} + 1 = 2$.

To antidifferentiate 2^x , we write $2^x = (e^{\log 2})^x = e^{x \log 2}$.

$$\begin{aligned} \text{Area} &= \int_0^1 [(\sqrt{x} + 1) - e^{x \log 2}] dx = \left[\frac{2}{3} x^{3/2} + x - \frac{1}{\log 2} 2^x \right]_0^1 \\ &= \frac{2}{3} + 1 - \frac{1}{\log 2} [2 - 1] = \frac{5}{3} - \frac{1}{\log 2} \end{aligned}$$

S-10: Here is a sketch of the specified region.



Both functions are even, so the region is symmetric about the y -axis. So, we will compute the area of the part with $x \geq 0$ and multiply by 2. The curves $y = \sqrt{2} \cos(\pi x/4)$ and $y = x$ intersect when $x = \sqrt{2} \cos(\pi x/4)$ or $\cos(\pi x/4) = \frac{x}{\sqrt{2}}$, which is the case⁴ when $x = 1$. So, using vertical strips as in the figure above, the area (including the multiplication by 2) is

$$2 \int_0^1 [\sqrt{2} \cos(\pi x/4) - x] dx = 2 \left[\sqrt{2} \frac{4}{\pi} \sin(\pi x/4) - \frac{x^2}{2} \right]_0^1 = 2 \left[\frac{4}{\pi} - \frac{1}{2} \right] = \frac{8}{\pi} - 1$$

S-11: For our computation, we will need an antiderivative of $x^2 \sqrt{x^3 + 1}$, which can be found using the substitution $u = x^3 + 1$, $du = 3x^2 dx$:

$$\int x^2 \sqrt{x^3 + 1} dx = \int \sqrt{u} \cdot \frac{1}{3} du = \frac{1}{3} \int u^{1/2} du = \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} + C = \frac{2}{9} (x^3 + 1)^{3/2} + C.$$

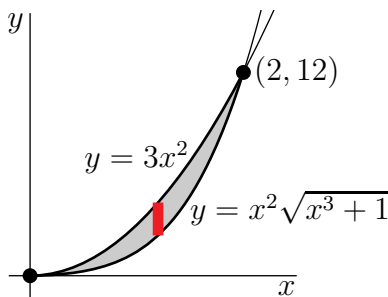
The two functions $f(x)$ and $g(x)$ are clearly equal at $x = 0$. If $x \neq 0$, then the functions

4 The solution $x = 1$ was found by guessing. To guess a solution to $\cos(\pi x/4) = \frac{x}{\sqrt{2}}$ just ask yourself what simple angle has a cosine that involves $\sqrt{2}$. This guessing strategy is essentially useless in the real world, but works great on problem sets and exams.

are equal when

$$\begin{aligned} 3x^2 &= x^2\sqrt{x^3+1} \\ 3 &= \sqrt{x^3+1} \\ 9 &= x^3+1 \\ 8 &= x^3 \\ 2 &= x. \end{aligned}$$

The function $g(x) = 3x^2$ is the larger of the two on the interval $[0, 2]$, as can be seen by plugging in $x = 1$, say, or by observing that when x is very small $f(x) = x^2\sqrt{x^3+1} \approx x^2$ and $g(x) = 3x^2$.



The area in question is therefore:

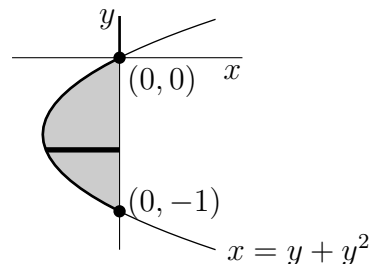
$$\begin{aligned} \int_0^2 (3x^2 - x^2\sqrt{x^3+1}) \, dx &= \left(x^3 - \frac{2}{9}(x^3+1)^{3/2} \right) \Big|_0^2 \\ &= \left(2^3 - \frac{2}{9}(2^3+1)^{3/2} \right) - \left(0^3 - \frac{2}{9}(0^3+1)^{3/2} \right) \\ &= (8 - 6) - \left(0 - \frac{2}{9} \right) = \frac{20}{9}. \end{aligned}$$

S-12: First, let's figure out what our curve $x = y^2 + y = y(y+1)$ looks like.

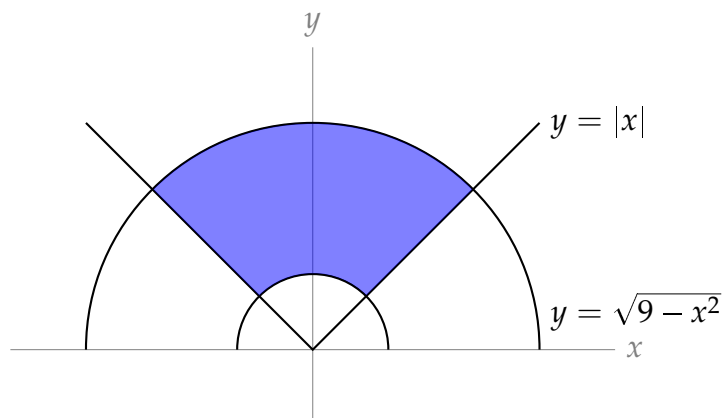
- The curve intercepts the y -axis when $y = 0$ and $y = -1$.
- The x -values of the curve are negative when $-1 < y < 0$, and positive elsewhere.

This leads to the figure below. We're evaluating the area from $y = -1$ to $y = 0$. Since $y^2 + y$ is negative there, the length of our (horizontal) slices are $0 - (y^2 + y)$.

$$\text{Area} = \int_{-1}^0 (0 - (y^2 + y)) \, dy = - \left[\frac{y^3}{3} + \frac{y^2}{2} \right]_{-1}^0 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$



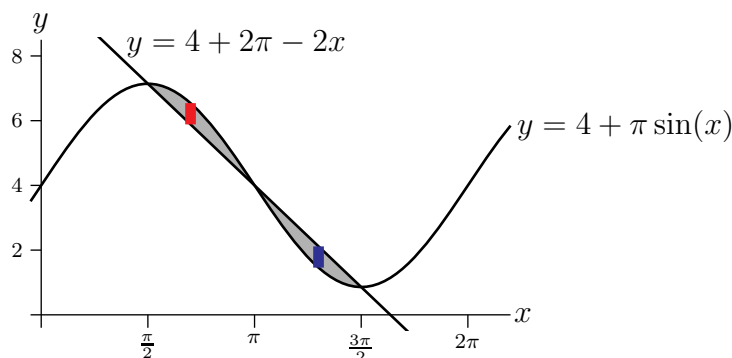
S-13: Let's begin by sketching our region. Note that $y = \sqrt{1 - x^2}$ and $y = \sqrt{9 - x^2}$ are the top halves of circles centred at the origin with radii 1 and 3, respectively.



Our region is the difference of two quarter-circles, so we find its area using geometry:

$$\text{Area} = \frac{1}{4} (\pi \cdot 3^2) - \frac{1}{4} (\pi \cdot 1^2) = 2\pi$$

S-14: We will compute the area by using thin vertical strips, as in the sketch below:



By looking at the sketch above, we guess the line $y = 4 + 2\pi - 2x$ intersects the curve $y = 4 + \pi \sin x$ when $x = \frac{\pi}{2}$, $x = \pi$, and $x = \frac{3\pi}{2}$. Let's make sure these are correct by plugging them into the two equations, and making sure the y -values match:

x	$4 + 2\pi - 2x$	$4 + \pi \sin(x)$	match?
$\frac{\pi}{2}$	$4 + \pi$	$4 + \pi$	✓
π	4	4	✓
$\frac{3\pi}{2}$	$4 - \pi$	$4 - \pi$	✓

Also from the sketch, we see that:

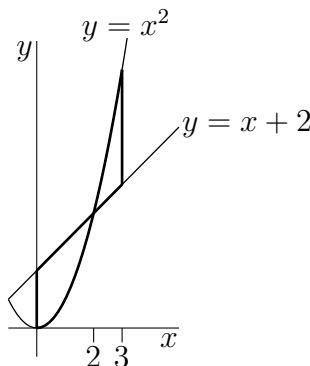
- When $\frac{\pi}{2} \leq x \leq \pi$, the top of the strip is at $y = 4 + \pi \sin x$ and the bottom of the strip is at $y = 4 + 2\pi - 2x$. So the strip has height $[(4 + \pi \sin x) - (4 + 2\pi - 2x)]$ and width dx , and hence area $[(4 + \pi \sin x) - (4 + 2\pi - 2x)] dx$.

- When $\pi \leq x \leq \frac{3\pi}{2}$, the top of the strip is at $y = 4 + 2\pi - 2x$ and the bottom of the strip is at $y = 4 + \pi \sin x$. So the strip has height $[(4 + 2\pi - 2x) - (4 + \pi \sin x)]$ and width dx , and hence area $[(4 + 2\pi - 2x) - (4 + \pi \sin x)]dx$.

Now we can calculate:

$$\begin{aligned}
 \text{Area} &= \int_{\pi/2}^{\pi} [(4 + \pi \sin x) - (4 + 2\pi - 2x)] dx + \int_{\pi}^{3\pi/2} [(4 + 2\pi - 2x) - (4 + \pi \sin x)] dx \\
 &= \int_{\pi/2}^{\pi} [\pi \sin x - 2\pi + 2x] dx + \int_{\pi}^{3\pi/2} [2\pi - 2x - \pi \sin x] dx \\
 &= \left[-\pi \cos x - 2\pi x + x^2 \right]_{\pi/2}^{\pi} + \left[2\pi x - x^2 + \pi \cos x \right]_{\pi}^{3\pi/2} \\
 &= \left[\pi - \pi^2 + \frac{3}{4}\pi^2 \right] + \left[\pi^2 - \frac{5}{4}\pi^2 + \pi \right] \\
 &= 2 \left[\pi - \frac{1}{4}\pi^2 \right]
 \end{aligned}$$

S-15: First, here is a sketch of the region. We are not asked for it, but it is crucial for understanding the question.



The two curves $y = x + 2$ and $y = x^2$ cross at $(2, 4)$. The area of the part between them with $0 \leq x \leq 2$ is:

$$\int_0^2 [x + 2 - x^2] dx = \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \right]_0^2 = 2 + 4 - \frac{8}{3} = \frac{10}{3}$$

The area of the part between the two curves with $2 \leq x \leq 3$ is:

$$\int_2^3 [x^2 - (x + 2)] dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x \right]_2^3 = 9 - \frac{9}{2} - 6 - \frac{8}{3} + 2 + 4 = \frac{11}{6}$$

The total area is $\frac{10}{3} + \frac{11}{6} = \frac{31}{6}$.

S-16: We need to figure out which curve is on top, when. To do this, set $h(x) = 3x - x\sqrt{25 - x^2}$. If $h(x) > 0$, then $y = 3x$ is the top curve; if $h(x) < 0$, then

$y = x\sqrt{25 - x^2}$ is the top curve.

$$h(x) = 3x - x\sqrt{25 - x^2} = x \left[3 - \sqrt{25 - x^2} \right]$$

We only care about values of x in $[0, 4]$, so x is nonnegative. Then $h(x)$ is positive when:

$$\begin{aligned} 3 &> \sqrt{25 - x^2} \\ 9 &> 25 - x^2 \\ x^2 &> 16 \\ x &> 4 \end{aligned}$$

That is, $h(x)$ is never positive over the interval $[0, 4]$. So, $y = x\sqrt{25 - x^2}$ lies above $y = 3x$ for all $0 \leq x \leq 4$.

The area we need to calculate is therefore:

$$\begin{aligned} A &= \int_0^4 \left[x\sqrt{25 - x^2} - 3x \right] dx \\ &= \int_0^4 x\sqrt{25 - x^2} dx - \int_0^4 3x dx \\ &= A_1 - A_2. \end{aligned}$$

To evaluate A_1 , we use the substitution $u(x) = 25 - x^2$, for which $du = u'(x) dx = -2x dx$; and $u(4) = 25 - 4^2 = 9$ when $x = 4$, while $u(0) = 25 - 0^2 = 25$ when $x = 0$. Therefore

$$A_1 = \int_{x=0}^{x=4} x\sqrt{25 - x^2} dx = -\frac{1}{2} \int_{u=25}^{u=9} \sqrt{u} du = \left[-\frac{1}{3} u^{3/2} \right]_{25}^9 = \frac{125 - 27}{3} = \frac{98}{3}$$

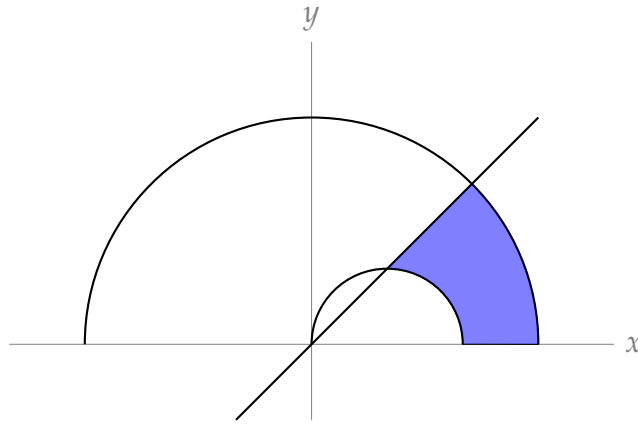
For A_2 we use the antiderivative directly:

$$A_2 = \int_0^4 3x dx = \left[\frac{3x^2}{2} \right]_0^4 = 24$$

Therefore the total area is:

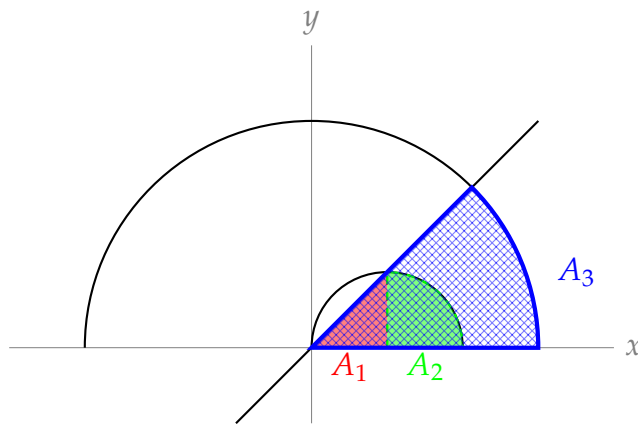
$$A = \frac{98}{3} - 24 = \frac{26}{3}$$

S-17: Let's begin by sketching our region. Note that $y = \sqrt{9 - x^2}$ is the top half of a circle centred at the origin with radius 3, while $y = \sqrt{1 - (x - 1)^2}$ is the top half of a circle of radius 1 centred at $(1, 0)$.



Note $y = x$ intersects $y = \sqrt{1 - (x - 1)^2}$ at $(1, 1)$, the highest part of the smaller half-circle.

We can easily take the area of triangles and sectors of circles. With that in mind, we cut up our region the following way:



- The desired area is $A_3 - (A_1 + A_2)$.
- A_1 is the area of right a triangle with base 1 and height 1, so $A_1 = \frac{1}{2}$.
- A_2 is the area of a quarter circle of radius 1, so $A_2 = \frac{\pi}{4}$.
- A_3 is the area of an eighth of a circle of radius 3, so $A_3 = \frac{9\pi}{8}$

So, the area of our region is $\frac{9\pi}{8} - \frac{1}{2} - \frac{\pi}{4} = \frac{7\pi}{8} - \frac{1}{2}$.

S-18: The first function is a cubic, with intercepts at $x = 0, \pm 2$. The second is a straight line with a positive slope.

We need to figure out what these functions look like in relation to one another, so let's

find their points of intersection.

$$\begin{aligned}
 x(x^2 - 4) &= x - 2 \\
 x(x + 2)(x - 2) &= x - 2 \\
 \boxed{x - 2 = 0} &\quad \text{or} \quad x(x + 2) = 1 \\
 x^2 + 2x - 1 &= 0 \\
 x &= \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2} \\
 x &= \boxed{-1 \pm \sqrt{2}}
 \end{aligned}$$

So, our three points of intersection are when $x = 2$ and when $x = -1 \pm \sqrt{2}$. We note

$$-1 - \sqrt{2} < -1 + \sqrt{2} < -1 + \sqrt{4} < 2.$$

So, we need to see which function is on top over the two intervals $[-1 - \sqrt{2}, -1 + \sqrt{2}]$ and $[-1 + \sqrt{2}, 2]$. It suffices to check points in these intervals.

x	$x(x^2 - 4)$	$x - 2$	top function:
0	0	-2	$x(x^2 - 4)$
1	-3	-1	$x - 2$

Since 0 is in the interval $[-1 - \sqrt{2}, -1 + \sqrt{2}]$, $x(x^2 - 4)$ is the top function in that interval. Since 1 is in the interval $[-1 + \sqrt{2}, 2]$, $x - 2$ is the top function in that interval. Now we can set up the integral to evaluate the area:

$$\begin{aligned}
 \text{Area} &= \int_{-1-\sqrt{2}}^{-1+\sqrt{2}} [x(x^2 - 4) - (x - 2)] \, dx + \int_{-1+\sqrt{2}}^2 [(x - 2) - x(x^2 - 4)] \, dx \\
 &= \int_{-1-\sqrt{2}}^{-1+\sqrt{2}} [x^3 - 5x + 2] \, dx + \int_{-1+\sqrt{2}}^2 [-x^3 + 5x - 2] \, dx \\
 &= \left[\frac{1}{4}x^4 - \frac{5}{2}x^2 + 2x \right]_{-1-\sqrt{2}}^{-1+\sqrt{2}} + \left[-\frac{1}{4}x^4 + \frac{5}{2}x^2 - 2x \right]_{-1+\sqrt{2}}^2
 \end{aligned}$$

After some taxing but rudimentary algebra:

$$= (8\sqrt{2}) + \left(4\sqrt{2} - \frac{13}{4}\right) = 12\sqrt{2} - \frac{13}{4}$$

Solutions to Exercises 1.7 — Jump to [TABLE OF CONTENTS](#)

S-1: If $u = \cos x$, then $du = -\sin x \, dx$. If $n \neq -1$, then

$$\int_0^{\pi/4} \sin x \cos^n x \, dx = -\int_1^{1/\sqrt{2}} u^n \, du = \left[-\frac{1}{n+1} u^{n+1} \right]_1^{1/\sqrt{2}} = \frac{1}{n+1} \left(1 - \frac{1}{\sqrt{2}^{n+1}} \right)$$

If $n = -1$, then

$$\begin{aligned}\int_0^{\pi/4} \sin x \cos^n x \, dx &= -\int_1^{1/\sqrt{2}} u^n \, du = -\int_1^{1/\sqrt{2}} \frac{1}{u} \, du = \left[-\log |u| \right]_1^{1/\sqrt{2}} \\ &= -\log \left(\frac{1}{\sqrt{2}} \right) = \frac{1}{2} \log 2\end{aligned}$$

So, (e) n can be any real number.

S-2: We use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$.

$$\int \sec^n x \tan x \, dx = \int \sec^{n-1} x \cdot \sec x \tan x \, dx = \int u^{n-1} \, du$$

Since n is positive, $n - 1 \neq -1$, so we antidifferentiate using the power rule.

$$= \frac{u^n}{n} + C = \frac{1}{n} \sec^n x + C$$

S-3: We divide both sides by $\cos^2 x$, and simplify.

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \frac{\sin^2 x + \cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\ \frac{\sin^2 x}{\cos^2 x} + 1 &= \sec^2 x \\ \tan^2 x + 1 &= \sec^2 x\end{aligned}$$

S-4: The power of cosine is odd, and the power of sine is even (zero). Following the strategy in the text, we make the substitution $u = \sin x$, so that $du = \cos x \, dx$ and $\cos^2 x = 1 - \sin^2 x = 1 - u^2$:

$$\begin{aligned}\int \cos^3 x \, dx &= \int (1 - \sin^2 x) \cos x \, dx = \int (1 - u^2) \, du \\ &= u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C\end{aligned}$$

S-5: Using the trig identity $\cos^2 x = \frac{1 + \cos(2x)}{2}$, we have

$$\int \cos^2 x \, dx = \frac{1}{2} \int_0^\pi [1 + \cos(2x)] \, dx = \frac{1}{2} \left[x + \frac{1}{2} \sin(2x) \right]_0^\pi = \frac{\pi}{2}$$

S-6: Since the power of cosine is odd, following the strategies in the text, we make the substitution $u = \sin t$, so that $du = \cos t \, dt$ and $\cos^2 t = 1 - \sin^2 t = 1 - u^2$.

$$\begin{aligned}\int \sin^{36} t \cos^3 t \, dt &= \int \sin^{36} t (1 - \sin^2 t) \cos t \, dt = \int u^{36} (1 - u^2) \, du \\ &= \frac{u^{37}}{37} - \frac{u^{39}}{39} + C = \frac{\sin^{37} t}{37} - \frac{\sin^{39} t}{39} + C\end{aligned}$$

S-7: Since the power of sine is odd (and positive), we can reserve one sine for du , and turn the rest into cosines using the identity $\sin^2 + \cos^2 x = 1$. This allows us to use the substitution $u = \cos x$, $du = -\sin x \, dx$, and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^4 x} \, dx &= \int \frac{\sin^2 x}{\cos^4 x} \sin x \, dx = \int -\frac{1 - u^2}{u^4} \, du \\ &= \int \left(-\frac{1}{u^4} + \frac{1}{u^2} \right) \, du = \frac{1}{3u^3} - \frac{1}{u} + C \\ &= \frac{1}{3 \cos^3 x} - \frac{1}{\cos x} + C\end{aligned}$$

S-8: Both sine and cosine have even powers (four and zero, respectively), so we don't have the option of using a substitution like $u = \sin x$ or $u = \cos x$. Instead, we use the identity $\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$.

$$\begin{aligned}\int_0^{\pi/3} \sin^4 x \, dx &= \int_0^{\pi/3} (\sin^2 x)^2 \, dx = \int_0^{\pi/3} \left(\frac{1 - \cos(2x)}{2} \right)^2 \, dx \\ &= \frac{1}{4} \int_0^{\pi/3} (1 - 2\cos(2x) + \cos^2(2x)) \, dx \\ &= \frac{1}{4} \int_0^{\pi/3} (1 - 2\cos(2x)) \, dx + \frac{1}{4} \int_0^{\pi/3} \cos^2(2x) \, dx\end{aligned}$$

We can antidifferentiate the first integral right away. For the second integral, we use the identity $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$, with $\theta = 2x$.

$$\begin{aligned}&= \frac{1}{4} [x - \sin(2x)]_0^{\pi/3} + \frac{1}{8} \int_0^{\pi/3} (1 + \cos(4x)) \, dx \\ &= \frac{1}{4} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right] + \frac{1}{8} \left[x + \frac{1}{4} \sin(4x) \right]_0^{\pi/3} \\ &= \frac{1}{4} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right] + \frac{1}{8} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{8} \right] \\ &= \frac{\pi}{8} - \frac{9\sqrt{3}}{64}\end{aligned}$$

S-9: Since the power of sine is odd, we can reserve one sine for du , and change the remaining four into cosines. This sets us up to use the substitution $u = \cos x$, $du = -\sin x \, dx$.

$$\begin{aligned}\int \sin^5 x \, dx &= \int \sin^4 x \cdot \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx \\ &= - \int (1 - u^2)^2 \, du = - \int (1 - 2u^2 + u^4) \, du \\ &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C\end{aligned}$$

S-10: If we use the substitution $u = \sin x$, then $du = \cos x \, dx$, which very conveniently shows up in the integrand.

$$\int \sin^{1.2} x \cos x \, dx = \int u^{1.2} \, du = \frac{u^{2.2}}{2.2} + C = \frac{1}{2.2} \sin^{2.2} x + C$$

Note this is exactly the strategy described in the text when the power of cosine is odd. The non-integer power of sine doesn't cause a problem.

S-11:

Solution 1: Let's use the substitution $u = \tan x$, $du = \sec^2 x \, dx$:

$$\int \tan x \sec^2 x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$$

Solution 2: We can also use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$:

$$\int \tan x \sec^2 x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sec^2 x + C$$

We note that because $\tan^2 x$ and $\sec^2 x$ only differ by a constant, the two answers are equivalent.

S-12:

Solution 1: Substituting $u = \cos x$, $du = -\sin x \, dx$, $\sin^2 x = 1 - \cos^2 x = 1 - u^2$, gives

$$\begin{aligned}\int \tan^3 x \sec^5 x \, dx &= \int \frac{\sin^3 x}{\cos^8 x} \, dx = \int \frac{(1 - \cos^2 x) \sin x}{\cos^8 x} \, dx = - \int \frac{1 - u^2}{u^8} \, du \\ &= - \left[\frac{u^{-7}}{-7} - \frac{u^{-5}}{-5} \right] + C = \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C\end{aligned}$$

Solution 2: Alternatively, substituting $u = \sec x$, $du = \sec x \tan x \, dx$,
 $\tan^2 x = \sec^2 x - 1 = u^2 - 1$, gives

$$\begin{aligned}\int \tan^3 x \sec^5 x \, dx &= \int \tan^2 x \sec^4 x (\tan x \sec x) \, dx = \int (u^2 - 1)u^4 \, du \\ &= \left[\frac{u^7}{7} - \frac{u^5}{5} \right] + C = \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C\end{aligned}$$

S-13: We use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$. Then
 $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.

$$\begin{aligned}\int \tan^3 x \sec^{1.5} x \, dx &= \int \tan^2 x \cdot \sec^{0.5} x \cdot \sec x \tan x \, dx \\ &= \int (u^2 - 1)u^{0.5} \, du = \int (u^{2.5} - u^{0.5}) \, du \\ &= \frac{u^{3.5}}{3.5} - \frac{u^{1.5}}{1.5} + C \\ &= \frac{1}{3.5} \sec^{3.5} x - \frac{1}{1.5} \sec^{1.5} x + C\end{aligned}$$

Note this solution used the same method as Example 1.7.14 in the text for the case that the power of tangent is odd and there is at least one secant.

S-14: We'll give two solutions.

Solution 1: As in Question 13, we have an odd power of tangent and at least one secant. So, as in strategy (2) of Section 1.7.2 in the text, we can use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$, and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.

$$\begin{aligned}\int \tan^3 x \sec^2 x \, dx &= \int \tan^2 x \sec x \cdot \sec x \tan x \, dx \\ &= \int (u^2 - 1)u \, du = \int (u^3 - u) \, du \\ &= \frac{1}{4}u^4 - \frac{1}{2}u^2 + C \\ &= \frac{1}{4} \sec^4 x - \frac{1}{2} \sec^2 x + C\end{aligned}$$

Solution 2: We have an even, strictly positive, power of sec x . So, as in strategy (3) of Section 1.7.2 in the text, we can use the substitution $u = \tan x$, $du = \sec^2 x \, dx$.

$$\begin{aligned}\int \tan^3 x \sec^2 x \, dx &= \int \tan^3 x \cdot \sec^2 x \, dx \\ &= \int u^3 \, du \\ &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4} \tan^4 x + C\end{aligned}$$

It looks like we have two different answers. But, because $\tan^2 x = \sec^2 x - 1$,

$$\frac{1}{4} \tan^4 = \frac{1}{4} (\sec^2 x - 1)^2 = \frac{1}{4} \sec^4 x - \frac{1}{2} \sec^2 x + \frac{1}{4}$$

and the two answers are really the same, except that the arbitrary constant C of Solution 1 is $\frac{1}{4}$ plus the arbitrary constant C of Solution 2.

S-15: In contrast to Questions 13 and 14, we do not have an odd power of tangent, so we should consider a different substitution. Luckily, if we choose $u = \tan x$, then $du = \sec^2 x \, dx$, and this fits our integrand nicely.

$$\int \tan^4 x \sec^2 x \, dx = \int u^4 \, du = \frac{1}{5} u^5 + C = \frac{1}{5} \tan^5 x + C$$

S-16:

Solution 1: Since the power of tangent is odd, let's try to use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$, and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$, as in Questions 13 and 14. In order to make this work, we need to see $\sec x \tan x \, dx$ in the integrand, so we do a little algebraic manipulation.

$$\begin{aligned} \int \tan^3 x \sec^{-0.7} x \, dx &= \int \frac{\tan^3 x}{\sec^{0.7x}} \, dx = \int \frac{\tan^3 x}{\sec^{1.7x}} \sec x \, dx \\ &= \int \frac{\tan^2 x}{\sec^{1.7} x} \cdot \sec x \tan x \, dx \\ &= \int \frac{u^2 - 1}{u^{1.7}} \, du = \int (u^{0.3} - u^{-1.7}) \, du \\ &= \frac{u^{1.3}}{1.3} + \frac{1}{0.7u^{0.7}} + C \\ &= \frac{1}{1.3} \sec^{1.3} x + \frac{1}{0.7 \sec^{0.7} x} + C \\ &= \frac{1}{1.3} \sec^{1.3} x + \frac{1}{0.7} \cos^{0.7} x + C \end{aligned}$$

Solution 2: Let's convert the secants and tangents to sines and cosines.

$$\begin{aligned} \int \tan^3 x \sec^{-0.7} x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} \cdot \cos^{0.7} x \, dx \\ &= \int \frac{\sin^3 x}{\cos^{2.3} x} \, dx = \int \frac{\sin^2 x}{\cos^{2.3} x} \cdot \sin x \, dx \end{aligned}$$

Using the substitution $u = \cos x$, $du = -\sin x \, dx$, and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$:

$$\begin{aligned} &= - \int \frac{1 - u^2}{u^{2.3}} \, du = \int (-u^{-2.3} + u^{-0.3}) \, du \\ &= \frac{1}{1.3} u^{-1.3} + \frac{1}{0.7} u^{0.7} + C \\ &= \frac{1}{1.3} \sec^{1.3} x + \frac{1}{0.7} \cos^{0.7} x + C \end{aligned}$$

S-17: We replace $\tan x$ with $\frac{\sin x}{\cos x}$.

$$\int \tan^5 x \, dx = \int \left(\frac{\sin x}{\cos x} \right)^5 dx = \int \frac{\sin^4 x}{\cos^5 x} \cdot \sin x \, dx$$

Now we use the substitution $u = \cos x$, $du = -\sin x \, dx$, and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned} &= - \int \frac{(1 - u^2)^2}{u^5} du = \int (-u^{-5} + 2u^{-3} - u^{-1}) du \\ &= \frac{1}{4}u^{-4} - u^{-2} - \log |u| + C \\ &= \frac{1}{4}\sec^4 x - \sec^2 x - \log |\cos x| + C \\ &= \frac{1}{4}\sec^4 x - \sec^2 x + \log |\sec x| + C \end{aligned}$$

where in the last line, we used the logarithm rule $\log(b^a) = a \log b$, with $b^a = \cos x = (\sec x)^{-1}$.

S-18: Recall $\tan x = \frac{\sin x}{\cos x}$.

$$\int \tan^5 x \cos^2 x \, dx = \int \frac{\sin^5 x}{\cos^5 x} \cos^2 x \, dx = \int \frac{\sin^5 x}{\cos^3 x} \, dx$$

Substitute $u = \cos x$, so $du = -\sin x \, dx$ and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned} &= \int \frac{\sin^4 x}{\cos^3 x} \sin x \, dx = - \int \frac{(1 - u^2)^2}{u^3} du \\ &= - \int \frac{1 - 2u^2 + u^4}{u^3} du = \int \left(-\frac{1}{u^3} + \frac{2}{u} - u \right) du \\ &= \frac{1}{2u^2} + 2 \log |u| - \frac{1}{2}u^2 + C \\ &= \frac{1}{2\cos^2 x} + 2 \log |\cos x| - \frac{1}{2}\cos^2 x + C \end{aligned}$$

S-19: Let's turn our secants and tangents into sines and cosines.

$$\int \tan x \sqrt{\sec x} \, dx = \int \frac{\sin x}{\cos x \cdot \sqrt{\cos x}} \, dx = \int \frac{\sin x}{\cos^{1.5} x} \, dx$$

We use the substitution $u = \cos x$, $du = -\sin x \, dx$.

$$\begin{aligned} &= \int -u^{-1.5} du = \frac{2}{\sqrt{u}} + C \\ &= 2\sqrt{\sec x} + C \end{aligned}$$

S-20: We can use the definition of secant to make this integral look more familiar.

$$\int \frac{1}{\cos^2 \theta} d\theta = \int \sec^2 \theta d\theta = \tan \theta + C$$

S-21: We re-write $\cot x = \frac{\cos x}{\sin x}$, and use the substitution $u = \sin x$, $du = \cos x dx$.

$$\begin{aligned} \int \cot x dx &= \int \frac{\cos x}{\sin x} dx = \int \frac{1}{u} du \\ &= \log |u| + C = \log |\sin x| + C \end{aligned}$$

S-22:

Solution 1: We begin with the obvious substitution, $w = e^x$, $dw = e^x dx$.

$$\int e^x \sin(e^x) \cos(e^x) dx = \int \sin w \cos w dw$$

Now we see another substitution, $u = \sin w$, $du = \cos w dw$.

$$\begin{aligned} &= \int u du = \frac{1}{2}u^2 + C = \frac{1}{2} \sin^2 w + C \\ &= \frac{1}{2} \sin^2(e^x) + C \end{aligned}$$

Solution 2: Notice that $\frac{d}{dx} \{\sin(e^x)\} = e^x \cos(e^x)$. This suggests to us the substitution $u = \sin(e^x)$, $du = e^x \cos(e^x) dx$.

$$\int e^x \sin(e^x) \cos(e^x) dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2} \sin^2(e^x) + C$$

S-23: (a) Using the trig identity $\tan^2 x = \sec^2 x - 1$ and the substitution $y = \tan x$, $dy = \sec^2 x dx$,

$$\begin{aligned} \int \tan^n x dx &= \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx \\ &= \int y^{n-2} dy - \int \tan^{n-2} x dx = \frac{y^{n-1}}{n-1} - \int \tan^{n-2} x dx \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx \end{aligned}$$

(b) By the reduction formula of part (a),

$$\begin{aligned}\int_0^{\pi/4} \tan^n(x) \, dx &= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2}(x) \, dx \\ &= \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2}(x) \, dx\end{aligned}$$

for all integers $n \geq 2$, since $\tan 0 = 0$ and $\tan \frac{\pi}{4} = 1$. We apply this reduction formula, with $n = 6, 4, 2$.

$$\begin{aligned}\int_0^{\pi/4} \tan^6(x) \, dx &= \frac{1}{5} - \int_0^{\pi/4} \tan^4(x) \, dx = \frac{1}{5} - \frac{1}{3} + \int_0^{\pi/4} \tan^2(x) \, dx = \frac{1}{5} - \frac{1}{3} + 1 - \int_0^{\pi/4} dx \\ &= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}\end{aligned}$$

Using a calculator, we see this is approximately 0.0813.

Solutions to Exercises 1.8 — Jump to [TABLE OF CONTENTS](#)

S-1: In the text, there is a template for choosing an appropriate substitution, but for this problem we will explain the logic of the choices.

The trig identities that we can use are:

$$1 - \sin^2 \theta = \cos^2 \theta \qquad \tan^2 \theta + 1 = \sec^2 \theta \qquad \sec^2 \theta - 1 = \tan^2 \theta$$

They have the following forms:

$$\text{constant} - \text{function} \qquad \text{function} + \text{constant} \qquad \text{function} - \text{constant}$$

In order to cancel out the square root, we should choose a substitution that will match the argument under the square root with the trig identity of the corresponding form.

(a) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the square root in the denominator. Under the square root is the function $9x^2 - 16$, which has the form (function) $-$ (constant). This form matches the trig identity $\sec^2 \theta - 1 = \tan^2 \theta$. We can set x to be whatever we need it to be, but we don't have the same control over the constant, 16. So, to make the substitution work, we use a different form of the trig identity: multiplying both sides by 16, we get

$$16 \sec^2 \theta - 16 = 16 \tan^2 \theta$$

What we want is a substitution that gives us

$$\begin{aligned}9x^2 - 16 &= 16 \sec^2 \theta - 16 \\ \text{So, } 9x^2 &= 16 \sec^2 \theta \\ x &= \frac{4}{3} \sec \theta\end{aligned}$$

Using this substitution,

$$\begin{aligned}\sqrt{9x^2 - 16} &= \sqrt{16 \sec^2 \theta - 16} \\ &= \sqrt{16 \tan^2 \theta} \\ &= 4|\tan \theta|\end{aligned}$$

So, we eliminated the square root.

(b) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the square root in the denominator. Under the square root is the function $1 - 4x^2$, which has the form (constant) - (function). This form matches the trig identity $1 - \sin^2 \theta = \cos^2 \theta$. What we want is a substitution that gives us

$$\begin{aligned}1 - 4x^2 &= 1 - \sin^2 \theta \\ \text{So, } 4x^2 &= \sin^2 \theta \\ x &= \frac{1}{2} \sin \theta\end{aligned}$$

Using this substitution,

$$\begin{aligned}\sqrt{1 - 4x^2} &= \sqrt{1 - \sin^2 \theta} \\ &= \sqrt{\cos^2 \theta} \\ &= |\cos \theta|\end{aligned}$$

So, we eliminated the square root. We remark that the absolute value signs are not needed in $|\cos \theta|$, because, for $-\frac{1}{2} \leq x \leq \frac{1}{2}$, we have $\theta = \arcsin(2x)$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and $\cos(\theta) \geq 0$ for those θ 's.

(c) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the fractional power. (That is, we want to eliminate the square root.) The function under the power is $25 + x^2$, which has the form (constant) + (function). This form matches the trig identity $\tan^2 \theta + 1 = \sec^2 \theta$. We can set x to be whatever we need it to be, but we don't have the same control over the constant, 25. So, to make the substitution work, we use a different form of the trig identity: multiplying both sides by 25, we get

$$25 \tan^2 \theta + 25 = 25 \sec^2 \theta$$

What we want is a substitution that gives us

$$\begin{aligned}25 + x^2 &= 25 \tan^2 \theta + 25 \\ \text{So, } x^2 &= 25 \tan^2 \theta \\ x &= 5 \tan \theta\end{aligned}$$

Using this substitution,

$$\begin{aligned}(25 + x^2)^{-5/2} &= (25 + 25 \tan^2 \theta)^{-5/2} \\ &= (25 \sec^2 \theta)^{-5/2} \\ &= (5|\sec \theta|)^{-5}\end{aligned}$$

So, we eliminated the square root. We remark that the absolute value signs are not needed in $|\sec \theta|$, because, for $-\infty < x < \infty$, we have $\theta = \arctan(x/5)$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and $\sec(\theta) \geq 0$ for those θ 's.

S-2: Just as in Question 1, we want to choose a trigonometric substitution that will allow us to eliminate the square roots. Before we can make that choice, though, we need to complete the square. In subsequent problems, we won't show the algebra behind completing the square, but for this problem we'll work it out explicitly. After some practice, you'll be able to do this step in your head for many cases.

After the squares are completed, the choice of trig substitution follows the logic outlined in the solutions to Question 1, or (equivalently) the template in the text.

- (a) The quadratic function under the square root is $x^2 - 4x + 1$. To complete the square, we match the non-constant terms to those of a perfect square.

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$

$$x^2 - 4x + 1 = a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-4 = 2ab = 2b$, so $b = -2$.
- Finally, the constant terms give us $1 = b^2 + c = 4 + c$, so $c = -3$.

$$\int \frac{1}{\sqrt{x^2 - 4x + 1}} dx = \int \frac{1}{\sqrt{(x-2)^2 - 3}} dx = \int \frac{1}{\sqrt{(x-2)^2 - \sqrt{3}^2}} dx$$

So we use the substitution $(x-2) = \sqrt{3} \sec u$, which eliminates the square root:

$$\sqrt{(x-2)^2 - 3} = \sqrt{3 \sec^2 u - 3} = \sqrt{3 \tan^2 u} = \sqrt{3} |\tan u|$$

- (b) The quadratic function under the square root is $-x^2 + 2x + 4 = -[x^2 - 2x - 4]$. To complete the square, we match the non-constant terms to those of a perfect square. We factored out the negative to make things a little easier—don't forget to put it back in before choosing a substitution!

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$

$$x^2 - 2x - 4 = a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-2 = 2ab = 2b$, so $b = -1$.
- Finally, the constant terms give us $-4 = b^2 + c = 1 + c$, so $c = -5$.
- Then $-x^2 + 2x + 4 = -[x^2 - 2x - 4] = -[(x-1)^2 - 5] = 5 - (x-1)^2$.

$$\int \frac{(x-1)^6}{(-x^2+2x+4)^{3/2}} dx = \int \frac{(x-1)^6}{(5-(x-1)^2)^{3/2}} dx = \int \frac{(x-1)^6}{(\sqrt{5}^2 - (x-1)^2)^{3/2}} dx$$

So we use the substitution $(x-1) = \sqrt{5} \sin u$, which eliminates the square root (fractional power):

$$(5 - (x-1)^2)^{3/2} = (5 - 5 \sin^2 u)^{3/2} = (5 \cos^2 u)^{3/2} = 5\sqrt{5} |\cos^3 u|$$

- (c) The quadratic function under the square root is $4x^2 + 6x + 10$. To complete the square, we match the non-constant terms to those of a perfect square.

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$

$$4x^2 + 6x + 10 = a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c$$

- Looking at the leading term tells us $a = 2$.
- Then the second term tells us $6 = 2ab = 4b$, so $b = \frac{3}{2}$.
- Finally, the constant terms give us $10 = b^2 + c = \frac{9}{4} + c$, so $c = \frac{31}{4}$.

$$\int \frac{1}{\sqrt{4x^2 + 6x + 10}} dx = \int \frac{1}{\sqrt{(2x + \frac{3}{2})^2 + \frac{31}{4}}} dx = \int \frac{1}{\sqrt{(2x + \frac{3}{2})^2 + (\frac{\sqrt{31}}{2})^2}} dx$$

So we use the substitution $(2x + \frac{3}{2}) = \frac{\sqrt{31}}{2} \tan u$, which eliminates the square root:

$$\sqrt{\left(2x + \frac{3}{2}\right)^2 + \frac{31}{4}} = \sqrt{\frac{31}{4} \tan^2 u + \frac{31}{4}} = \sqrt{\frac{31}{4} \sec^2 u} = \frac{\sqrt{31}}{2} |\sec u|$$

- (d) The quadratic function under the square root is $x^2 - x$. To complete the square, we match the non-constant terms to those of a perfect square.

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$

$$x^2 - x = a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-1 = 2ab = 2b$, so $b = -\frac{1}{2}$.
- Finally, the constant terms give us $0 = b^2 + c = \frac{1}{4} + c$, so $c = -\frac{1}{4}$.

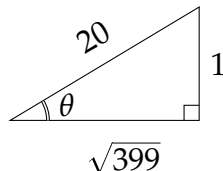
$$\int \sqrt{x^2 - x} dx = \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}} dx = \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx$$

So we use the substitution $(x - 1/2) = \frac{1}{2} \sec u$, which eliminates the square root:

$$\sqrt{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}} = \sqrt{\frac{1}{4} \sec^2 u - \frac{1}{4}} = \sqrt{\frac{1}{4} \tan^2 u} = \frac{1}{2} |\tan u|$$

S-3:

- (a) If $\sin \theta = \frac{1}{20}$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has opposite side length 1, and hypotenuse length 20. By the Pythagorean Theorem, the adjacent side has length $\sqrt{20^2 - 1^2} = \sqrt{399}$. So,
- $$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{399}}{20}.$$



We can do a quick “reasonableness” check here: $\frac{1}{20}$ is pretty close to 0, so we might expect θ to be pretty close to 0, and so $\cos \theta$ should be pretty close to 1. Indeed it is: $\frac{\sqrt{399}}{20} \approx \frac{\sqrt{400}}{20} = \frac{20}{20} = 1$.

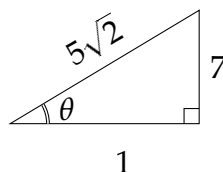
Alternately, we can solve this problem using identities.

$$\begin{aligned}\sin^2 \theta + \cos^2 \theta &= 1 \\ \left(\frac{1}{20}\right)^2 + \cos^2 \theta &= 1 \\ \cos \theta &= \pm \sqrt{1 - \frac{1}{400}} = \pm \frac{\sqrt{399}}{20}\end{aligned}$$

Since $0 \leq \theta \leq \frac{\pi}{2}$, $\cos \theta \geq 0$, so

$$\cos \theta = \frac{\sqrt{399}}{20}$$

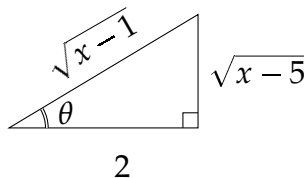
- (b) If $\tan \theta = 7$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has opposite side length 7 and adjacent side length 1. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$. So, $\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{5\sqrt{2}}{7}$.



Again, we can do a quick reasonableness check. Since 7 is much larger than 1, the triangle we’re thinking of doesn’t look much like the triangle in our standardized picture above: it’s really quite tall, with a small base. So, the opposite side and

hypotenuse are pretty close in length. Indeed, $\frac{5\sqrt{2}}{7} \approx 7.071$, so this dimension seems reasonable.

- (c) If $\sec \theta = \frac{\sqrt{x-1}}{2}$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has hypotenuse length $\sqrt{x-1}$ and adjacent side length 2. By the Pythagorean Theorem, the opposite side has length $\sqrt{\sqrt{x-1}^2 - 2^2} = \sqrt{x-1-4} = \sqrt{x-5}$. So, $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x-5}}{2}$.



We can also solve this using identities. Note that since $\sec \theta$ exists, $\theta \neq \frac{\pi}{2}$.

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\tan^2 \theta + 1 = \left(\frac{\sqrt{x-1}}{2} \right)^2 = \frac{x-1}{4}$$

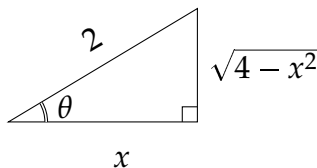
$$\tan \theta = \pm \sqrt{\frac{x-1}{4} - 1} = \pm \frac{\sqrt{x-5}}{2}$$

Since $0 \leq \theta < \frac{\pi}{2}$, $\tan \theta \geq 0$, so

$$\tan \theta = \frac{\sqrt{x-5}}{2}$$

S-4:

- (a) Let $\theta = \arccos\left(\frac{x}{2}\right)$. That is, $\cos(\theta) = \frac{x}{2}$, and $0 \leq \theta \leq \pi$. Then we can draw the corresponding right triangle with angle θ with adjacent side of signed length x (we note that if $\theta > \frac{\pi}{2}$, then x is negative) and hypotenuse of length 2. By the Pythagorean Theorem, the opposite side of the triangle has length $\sqrt{4-x^2}$.



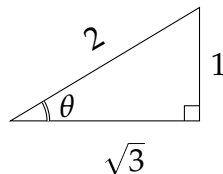
So,

$$\sin\left(\arccos\left(\frac{x}{2}\right)\right) = \sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$$

- (b) Let $\theta = \arctan\left(\frac{1}{\sqrt{3}}\right)$. That is, $\tan(\theta) = \frac{1}{\sqrt{3}}$, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Solution 1: Then $\theta = \frac{\pi}{6}$, so $\sin \theta = \frac{1}{2}$.

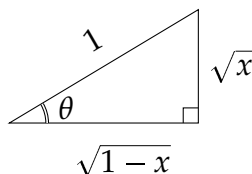
Solution 2: Then we can draw the corresponding right triangle with angle θ with opposite side of length 1 and adjacent side of length $\sqrt{3}$. By the Pythagorean Theorem, the hypotenuse of the triangle has length $\sqrt{\sqrt{3}^2 + 1^2} = 2$.



So,

$$\sin \left(\arctan \left(\frac{1}{\sqrt{3}} \right) \right) = \sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$$

(c) Let $\theta = \arcsin(\sqrt{x})$. That is, $\sin(\theta) = \sqrt{x}$, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then we can draw the corresponding right triangle with angle θ with opposite side of length $\sqrt{x}$ and hypotenuse of length 1. By the Pythagorean Theorem, the adjacent side of the triangle has length $\sqrt{1-x}$.

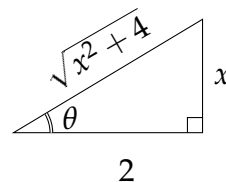


So,

$$\sec(\arcsin(\sqrt{x})) = \sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\sqrt{1-x}}$$

S-5: Let $x = 2 \tan \theta$, so that $x^2 + 4 = 4 \tan^2 \theta + 4 = 4 \sec^2 \theta$ and $dx = 2 \sec^2 \theta d\theta$. Then

$$\begin{aligned} \int \frac{1}{(x^2 + 4)^{3/2}} dx &= \int \frac{1}{(4 \sec^2 \theta)^{3/2}} \cdot 2 \sec^2 \theta d\theta \\ &= \int \frac{2 \sec^2 \theta}{8 \sec^3 \theta} d\theta \\ &= \frac{1}{4} \int \cos \theta d\theta \\ &= \frac{1}{4} \sin \theta + C = \frac{1}{4} \frac{x}{\sqrt{x^2 + 4}} + C \end{aligned}$$



To find $\sin \theta$ in terms of x , we construct the right triangle above. Since $\tan \theta = \frac{x}{2} = \frac{\text{opp}}{\text{adj}}$, we label the opposite side x and the adjacent side 2. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{x^2 + 4}$. Then $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2 + 4}}$.

To see why we could write $(\sec^2 \theta)^{3/2} = \sec^3 \theta$, as opposed to $(\sec^2 \theta)^{3/2} = |\sec^3 \theta|$, in the second line above, see Example 1.8.5 in the text.

As a check, we observe that the derivative of the answer

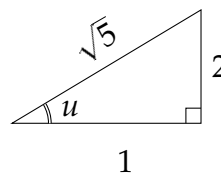
$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{4} \frac{x}{\sqrt{x^2 + 4}} + C \right) &= \frac{1}{4} \frac{1}{\sqrt{x^2 + 4}} - \frac{1}{2 \times 4} \frac{x(2x)}{(x^2 + 4)^{3/2}} = \frac{\frac{x^2}{4} + 1 - \frac{x^2}{4}}{(x^2 + 4)^{3/2}} \\ &= \frac{1}{(x^2 + 4)^{3/2}} \end{aligned}$$

is exactly the integrand.

S-6:

Solution 1: As in Question 5, substitute $x = 2 \tan u$, $dx = 2 \sec^2 u \, du$. Note that when $x = 4$ we have $4 = 2 \tan u$, so that $\tan u = 2$.

$$\begin{aligned} \int_0^4 \frac{1}{(4 + x^2)^{3/2}} dx &= \int_0^{\arctan 2} \frac{1}{(4 + 4 \tan^2 u)^{3/2}} 2 \sec^2 u \, du \\ &= \int_0^{\arctan 2} \frac{2 \sec^2 u}{(2 \sec u)^3} du \\ &= \frac{1}{4} \int_0^{\arctan 2} \frac{\sec^2 u}{\sec^3 u} du \\ &= \frac{1}{4} \int_0^{\arctan 2} \cos u \, du \\ &= \left[\frac{1}{4} \sin u \right]_0^{\arctan 2} \\ &= \frac{1}{4} (\sin(\arctan 2) - 0) = \frac{1}{2\sqrt{5}} \end{aligned}$$



To find $\sin(\arctan 2)$, we use the right triangle above, with angle $u = \arctan 2$. Since $\tan u = 2 = \frac{\text{opp}}{\text{adj}}$, we label the opposite side as 2, and the adjacent side as 1. The

Pythagorean Theorem tells us the hypotenuse has length $\sqrt{5}$, so $\sin u = \frac{\text{opp}}{\text{hyp}} = \frac{2}{\sqrt{5}}$.

Solution 2: Using our result from Question 5,

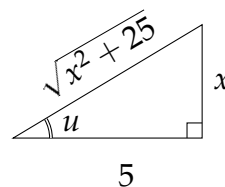
$$\begin{aligned} \int_0^4 \frac{1}{(4 + x^2)^{3/2}} dx &= \frac{1}{4} \left[\frac{x}{\sqrt{x^2 + 4}} \right]_0^4 \\ &= \frac{1}{4} \cdot \frac{4}{\sqrt{4^2 + 4}} = \frac{1}{2\sqrt{5}} \end{aligned}$$

S-7: Make the change of variables $x = 5 \sin \theta$, $dx = 5 \cos \theta d\theta$. Since $x = 0$ corresponds to $\theta = 0$ and $x = \frac{5}{2}$ corresponds to $\sin \theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$,

$$\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}} = \int_0^{\pi/6} \frac{5 \cos \theta d\theta}{\sqrt{25-25 \sin^2 \theta}} = \int_0^{\pi/6} d\theta = \frac{\pi}{6}$$

S-8: Substitute $x = 5 \tan u$, so that $dx = 5 \sec^2 u du$.

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+25}} dx &= \int \frac{1}{\sqrt{25 \tan^2 u + 25}} 5 \sec^2 u du \\ &= \int \frac{5 \sec^2 u}{5 \sec u} du = \int \sec u du \\ &= \log |\sec u + \tan u| + C \\ &= \log \left| \sqrt{1 + \frac{x^2}{25}} + \frac{x}{5} \right| + C \end{aligned}$$



To find $\sec u$ and $\tan u$, we have two options. One is to set up a right triangle with angle u and $\tan u = \frac{x}{5}$. Then we can label the opposite side x and the adjacent side 5 , and use Pythagoras to find that the hypotenuse is $\sqrt{x^2 + 25}$.

Another option is to look back at our work a little more closely—in fact, we’ve already found what we’re looking for. Since we used the substitution $x = 5 \tan u$, this gives us $\tan u = \frac{x}{5}$. In the denominator of the integrand, we simplified $\sqrt{x^2 + 25} = 5 \sec u$, so $\sec u = \frac{1}{5} \sqrt{x^2 + 25} = \sqrt{1 + \frac{x^2}{25}}$.

To see why we could write $\sqrt{x^2 + 25} = 5 \sec u$, as opposed to $\sqrt{x^2 + 25} = 5 |\sec u|$, see Example 1.8.5 in the text.

S-9: The quadratic formula underneath the square root makes us think of a trig substitution, but in the interest of developing good habits, let’s check for an easier way first. If we let $u = 2x^2 + 4x$, then $du = (4x + 4) dx$, so $\frac{1}{4} du = (x + 1) dx$. This substitution looks easier than a trig substitution (which would start with completing the square).

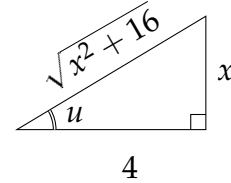
$$\int \frac{x+1}{\sqrt{2x^2+4x}} dx = \frac{1}{4} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} \sqrt{u} + C = \frac{1}{2} \sqrt{2x^2+4x} + C$$

S-10: Substitute $x = 4 \tan u$, $dx = 4 \sec^2 u \, du$.

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{x^2 + 16}} dx &= \int \frac{1}{16 \tan^2 u \sqrt{16 \tan^2 u + 16}} 4 \sec^2 u \, du \\ &= \int \frac{\sec^2 u}{16 \tan^2 u \sec u} du = \frac{1}{16} \int \frac{\sec u}{\tan^2 u} du \\ &= \frac{1}{16} \int \frac{\cos u}{\sin^2 u} du \end{aligned}$$

To finish off the integral, we'll substitute $v = \sin u$, $dv = \cos u \, du$.

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{x^2 + 16}} dx &= \frac{1}{16} \int \frac{\cos u}{\sin^2 u} du = \frac{1}{16} \int \frac{dv}{v^2} = -\frac{1}{16v} + C \\ &= -\frac{1}{16 \sin u} + C = -\frac{1}{16} \frac{\sqrt{x^2 + 16}}{x} + C \end{aligned}$$



To find $\sin u$, we draw a right triangle with angle u and $\tan u = \frac{x}{4}$. We label the opposite side x and the adjacent side 4, and then from Pythagoras we find that the hypotenuse has length $\sqrt{x^2 + 16}$. So, $\sin u = \frac{\sqrt{x^2 + 16}}{x}$.

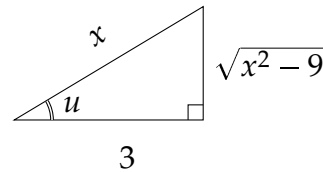
As a check, we observe that the derivative of the answer

$$\begin{aligned} \frac{d}{dx} \left(-\frac{1}{16} \frac{\sqrt{x^2 + 16}}{x} + C \right) &= \frac{1}{16} \frac{\sqrt{x^2 + 16}}{x^2} - \frac{1}{16} \frac{x}{x \sqrt{x^2 + 16}} = \frac{1}{16} \frac{(x^2 + 16) - x^2}{x^2 \sqrt{x^2 + 16}} \\ &= \frac{1}{x^2 \sqrt{x^2 + 16}} \end{aligned}$$

is exactly the integrand.

S-11: Substitute $x = 3 \sec u$ with $0 \leq u < \frac{\pi}{2}$. Then $dx = 3 \sec u \tan u \, du$ and $\sqrt{x^2 - 9} = \sqrt{9 \sec^2 u - 9} = \sqrt{9 \tan^2 u} = 3 \tan u$, so that

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 - 9}} &= \int \frac{3 \sec u \tan u \, du}{9 \sec^2 u \sqrt{9 \tan^2 u}} \\ &= \frac{1}{9} \int \frac{du}{\sec u} \\ &= \frac{1}{9} \int \cos u \, du = \frac{1}{9} \sin u + C. \end{aligned}$$



To evaluate $\sin u$, we make a right triangle with angle u . Since $\sec u = \frac{x}{3} = \frac{\text{hyp}}{\text{adj}}$, we label the hypotenuse x and the adjacent side 3. Using the Pythagorean Theorem, the opposite side has length $\sqrt{x^2 - 9}$. So, $\sin u = \frac{\sqrt{x^2 - 9}}{x}$ and

$$\int \frac{dx}{x^2 \sqrt{x^2 - 9}} = \frac{\sqrt{x^2 - 9}}{9x} + C.$$

As a check, we observe that the derivative of the answer

$$\begin{aligned}\frac{d}{dx} \left(\frac{\sqrt{x^2-9}}{9x} + C \right) &= -\frac{\sqrt{x^2-9}}{9x^2} + \frac{x}{9x\sqrt{x^2-9}} = \frac{1}{9} \frac{-(x^2-9) + x^2}{x^2\sqrt{x^2-9}} \\ &= \frac{1}{x^2\sqrt{x^2-9}}\end{aligned}$$

is exactly the integrand. (We remark that this is the case even for $x \leq -3$.)

S-12: (a) We'll use the trig identity $\cos 2\theta = 2\cos^2 \theta - 1$. It implies that

$$\begin{aligned}\cos^2 \theta = \frac{\cos 2\theta + 1}{2} \implies \cos^4 \theta &= \frac{1}{4} [\cos^2 2\theta + 2\cos 2\theta + 1] = \frac{1}{4} \left[\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1 \right] \\ &= \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\end{aligned}$$

So,

$$\begin{aligned}\int_0^{\pi/4} \cos^4 \theta \, d\theta &= \int_0^{\pi/4} \left(\frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8} \right) d\theta \\ &= \left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3}{8}\theta \right]_0^{\pi/4} \\ &= \frac{1}{4} + \frac{3}{8} \cdot \frac{\pi}{4} \\ &= \frac{8+3\pi}{32}\end{aligned}$$

as required.

(b) We'll use the trig substitution $x = \tan \theta$, $dx = \sec^2 \theta \, d\theta$. Note that when $\theta = \pm \frac{\pi}{4}$, we have $x = \pm 1$. Also note that dividing the trig identity $\sin^2 \theta + \cos^2 \theta = 1$ by $\cos^2 \theta$ gives the trig identity $\tan^2 \theta + 1 = \sec^2 \theta$. So

$$\begin{aligned}\int_{-1}^1 \frac{dx}{(x^2+1)^3} &= 2 \int_0^1 \frac{dx}{(x^2+1)^3} && \text{(even integrand)} \\ &= 2 \int_0^{\pi/4} \frac{\sec^2 \theta \, d\theta}{(\tan^2 \theta + 1)^3} \\ &= 2 \int_0^{\pi/4} \frac{\sec^2 \theta \, d\theta}{(\sec^2 \theta)^3} \\ &= 2 \int_0^{\pi/4} \cos^4 \theta \, d\theta \\ &= \frac{8+3\pi}{16}\end{aligned}$$

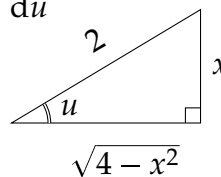
by part (a).

S-13: The integrand is an odd function, and the limits of integration are symmetric, so

$$\int_{-\pi/12}^{\pi/12} \frac{15x^3}{(x^2 + 1)\sqrt{9 - x^2}^5} dx = 0.$$

S-14: Substitute $x = 2 \sin u$, so that $dx = 2 \cos u \, du$.

$$\begin{aligned} \int \sqrt{4 - x^2} \, dx &= \int \sqrt{4 - 4 \sin^2 u} \, 2 \cos u \, du \\ &= \int \sqrt{4 \cos^2 u} \, 2 \cos u \, du \\ &= \int 4 \cos^2 u \, du = 2 \int (1 + \cos(2u)) \, du \\ &= 2u + \sin(2u) + C \\ &= 2u + 2 \sin u \cos u + C \\ &= 2 \arcsin \frac{x}{2} + \frac{x}{2} \sqrt{4 - x^2} + C \end{aligned}$$



To see why we could write $\sqrt{4 \cos^2 u} = 2 \cos u$, as opposed to $\sqrt{4 \cos^2 u} = 2|\cos u|$, in the third line above, see Example 1.8.2 in the text.

We used the substitution $x = 2 \sin u$, so we know $\sin u = \frac{x}{2}$ and $u = \arcsin(\frac{x}{2})$. We have three options for finding $\cos u$.

First, we can draw a right triangle with angle u . Since $\sin u = \frac{x}{2}$, we label the opposite side x and the hypotenuse 2, then by the Pythagorean Theorem the adjacent side has length $\sqrt{4 - x^2}$. So, $\cos u = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{4 - x^2}}{2}$.

Second, we can look back carefully at our work. We simplified $\sqrt{4 - x^2} = 2 \cos u$, so $\cos u = \frac{\sqrt{4 - x^2}}{2}$.

Third, we could use the identity $\sin^2 u + \cos^2 u = 1$. Then $\cos u = \pm \sqrt{1 - \sin^2 u} = \pm \sqrt{1 - \frac{x^2}{4}}$. Since $u = \arcsin(x/2)$, u is in the range of arcsine, which means $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$. Therefore, $\cos u \geq 0$, so $\cos u = \sqrt{1 - \frac{x^2}{4}} = \frac{\sqrt{4 - x^2}}{2}$.

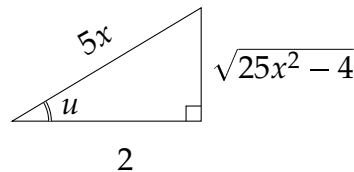
So,

$$\int \sqrt{4 - x^2} \, dx = 2u + 2 \sin u \cos u + C = 2 \arcsin \frac{x}{2} + x \cdot \frac{\sqrt{4 - x^2}}{2} + C$$

S-15: Substitute $x = \frac{2}{5} \sec u$ with $0 < u < \frac{\pi}{2}$, so that $dx = \frac{2}{5} \sec u \tan u \, du$ and

$\sqrt{25x^2 - 4} = \sqrt{4(\sec^2 u - 1)} = \sqrt{4 \tan^2 u} = 2 \tan u$. Then

$$\begin{aligned} \int \frac{\sqrt{25x^2 - 4}}{x} dx &= \int \frac{2 \tan u}{\frac{2}{5} \sec u} \cdot \frac{2}{5} \sec u \tan u du \\ &= 2 \int \tan^2 u du = 2 \int (\sec^2 u - 1) du \\ &= 2 \tan u - 2u + C \\ &= \sqrt{25x^2 - 4} - 2 \operatorname{arcsec} \frac{5x}{2} + C \end{aligned}$$



To find $\tan u$, we draw a right triangle with angle u . Since $\sec u = \frac{5x}{2}$, we label the hypotenuse $5x$ and the adjacent side 2 . Then the Pythagorean Theorem gives us the opposite side as length $\sqrt{25x^2 - 4}$. Then $\tan u = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{25x^2 - 4}}{2}$.

Alternately, we can notice that in our work, we already showed $2 \tan u = \sqrt{25x^2 - 4}$, so $\tan u = \frac{1}{2} \sqrt{25x^2 - 4}$.

As a check, we observe that the derivative of the answer

$$\begin{aligned} \frac{d}{dx} \left(\sqrt{25x^2 - 4} - 2 \operatorname{arcsec} \frac{5x}{2} + C \right) &= \frac{25x}{\sqrt{25x^2 - 4}} - 2 \frac{\frac{5}{2}}{\left| \frac{5x}{2} \right| \sqrt{\frac{25x^2}{4} - 1}} \\ &= \frac{25x}{\sqrt{25x^2 - 4}} - \frac{4}{x \sqrt{25x^2 - 4}} \quad \text{since } x > 0 \\ &= \frac{25x^2 - 4}{x \sqrt{25x^2 - 4}} \\ &= \frac{\sqrt{25x^2 - 4}}{x} \end{aligned}$$

is exactly the integrand (provided $x > \frac{2}{5}$).

S-16: The integrand has a quadratic polynomial under a square root, which makes us think of trig substitutions. However, it's good practice to look for simpler methods before we jump into more complicated ones, and in this case we find something nicer than a trig substitution: the substitution $u = x^2 - 1$, $du = 2x dx$. Then $x dx = \frac{1}{2} du$, and $x^2 = u + 1$. When $x = \sqrt{10}$, $u = 9$, and when $x = \sqrt{17}$, $u = 16$.

$$\begin{aligned}
\int_{\sqrt{10}}^{\sqrt{17}} \frac{x^3}{\sqrt{x^2-1}} dx &= \int_{\sqrt{10}}^{\sqrt{17}} \frac{x^2}{\sqrt{x^2-1}} \cdot x dx \\
&= \frac{1}{2} \int_9^{16} \frac{u+1}{\sqrt{u}} du \\
&= \frac{1}{2} \int_9^{16} \left(u^{1/2} + u^{-1/2} \right) du \\
&= \frac{1}{2} \left[\frac{2}{3} u^{3/2} + 2u^{1/2} \right]_9^{16} \\
&= \frac{1}{2} \left[\frac{2}{3} \cdot 4^3 + 2 \cdot 4 - \frac{2}{3} \cdot 3^3 - 2 \cdot 3 \right] \\
&= \frac{40}{3}
\end{aligned}$$

S-17: This integrand looks very different from those above. But it is only slightly disguised. If we complete the square

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(x+1)^2}}$$

and make the substitution $y = x + 1$, $dy = dx$

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(x+1)^2}} = \int \frac{dy}{\sqrt{4-y^2}}$$

we get a typical trig substitution integral. So, we substitute $y = 2 \sin \theta$, $dy = 2 \cos \theta d\theta$ to get

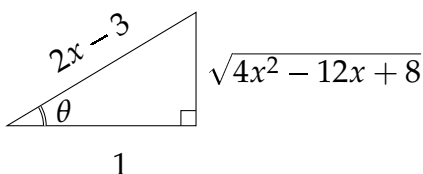
$$\begin{aligned}
\int \frac{dx}{\sqrt{3-2x-x^2}} &= \int \frac{dy}{\sqrt{4-y^2}} = \int \frac{2 \cos \theta d\theta}{\sqrt{4-4 \sin^2 \theta}} = \int \frac{2 \cos \theta d\theta}{\sqrt{4 \cos^2 \theta}} \\
&= \int d\theta = \theta + C = \arcsin \frac{y}{2} + C \\
&= \arcsin \frac{x+1}{2} + C
\end{aligned}$$

An experienced integrator would probably substitute $x + 1 = 2 \sin \theta$ directly, without going through y .

S-18: Completing the square, we see $4x^2 - 12x + 8 = (2x - 3)^2 - 1$.

$$\int \frac{1}{(2x-3)^3 \sqrt{4x^2-12x+8}} dx = \int \frac{1}{(2x-3)^3 \sqrt{(2x-3)^2-1}} dx$$

As $x > 2$, we have $2x - 3 > 1$. We use the substitution $2x - 3 = \sec \theta$ with $0 \leq \theta < \frac{\pi}{2}$. So $2 dx = \sec \theta \tan \theta d\theta$ and $\sqrt{(2x - 3)^2 - 1} = \sqrt{\sec^2 \theta - 1} = \sqrt{\tan^2 \theta} = \tan \theta$.



$$\begin{aligned}
 &= \frac{1}{2} \int \frac{1}{\sec^3 \theta \sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta \\
 &= \frac{1}{2} \int \frac{1}{\sec^3 \theta \tan \theta} \sec \theta \tan \theta d\theta \\
 &= \frac{1}{2} \int \frac{1}{\sec^2 \theta} d\theta \\
 &= \frac{1}{2} \int \cos^2 \theta d\theta \\
 &= \frac{1}{4} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{1}{4} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= \frac{1}{4} (\theta + \sin \theta \cos \theta) + C \\
 &= \frac{1}{4} \left(\arccos \left(\frac{1}{2x - 3} \right) + \frac{\sqrt{4x^2 - 12x + 8}}{(2x - 3)^2} \right) + C
 \end{aligned}$$

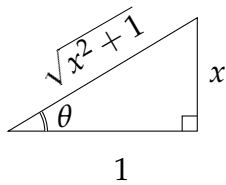
Since $2x - 3 = \sec \theta$, we know $\cos \theta = \frac{1}{2x - 3}$ and $\theta = \arccos \left(\frac{1}{2x - 3} \right)$. (Equivalently, $\theta = \operatorname{arcsec}(2x - 3)$.) To find $\sin \theta$, we draw a right triangle with adjacent side of length 1, and hypotenuse of length $2x - 3$. By the Pythagorean Theorem, the opposite side has length $\sqrt{4x^2 - 12x + 8}$.

S-19: We use the substitution $x = \tan u$, $dx = \sec^2 u du$. Note $\tan 0 = 0$ and $\tan \frac{\pi}{4} = 1$.

$$\begin{aligned}
 \int_0^1 \frac{x^2}{\sqrt{x^2 + 1}^3} dx &= \int_0^{\pi/4} \frac{\tan^2 u}{\sqrt{\tan^2 u + 1}^3} \sec^2 u du \\
 &= \int_0^{\pi/4} \frac{\tan^2 u}{\sqrt{\sec^2 u}^3} \sec^2 u du \\
 &= \int_0^{\pi/4} \frac{\tan^2 u}{\sec u} du \\
 &= \int_0^{\pi/4} \frac{\sec^2 u - 1}{\sec u} du \\
 &= \int_0^{\pi/4} (\sec u - \cos u) du \\
 &= \left[\log |\sec u + \tan u| - \sin u \right]_0^{\pi/4} \\
 &= \left(\log \left| \sqrt{2} + 1 \right| - \frac{1}{\sqrt{2}} \right) - (\log |1 + 0| - 0) \\
 &= \log(1 + \sqrt{2}) - \frac{1}{\sqrt{2}}
 \end{aligned}$$

S-20: There's no square root, but we can still make use of the substitution $x = \tan \theta$, $dx = \sec^2 \theta \, d\theta$.

$$\begin{aligned}
 \int \frac{1}{(x^2 + 1)^2} \, dx &= \int \frac{1}{(\tan^2 \theta + 1)^2} \sec^2 \theta \, d\theta \\
 &= \int \frac{1}{\sec^4 \theta} \sec^2 \theta \, d\theta = \int \cos^2 \theta \, d\theta \\
 &= \frac{1}{2} \int (1 + \cos(2\theta)) \, d\theta \\
 &= \frac{1}{2} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\
 &= \frac{1}{2} \left(\arctan x + \frac{x}{x^2 + 1} \right) + C
 \end{aligned}$$



Since $x = \tan \theta$, we can draw a right triangle with angle θ , opposite side x , and adjacent side 1. Then by the Pythagorean Theorem, its hypotenuse has length $\sqrt{x^2 + 1}$, which allows us to find $\sin \theta$ and $\cos \theta$.

S-21: First, we complete the square. The constants aren't integers, but we can still use the same method as in Question 2. The quadratic function under the square root is $3x^2 + 5x$. We match the non-constant terms to those of a perfect square.

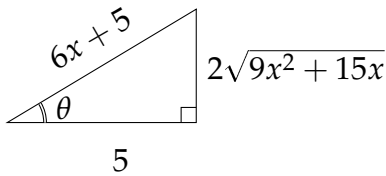
$$\begin{aligned}
 (ax + b)^2 &= a^2x^2 + 2abx + b^2 \\
 3x^2 + 5x &= a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c
 \end{aligned}$$

- Looking at the leading term tells us $a = \sqrt{3}$.
- Then the second term tells us $5 = 2ab = 2\sqrt{3}b$, so $b = \frac{5}{2\sqrt{3}}$.
- Finally, the constant terms give us $0 = b^2 + c = \frac{25}{12} + c$, so $c = -\frac{25}{12}$.

$$\text{So, } 3x^2 + 5x = \left(\sqrt{3}x + \frac{5}{2\sqrt{3}} \right)^2 - \frac{25}{12}.$$

$$\int \frac{1}{\sqrt{3x^2 + 5x}} \, dx = \int \frac{1}{\sqrt{\left(\sqrt{3}x + \frac{5}{2\sqrt{3}} \right)^2 - \frac{25}{12}}} \, dx$$

We use the substitution $\sqrt{3}x + \frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \sec \theta$, which leads to $\sqrt{3}dx = \frac{5}{2\sqrt{3}} \sec \theta \tan \theta d\theta$, i.e. $dx = \frac{5}{6} \sec \theta \tan \theta d\theta$.



$$\begin{aligned}
 &= \int \frac{1}{\sqrt{\left(\frac{5}{2\sqrt{3}} \sec \theta\right)^2 - \frac{25}{12}}} \cdot \frac{5}{6} \sec \theta \tan \theta d\theta \\
 &= \int \frac{1}{\sqrt{\frac{25}{12} \sec^2 \theta - \frac{25}{12}}} \cdot \frac{5}{6} \sec \theta \tan \theta d\theta \\
 &= \int \frac{1}{\sqrt{\frac{25}{12} \tan^2 \theta}} \cdot \frac{5}{6} \sec \theta \tan \theta d\theta \\
 &= \int \frac{1}{\frac{5}{2\sqrt{3}} \tan \theta} \cdot \frac{5}{6} \sec \theta \tan \theta d\theta \\
 &= \frac{1}{\sqrt{3}} \int \sec \theta d\theta \\
 &= \frac{1}{\sqrt{3}} \log |\sec \theta + \tan \theta| + C \\
 &= \frac{1}{\sqrt{3}} \log \left| \left(\frac{6}{5}x + 1 \right) + \frac{2}{5} \sqrt{9x^2 + 15x} \right| + C
 \end{aligned}$$

Since we used the substitution $\sqrt{3}x + \frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \sec \theta$, we have $\sec \theta = \frac{6}{5}x + 1 = \frac{6x+5}{5}$. To find $\tan \theta$ in terms of x , we have two options. We can make a right triangle with angle θ , hypotenuse $6x + 5$, and adjacent side 5, then use the Pythagorean Theorem to find the opposite side. Or, we can look through our work and see that $\sqrt{3x^2 + 5} = \frac{5}{2\sqrt{3}} \tan \theta$, so $\tan \theta = \frac{2\sqrt{3}}{5} \sqrt{3x^2 + 5} = \frac{2}{5} \sqrt{9x^2 + 15x}$.

As a check, we observe that the derivative of the answer

$$\begin{aligned}
 \frac{d}{dx} \left(\frac{1}{\sqrt{3}} \log \left| \left(\frac{6}{5}x + 1 \right) + \frac{2}{5} \sqrt{9x^2 + 15x} \right| + C \right) &= \frac{1}{\sqrt{3}} \frac{\frac{6}{5} + \frac{1}{5} \frac{18x+15}{\sqrt{9x^2+15x}}}{\left(\frac{6}{5}x + 1 \right) + \frac{2}{5} \sqrt{9x^2 + 15x}} \\
 &= \frac{1}{\sqrt{3}} \frac{6 + 3 \frac{6x+5}{\sqrt{9x^2+15x}}}{(6x+5) + 2\sqrt{9x^2+15x}} = \sqrt{3} \frac{2 + \frac{6x+5}{\sqrt{9x^2+15x}}}{(6x+5) + 2\sqrt{9x^2+15x}} = \frac{\sqrt{3}}{\sqrt{9x^2+15x}} \\
 &= \frac{1}{\sqrt{3x^2+5x}}
 \end{aligned}$$

is exactly the integrand.

Remark: in applications, often the numbers involved are messier than they are in textbooks. The ideas of this problem are similar to other problems in this section, but it's good practice to apply them in a slightly messy context.

S-22: The half of the ellipse to the right of the y -axis is given by the equation

$$x = f(y) = 4\sqrt{1 - \left(\frac{y}{2}\right)^2}$$

The area we want is twice the area between the right-hand side of the curve and the y -axis, from $y = -1$ to $y = 1$. In other words,

$$\text{Area} = 2 \int_{-1}^1 4\sqrt{1 - \left(\frac{y}{2}\right)^2} dy = 8 \int_{-1}^1 \sqrt{1 - \left(\frac{y}{2}\right)^2} dy$$

Since the integrand $\sqrt{1 - \left(\frac{y}{2}\right)^2}$ is an even function of y ,

$$\int_{-1}^1 \sqrt{1 - \left(\frac{y}{2}\right)^2} dy = 2 \int_0^1 \sqrt{1 - \left(\frac{y}{2}\right)^2} dy \implies \text{Area} = 16 \int_0^1 \sqrt{1 - \left(\frac{y}{2}\right)^2} dy$$

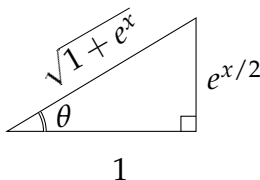
We use the substitution $\frac{y}{2} = \sin \theta$, $\frac{1}{2}dy = \cos \theta d\theta$. When $y = 0$, $\sin \theta = 0$ so that $\theta = 0$, and when $y = 1$, $\sin \theta = \frac{1}{2}$ so that $\theta = \frac{\pi}{6}$. Hence

$$\begin{aligned} \text{Area} &= 16 \int_0^{\pi/6} \sqrt{1 - (\sin \theta)^2} \cdot 2 \cos \theta d\theta \\ &= 32 \int_0^{\pi/6} \sqrt{\cos^2 \theta} \cos \theta d\theta \\ &= 32 \int_0^{\pi/6} \cos^2 \theta d\theta \\ &= 16 \int_0^{\pi/6} (1 + \cos(2\theta)) d\theta \\ &= 16 \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{\pi/6} \\ &= 16 \left(\frac{\pi}{6} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right) \\ &= \frac{8\pi}{3} + 4\sqrt{3} \end{aligned}$$

Remark: we also investigated areas of ellipses in Question 16, Section 1.2.

S-23: If we think of e^x as $(e^{x/2})^2$, the function under the square root suggests the

substitution $e^{x/2} = \tan \theta$. Then $\frac{1}{2}e^{x/2} dx = \sec^2 \theta d\theta$, so $dx = \frac{2}{e^{x/2}} \sec^2 \theta d\theta = \frac{2}{\tan \theta} \sec \theta d\theta$.



$$\begin{aligned}
 \int \sqrt{1 + e^x} dx &= \int \frac{2\sqrt{1 + \tan^2 \theta}}{\tan \theta} \sec^2 \theta d\theta \\
 &= 2 \int \frac{\sec^3 \theta}{\tan \theta} d\theta \\
 &= 2 \int \frac{\sec \theta (\tan^2 \theta + 1)}{\tan \theta} d\theta \\
 &= 2 \int \left(\sec \theta \tan \theta + \frac{\sec \theta}{\tan \theta} \right) d\theta \\
 &= 2 \int (\sec \theta \tan \theta + \csc \theta) d\theta \\
 &= 2 \sec \theta + 2 \log |\cot \theta - \csc \theta| + C \\
 &= 2\sqrt{1 + e^x} + 2 \log \left| \frac{1}{e^{x/2}} - \frac{\sqrt{1 + e^x}}{e^{x/2}} \right| + C \\
 &= 2\sqrt{1 + e^x} + 2 \log |1 - \sqrt{1 + e^x}| - 2 \log(e^{x/2}) + C \\
 &= 2\sqrt{1 + e^x} + 2 \log |1 - \sqrt{1 + e^x}| - x + C
 \end{aligned}$$

We used the substitution $e^{x/2} = \tan \theta$, so $\cot \theta = \frac{1}{e^{x/2}}$. To find $\sec \theta$ and $\csc \theta$, we draw a right triangle with opposite side $e^{x/2}$ and adjacent side 1. They by the Pythagorean Theorem, the hypotenuse has length $\sqrt{1 + e^x}$.

Remark: if we use the substitution $u = \sqrt{1 + e^x}$, then we can change the integral to $\int \frac{2u^2}{u^2 - 1} du$. We can integrate this using the method of partial fractions, which we'll learn in the next section. You can explore this option in Question 21, Section 1.10.

S-24: Remember that for any value X ,

$$|X| = \begin{cases} X & \text{if } X \geq 0 \\ -X & \text{if } X \leq 0 \end{cases}$$

So, $|X| \neq X$ precisely when $X < 0$.

(a) The range of arcsine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So, since $u = \arcsin(x/a)$, u is in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore $\cos u \geq 0$. Since a is positive, $a \cos u \geq 0$, so $a \cos u = |a \cos u|$. That is,

$$\sqrt{a^2 - x^2} = |a \cos u| = a \cos u$$

all the time.

(b) The range of arctangent is $(-\frac{\pi}{2}, \frac{\pi}{2})$. So, since $u = \arctan(x/a)$, u is in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore $\sec u = \frac{1}{\cos u} > 0$. Since a is positive, $a \sec u > 0$, so $a \sec u = |a \sec u|$. That is,

$$\sqrt{a^2 + x^2} = |a \sec u| = a \sec u$$

all the time.

(c) The range of arccosine is $[0, \pi]$. So, since $u = \operatorname{arcsec}(x/a) = \arccos(a/x)$, u is in the range $[0, \pi]$. (Actually, it's in the range $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, since secant is undefined at $\pi/2$.) If $|a \tan u| \neq a \tan u$, then $\tan u < 0$, which happens when u is in the range $(\frac{\pi}{2}, \pi)$. This is the same range over which $-1 < \cos u < 0$, and so $-1 < \frac{a}{x} < 0$. Since $\frac{a}{x} < 0$, a and x have different signs, so $x < 0$. Then since $-1 < \frac{a}{x}$, also $x < -a$.

So,

$$\sqrt{x^2 - a^2} = |a \tan u| = -a \tan u \neq a \tan u$$

happens precisely when $x < -a$.

Solutions to Exercises 1.9 — Jump to [TABLE OF CONTENTS](#)

S-1: If a quadratic function can be factored as $(ax + b)(cx + d)$ for some constants a, b, c, d , then it has roots $-\frac{b}{a}$ and $-\frac{d}{c}$. So, if a quadratic function has no roots, it is irreducible: this is the case for the function in graph (d).

If a quadratic function has two different roots, then $(ax + b) \neq \alpha(cx + d)$ for any constant α . That is, the quadratic function is the product of distinct linear factors. This is the case for the functions graphed in (b) and (c), since these each have two distinct places where they cross the x -axis.

Finally, if a quadratic function has precisely one root, then $\frac{b}{a} = \frac{d}{c}$, so:

$$(ax + b)(cx + d) = a(x + \frac{b}{a})(cx + d) = a(x + \frac{d}{c})(cx + d) = \frac{a}{c}(cx + d)(cx + d)$$

That is, the quadratic function is the product of a repeated linear factor, and a constant $\frac{a}{c}$ (which might simply be $\frac{a}{c} = 1$).

S-2: The partial fraction decomposition has the form

$$\frac{3x^3 - 2x^2 + 11}{x^2(x-1)(x^2+3)} = \frac{A}{x-1} + \text{various terms}$$

When we multiply through by the original denominator, this becomes

$$3x^3 - 2x^2 + 11 = x^2(x^2 + 3)A + (x - 1)(\text{other terms}).$$

Evaluating both sides at $x = 1$ yields $3 \cdot 1^3 - 2 \cdot 1^2 + 11 = 1^2(1^2 + 3)A + 0$, or $A = 3$.

S-3:

(a) We start by dividing. The leading term of the numerator is x times the leading term of the denominator. The remainder is $x + 2$.

$$\begin{array}{r} x \\ x^2 + 1 \overline{) x^3 + 2x + 2} \\ \underline{-x^3} \\ x + 2 \end{array}$$

That is, $x^3 + 2x + 2 = x(x^2 + 1) + (x + 2)$. So,

$$\frac{x^3 + 2x + 2}{x^2 + 1} = x + \frac{x + 2}{x^2 + 1}$$

- (b) We start by dividing. The leading term of the numerator is $3x^2$ times the leading term of the denominator.

$$\begin{array}{r} 3x^2 \\ 5x^2 + 2x + 8 \overline{) 15x^4 + 6x^3 + 34x^2 + 4x + 20} \\ \underline{- 15x^4 - 6x^3 - 24x^2} \\ 10x^2 + 4x + 20 \end{array}$$

Then $5x^2$ goes into $10x^2$ twice, so

$$\begin{array}{r} 3x^2 \\ 5x^2 + 2x + 8 \overline{) 15x^4 + 6x^3 + 34x^2 + 4x + 20} \\ \underline{- 15x^4 - 6x^3 - 24x^2} \\ 10x^2 + 4x + 20 \\ \underline{- 10x^2 - 4x - 16} \\ 4 \end{array}$$

Our remainder is 4. That is,

$$\frac{15x^4 + 6x^3 + 34x^2 + 4x + 20}{5x^2 + 2x + 8} = 3x^2 + 2 + \frac{4}{5x^2 + 2x + 8}.$$

- (c) We start by dividing. The leading term of the numerator is x^3 times the leading term of the denominator.

$$\begin{array}{r} x^3 \\ 2x^2 + 5 \overline{) 2x^5 + 9x^3 + 12x^2 + 10x + 30} \\ \underline{- 2x^5 - 5x^3} \\ 4x^3 + 12x^2 + 10x \end{array}$$

Then $2x^2(2x)$ gives us $4x^3$.

$$\begin{array}{r} x^3 \\ 2x^2 + 5 \overline{) 2x^5 + 9x^3 + 12x^2 + 10x + 30} \\ \underline{- 2x^5 - 5x^3} \\ 4x^3 + 12x^2 + 10x \\ \underline{- 4x^3 - 10x} \\ 12x^2 + 30 \end{array}$$

Finally, $2x^2$ goes into $12x^2$ six times.

We'd like to use the difference of two squares to factor these quadratic expressions. For this to work, the constants must be positive (so their square roots are real). Since $\sqrt{29} > 3$, only the first quadratic is factorable. The other is irreducible—it's always positive, so it had no roots.

$$x^4 - 3x^2 - 5 = \left(x + \sqrt{\frac{3 + \sqrt{29}}{2}}\right) \left(x - \sqrt{\frac{3 + \sqrt{29}}{2}}\right) \left(x^2 + \frac{\sqrt{29} - 3}{2}\right)$$

- (c) Without seeing any obvious patterns, we start hunting for roots. Since we have all integer coefficients, if there are any integer roots, they will divide our constant term, -6 . So, our candidates for roots are ± 1 , ± 2 , ± 3 , and ± 6 . To save time, we don't need to know exactly the value of our polynomial at these points: only whether or not it is 0. Write $f(x) = x^4 - 4x^3 - 10x^2 - 11x - 6$.

$$\begin{array}{cccc} f(-1) = 0 & f(-2) \neq 0 & f(-3) \neq 0 & f(-6) \neq 0 \\ f(1) \neq 0 & f(2) \neq 0 & f(3) \neq 0 & f(6) = 0 \end{array}$$

Since $x = -1$ and $x = 6$ are roots of our polynomial, it has factors $(x + 1)$ and $(x - 6)$. Note $(x + 1)(x - 6) = x^2 - 5x - 6$. We use long division to figure out what else is lurking in our polynomial.

$$\begin{array}{r} x^2 - 5x - 6 \overline{) x^4 - 4x^3 - 10x^2 - 11x - 6} \\ \underline{-x^4 + 5x^3 + 6x^2} \\ x^3 - 4x^2 - 11x \\ \underline{-x^3 + 5x^2 + 6x} \\ x^2 - 5x - 6 \\ \underline{-x^2 + 5x + 6} \\ 0 \end{array}$$

$$\text{So, } x^4 - 4x^3 - 10x^2 - 11x - 6 = (x + 1)(x - 6)(x^2 + x + 1).$$

We should check whether $x^2 + x + 1$ is reducible or not. If we try to find its roots with the quadratic equation, we get $\frac{-1 \pm \sqrt{-3}}{2}$, which are not real numbers. So, we're at the end of our factoring.

- (d) Without seeing any obvious patterns, we start hunting for roots. Since we have all integer coefficients, if there are any integer roots, they will divide our constant term, -15 . So, our candidates for roots are ± 1 , ± 3 , ± 5 , and ± 15 . Write $f(x) = 2x^4 + 12x^3 - x^2 - 52x + 15$.

$$\begin{array}{cccc} f(-1) \neq 0 & f(-3) = 0 & f(-5) = 0 & f(-15) \neq 0 \\ f(1) \neq 0 & f(3) \neq 0 & f(5) \neq 0 & f(15) \neq 0 \end{array}$$

Since $x = -3$ and $x = -5$ are roots of our polynomial, it has factors $(x + 3)$ and $(x + 5)$. Note $(x + 3)(x + 5) = x^2 + 8x + 15$. We use long division to move forward.

$$\begin{array}{r}
 x^2 + 8x + 15 \overline{) \begin{array}{r} 2x^4 + 12x^3 - x^2 - 52x + 15 \\ - 2x^4 - 16x^3 - 30x^2 \\ \hline -4x^3 - 31x^2 - 52x \\ 4x^3 + 32x^2 + 60x \\ \hline x^2 + 8x + 15 \\ - x^2 - 8x - 15 \\ \hline 0 \end{array}}
 \end{array}$$

So, $2x^4 + 12x^3 - x^2 - 52x + 15 = (x + 3)(x + 5)(2x^2 - 4x + 1)$.

We should check whether $2x^2 - 4x + 1$ is reducible or not. There's not an obvious way to factor it, but we can use the quadratic equation. This gives us roots

$$\frac{4 \pm \sqrt{16 - 8}}{4} = 1 \pm \frac{\sqrt{2}}{2}. \text{ So, we have two more linear factors.}$$

Specifically:

$$2x^4 + 12x^3 - x^2 - 52x + 15 = (x + 3)(x + 5) \left(x - \left(1 + \frac{\sqrt{2}}{2} \right) \right) \left(x - \left(1 - \frac{\sqrt{2}}{2} \right) \right).$$

S-5: The goal of partial fraction decomposition is to write our integrand in a form that is easy to integrate. The antiderivative of (1) can be easily determined with the substitution $u = (ax + b)$. It's less clear how to find the antiderivative of (2).

S-6: The integrand is a rational function, so it's a candidate for partial fraction. We quickly rule out any obvious substitution so we go ahead with the decomposition.

We start by expressing the integrand, i.e. the fraction $\frac{1}{x+x^2} = \frac{1}{x(1+x)}$, as a linear combination of the simpler fractions $\frac{1}{x}$ and $\frac{1}{x+1}$ (which we already know how to integrate). We will have

$$\frac{1}{x+x^2} = \frac{1}{x(1+x)} = \frac{a}{x} + \frac{b}{x+1} = \frac{a(x+1) + bx}{x(1+x)}$$

The fraction on the left hand side is the same as the fraction on the right hand side if and only if the numerator on the left hand side, which is $1 = 0x + 1$, is equal to the numerator on the right hand side, which is $a(x+1) + bx = (a+b)x + a$. This in turn is the case if and only if $a = 1$ (i.e. the constant terms are the same in the two numerators) and $a + b = 0$ (i.e. the coefficients of x are the same in the two numerators). So $a = 1$ and $b = -1$. Now we can easily evaluate the integral

$$\begin{aligned}
 \int_1^2 \frac{dx}{x+x^2} &= \int_1^2 \frac{dx}{x(x+1)} = \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1} \right) dx = \left[\log x - \log(x+1) \right]_1^2 \\
 &= \log 2 - \log \frac{3}{2} = \log \frac{4}{3}
 \end{aligned}$$

S-7: We'll first do a partial fraction decomposition. The sneaky way is to temporarily rename x^2 to y . Then $x^4 + x^2 = y^2 + y$ and

$$\frac{1}{x^4 + x^2} = \frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

as we found in Question 6. Now we restore y to x^2 .

$$\int \frac{1}{x^4 + x^2} dx = \int \left(\frac{1}{x^2} - \frac{1}{x^2 + 1} \right) dx = -\frac{1}{x} - \arctan x + C$$

S-8: The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. We immediately look for a partial fraction decomposition:

$$\frac{12x + 4}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}.$$

Multiplying through by the denominator yields

$$12x + 4 = A(x^2 + 1) + (Bx + C)(x - 3) \quad (*)$$

Setting $x = 3$ we find:

$$36 + 4 = A(9 + 1) + 0 \implies 40 = 10A \implies A = 4$$

Substituting $A = 4$ in $(*)$ gives

$$\begin{aligned} 12x + 4 &= 4(x^2 + 1) + (Bx + C)(x - 3) \implies -4x^2 + 12x = (x - 3)(Bx + C) \\ &\implies (-4x)(x - 3) = (Bx + C)(x - 3) \\ &\implies B = -4, C = 0 \end{aligned}$$

So we have found that $A = 4$, $B = -4$, and $C = 0$. Therefore

$$\begin{aligned} \int \frac{12x + 4}{(x-3)(x^2+1)} dx &= \int \left(\frac{4}{x-3} - \frac{4x}{x^2+1} \right) dx \\ &= 4 \log |x-3| - 2 \log(x^2+1) + C \end{aligned}$$

The second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2 + 1$, $du = 2x dx$.

S-9: The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. With no obvious substitution available, we look for a partial fraction decomposition.

$$\frac{3x^2 - 4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

Multiplying through by the denominator gives

$$3x^2 - 4 = A(x^2 + 4) + (Bx + C)(x - 2) \quad (*)$$

Setting $x = 2$ we find:

$$12 - 4 = A(4 + 4) + 0 \implies 8 = 8A \implies A = 1$$

Substituting $A = 1$ in (*) gives

$$\begin{aligned} 3x^2 - 4 &= (x^2 + 4) + (x - 2)(Bx + C) \implies 2x^2 - 8 = (x - 2)(Bx + C) \\ &\implies (x - 2)(2x + 4) = (x - 2)(Bx + C) \\ &\implies B = 2, C = 4 \end{aligned}$$

Thus, we have:

$$\frac{3x^2 - 4}{(x - 2)(x^2 + 4)} = \frac{1}{x - 2} + \frac{2x + 4}{x^2 + 4} = \frac{1}{x - 2} + \frac{2x}{x^2 + 4} + \frac{4}{x^2 + 4}$$

The first two of these are directly integrable:

$$F(x) = \log|x - 2| + \log|x^2 + 4| + \int \frac{4}{x^2 + 4} dx$$

(The second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2 + 4$, $du = 2x dx$.) For the final integral, we substitute: $x = 2y$, $dx = 2dy$, and see that:

$$\int \frac{4}{x^2 + 4} dx = 2 \int \frac{1}{y^2 + 1} dy = 2 \arctan y + D = 2 \arctan(x/2) + D$$

for any constant D . All together we have:

$$F(x) = \log|x - 2| + \log(x^2 + 4) + 2 \arctan(x/2) + D$$

S-10: This integrand is a rational function, with no obvious substitution. This sure looks like a partial fraction problem. Let's go through our protocol.

- The degree of the numerator $x - 13$ is one, which is strictly smaller than the degree of the denominator $x^2 - x - 6$, which is two. So we don't need long division to pull out a polynomial.
- Next we factor the denominator.

$$x^2 - x - 6 = (x - 3)(x + 2)$$

- Next we find the partial fraction decomposition of the integrand. It is of the form

$$\frac{x - 13}{(x - 3)(x + 2)} = \frac{A}{x - 3} + \frac{B}{x + 2}$$

To find A and B , using the sneaky method, we cross multiply by the denominator.

$$x - 13 = A(x + 2) + B(x - 3)$$

Now we can find A by evaluating at $x = 3$

$$3 - 13 = A(3 + 2) + B(3 - 3) \implies A = -2$$

and find B by evaluating at $x = -2$.

$$-2 - 13 = A(-2 + 2) + B(-2 - 3) \implies B = 3$$

(Hmmm. A and B are nice round numbers. Sure looks like a rigged exam or homework question.) Our partial fraction decomposition is

$$\frac{x - 13}{(x - 3)(x + 2)} = \frac{-2}{x - 3} + \frac{3}{x + 2}$$

As a check, we recombine the right hand side and make sure that it matches the left hand side.

$$\frac{-2}{x - 3} + \frac{3}{x + 2} = \frac{-2(x + 2) + 3(x - 3)}{(x - 3)(x + 2)} = \frac{x - 13}{(x - 3)(x + 2)}$$

- Finally, we evaluate the integral.

$$\int \frac{x - 13}{x^2 - x - 6} dx = \int \left(\frac{-2}{x - 3} + \frac{3}{x + 2} \right) dx = -2 \log |x - 3| + 3 \log |x + 2| + C$$

S-11: Again, this sure looks like a partial fraction problem. So let's go through our protocol.

- The degree of the numerator $5x + 1$ is one, which is strictly smaller than the degree of the denominator $x^2 + 5x + 6$, which is two. So we do not long divide to pull out a polynomial.
- Next we factor the denominator.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

- Next we find the partial fraction decomposition of the integrand. It is of the form

$$\frac{5x + 1}{(x + 2)(x + 3)} = \frac{A}{x + 2} + \frac{B}{x + 3}$$

To find A and B , using the sneaky method, we cross multiply by the denominator.

$$5x + 1 = A(x + 3) + B(x + 2)$$

Now we can find A by evaluating at $x = -2$

$$-10 + 1 = A(-2 + 3) + B(-2 + 2) \implies A = -9$$

and find B by evaluating at $x = -3$.

$$-15 + 1 = A(-3 + 3) + B(-3 + 2) \implies B = 14$$

So our partial fraction decomposition is

$$\frac{5x + 1}{(x + 2)(x + 3)} = \frac{-9}{x + 2} + \frac{14}{x + 3}$$

As a check, we recombine the right hand side and make sure that it matches the left hand side

$$\frac{-9}{x + 2} + \frac{14}{x + 3} = \frac{-9(x + 3) + 14(x + 2)}{(x + 2)(x + 3)} = \frac{5x + 1}{(x + 2)(x + 3)}$$

- Finally, we evaluate the integral

$$\int \frac{5x + 1}{x^2 + 5x + 6} dx = \int \left(\frac{-9}{x + 2} + \frac{14}{x + 3} \right) dx = -9 \log |x + 2| + 14 \log |x + 3| + C$$

S-12: We have a rational function with no obvious substitution, so let's use partial fraction decomposition.

- Since the degree of the numerator is the same as the degree of the denominator, we need to pull out a polynomial.

$$\begin{array}{r} 5 \\ x^2 - 1 \overline{) 5x^2 - 3x - 1} \\ \underline{-5x^2} + 5 \\ -3x + 4 \end{array}$$

That is,

$$\int \frac{5x^2 - 3x - 1}{x^2 - 1} dx = \int \left(5 + \frac{-3x + 4}{x^2 - 1} \right) dx = 5x + \int \frac{-3x + 4}{x^2 - 1} dx$$

- Again, there's no obvious substitution for the new integrand, so we want to use partial fraction. The denominator factors as $(x - 1)(x + 1)$, so our decomposition has this form:

$$\frac{-3x + 4}{x^2 - 1} = \frac{-3x + 4}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} = \frac{(A + B)x + (A - B)}{(x - 1)(x + 1)}$$

So, (1) $A + B = -3$ and (2) $A - B = 4$.

- We solve (2) for A in terms of B , namely $A = 4 + B$. Plugging this into (1), we see $(4 + B) + B = -3$. So, $B = -\frac{7}{2}$, and $A = \frac{1}{2}$.

- Now we can write our integral in a friendlier form and evaluate.

$$\begin{aligned}\int \frac{5x^2 - 3x - 1}{x^2 - 1} dx &= 5x + \int \frac{-3x + 4}{x^2 - 1} dx = 5x + \int \frac{1/2}{x-1} - \frac{7/2}{x+1} dx \\ &= 5x + \frac{1}{2} \log |x-1| - \frac{7}{2} \log |x+1| + C\end{aligned}$$

S-13: The integrand is a rational function with no obvious substitution, so we use partial fraction decomposition.

- The degree of the numerator is the same as the degree of the denominator. Since it's not smaller, we need to re-write our integrand. We could do this using long division, but this case is simple enough to do more informally.

$$\begin{aligned}\frac{4x^4 + 14x^2 + 2}{4x^4 + x^2} &= \frac{4x^4 + x^2 + 13x^2 + 2}{4x^4 + x^2} \\ &= \frac{4x^4 + x^2}{4x^4 + x^2} + \frac{13x^2 + 2}{4x^4 + x^2} \\ &= 1 + \frac{13x^2 + 2}{4x^4 + x^2}\end{aligned}$$

- The denominator factors as $x^2(4x^2 + 1)$.
- We want to find the partial fraction decomposition of the fractional part of our simplified integrand.

$$\frac{13x^2 + 2}{4x^4 + x^2} = \frac{13x^2 + 2}{x^2(4x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{4x^2 + 1}$$

Multiply through by the original denominator.

$$13x^2 + 2 = Ax(4x^2 + 1) + B(4x^2 + 1) + (Cx + D)x^2 \quad (1)$$

Setting $x = 0$ gives us:

$$2 = B$$

We use $B = 2$ to simplify Equation (1).

$$\begin{aligned}13x^2 + 2 &= Ax(4x^2 + 1) + 2(4x^2 + 1) + (Cx + D)x^2 \\ 5x^2 &= Ax(4x^2 + 1) + (Cx + D)x^2 \\ 5x &= A(4x^2 + 1) + (Cx + D)x\end{aligned} \quad (2)$$

Again, let $x = 0$.

$$0 = A$$

Using $A = 0$, simplify Equation (2).

$$\begin{aligned}5x &= (Cx + D)x \\ 5 &= Cx + D \\ C = 0, \quad D &= 5\end{aligned}$$

- Now we can write our integral in pieces.

$$\begin{aligned}\int \frac{4x^4 + 14x^2 + 2}{4x^4 + x^2} dx &= \int \left(1 + \frac{13x^2 + 2}{4x^4 + x^2} \right) dx \\ &= \int \left(1 + \frac{2}{x^2} + \frac{5}{4x^2 + 1} \right) dx \\ &= x - \frac{2}{x} + \int \frac{5}{(2x)^2 + 1} dx\end{aligned}$$

Substitute $u = 2x$, $du = 2dx$.

$$\begin{aligned}&= x - \frac{2}{x} + \int \frac{5/2}{u^2 + 1} du \\ &= x - \frac{2}{x} + \frac{5}{2} \arctan u + C \\ &= x - \frac{2}{x} + \frac{5}{2} \arctan(2x) + C\end{aligned}$$

S-14: Our integrand is a rational function with no obvious substitution, so we'll use the method of partial fractions.

- The degree of the numerator is less than the degree of the denominator.
- We need to factor the denominator. The first two terms have the same ratio as the last two terms.

$$\begin{aligned}2x^3 - x^2 - 8x + 4 &= x^2(2x - 1) - 4(2x - 1) \\ &= (x^2 - 4)(2x - 1) \\ &= (x - 2)(x + 2)(2x - 1)\end{aligned}$$

- Now we find our partial fraction decomposition.

$$\frac{3x^2 - 4x - 10}{2x^3 - x^2 - 8x + 4} = \frac{3x^2 - 4x - 10}{(x - 2)(x + 2)(2x - 1)} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{C}{2x - 1}$$

Multiply both sides by the original denominator.

$$3x^2 - 4x - 10 = A(x + 2)(2x - 1) + B(x - 2)(2x - 1) + C(x - 2)(x + 2)$$

Distinct linear factors is the best possible scenario for the sneaky method. First, let's set $x = 2$.

$$3(4) - 4(2) - 10 = A(4)(3) + B(0) + C(0)$$

$$A = -\frac{1}{2}$$

Now, let $x = -2$.

$$3(4) - 4(-2) - 10 = A(0) + B(-4)(-5) + C(0)$$

$$B = \frac{1}{2}$$

Finally, let $x = \frac{1}{2}$.

$$\frac{3}{4} - 2 - 10 = A(0) + B(0) + C \left(-\frac{3}{2} \right) \left(\frac{5}{2} \right)$$

$$C = 3$$

- Now we can evaluate our integral in its new form.

$$\begin{aligned} \int \frac{3x^2 - 4x - 10}{2x^3 - x^2 - 8x + 4} dx &= \int \left(\frac{-1/2}{x-2} + \frac{1/2}{x+2} + \frac{3}{2x-1} \right) dx \\ &= -\frac{1}{2} \log |x-2| + \frac{1}{2} \log |x+2| + \frac{3}{2} \log |2x-1| + C \\ &= \frac{1}{2} \log \left| \frac{x+2}{x-2} \right| + \frac{3}{2} \log |2x-1| + C \end{aligned}$$

S-15: The integrand is a rational function with no obvious substitution, so we use the method of partial fractions.

- The numerator has smaller degree than the denominator.
- We need to factor the denominator. In the absence of any clues, we look for an integer root. The constant term is 5, so the possible integer roots are ± 1 and ± 5 . Name $f(x) = 2x^3 + 11x^2 + 6x + 5$.

$$f(-1) \neq 0 \quad f(-5) = 0 \quad f(1) \neq 0 \quad f(5) \neq 0$$

So, $(x + 5)$ is a factor of the denominator.

- We use long division to pull out the factor of $(x + 5)$.

$$\begin{array}{r} 2x^2 + x + 1 \\ x+5 \overline{) 2x^3 + 11x^2 + 6x + 5} \\ \underline{-2x^3 - 10x^2} \\ x^2 + 6x \\ \underline{-x^2 - 5x} \\ x + 5 \\ \underline{-x - 5} \\ 0 \end{array}$$

That is, our denominator is $(x + 5)(2x^2 + x + 1)$.

- The quadratic function $2x^2 + x + 1$ is irreducible: we can see this by using the quadratic equation, and finding no real roots. So, we are ready to find our partial fraction decomposition.

$$\frac{10x^2 + 24x + 8}{2x^3 + 11x^2 + 6x + 5} = \frac{10x^2 + 24x + 8}{(x + 5)(2x^2 + x + 1)} = \frac{A}{x + 5} + \frac{Bx + C}{2x^2 + x + 1}$$

Multiply through by the original denominator.

$$10x^2 + 24x + 8 = A(2x^2 + x + 1) + (Bx + C)(x + 5) \quad (1)$$

Set $x = -5$.

$$10(25) - 24(5) + 8 = A(2(25) - 5 + 1) + (B(-5) + C)(0)$$
$$A = 3$$

Using our value of A , we simplify Equation (1).

$$10x^2 + 24x + 8 = 3(2x^2 + x + 1) + (Bx + C)(x + 5)$$
$$4x^2 + 21x + 5 = (Bx + C)(x + 5)$$

We factor the left side. We know $(x + 5)$ must be one of its factors.

$$(4x + 1)(x + 5) = (Bx + C)(x + 5)$$
$$4x + 1 = Bx + C$$

So, $B = 4$ and $C = 1$.

- Now we can write our integral in smaller pieces.

$$\int_0^1 \frac{10x^2 + 24x + 8}{2x^3 + 11x^2 + 6x + 5} dx = \int_0^1 \left(\frac{3}{x + 5} + \frac{4x + 1}{2x^2 + x + 1} \right) dx$$

The antiderivative of the left fraction is $3 \log |x + 5|$. For the right fraction, we use the substitution $u = 2x^2 + x + 1$, $du = (4x + 1)dx$ to antidifferentiate.

$$= [3 \log |x + 5| + \log |2x^2 + x + 1|]_0^1$$
$$= 3 \log 6 + \log 4 - 3 \log 5 - \log 1$$
$$= \log \left(\frac{4 \cdot 6^3}{5^3} \right)$$

S-16: We follow the example in the text.

$$\int \csc x \, dx = \int \frac{1}{\sin x} \, dx = \int \frac{\sin x}{\sin^2 x} \, dx = \int \frac{\sin x}{1 - \cos^2 x} \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$.

$$= \int \frac{-1}{1 - u^2} \, du = \int \frac{-1}{(1 + u)(1 - u)} \, du$$

We see an opportunity for partial fraction.

$$\frac{-1}{(1 + u)(1 - u)} = \frac{A}{1 + u} + \frac{B}{1 - u}$$

Multiply both sides by the original denominator.

$$-1 = A(1 - u) + B(1 + u)$$

Let $u = 1$.

$$-1 = 2B \quad \Rightarrow \quad B = -\frac{1}{2}$$

Let $u = -1$.

$$-1 = 2A \quad \Rightarrow \quad A = -\frac{1}{2}$$

We can now re-write our integral.

$$\begin{aligned} \int \csc x \, dx &= \int \frac{-1}{(1+u)(1-u)} \, du = \int \left(\frac{-1/2}{1+u} + \frac{-1/2}{1-u} \right) \, du \\ &= -\frac{1}{2} \log |1+u| + \frac{1}{2} \log |1-u| + C \\ &= \frac{1}{2} \log \left| \frac{1-u}{1+u} \right| + C \\ &= \frac{1}{2} \log \left| \frac{1-\cos x}{1+\cos x} \right| + C \end{aligned}$$

Remark: Elsewhere in the text, and in many tables of integrals, the antiderivative of cosecant is given as $\log |\csc x - \cot x|$. We show that this is equivalent to our result.

$$\begin{aligned} \log |\csc x - \cot x| &= \frac{1}{2} \log |(\csc x - \cot x)^2| = \frac{1}{2} \log |\csc^2 x - 2 \csc x \cot x + \cot^2 x| \\ &= \frac{1}{2} \log \left| \frac{1}{\sin^2 x} - \frac{2 \cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \right| \\ &= \frac{1}{2} \log \left| \frac{1 - 2 \cos x + \cos^2 x}{\sin^2 x} \right| = \frac{1}{2} \log \left| \frac{(1 - \cos x)^2}{1 - \cos^2 x} \right| \\ &= \frac{1}{2} \log \left| \frac{(1 - \cos x)^2}{(1 - \cos x)(1 + \cos x)} \right| = \frac{1}{2} \log \left| \frac{1 - \cos x}{1 + \cos x} \right| \end{aligned}$$

S-17: We follow the example in the text.

$$\int \csc^3 x \, dx = \int \frac{1}{\sin^3 x} \, dx = \int \frac{\sin x}{\sin^4 x} \, dx = \int \frac{\sin x}{(1 - \cos^2 x)^2} \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$.

$$= \int \frac{-1}{(1 - u^2)^2} \, du$$

In Question 16, we saw $\frac{1}{1-u^2} = \frac{1/2}{1+u} + \frac{1/2}{1-u}$, so

$$\begin{aligned}
 \int \frac{-1}{(1-u^2)^2} du &= - \int \left(\frac{1}{1-u^2} \right)^2 du = - \int \left(\frac{1/2}{1+u} + \frac{1/2}{1-u} \right)^2 du \\
 &= -\frac{1}{4} \int \left(\frac{1}{(1+u)^2} + \frac{2}{1-u^2} + \frac{1}{(1-u)^2} \right) du \\
 &= -\frac{1}{4} \int \left(\frac{1}{(1+u)^2} + \frac{1}{1+u} + \frac{1}{1-u} + \frac{1}{(1-u)^2} \right) du \\
 &= -\frac{1}{4} \left(-\frac{1}{1+u} + \log|1+u| - \log|1-u| + \frac{1}{1-u} \right) + C \\
 &= -\frac{1}{4} \left(\frac{2u}{1-u^2} + \log \left| \frac{1+u}{1-u} \right| \right) + C \\
 &= \frac{-u}{2(1-u^2)} + \frac{1}{4} \log \left| \frac{1-u}{1+u} \right| + C \\
 &= \frac{-\cos x}{2 \sin^2 x} + \frac{1}{4} \log \left| \frac{1-\cos x}{1+\cos x} \right| + C
 \end{aligned}$$

Remark: In many tables of integrals, the antiderivative of $\csc^3 x$ is given as $-\frac{1}{2} \cot x \csc x + \frac{1}{2} \log |\csc x - \cot x| + C$. This is equivalent to our result. Recall in the remark after the solution to Question 16, we saw $\frac{1}{2} \log \left| \frac{1-\cos x}{1+\cos x} \right| = \log |\csc x - \cot x|$.

$$\begin{aligned}
 -\frac{1}{2} \cot x \csc x + \frac{1}{2} \log |\csc x - \cot x| &= -\frac{1}{2} \cot x \csc x + \frac{1}{4} \log \left| \frac{1-\cos x}{1+\cos x} \right| \\
 &= -\frac{1}{2} \left(\frac{\cos x}{\sin x} \right) \left(\frac{1}{\sin x} \right) + \frac{1}{4} \log \left| \frac{1-\cos x}{1+\cos x} \right| \\
 &= \frac{-\cos x}{2 \sin^2 x} + \frac{1}{4} \log \left| \frac{1-\cos x}{1+\cos x} \right|
 \end{aligned}$$

S-18: This is a rational function, and there's no obvious substitution, so we'll use partial fraction decomposition.

- First, we check that the numerator has strictly smaller degree than the denominator, so we don't have to use long division.
- Second, we factor the denominator. We can immediately pull out a factor of x^2 ; then we're left with the quadratic polynomial $x^2 + 5x + 10$. Using the quadratic equation, we check that this has no real roots, so it is irreducible.
- Once we know the factorization of the denominator, we can set up our decomposition.

$$\begin{aligned}
 \frac{3x^3 + 15x^2 + 35x + 10}{x^4 + 5x^3 + 10x^2} &= \frac{3x^3 + 15x^2 + 35x + 10}{x^2(x^2 + 5x + 10)} \\
 &= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 5x + 10}
 \end{aligned}$$

We multiply both sides by the original denominator.

$$3x^3 + 15x^2 + 35x + 10 = Ax(x^2 + 5x + 10) + B(x^2 + 5x + 10) + (Cx + D)x^2 \quad (1)$$

Following the “Sneaky Method,” we plug in $x = 0$.

$$0 + 10 = A(0) + B(10) + (C(0) + D)(0)$$

$$B = 1$$

- Knowing B allows us to simplify our Equation (1).

$$\begin{aligned} 3x^3 + 15x^2 + 35x + 10 &= Ax(x^2 + 5x + 10) + 1(x^2 + 5x + 10) + (Cx + D)x^2 \\ 3x^3 + 14x^2 + 30x &= Ax(x^2 + 5x + 10) + (Cx + D)x^2 \end{aligned}$$

We can factor x out of both sides of the equation.

$$3x^2 + 14x + 30 = A(x^2 + 5x + 10) + (Cx + D)x \quad (2)$$

- Again, we set $x = 0$.

$$0 + 30 = A(10) + (C(0) + D)(0)$$

$$A = 3$$

- We simplify Equation (2), using $A = 3$.

$$\begin{aligned} 3x^2 + 14x + 30 &= 3(x^2 + 5x + 10) + (Cx + D)x \\ -x &= Cx^2 + Dx \\ C &= 0, \quad D = -1 \end{aligned}$$

- Now that we have our coefficients, we can re-write our integral in a friendlier form.

$$\begin{aligned} \int_1^2 \frac{3x^3 + 15x^2 + 35x + 10}{x^4 + 5x + 10x^2} dx &= \int_1^2 \left(\frac{3}{x} + \frac{1}{x^2} - \frac{1}{x^2 + 5x + 10} \right) dx \\ &= \left[3 \log |x| - \frac{1}{x} \right]_1^2 - \int_1^2 \frac{1}{x^2 + 5x + 10} dx \\ &= 3 \log 2 + \frac{1}{2} - \int_1^2 \frac{1}{x^2 + 5x + 10} dx \end{aligned}$$

The remaining integral is the reciprocal of a quadratic polynomial, much like $\frac{1}{1+x^2}$, whose antiderivative is arctangent. We complete the square and use the substitution $u = \left(\frac{2x+5}{\sqrt{15}}\right)$, $du = \frac{2}{\sqrt{15}} dx$.

$$\begin{aligned}\int_1^2 \frac{1}{x^2 + 5x + 10} dx &= \int_1^2 \frac{1}{\left(x + \frac{5}{2}\right)^2 + \frac{15}{4}} dx \\&= \frac{4}{15} \int_1^2 \frac{1}{\left(\frac{2x+5}{\sqrt{15}}\right)^2 + 1} dx \\&= \frac{2}{\sqrt{15}} \int_{7/\sqrt{15}}^{9/\sqrt{15}} \frac{1}{u^2 + 1} du \\&= \frac{2}{\sqrt{15}} \left[\arctan u \right]_{7/\sqrt{15}}^{9/\sqrt{15}} \\&= \frac{2}{\sqrt{15}} \left(\arctan \left(\frac{9}{\sqrt{15}} \right) - \arctan \left(\frac{7}{\sqrt{15}} \right) \right)\end{aligned}$$

So, all together,

$$\int_1^2 \frac{3x^3 + 15x^2 + 35x + 10}{x^4 + 5x^3 + 10x^2} dx = 3 \log 2 + \frac{1}{2} - \frac{2}{\sqrt{15}} \left(\arctan \left(\frac{9}{\sqrt{15}} \right) - \arctan \left(\frac{7}{\sqrt{15}} \right) \right)$$

S-19: If our denominator were all sines, we could use the substitution $x = \sin \theta$. To that end, we apply the identity $\cos^2 \theta = 1 - \sin^2 \theta$.

$$\int \frac{\cos \theta}{3 \sin \theta + \cos^2 \theta - 3} d\theta = \int \frac{\cos \theta}{3 \sin \theta + 1 - \sin^2 \theta - 3} d\theta = \int \frac{\cos \theta}{3 \sin \theta - \sin^2 \theta - 2} d\theta$$

We use the substitution $x = \sin \theta$, $dx = \cos \theta d\theta$.

$$= \int \frac{1}{3x - x^2 - 2} dx = \int \frac{-1}{x^2 - 3x + 2} dx = \int \frac{-1}{(x-1)(x-2)} dx$$

Now we can find a partial fraction decomposition.

$$\begin{aligned}\frac{-1}{(x-1)(x-2)} &= \frac{A}{x-1} + \frac{B}{x-2} \\-1 &= A(x-2) + B(x-1)\end{aligned}$$

Setting $x = 1$ and $x = 2$, we see

$$A = 1, \quad B = -1$$

Now, we can evaluate our integral.

$$\begin{aligned}\int \frac{\cos \theta}{3 \sin \theta + \cos^2 \theta - 3} d\theta &= \int \frac{-1}{(x-1)(x-2)} dx = \int \left(\frac{1}{x-1} - \frac{1}{x-2} \right) dx \\&= \log |x-1| - \log |x-2| + C = \log \left| \frac{x-1}{x-2} \right| + C \\&= \log \left| \frac{\sin \theta - 1}{\sin \theta - 2} \right| + C\end{aligned}$$

S-20: This looks a lot like a rational function, but with the function e^t in place of the variable. So, we would like to make the substitution $x = e^t$, $dx = e^t dt$. Then $dt = \frac{1}{e^t} dx = \frac{1}{x} dx$.

$$\int \frac{1}{e^{2t} + e^t + 1} dt = \int \frac{1}{x(x^2 + x + 1)} dx$$

The factor $x^2 + x + 1$ is an irreducible quadratic, so the denominator is completely factored. Now we can use partial fraction decomposition.

$$\begin{aligned} \frac{1}{x(x^2 + x + 1)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1} \\ 1 &= A(x^2 + x + 1) + (Bx + C)x \\ 1 &= (A + B)x^2 + (A + C)x + A \end{aligned}$$

The constant terms tell us $A = 1$; then the coefficient of x tells us $C = -A = -1$. Finally, the coefficient of x^2 tells us $B = -A = -1$. Now we can evaluate our integral.

$$\begin{aligned} \int \frac{1}{e^{2t} + e^t + 1} dt &= \int \frac{1}{x(x^2 + x + 1)} dx \\ &= \int \left(\frac{1}{x} - \frac{x+1}{x^2 + x + 1} \right) dx \\ &= \int \left(\frac{1}{x} - \frac{x+1/2+1/2}{x^2 + x + 1} \right) dx \quad (*) \\ &= \int \frac{1}{x} dx - \int \frac{x+1/2}{x^2 + x + 1} dx - \int \frac{1/2}{x^2 + x + 1} dx \\ &= \log|x| - \frac{1}{2} \log|x^2 + x + 1| - \int \frac{1/2}{x^2 + x + 1} dx \end{aligned}$$

In step (*), we set ourselves up so that we could evaluate the second integral with the substitution $u = x^2 + x + 1$. For the remaining integral, we complete the square, so that the integrand looks something like the derivative of arctangent.

$$\begin{aligned} &= \log|x| - \frac{1}{2} \log|x^2 + x + 1| - \int \frac{1/2}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx \\ &= \log|x| - \frac{1}{2} \log|x^2 + x + 1| - \frac{2}{3} \int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} dx \end{aligned}$$

We use the substitution $u = \frac{2x+1}{\sqrt{3}}$, $du = \frac{2}{\sqrt{3}}$.

$$\begin{aligned}
 &= \log |x| - \frac{1}{2} \log |x^2 + x + 1| - \frac{1}{\sqrt{3}} \int \frac{1}{u^2 + 1} du \\
 &= \log |x| - \frac{1}{2} \log |x^2 + x + 1| - \frac{1}{\sqrt{3}} \arctan u + C \\
 &= \log |x| - \frac{1}{2} \log |x^2 + x + 1| - \frac{1}{\sqrt{3}} \arctan \left(\frac{2x+1}{\sqrt{3}} \right) + C \\
 &= \log |e^t| - \frac{1}{2} \log |e^{2t} + e^t + 1| - \frac{1}{\sqrt{3}} \arctan \left(\frac{2e^t + 1}{\sqrt{3}} \right) + C \\
 &= t - \frac{1}{2} \log |e^{2t} + e^t + 1| - \frac{1}{\sqrt{3}} \arctan \left(\frac{2e^t + 1}{\sqrt{3}} \right) + C
 \end{aligned}$$

S-21:

Solution 1: We use the substitution $u = \sqrt{1 + e^x}$.

Then $du = \frac{e^x}{2\sqrt{1 + e^x}} dx$, so $dx = \frac{2u}{u^2 - 1} du$.

$$\begin{aligned}
 \int \sqrt{1 + e^x} dx &= \int u \cdot \frac{2u}{u^2 - 1} du = \int \frac{2u^2}{u^2 - 1} du \\
 &= \int \frac{2(u^2 - 1) + 2}{u^2 - 1} du = \int \left(2 + \frac{2}{u^2 - 1} \right) du
 \end{aligned}$$

We use a partial fraction decomposition on the fractional part of the integrand.

$$\begin{aligned}
 \frac{2}{u^2 - 1} &= \frac{2}{(u - 1)(u + 1)} = \frac{A}{u - 1} + \frac{B}{u + 1} = \frac{(A + B)u + (A - B)}{(u - 1)(u + 1)} \\
 A + B &= 0, \quad A - B = 2 \\
 A &= 1, \quad B = -1
 \end{aligned}$$

$$\begin{aligned}
 \int \sqrt{1 + e^x} dx &= \int \left(2 + \frac{2}{u^2 - 1} \right) du = \int \left(2 + \frac{1}{u - 1} - \frac{1}{u + 1} \right) du \\
 &= 2u + \log |u - 1| - \log |u + 1| + C = 2u + \log \left| \frac{u - 1}{u + 1} \right| + C \\
 &= 2\sqrt{1 + e^x} + \log \left| \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right| + C
 \end{aligned}$$

Solution 2: It might not occur to us right away to use the fruitful substitution in

Solution 1. More realistically, we might start with the “inside function,” $u = 1 + e^x$.

Then $du = e^x dx$, so $dx = \frac{1}{u-1} du$.

$$\int \sqrt{1 + e^x} dx = \int \frac{\sqrt{u}}{u - 1} du$$

This isn't quite a rational function, because we have a square root on top. If we could turn it into a rational function, we could use partial fraction. To that end, let $w = \sqrt{u}$, $dw = \frac{1}{2\sqrt{u}} du$, so $du = 2w dw$.

$$\begin{aligned} &= \int \frac{w}{w^2 - 1} 2w dw = \int \frac{2w^2}{w^2 - 1} dw \\ &= \int \frac{2(w^2 - 1) + 2}{w^2 - 1} dw = \int 2 + \frac{2}{w^2 - 1} dw \end{aligned}$$

Now we can use partial fraction decomposition.

$$\begin{aligned} \frac{2}{w^2 - 1} &= \frac{2}{(w - 1)(w + 1)} = \frac{A}{w - 1} + \frac{B}{w + 1} = \frac{(A + B)w + (A - B)}{(w - 1)(w + 1)} \\ A + B &= 0, \quad A - B = 2 \\ \mathbf{A = 1, \quad B = -1} \end{aligned}$$

This allows us to antidifferentiate.

$$\begin{aligned} \int \sqrt{1 + e^x} dx &= \int \left(2 + \frac{2}{w^2 - 1} \right) dw = \int \left(2 + \frac{1}{w - 1} - \frac{1}{w + 1} \right) dw \\ &= 2w + \log |w - 1| - \log |w + 1| + C \\ &= 2w + \log \left| \frac{w - 1}{w + 1} \right| + C \\ &= 2\sqrt{u} + \log \left| \frac{\sqrt{u} - 1}{\sqrt{u} + 1} \right| + C \\ &= 2\sqrt{1 + e^x} + \log \left| \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right| + C \end{aligned}$$

Remark: we also evaluated this integral using trigonometric substitution in Section 1.9, Question 23. In that question, we found the antiderivative to be $2\sqrt{1 + e^x} + 2 \log |1 - \sqrt{1 + e^x}| - x + C$. These expressions are equivalent:

$$\begin{aligned} \log \left| \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right| &= \log |\sqrt{1 + e^x} - 1| + \log \left| \frac{1}{\sqrt{1 + e^x} + 1} \right| \\ &= \log |\sqrt{1 + e^x} - 1| + \log \left| \left(\frac{1}{\sqrt{1 + e^x} + 1} \right) \left(\frac{1 - \sqrt{1 + e^x}}{1 - \sqrt{1 + e^x}} \right) \right| \\ &= \log |\sqrt{1 + e^x} - 1| + \log \left| \frac{1 - \sqrt{1 + e^x}}{1 - (1 + e^x)} \right| \\ &= \log |\sqrt{1 + e^x} - 1| + \log \left| \frac{1 - \sqrt{1 + e^x}}{-e^x} \right| \\ &= \log |\sqrt{1 + e^x} - 1| + \log |1 - \sqrt{1 + e^x}| - \log |-e^x| \\ &= 2 \log |\sqrt{1 + e^x} - 1| - x \end{aligned}$$

S-22: In order to find the area between the curves, we need to know which one is on top, and which on the bottom. Let's start by finding where they meet.

$$\begin{aligned}\frac{4}{3+x^2} &= \frac{2}{x(x+1)} \\ 2x^2 + 2x &= 3 + x^2 \\ x^2 + 2x - 3 &= 0 \\ (x-1)(x+3) &= 0\end{aligned}$$

In the interval $[\frac{1}{4}, 3]$, the curves only meet at $x = 1$. So, to find which is on top and on bottom in the intervals $[\frac{1}{4}, 1)$ and $(1, 3]$, it suffices to check some point in each interval.

x	$\frac{4}{3+x^2}$	$\frac{2}{x(x+1)}$	Top:
$1/2$	$16/13$	$8/3$	$\frac{2}{x(x+1)}$
2	$4/7$	$1/3$	$\frac{4}{3+x^2}$

So, $\frac{2}{x(x+1)}$ is the top function when $\frac{1}{4} \leq x < 1$, and $\frac{4}{3+x^2}$ is the top function when $1 < x \leq 3$. Then the area we want to find is:

$$\text{Area} = \int_{\frac{1}{4}}^1 \left(\frac{2}{x(x+1)} - \frac{4}{3+x^2} \right) dx + \int_1^3 \left(\frac{4}{3+x^2} - \frac{2}{x(x+1)} \right) dx$$

We'll need to antidifferentiate both these functions. We can antidifferentiate $\frac{2}{x(x+1)}$ using partial fraction decomposition.

$$\begin{aligned}\frac{2}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} = \frac{(A+B)x + A}{x(x+1)} \\ A &= 2, \quad B = -2 \\ \int \frac{2}{x(x+1)} dx &= \int \left(\frac{2}{x} - \frac{2}{x+1} \right) dx = 2 \log |x| - 2 \log |x+1| + C \\ &= 2 \log \left| \frac{x}{x+1} \right| + C\end{aligned}$$

We can antidifferentiate $\frac{4}{3+x^2}$ using the substitution $u = \frac{x}{\sqrt{3}}$, $du = \frac{1}{\sqrt{3}} dx$.

$$\begin{aligned}\int \frac{4}{3+x^2} dx &= \int \frac{4}{3 \left(1 + \left(\frac{x}{\sqrt{3}} \right)^2 \right)} dx = \int \frac{4\sqrt{3}}{3(1+u^2)} du \\ &= \frac{4}{\sqrt{3}} \arctan u + C = \frac{4}{\sqrt{3}} \arctan \left(\frac{x}{\sqrt{3}} \right) + C\end{aligned}$$

Now, we can find our area.

$$\begin{aligned}
 \text{Area} &= \int_{\frac{1}{4}}^1 \left(\frac{2}{x(x+1)} - \frac{4}{3+x^2} \right) dx + \int_1^3 \left(\frac{4}{3+x^2} - \frac{2}{x(x+1)} \right) dx \\
 &= \left[2 \log \left| \frac{x}{x+1} \right| - \frac{4}{\sqrt{3}} \arctan \left(\frac{x}{\sqrt{3}} \right) \right]_{1/4}^1 + \left[\frac{4}{\sqrt{3}} \arctan \left(\frac{x}{\sqrt{3}} \right) - 2 \log \left| \frac{x}{x+1} \right| \right]_1^3 \\
 &= \left(2 \log \frac{1}{2} - \frac{4}{\sqrt{3}} \cdot \frac{\pi}{6} - 2 \log \frac{1}{5} + \frac{4}{\sqrt{3}} \arctan \frac{1}{4\sqrt{3}} \right) + \\
 &\quad \left(\frac{4}{\sqrt{3}} \cdot \frac{\pi}{3} - 2 \log \frac{3}{4} - \frac{4}{\sqrt{3}} \cdot \frac{\pi}{6} + 2 \log \frac{1}{2} \right) \\
 &= 2 \log \frac{5}{3} + \frac{4}{\sqrt{3}} \arctan \frac{1}{4\sqrt{3}}
 \end{aligned}$$

S-23: (a) To antidifferentiate $\frac{1}{t^2-9}$, we use a partial fraction decomposition.

$$\begin{aligned}
 \frac{1}{t^2-9} &= \frac{1}{(t-3)(t+3)} = \frac{A}{t-3} + \frac{B}{t+3} = \frac{(A+B)t + 3(A-B)}{(t-3)(t+3)} \\
 A+B &= 0, \quad A-B = \frac{1}{3} \\
 A &= \frac{1}{6}, \quad B = -\frac{1}{6} \\
 F(x) &= \int_1^x \frac{1}{t^2-9} dx = \int_1^x \left(\frac{1/6}{t-3} - \frac{1/6}{t+3} \right) dx \\
 &= \left[\frac{1}{6} \log |t-3| - \frac{1}{6} \log |t+3| \right]_1^x \\
 &= \left(\frac{1}{6} \log |x-3| - \frac{1}{6} \log |x+3| - \frac{1}{6} \log 2 + \frac{1}{6} \log 4 \right) \\
 &= \frac{1}{6} \left(\log \left| 2 \cdot \frac{x-3}{x+3} \right| \right)
 \end{aligned}$$

(b) Rather than differentiate our answer from (a), we use the Fundamental Theorem of Calculus Part 1 to conclude

$$F'(x) = \frac{d}{dx} \left\{ \int_1^x \frac{1}{t^2-9} dt \right\} = \frac{1}{x^2-9}$$

Solutions to Exercises 1.10 — Jump to [TABLE OF CONTENTS](#)

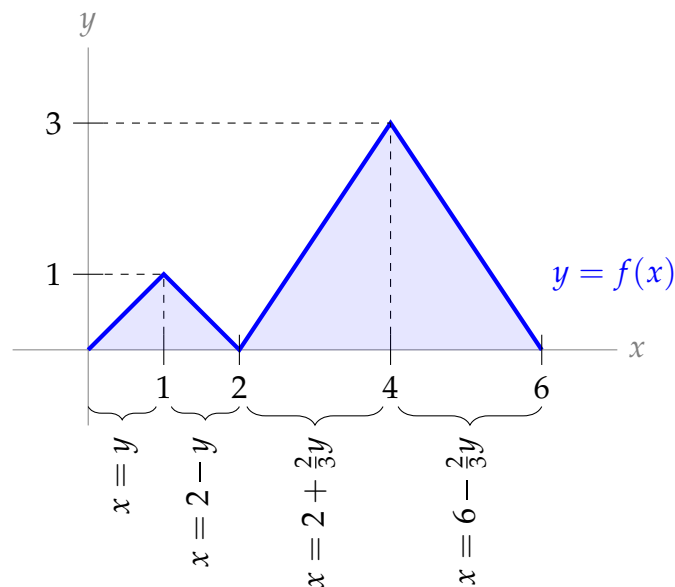
S-1: If we take a horizontal slice of a cone, we get a circle. If we take a vertical cross-section, the base is flat (it's a chord on the circular base of the cone), so we know right away it isn't a circle. Indeed, if we slice down through the very centre, we get a

triangle. (Other vertical slices have a curvy top, corresponding to a class of curves known as hyperbolas.)

S-2: The columns have the same volume. We can see this by chopping up the columns into horizontal cross-sections. Each cross-section has the same area as the cookie cutter, A , and height dy . Then in both cases, the volume of the column is

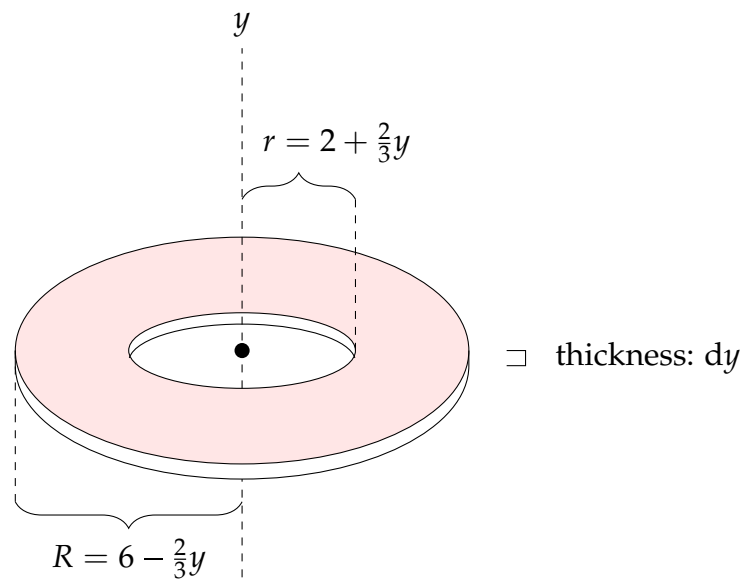
$$\int_0^h A \, dy = hA \text{ cubic units}$$

S-3: Notice $f(x)$ is a piecewise linear function, so we can find explicit equations for each of its pieces from the graph. The radii will be determined by the x -values, so below we give the x -values as functions of y .

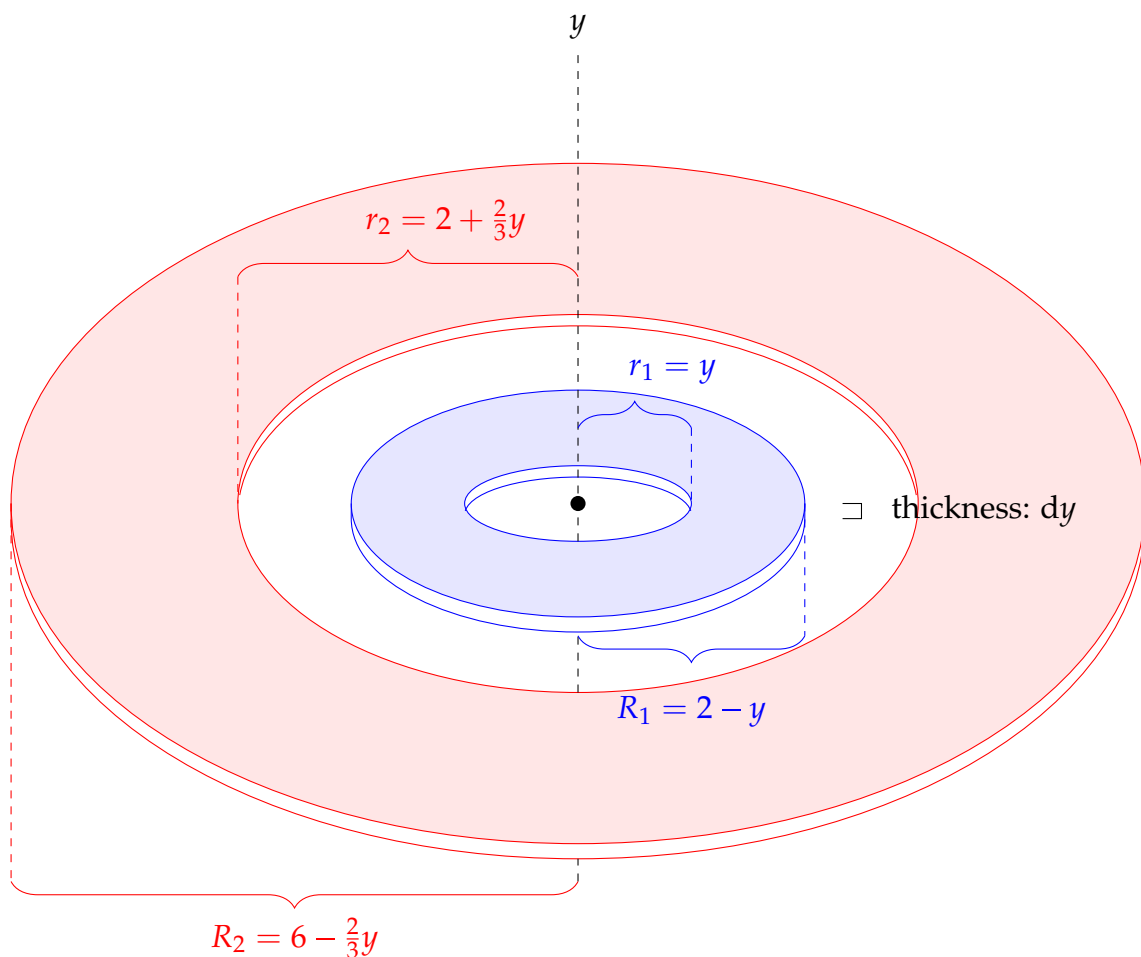


If we imagine rotating the region from the picture about the y -axis, there will be two kinds of washers formed: when $y < 1$, we have a “double washer,” two concentric rings. When $y > 1$, we have a single ring.

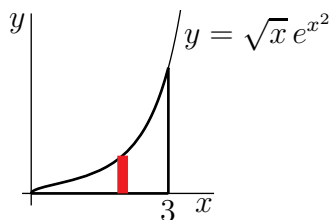
Washers when $1 < y \leq 6$: If $y > 1$, then our washer has inner radius $2 + \frac{2}{3}y$, outer radius $6 - \frac{2}{3}y$, and height dy .



Washers when $0 \leq y < 1$: When $0 \leq y < 1$, we have a “double washer,” two concentric rings corresponding to the two “humps” in the function. The inner washer has inner radius $r_1 = y$ and outer radius $R_1 = 2 - y$. The outer washer has inner radius $r_2 = 2 + \frac{2}{3}y$ and outer radius $R_2 = 6 - \frac{2}{3}y$. The thickness of the washers is dy .



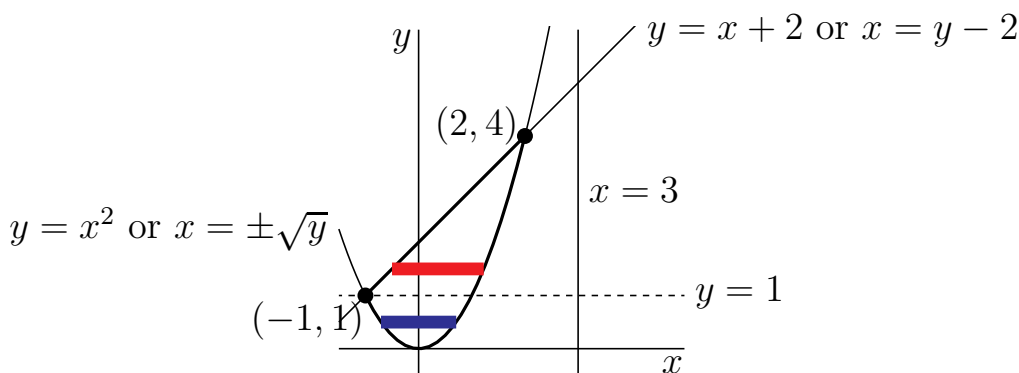
S-4: (a) When the strip shown in the figure



is rotated about the x -axis, it forms a thin disk of radius $\sqrt{x}e^{x^2}$ and thickness dx and hence of cross sectional area $\pi x e^{2x^2}$ and volume $\pi x e^{2x^2} dx$. So the volume of the solid is

$$\pi \int_0^3 x e^{2x^2} dx$$

(b) The curves intersect at $(-1, 1)$ and $(2, 4)$.

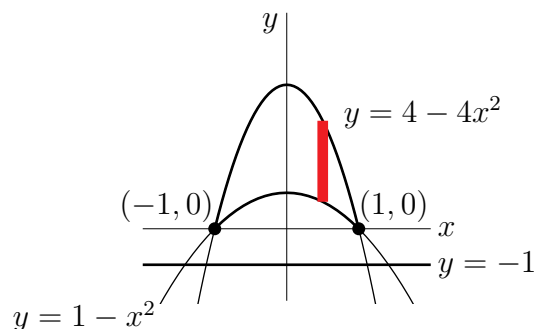


We'll use horizontal washers as in Example 1.10.5 of the text.

- We use thin horizontal strips of width dy as in the figure above.
- When we rotate about the line $x = 3$, each strip sweeps out a thin washer
 - whose inner radius is $r_{in} = 3 - \sqrt{y}$, and
 - whose outer radius is $r_{out} = 3 - (y - 2) = 5 - y$ when $y \geq 1$ (see the red strip in the figure on the right above), and whose outer radius is $r_{out} = 3 - (-\sqrt{y}) = 3 + \sqrt{y}$ when $y \leq 1$ (see the blue strip in the figure on the right above) and
 - whose thickness is dy and hence
 - whose volume is $\pi(r_{out}^2 - r_{in}^2)dy = \pi[(5 - y)^2 - (3 - \sqrt{y})^2]dy$ when $y \geq 1$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)dy = \pi[(3 + \sqrt{y})^2 - (3 - \sqrt{y})^2]dy$ when $y \leq 1$ and
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 4$, the total volume is

$$\int_0^1 \pi[(3 + \sqrt{y})^2 - (3 - \sqrt{y})^2]dy + \int_1^4 \pi[(5 - y)^2 - (3 - \sqrt{y})^2]dy$$

S-5: (a) The curves intersect at $(1, 0)$ and $(-1, 0)$. When the strip shown in the figure



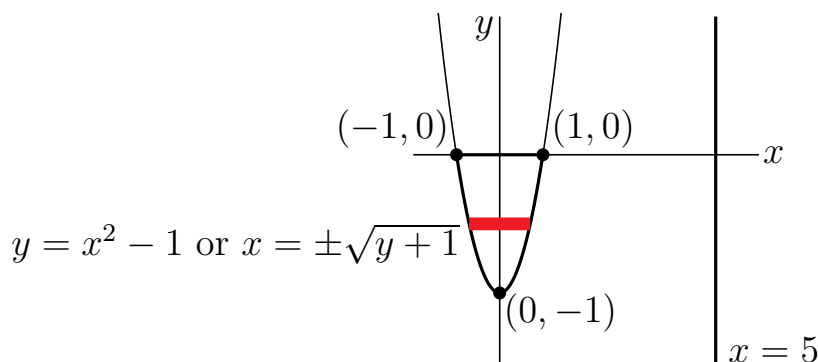
is rotated about the line $y = -1$, it forms a thin washer with:

- inner radius $(1 - x^2) - (-1) = 2 - x^2$,
- outer radius $(4 - 4x^2) - (-1) = 5 - 4x^2$ and
- thickness dx ; so, it has
- cross sectional area $\pi[(5 - 4x^2)^2 - (2 - x^2)^2]$ and
- volume $\pi[(5 - 4x^2)^2 - (2 - x^2)^2] dx$.

So the volume of the solid is

$$\int_{-1}^1 \pi[(5 - 4x^2)^2 - (2 - x^2)^2] dx$$

(b) The curve $y = x^2 - 1$ intersects $y = 0$ at $(1, 0)$ and $(-1, 0)$.



We'll use horizontal washers.

- We use thin horizontal strips of height dy as in the figure above.
- When we rotate about the line $x = 5$, each strip sweeps out a thin washer
 - whose inner radius is $r_{in} = 5 - \sqrt{y + 1}$, and
 - whose outer radius is $r_{out} = 5 - (-\sqrt{y + 1}) = 5 + \sqrt{y + 1}$ and

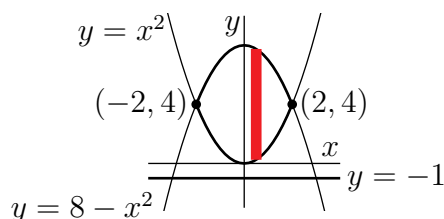
– whose thickness is dy and hence

– whose volume is $\pi(r_{out}^2 - r_{in}^2) dy = \pi[(5 + \sqrt{y+1})^2 - (5 - \sqrt{y+1})^2] dy$

- As our topmost strip is at $y = 0$ and our bottommost strip is at $y = -1$ (when $x = 0$), the total volume is

$$\int_{-1}^0 \pi [(5 + \sqrt{y+1})^2 - (5 - \sqrt{y+1})^2] dy$$

S-6: The curves intersect at $(-2, 4)$ and $(2, 4)$. When the strip shown in the figure



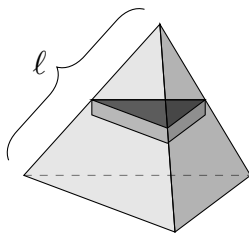
is rotated about the line $y = -1$, it forms a thin washer (punctured disc) of

- inner radius $x^2 + 1$,
- outer radius $9 - x^2$ and
- thickness dx and hence of
- cross sectional area $\pi[(9 - x^2)^2 - (x^2 + 1)^2]$ and
- volume $\pi[(9 - x^2)^2 - (x^2 + 1)^2] dx$.

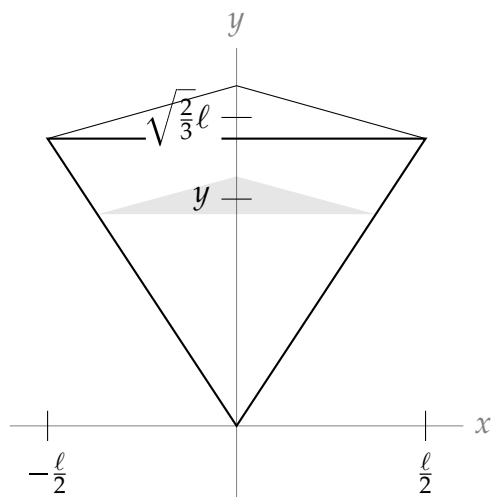
So the volume of the solid is

$$\pi \int_{-2}^2 [(9 - x^2)^2 - (x^2 + 1)^2] dx$$

S-7: We'll make horizontal slices, parallel to one of the faces of the tetrahedron. Then our slices will be equilateral triangles, of varying sizes.



For the sake of ease, as in Example 1.10.1 of the text, we picture the tetrahedron perched on a tip, one base horizontal on top.



Notice our slice forms the horizontal top of a smaller tetrahedron. The horizontal top of the full tetrahedron has side length ℓ , which is $\sqrt{\frac{3}{2}}$ times the height of the full tetrahedron. Our slice is the horizontal top of a tetrahedron of height y and so has side length $\sqrt{\frac{3}{2}}y$. An equilateral triangle with side length L has base L and height $\frac{\sqrt{3}}{2}L$, and hence area $\frac{\sqrt{3}}{4}L^2$. So, the area of our slice with side length $\sqrt{\frac{3}{2}}y$ is

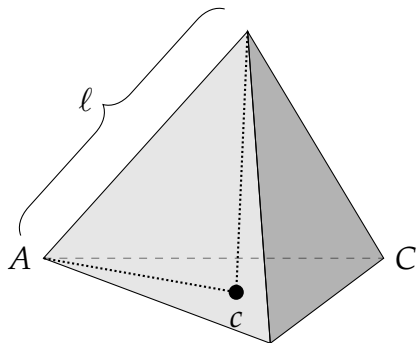
$$A = \frac{\sqrt{3}}{4} \left(\sqrt{\frac{3}{2}}y \right)^2 = \frac{3\sqrt{3}}{8}y^2$$

So, the volume of a tetrahedron with side length ℓ is:

$$\begin{aligned} \text{Volume} &= \int_0^{\sqrt{\frac{2}{3}}\ell} \frac{3\sqrt{3}}{8}y^2 \, dy \\ &= \frac{\sqrt{3}}{8} \cdot \left(\sqrt{\frac{2}{3}}\ell \right)^3 = \frac{\sqrt{2}}{12}\ell^3 \end{aligned}$$

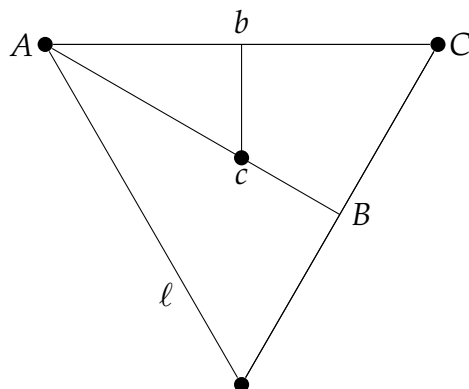
You were given the height of a tetrahedron, but for completeness we calculate it here.

Draw a line starting at one tip, and dropping straight down to the middle of the opposite face. It forms a right triangle with one edge of the tetrahedron, and a line from the middle of the face to the corner.



We know the length of the hypotenuse of this right triangle (it's ℓ), so if we know the length of its base (labeled Ac in the diagram), we can figure out its third side, the height of our tetrahedron. Note by using the Pythagorean theorem, we see that the height of an equilateral triangle with edge length ℓ is $\sqrt{\frac{3}{2}}\ell$.

Here is a sketch of the base of the pyramid:



The triangles ABC and Abc are similar (since b and B are right angles, and also A has the same angle in both). Therefore,

$$\begin{aligned}\frac{Ac}{Ab} &= \frac{AC}{AB} \\ \frac{Ac}{\ell/2} &= \frac{\ell}{\sqrt{3}\ell/2} \\ Ac &= \frac{1}{\sqrt{3}}\ell\end{aligned}$$

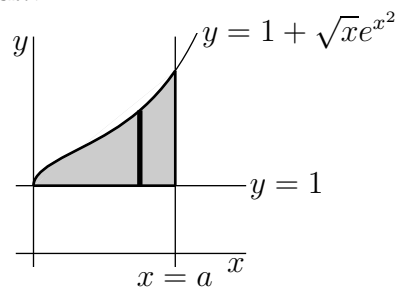
With this in our pocket, we can find the height of the tetrahedron: $\sqrt{\ell^2 - \left(\frac{1}{\sqrt{3}}\ell\right)^2} = \sqrt{\frac{2}{3}}\ell$.

S-8: Let $f(x) = 1 + \sqrt{x}e^{x^2}$. On the vertical slice a distance x from the y -axis, sketched in the figure below, y runs from 1 to $f(x)$. Upon rotation about the line $y = 1$, this thin slice sweeps out a thin disk of thickness dx and radius $f(x) - 1$ and hence of volume $\pi[f(x) - 1]^2 dx$. The full volume generated (for any fixed $a > 0$) is

$$\int_0^a \pi[f(x) - 1]^2 dx = \pi \int_0^a xe^{2x^2} dx.$$

Using the substitution $u = 2x^2$, so that $du = 4x dx$:

$$\text{Volume} = \pi \int_0^{2a^2} e^u \frac{du}{4} = \frac{\pi}{4} e^u \Big|_0^{2a^2} = \frac{\pi}{4} (e^{2a^2} - 1)$$



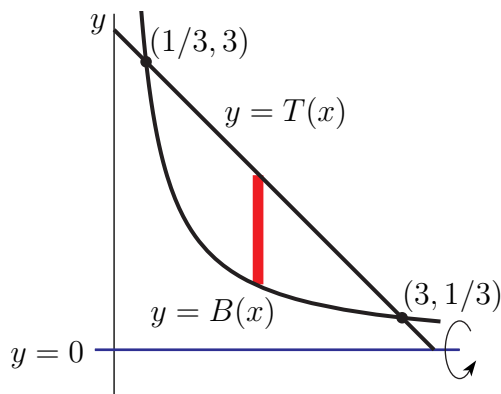
Remark: we spent a good deal of time last semester developing highly accurate but time-consuming methods for sketching common functions. For the purposes of

questions like this, we don't need a detailed picture of a function—broad outlines suffice. Notice that $\sqrt{x} > 0$ whenever $x > 0$, and $e^{x^2} > 0$ for all x . Therefore, $\sqrt{x}e^{x^2}$ is nonnegative over its entire domain, and so the graph $y = 1 + \sqrt{x}e^{x^2}$ is always the top function, above the bottom function $y = 1$. That is the only information we needed to perform our calculation.

S-9:

The curves $y = 1/x$ and $3x + 3y = 10$, i.e. $y = \frac{10}{3} - x$ intersect when

$$\begin{aligned}\frac{1}{x} &= \frac{10}{3} - x \iff 3 = 10x - 3x^2 \iff 3x^2 - 10x + 3 = 0 \\ &\iff (3x - 1)(x - 3) = 0 \\ &\iff x = 3, \frac{1}{3}\end{aligned}$$



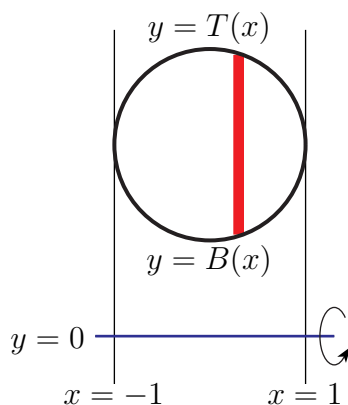
When the region is rotated about the x -axis, the vertical strip in the figure above sweeps out a washer with thickness dx , outer radius $T(x) = \frac{10}{3} - x$ and inner radius $B(x) = \frac{1}{x}$. This washer has volume

$$\pi(T(x)^2 - B(x)^2) dx = \pi\left(\frac{100}{9} - \frac{20}{3}x + x^2 - \frac{1}{x^2}\right) dx$$

Hence the volume of the solid is

$$\begin{aligned}\pi \int_{1/3}^3 \left(\frac{100}{9} - \frac{20}{3}x + x^2 - \frac{1}{x^2}\right) dx &= \pi \left[\frac{100x}{9} - \frac{10}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{x}\right]_{1/3}^3 \\ &= \pi \left[\frac{38}{3} - \frac{514}{3^4}\right] = \pi \frac{512}{81}\end{aligned}$$

S-10: (a) The top and the bottom of the circle have equations $y = T(x) = 2 + \sqrt{1 - x^2}$ and $y = B(x) = 2 - \sqrt{1 - x^2}$, respectively.



When R is rotated about the x -axis, the vertical strip of R in the figure above sweeps out a washer with thickness dx , outer radius $T(x)$ and inner radius $B(x)$. This washer has volume

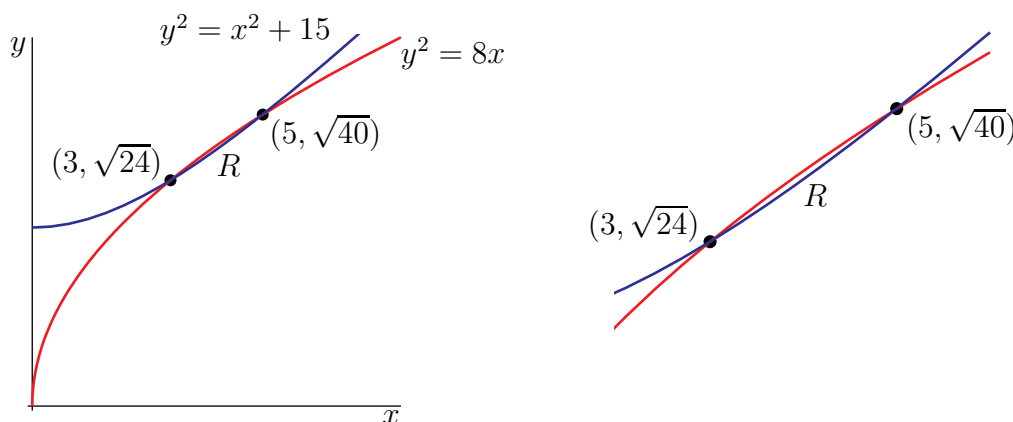
$$\pi(T(x)^2 - B(x)^2) dx = \pi(T(x) + B(x))(T(x) - B(x)) dx = \pi \times 4 \times 2\sqrt{1 - x^2} dx$$

Hence the volume of the solid is

$$8\pi \int_{-1}^1 \sqrt{1 - x^2} dx$$

(b) Since $y = \sqrt{1 - x^2}$ is equivalent to $x^2 + y^2 = 1$, $y \geq 0$, the integral is 8π times the area of the upper half of the circle $x^2 + y^2 = 1$ and hence is $8\pi \times \frac{1}{2}\pi 1^2 = 4\pi^2$.

S-11: (a) The two curves intersect when x obeys $8x = x^2 + 15$ or $x^2 - 8x + 15 = (x - 5)(x - 3) = 0$. The points of intersection, in the first quadrant, are $(3, \sqrt{24})$ and $(5, \sqrt{40})$. The region R is the region between the blue and red curves, with $3 \leq x \leq 5$, in the figures below.

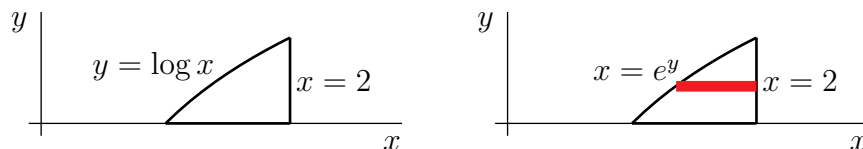


(b) The part of the solid with x coordinate between x and $x + dx$ is a “washer” shaped region with inner radius $\sqrt{x^2 + 15}$, outer radius $\sqrt{8x}$ and thickness dx . The surface area of the washer is $\pi(\sqrt{8x})^2 - \pi(\sqrt{x^2 + 15})^2 = \pi(8x - x^2 - 15)$ and its volume is

$\pi(8x - x^2 - 15) dx$. The total volume is

$$\begin{aligned} \int_3^5 \pi(8x - x^2 - 15) dx &= \pi \left[4x^2 - \frac{1}{3}x^3 - 15x \right]_3^5 = \pi \left[100 - \frac{125}{3} - 75 - 36 + 9 + 45 \right] \\ &= \frac{4}{3}\pi \approx 4.19 \end{aligned}$$

S-12: (a) The region R is sketched in the figure on the left below. (The bound $y = 0$ renders the bound $x = 1$ unnecessary, since the graph $y = \log x$ hits the x -axis when $x = 1$.)



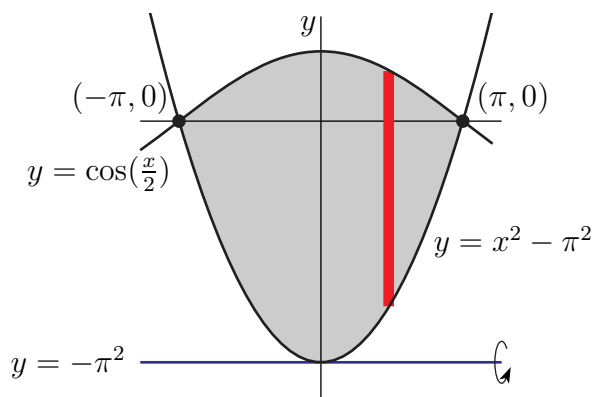
(b) We'll use horizontal washers as in Example 1.10.5 of the text.

- We cut R into thin horizontal strips of height dy as in the figure on the right above.
- When we rotate R about the y -axis, i.e. about the line $x = 0$, each strip sweeps out a thin washer
 - whose inner radius is $r_{in} = e^y$ and outer radius is $r_{out} = 2$, and
 - whose thickness is dy and hence
 - whose volume $\pi(r_{out}^2 - r_{in}^2)dy = \pi(4 - e^{2y})dy$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \log 2$ (since at the top $x = 2$ and $x = e^y$), the total

$$\begin{aligned} \text{Volume} &= \int_0^{\log 2} \pi(4 - e^{2y}) dy = \pi \left[4y - e^{2y}/2 \right]_0^{\log 2} = \pi \left[4 \log 2 - 2 + \frac{1}{2} \right] \\ &= \pi \left[4 \log 2 - \frac{3}{2} \right] \end{aligned}$$

Using a calculator, we see this is approximately 3.998.

S-13: Here is a sketch of the curves $y = \cos(\frac{x}{2})$ and $y = x^2 - \pi^2$.



By inspection, the curves meet at $x = \pm\pi$ where both $\cos(\frac{x}{2})$ and $x^2 - \pi^2$ take the value zero. We'll use vertical washers as specified in the question.

- We cut the specified region into thin vertical strips of width dx as in the figure above.
- When we rotate about the line $y = -\pi^2$, each strip sweeps out a thin washer
 - whose inner radius is $r_{in} = (x^2 - \pi^2) - (-\pi^2) = x^2$ and outer radius is $r_{out} = \cos(\frac{x}{2}) - (-\pi^2) = \cos(\frac{x}{2}) + \pi^2$, and
 - whose thickness is dx and hence
 - whose volume $\pi(r_{out}^2 - r_{in}^2)dx = \pi((\cos(\frac{x}{2}) + \pi^2)^2 - (x^2)^2)dx$.
- As our leftmost strip is at $x = -\pi$ and our rightmost strip is at $x = \pi$,

the total volume is

$$\begin{aligned} \pi \int_{-\pi}^{\pi} \left(\cos^2\left(\frac{x}{2}\right) + 2\pi^2 \cos\left(\frac{x}{2}\right) + \pi^4 - x^4 \right) dx \\ = \pi \int_{-\pi}^{\pi} \left(\frac{1 + \cos(x)}{2} + 2\pi^2 \cos\left(\frac{x}{2}\right) + \pi^4 - x^4 \right) dx \end{aligned}$$

Because the integrand is even,

$$\begin{aligned} &= 2\pi \int_0^{\pi} \left(\frac{1 + \cos(x)}{2} + 2\pi^2 \cos\left(\frac{x}{2}\right) + \pi^4 - x^4 \right) dx \\ &= 2\pi \left[\frac{1}{2}x + \frac{1}{2} \sin(x) + 4\pi^2 \sin\left(\frac{x}{2}\right) + \pi^4 x - \frac{1}{5}x^5 \right]_0^{\pi} \\ &= 2\pi \left[\frac{\pi}{2} + 0 + 4\pi^2 + \pi^5 - \frac{\pi^5}{5} \right] \\ &= \pi^2 + 8\pi^3 + \frac{8\pi^6}{5} \end{aligned}$$

We used the fact that the integrand is an even function and the interval of integration $[-\pi, \pi]$ is symmetric, but one can also compute directly.

S-14: As in Example 1.10.6 of the text notes, we slice V into thin horizontal “square pancakes”.

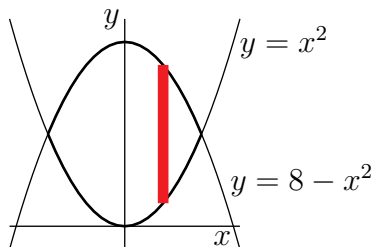
- We are told that the pancake at height x is a square of side $\frac{2}{1+x}$ and so
- has cross-sectional area $\left(\frac{2}{1+x}\right)^2$ and thickness dx and hence
- has volume $\left(\frac{2}{1+x}\right)^2 dx$.

Hence the volume of V is

$$\int_0^2 \left[\frac{2}{1+x} \right]^2 dx = \int_1^3 \frac{4}{u^2} du = 4 \frac{u^{-1}}{-1} \Big|_1^3 = -4 \left[\frac{1}{3} - 1 \right] = \frac{8}{3}$$

We made the change of variables $u = 1 + x$, $du = dx$.

S-15: Here is a sketch of the base region.

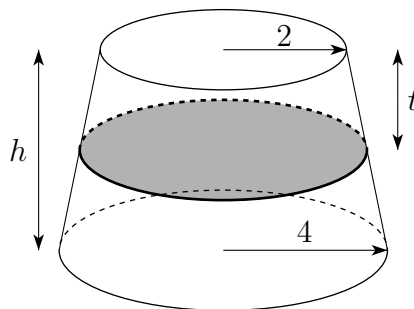


Consider the thin vertical cross-section resting on the heavy red line in the figure above. It has thickness dx . Its face is a square whose side runs from $y = x^2$ to $y = 8 - x^2$, a distance of $8 - 2x^2$. So the face has area $(8 - 2x^2)^2$ and the slice has volume $(8 - 2x^2)^2 dx$. The two curves cross when $x^2 = 8 - x^2$, i.e. when $x^2 = 4$ or $x = \pm 2$. So x runs from -2 to 2 and the total volume is

$$\begin{aligned} \int_{-2}^2 (8 - 2x^2)^2 dx &= 2 \int_0^2 4(4 - x^2)^2 dx = 8 \int_0^2 [16 - 8x^2 + x^4] dx \\ &= 8 \left[16 \times 2 - \frac{8}{3} 2^3 + \frac{1}{5} 2^5 \right] = \frac{256 \times 8}{15} = 136.5\bar{3} \end{aligned}$$

In the first simplification step, we used the fact that our integrand was even, but we also could have finished our computation without this step.

S-16: Slice the frustum into horizontal discs. When the disc is a distance t from the top of the frustum it has radius $2 + 2t/h$. Note that as t runs from 0 (the top of the frustum) to $t = h$ (the bottom of the frustum) the radius $2 + 2t/h$ increases linearly from 2 to 4 .



Thus the disk has volume $\pi(2 + 2t/h)^2 dt$. The total volume of the frustum is

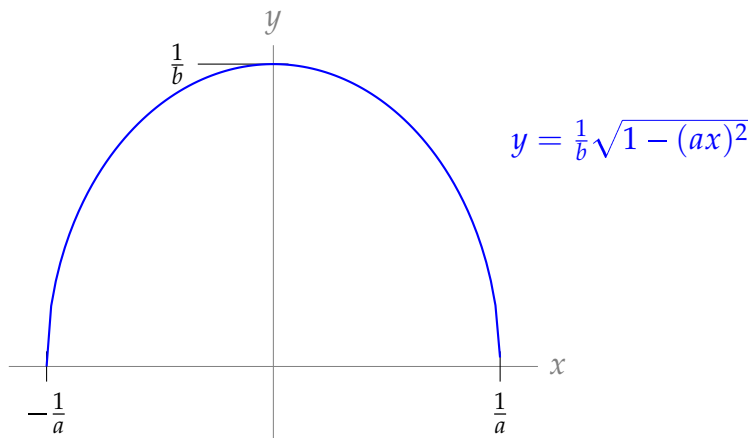
$$\pi \int_0^h (2 + 2t/h)^2 dt = 4\pi \int_0^h (1 + t/h)^2 dt = 4\pi \left[\frac{(1 + t/h)^3}{3/h} \right]_0^h = \frac{4}{3}\pi h \times 7 = \frac{28}{3}\pi h$$

Remark: we could also solve this problem using the formula for the volume of a cone. Using similar triangles, the frustum in question is shaped like a right circular cone of height $2h$ and base radius 4 (and hence of volume $\frac{1}{3}\pi(4^2)(2h)$), but missing its top, which is a right circular cone of height h and base radius 2 (and hence volume $\frac{1}{3}\pi(2^2)h$).

So, the volume of the frustum is $\frac{1}{3}\pi(4^2)(2h) - \frac{1}{3}\pi(2^2)h = \frac{28}{3}\pi h$.

S-17: (a)

We'll want to start by graphing the upper half of the ellipse $(ax)^2 + (by)^2 = 1$. Its intercepts will be enough to get us an idea: $(0, \frac{1}{b})$ and $(\pm\frac{1}{a}, 0)$:



We note a few things at the outset: first, since $a \geq b$, then $\frac{1}{a} \leq \frac{1}{b}$, so indeed the x -axis is the minor axis. That is, we're rotating about the proper axis to create an oblate spheroid.

Second, if we solve our equation for y , we get $y = \frac{1}{b}\sqrt{1 - (ax)^2}$. (Since we only want the upper half of the ellipse, we only need to consider the positive square root.)

Now, we have a standard volume-of-revolution problem. We make vertical slices, of width dx and height $y = \frac{1}{b}\sqrt{1 - (ax)^2}$. When we rotate these slices about the x -axis, they form thin disks of volume $\pi \left[\frac{1}{b}\sqrt{1 - (ax)^2} \right]^2 dx$. Since x runs from $-\frac{1}{a}$ to $\frac{1}{a}$, the volume of our oblate spheroid is:

$$\begin{aligned}
 \text{Volume} &= \int_{-\frac{1}{a}}^{\frac{1}{a}} \pi \left[\frac{1}{b}\sqrt{1 - (ax)^2} \right]^2 dx \\
 &= \frac{\pi}{b^2} \int_{-\frac{1}{a}}^{\frac{1}{a}} 1 - (ax)^2 dx \\
 &= \frac{2\pi}{b^2} \int_0^{\frac{1}{a}} 1 - (ax)^2 dx && \text{(even function)} \\
 &= \frac{2\pi}{b^2} \left[x - \frac{a^2 x^3}{3} \right]_0^{\frac{1}{a}} \\
 &= \frac{2\pi}{b^2} \left[\frac{1}{a} - \frac{1}{3a} \right] = \frac{4\pi}{3b^2 a}
 \end{aligned}$$

(b) As we saw in the sketch from part (a), the shortest radius of the ellipse is $\frac{1}{a}$, while the largest is $\frac{1}{b}$. So, $\frac{1}{a} = 6356.752$, and $\frac{1}{b} = 6378.137$. That is, $a = \frac{1}{6356.752}$ and $b = \frac{1}{6378.137}$.

Note $a \geq b$, as specified in part (a).

(c) Combining our answers from (a) and (b), the volume of an oblate spheroid with approximately the same dimensions as the earth is:

$$\begin{aligned}\frac{4\pi}{3b^2a} &= \frac{4\pi}{3} \left(\frac{1}{b}\right)^2 \left(\frac{1}{a}\right) \\ &= \frac{4\pi}{3} (6378.137)^2 (6356.752) \\ &\approx 1.08321 \times 10^{12} \text{ km}^3 \\ &\approx 1.08321 \times 10^{21} \text{ m}^3\end{aligned}$$

(d) A sphere of radius 6378.137 has volume

$$\frac{4}{3}\pi (6378.137)^3$$

So, our absolute error is:

$$\begin{aligned}&\left| \frac{4\pi}{3} (6378.137)^2 (6356.752) - \frac{4}{3}\pi (6378.137)^3 \right| \\ &= \frac{4\pi}{3} (6378.137)^2 |6356.752 - 6378.137| \\ &= \frac{4\pi}{3} (6378.137)^2 (21.385) \\ &\approx 3.64 \times 10^9 \text{ km}^3\end{aligned}$$

And our relative error is:

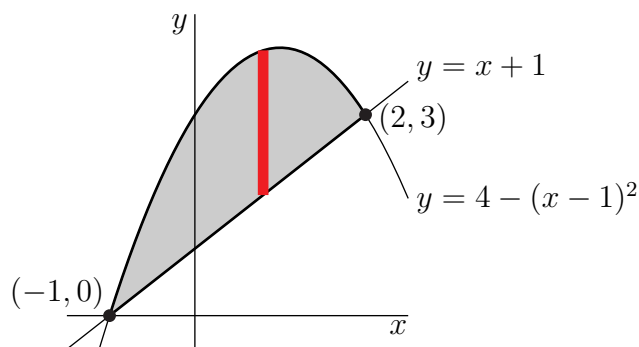
$$\begin{aligned}\frac{\text{abs error}}{\text{actual value}} &= \frac{\frac{4\pi}{3} (6378.137)^2 |6356.752 - 6378.137|}{\frac{4\pi}{3} (6378.137)^2 (6356.752)} \\ &= \frac{|6356.752 - 6378.137|}{6356.752} \\ &= \frac{6378.137}{6356.752} - 1 \\ &\approx 0.00336\end{aligned}$$

That is, about 0.336%, or about one-third of one percent.

S-18: (a) The curve $y = 4 - (x - 1)^2$ is an “upside down parabola” and line $y = x + 1$ has slope 1. They intersect at points (x, y) which satisfy both $y = x + 1$ and $y = 4 - (x - 1)^2$. That is, when x obeys

$$\begin{aligned}x + 1 &= 4 - (x - 1)^2 \\ x + 1 &= 4 - x^2 + 2x - 1 \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x &= -1 \quad \text{or} \quad x = 2\end{aligned}$$

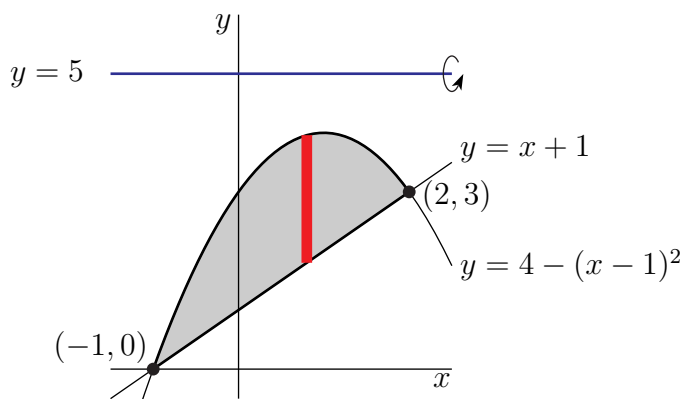
Thus the intersection points are $(-1, 0)$ and $(2, 3)$. Here is a sketch of R :



The red strip in the sketch above runs from $y = x + 1$ to $y = 4 - (x - 1)^2$ and so has area $[4 - (x - 1)^2 - (x + 1)] dx = [2 + x - x^2] dx$. All together R has

$$\begin{aligned} \text{Area} &= \int_{-1}^2 [2 + x - x^2] dx \\ &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 \\ &= 6 + \frac{3}{2} - \frac{9}{3} = \frac{9}{2} \end{aligned}$$

(b) We'll use vertical washers as in Example 1.10.3 of the text. Note that the highest point achieved by $y = 4 - (x - 1)^2$ is $y = 4$, so rotating around the line $y = 5$ causes no unexpected problems.

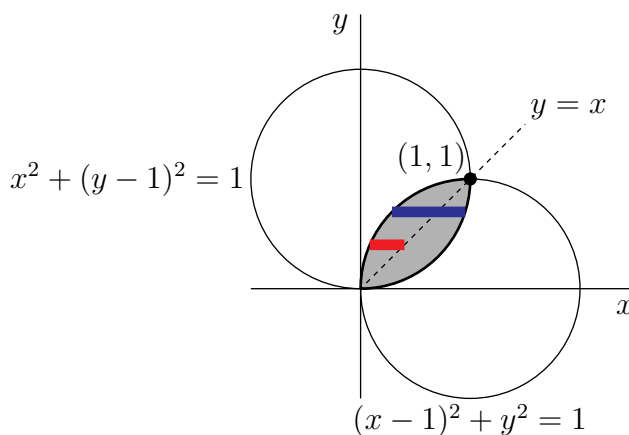


- We cut R into thin vertical strips of width dx like the red strip in the figure above.
- When we rotate R about the horizontal line $y = 5$, each strip sweeps out a thin washer
 - whose inner radius is $r_{in} = 5 - [4 - (x - 1)^2] = 1 + (x - 1)^2$, and
 - whose outer radius is $r_{out} = 5 - [x + 1] = 4 - x$ and
 - whose thickness is dx and hence
 - whose volume is $\pi [r_{out}^2 - r_{in}^2] dx = \pi [(4 - x)^2 - (1 + (x - 1)^2)^2] dx$

- As our leftmost strip is at $x = -1$ and our rightmost strip is at $x = 2$, the total

$$\text{Volume} = \pi \int_{-1}^2 [(4-x)^2 - (1+(x-1)^2)^2] dx$$

S-19: (a) The curves $(x-1)^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$ are circles of radius 1 centered on $(1,0)$ and $(0,1)$ respectively. Both circles pass through $(0,0)$ and $(1,1)$. They are sketched below.



The region $\mathcal{R}$ is symmetric about the line $y = x$, so the area of $\mathcal{R}$ is twice the area of the part of $\mathcal{R}$ to the left of the line $y = x$. The red strip in the sketch above runs from the edge of the lower circle to $x = y$. So, given a value of y in $[0, 1]$, we need to find the corresponding value of x along the circle. We solve $(x-1)^2 + y^2 = 1$ for x , keeping in mind that $0 \leq x \leq 1$:

$$\begin{aligned} (x-1)^2 + y^2 &= 1 \\ (x-1)^2 &= 1 - y^2 \\ |x-1| &= \sqrt{1-y^2} \\ 1-x &= \sqrt{1-y^2} \\ x &= 1 - \sqrt{1-y^2} \end{aligned}$$

Now, we calculate:

$$\begin{aligned} \text{Area} &= 2 \int_0^1 [y - (1 - \sqrt{1-y^2})] dy \\ &= 2 \left\{ \int_0^1 y - 1 dy + \int_0^1 \sqrt{1-y^2} dy \right\} \\ &= 2 \left\{ \left[\frac{y^2}{2} - y \right]_0^1 + \int_0^1 \sqrt{1-y^2} dy \right\} \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

Here the integral $\int_0^1 \sqrt{1-y^2} \, dy$ was evaluated simply as the area of one quarter of a circular disk of radius 1. It can also be evaluated by substituting $y = \sin \theta$, a technique we'll learn more about in Section 1.8 of the text.

(b) We'll use horizontal washers as in Example 1.10.5 of the in the text.

- We cut $\mathcal{R}$ into thin horizontal strips of width dy like the blue strip in the figure above.
- When we rotate $\mathcal{R}$ about the y -axis, each strip sweeps out a thin washer
 - whose inner radius is $r_{in} = 1 - \sqrt{1-y^2}$, and
 - whose outer radius is $r_{out} = \sqrt{1-(y-1)^2}$ and
 - whose thickness is dy and hence
 - whose volume is

$$\begin{aligned} & \pi \left[\left(\sqrt{1-(y-1)^2} \right)^2 - \left(1 - \sqrt{1-y^2} \right)^2 \right] dy \\ &= \pi \left[1 - (y-1)^2 - 1 + 2\sqrt{1-y^2} - (1-y^2) \right] \\ &= 2\pi \left[\sqrt{1-y^2} + y - 1 \right] dy \end{aligned}$$

- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 1$, the total

$$\begin{aligned} \text{Volume} &= 2\pi \int_0^1 \left[\sqrt{1-y^2} + y - 1 \right] dy = 2\pi \left[\frac{\pi}{4} + \frac{1}{2} - 1 \right] \\ &= \frac{\pi^2}{2} - \pi \approx 1.793 \end{aligned}$$

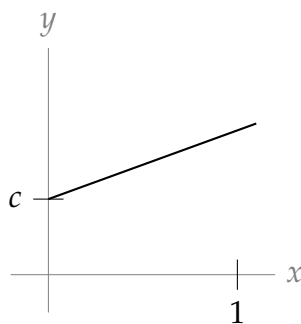
Here, we again used that $\int_0^1 \sqrt{1-y^2} \, dy$ is the area of a quarter circle of radius one, and we used a calculator to approximate the final answer.

S-20: Before we start, it will be useful to have a reasonable sketch of the graph $y = c\sqrt{1+x^2}$ over the interval $[0, 1]$. Its endpoints are $(0, c)$ and $(1, c\sqrt{2})$. The function is entirely above the x -axis, which we need to know for part (a). For part (b), we need to know whether it is always increasing or not: when we're drawing horizontal strips, we need to know their endpoints, and if the function has "humps," the right endpoint will not be simply the line $x = 1$.

If you're comfortable noticing that $1+x^2$ increases as x increases because we only consider nonnegative values of x , then you can also be confident that $\sqrt{1+x^2}$ is simply increasing. Alternately, we can consider the derivative:

$$\frac{d}{dx} \left\{ c\sqrt{1+x^2} \right\} = c \cdot \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{cx}{\sqrt{1+x^2}}$$

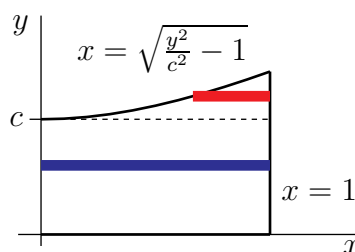
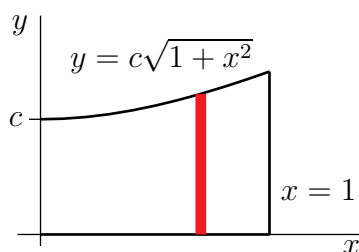
Since we only consider positive values of x , this derivative is never negative, so the function is never decreasing. This gives us the following basic sketch:



The figures in the solution below use a slightly more detailed rendering of our function, but so much accuracy is not necessary.

(a) Let $\mathcal{V}_1$ be the solid obtained by revolving $\mathcal{R}$ about the x -axis. The portion of $\mathcal{V}_1$ with x -coordinate between x and $x + dx$ is obtained by rotating the red vertical strip in the figure on the left below about the x -axis. That portion is a disk of radius $c\sqrt{1+x^2}$ and thickness dx . The volume of this disk is $\pi(c\sqrt{1+x^2})^2 dx = \pi c^2(1+x^2) dx$. So the total volume of $\mathcal{V}_1$ is

$$V_1 = \int_0^1 \pi c^2(1+x^2) dx = \pi c^2 \left[x + \frac{x^3}{3} \right]_0^1 = \frac{4}{3} \pi c^2$$



(b) We'll use horizontal washers as in Example 1.10.5 of the in the text.

- We cut $\mathcal{R}$ into thin horizontal strips of width dy as in the figure on the right above.
- When we rotate $\mathcal{R}$ about the y -axis, i.e. about the line $x = 0$, each strip sweeps out a thin washer
 - whose outer radius is $r_{out} = 1$, and
 - whose inner radius is $r_{in} = \sqrt{\frac{y^2}{c^2} - 1}$ when $y \geq c\sqrt{1+0^2} = c$ (see the red strip in the figure on the right above), and whose inner radius is $r_{in} = 0$ when $y \leq c$ (see the blue strip in the figure on the right above) and
 - whose thickness is dy and hence
 - whose volume is $\pi(r_{out}^2 - r_{in}^2)dy = \pi(2 - \frac{y^2}{c^2})dy$ when $y \geq c$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)dy = \pi dy$ when $y \leq c$ and
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \sqrt{2}c$ (since at

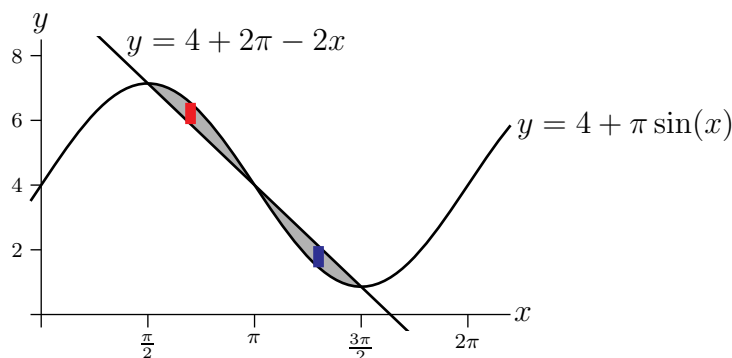
the top $x = 1$ and $y = c\sqrt{1+x^2}$, the total

$$\begin{aligned} V_2 &= \int_c^{\sqrt{2}c} \pi \left(2 - \frac{y^2}{c^2} \right) dy + \int_0^c \pi dy \\ &= \pi \left[2y - \frac{y^3}{3c^2} \right]_c^{\sqrt{2}c} + \pi c \\ &= \pi c \left[\frac{4\sqrt{2}}{3} - \frac{5}{3} \right] + \pi c \\ &= \frac{\pi c}{3} [4\sqrt{2} - 2] \end{aligned}$$

(c) We have $V_1 = V_2$ if and only if

$$\begin{aligned} \frac{4}{3}\pi c^2 &= \frac{\pi c}{3} [4\sqrt{2} - 2] \\ 4c^2 &= c (4\sqrt{2} - 2) \\ 4c^2 - c (4\sqrt{2} - 2) &= 0 \\ 4c \left(c - \left(\sqrt{2} - \frac{1}{2} \right) \right) &= 0 \\ c = 0 \quad \text{or} \quad c &= \sqrt{2} - \frac{1}{2} \end{aligned}$$

S-21: We will compute the volume by rotating thin vertical strips as in the sketch



about the line $y = -1$ to generate thin washers. We need to know when the line $y = 4 + 2\pi - 2x$ intersects the curve $y = 4 + \pi \sin x$. Looking at the graph, it appears to be at $\frac{\pi}{2}$, π , and $\frac{3\pi}{2}$. By plugging in these values of x to both functions, we see they are indeed the points of intersection.

- When $\frac{\pi}{2} \leq x \leq \pi$, the top of the strip is at $y = 4 + \pi \sin x$ and the bottom of the strip is at $y = 4 + 2\pi - 2x$. When the strip is rotated, we get a thin washer with outer radius $R_1(x) = 1 + 4 + \pi \sin x = 5 + \pi \sin x$ and inner radius $r_1(x) = 1 + 4 + 2\pi - 2x = 5 + 2\pi - 2x$.
- When $\pi \leq x \leq \frac{3\pi}{2}$, the top of the strip is at $y = 4 + 2\pi - 2x$ and the bottom of the strip is at $y = 4 + \pi \sin x$. When the strip is rotated, we get a thin washer with outer

radius $R_2(x) = 1 + 4 + 2\pi - 2x = 5 + 2\pi - 2x$ and inner radius $r_2(x) = 1 + 4 + \pi \sin x = 5 + \pi \sin x$.

So, the total

$$\begin{aligned} \text{Volume} &= \int_{\pi/2}^{\pi} \pi [R_1(x)^2 - r_1(x)^2] dx + \int_{\pi}^{3\pi/2} \pi [R_2(x)^2 - r_2(x)^2] dx \\ &= \int_{\pi/2}^{\pi} \pi [(5 + \pi \sin x)^2 - (5 + 2\pi - 2x)^2] dx \\ &\quad + \int_{\pi}^{3\pi/2} \pi [(5 + 2\pi - 2x)^2 - (5 + \pi \sin x)^2] dx \end{aligned}$$

S-22: (a)

We use the same ideas for volume, and apply them to mass. We want to take slices of the column, approximate their mass, then add them up. To reconcile our units, let $k = 1000h$, so k is the height in metres. Then the density of air at height k is $c2^{-k/6000} \frac{\text{kg}}{\text{m}^3}$.

A horizontal slice of the column is a circular disk with height dk and radius 1 m. So, its volume is $\pi dk \text{ m}^3$. What we're interested in, though, is its mass. At height k , its mass is

$$\begin{aligned} (\text{volume}) \times (\text{density}) &= \left(\pi dk \text{ m}^3 \right) \times \left(c2^{-k/6000} \frac{\text{kg}}{\text{m}^3} \right) \\ &= c\pi 2^{-k/6000} dk \text{ kg} \end{aligned}$$

Since k runs from 0 to 60,000, the total mass is given by

$$\int_0^{60000} c\pi 2^{-k/6000} dk = c\pi \int_0^{60000} 2^{-k/6000} dk$$

To facilitate integration, we can write our exponential function in terms of e , then use the substitution $u = -\frac{k}{6000} \log 2$, $du = -\frac{1}{6000} \log 2 dk$.

$$\begin{aligned} &= c\pi \int_0^{60000} \left(e^{\log 2} \right)^{-k/6000} dk \\ &= c\pi \int_0^{60000} e^{-\frac{k}{6000} \log 2} dk \\ &= -\frac{6000c\pi}{\log 2} \int_0^{-10 \log 2} e^u du \\ &= \frac{6000c\pi}{\log 2} \int_{-10 \log 2}^0 e^u du \\ &= \frac{6000c\pi}{\log 2} \left(1 - \frac{1}{2^{10}} \right) \end{aligned}$$

We note this is fairly close to $\frac{6000c\pi}{\log 2}$.

We also remark that this is a demonstration of the usefulness of integrals. We wanted to know how much of *something* there was, but the amount of that something was different everywhere: more in some places, less in others. Integration allowed us to account for this gradient. You've seen this behaviour exploited to find distances travelled, areas, volumes, and now mass. In your studies, you will doubtless learn to use it to find still more quantities.

(b) We want to find the value of k that gives a mass of $\frac{3000c\pi}{\log 2}$. By following our reasoning above, the mass of air in the column from the ground to height k is

$$\frac{6000c\pi}{\log 2} \left(1 - \frac{1}{2^{k/6000}}\right)$$

So, we set this equal to the mass we want, and solve for k .

$$\begin{aligned} \frac{6000c\pi}{\log 2} \left(1 - \frac{1}{2^{k/6000}}\right) &= \frac{3000c\pi}{\log 2} \\ 2 \left(1 - \frac{1}{2^{k/6000}}\right) &= 1 \\ 1 &= \frac{2}{2^{k/6000}} \\ 2^{k/6000} &= 2^1 \\ k &= 6000 \\ h &= 6 \end{aligned}$$

This means that there is roughly the same mass of air in the lowest 6 km of the column as there is in the remaining 54 km.

S-23: Note that $f(x)$ is an even function, nonnegative over its entire domain.

(a) To find the area of R , we evaluate

$$\text{Area} = \int_{-1/2}^{1/2} \frac{|x|}{\sqrt[4]{1-x^2}} dx = 2 \int_0^{1/2} \frac{x}{\sqrt[4]{1-x^2}} dx$$

We use the substitution $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned} &= - \int_1^{3/4} \frac{1}{u^{1/4}} du \\ &= - \left[\frac{4}{3} u^{3/4} \right]_1^{3/4} = -\frac{4}{3} \left(\left(\frac{3}{4} \right)^{3/4} - 1 \right) \\ &= \frac{4}{3} - \sqrt[4]{\frac{4}{3}} \end{aligned}$$

(b) We slice the solid of rotation into circular disks of width dx and radius $\frac{|x|}{\sqrt[4]{1-x^2}}$.

$$\begin{aligned}\text{Volume} &= \int_{-1/2}^{1/2} \pi \left(\frac{|x|}{\sqrt[4]{1-x^2}} \right)^2 dx \\ &= 2\pi \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx\end{aligned}$$

We use the substitution $x = \sin \theta$, $dx = \cos \theta d\theta$, so $\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \cos \theta$. Note $\sin 0 = 0$ and $\sin \frac{\pi}{6} = \frac{1}{2}$.

$$\begin{aligned}&= 2\pi \int_0^{\pi/6} \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \\ &= 2\pi \int_0^{\pi/6} \sin^2 \theta d\theta \\ &= \pi \int_0^{\pi/6} (1 - \cos(2\theta)) d\theta \\ &= \pi \left[\theta - \frac{1}{2} \sin(2\theta) \right]_0^{\pi/6} \\ &= \pi \left(\frac{\pi}{6} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right) \\ &= \frac{\pi^2}{6} - \frac{\sqrt{3}\pi}{4}\end{aligned}$$

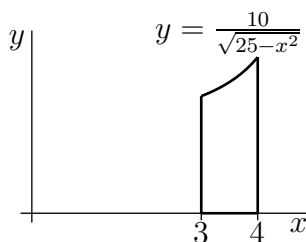
S-24: (a) Let's graph $y = \frac{10}{\sqrt{25-x^2}}$. We start with the endpoints: $(3, \frac{5}{2})$ and $(4, \frac{10}{3})$. Then we consider the first derivative:

$$\frac{d}{dx} \left\{ \frac{10}{\sqrt{25-x^2}} \right\} = \frac{10x}{\sqrt{25-x^2}^3}$$

Over the interval $[3, 4]$, this is always positive, so our function is increasing over the entire interval. The second derivative,

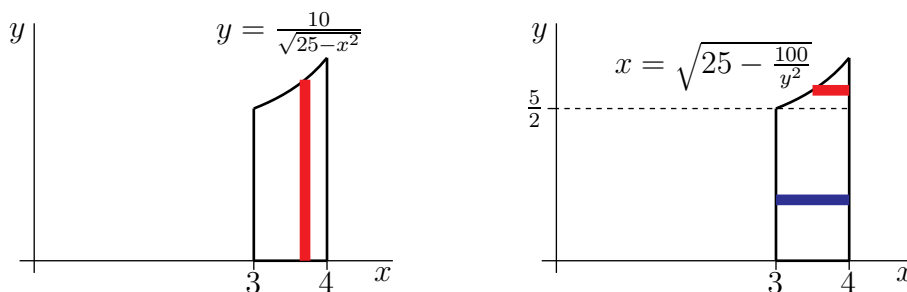
$$\frac{d^2}{dx^2} \left\{ \frac{10}{\sqrt{25-x^2}} \right\} = \frac{d}{dx} \left\{ \frac{10x}{\sqrt{25-x^2}^3} \right\} = \frac{10(2x^2+25)}{\sqrt{25-x^2}^5},$$

is always positive, so our function is concave up over the entire interval. So, the region R is:



(b) Let $\mathcal{V}_1$ be the solid obtained by revolving $\mathcal{R}$ about the x -axis. The portion of $\mathcal{V}_1$ with x -coordinate between x and $x + dx$ is obtained by rotating the red vertical strip in the figure on the left below about the x -axis. That portion is a disk of radius $\frac{10}{\sqrt{25-x^2}}$ and thickness dx . The volume of this disk is $\pi\left(\frac{10}{\sqrt{25-x^2}}\right)^2 dx$. So the total volume of $\mathcal{V}_1$ is

$$\begin{aligned}\int_3^4 \pi \left(\frac{10}{\sqrt{25-x^2}} \right)^2 dx &= 100\pi \int_3^4 \frac{1}{25-x^2} dx = 100\pi \int_3^4 \frac{1}{(5-x)(5+x)} dx \\ &= 10\pi \int_3^4 \left(\frac{1}{5-x} + \frac{1}{5+x} \right) dx = 10\pi \left[-\log(5-x) + \log(5+x) \right]_3^4 \\ &= 10\pi \left[-\log 1 + \log 9 + \log 2 - \log 8 \right] = 10\pi \log \frac{9}{4} = 20\pi \log \frac{3}{2}\end{aligned}$$



(c) We'll use horizontal washers as in Example 1.10.5 of the text.

- We cut $\mathcal{R}$ into thin horizontal strips of width dy as in the figure on the right above.
- When we rotate $\mathcal{R}$ about the y -axis, each strip sweeps out a thin washer
 - whose outer radius is $r_{out} = 4$, and
 - whose inner radius is $r_{in} = \sqrt{25 - \frac{100}{y^2}}$ when $y \geq \frac{10}{\sqrt{25-3^2}} = \frac{10}{4} = \frac{5}{2}$ (see the red strip in the figure on the right above), and whose inner radius is $r_{in} = 3$ when $y \leq \frac{5}{2}$ (see the blue strip in the figure on the right above) and
 - whose thickness is dy and hence
 - whose volume is $\pi(r_{out}^2 - r_{in}^2)dy = \pi\left(\frac{100}{y^2} - 9\right)dy$ when $y \geq \frac{5}{2}$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)dy = 7\pi dy$ when $y \leq \frac{5}{2}$ and
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \frac{10}{3}$ (since at the top $x = 4$ and $y = \frac{10}{\sqrt{25-4^2}} = \frac{10}{\sqrt{25-16}} = \frac{10}{3}$), the volume is

$$\begin{aligned}&\int_{5/2}^{10/3} \pi \left(\frac{100}{y^2} - 9 \right) dy + \int_0^{5/2} 7\pi dy \\ &= \pi \left[-\frac{100}{y} - 9y \right]_{5/2}^{10/3} + \frac{35}{2}\pi \\ &= \pi \left[-30 + 40 - 30 + \frac{45}{2} \right] + \frac{35}{2}\pi \\ &= 20\pi\end{aligned}$$

Solutions to Exercises 1.11 — Jump to [TABLE OF CONTENTS](#)

S-1: Integration by substitution is just using the chain rule, backwards:

$$\begin{aligned}\frac{d}{dx}\{f(g(x))\} &= f'(g(x))g'(x) \\ \Leftrightarrow \int \frac{d}{dx}\{f(g(x))\}dx &= \int f'(g(x))g'(x)dx \\ \Leftrightarrow \underbrace{f(g(x))}_{f(u)} + C &= \int \underbrace{f'(g(x))}_{f'(u)} \underbrace{g'(x)dx}_{du}\end{aligned}$$

Similarly, integration by parts comes from the product rule:

$$\begin{aligned}\frac{d}{dx}\{f(x)g(x)\} &= f'(x)g(x) + f(x)g'(x) \\ \Leftrightarrow \int \frac{d}{dx}\{f(x)g(x)\}dx &= \int f'(x)g(x) + f(x)g'(x)dx \\ \Leftrightarrow f(x)g(x) + C &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \\ \Leftrightarrow \int \underbrace{f(x)}_u \underbrace{g'(x)dx}_{dv} &= \underbrace{f(x)}_u \underbrace{g(x)}_v - \int \underbrace{g(x)}_v \underbrace{f'(x)dx}_{du}\end{aligned}$$

In the last line, the “+C” has been absorbed into the indefinite integral on the right hand side.

S-2: Remember our rule: $\int u dv = uv - \int v du$. So, we take u and use it to make du —that is, we differentiate it. We take dv and use it to make v —that is, we antidifferentiate it.

S-3: We’ll use the same ideas that led to the method of integration by parts. (You can review this in your text, or see the solution to Question 1 in this section.) According to the quotient rule,

$$\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Antidifferentiating both sides gives us:

$$\begin{aligned}\int \frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} dx &= \int \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} dx \\ \frac{f(x)}{g(x)} + C &= \int \frac{f'(x)}{g(x)} dx - \int \frac{f(x)g'(x)}{g^2(x)} dx \\ \int \frac{f'(x)}{g(x)} dx &= \frac{f(x)}{g(x)} + \int \frac{f(x)g'(x)}{g^2(x)} dx\end{aligned}$$

In the last line, the “+C” has been absorbed into the indefinite integral on the right hand side.

This is exactly the integration by parts formula for the functions $u = 1/g$ and $v = f$.

S-4: All the antiderivatives differ only by a constant, so we can write them all as $v(x) + C$ for some C . Then, using the formula for integration by parts,

$$\begin{aligned}\int u(x) \cdot v'(x) dx &= \underbrace{u(x)}_u \underbrace{[v(x) + C]}_v - \int \underbrace{[v(x) + C]}_v \underbrace{u'(x) dx}_{du} \\ &= u(x)v(x) + Cu(x) - \int v(x)u'(x) dx - \int Cu'(x) dx \\ &= u(x)v(x) + Cu(x) - \int v(x)u'(x) dx - Cu(x) + D \\ &= u(x)v(x) - \int v(x)u'(x) dx + D\end{aligned}$$

where D is any constant.

Since the terms with C cancel out, it didn't matter what we chose for C —all choices end up the same.

S-5: Suppose we choose $dv = f(x)dx$, $u = 1$. Then $v = \int f(x)dx$, and $du = dx$. So, our integral becomes:

$$\int \underbrace{(1)}_u \underbrace{f(x)dx}_{dv} = \underbrace{(1)}_u \underbrace{\int f(x)dx}_v - \int \underbrace{\left(\int f(x)dx\right)}_v \underbrace{dx}_{du}$$

In order to figure out the first product (and the second integrand), you need to know the antiderivative of $f(x)$ —but that's exactly what you're trying to figure out! So, using integration by parts has not eased your pain.

We note here that in certain cases, such as $\int \log x \, dx$ (Example 1.11.8 in the text), it is useful to choose $dv = 1dx$. This is similar to, but crucially different from, the do-nothing method in this problem.

S-6: For integration by parts, we want to break the integrand into two pieces, multiplied together. There is an obvious choice for how to do this: one piece is x , and the other is $\log x$. Remember that one piece will be integrated, while the other is differentiated. The question is, which choice will be more helpful. After some practice, you'll get the hang of making the choice. For now, we'll present both choices—but when you're writing a solution, you don't have to write this part down. It's enough to present your choice, and then a successful computation is justification enough.

Option 1:

$u = x$	$du = 1 \, dx$
$dv = \log x \, dx$	$v = ??$

Option 2:

$u = \log x$	$du = \frac{1}{x} \, dx$
$dv = x \, dx$	$v = \frac{1}{2}x^2$

In Example 1.11.8 of the text, we found the antiderivative of logarithm, but it wasn't trivial. We might reasonably want to avoid using this complicated antiderivative. Indeed, Option 2 (differentiating logarithm, antidifferentiating x) looks promising—when we multiply the blue equations, we get something easily integrable—so let's not even bother going deeper into Option 1.

That is, we perform integration by parts with $u = \log x$ and $dv = x \, dx$, so that $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$.

$$\begin{aligned} \int \underbrace{\log x}_u \underbrace{x \, dx}_{dv} &= \underbrace{\frac{x^2 \log x}{2}}_{uv} - \int \underbrace{\frac{x^2}{2}}_v \underbrace{\frac{dx}{x}}_{du} = \frac{x^2 \log x}{2} - \frac{1}{2} \int x \, dx \\ &= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C \end{aligned}$$

S-7: Our integrand is the product of two functions, and there's no clear substitution. So, we might reasonably want to try integration by parts. Again, we have two obvious pieces: $\log x$, and x^{-7} . We'll consider our options for assigning these to u and dv :

Option 1:

$u = \log x$	$du = \frac{1}{x} \, dx$
$dv = x^{-7} \, dx$	$v = \frac{1}{-6}x^{-6}$

Option 2:

$u = x^{-7}$	$du = -7x^{-8} \, dx$
$dv = \log x \, dx$	$v = ??$

Again, we remember that logarithm has some antiderivative we found in Example 1.11.8 of the text, but it was something complicated. Luckily, we don't need to bother with it: when we multiply the red equations in Option 1, we get a perfectly workable integral.

We perform integration by parts with $u = \log x$ and $dv = x^{-7} dx$, so that $du = \frac{dx}{x}$ and $v = -\frac{x^{-6}}{6}$.

$$\begin{aligned}\int \frac{\log x}{x^7} dx &= \underbrace{-\log x \frac{x^{-6}}{6}}_{uv} + \int \underbrace{\frac{x^{-6}}{6}}_{-v} \underbrace{\frac{dx}{x}}_{du} = -\frac{\log x}{6x^6} + \frac{1}{6} \int x^{-7} dx \\ &= -\frac{\log x}{6x^6} - \frac{1}{36x^6} + C\end{aligned}$$

S-8: To integrate by parts, we need to decide what to use as u , and what to use as dv . The salient parts of this integrand are x and $\sin x$, so we only need to decide which is u and which dv . Again, this process will soon become familiar, but to help you along we show you both options below.

Option 1:	$u = x$	$du = 1 dx$
	$dv = \sin x dx$	$v = -\cos x$
Option 2:	$u = \sin x$	$du = \cos x dx$
	$dv = x dx$	$v = \frac{1}{2}x^2$

The derivative and antiderivative of sine are almost the same, but x turns into something simpler when we differentiate it. So, we choose Option 1.

We integrate by parts, using $u = x$, $dv = \sin x dx$ so that $v = -\cos x$ and $du = dx$:

$$\int_0^\pi x \sin x dx = \underbrace{-x \cos x}_{uv} \Big|_0^\pi - \int_0^\pi \underbrace{(-\cos x)}_v \underbrace{dx}_{du} = \left[-x \cos x + \sin x \right]_0^\pi = -\pi(-1) = \pi$$

S-9: When we have two functions multiplied like this, and there's no obvious substitution, our minds turn to integration by parts. We hope that our integral will be improved by differentiating one part and antidifferentiating the other. Let's see what our choices are:

Option 1:	$u = x$	$du = 1 \, dx$
	$dv = \cos x \, dx$	$v = \sin x$
Option 2:	$u = \cos x$	$du = -\sin x \, dx$
	$dv = x \, dx$	$v = \frac{1}{2}x^2$

Option 1 seems preferable. We integrate by parts, using $u = x$, $dv = \cos x \, dx$ so that $v = \sin x$ and $du = dx$:

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx = \underbrace{x}_u \underbrace{\sin x}_v \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \underbrace{\sin x}_v \underbrace{dx}_{du} = \left[x \sin x + \cos x \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

S-10: This integrand is the product of two functions, with no obvious substitution. So, let's try integration by parts, with one part e^x and one part x^3 .

Option 1:	$u = e^x$	$du = e^x \, dx$
	$dv = x^3 \, dx$	$v = \frac{1}{4}x^4$
Option 2:	$u = x^3$	$du = 3x^2 \, dx$
	$dv = e^x \, dx$	$v = e^x$

At first glance, multiplying the red functions and multiplying the blue functions give largely equivalent integrands to what we started with—none of them with obvious antiderivatives. In previous questions, we were able to choose $u = x$, and then $du = dx$, so the “ x ” in the integrand effectively went away. Here, we see that choosing $u = x^3$ will lead to $du = 3x^2 dx$, which has a lower power. If we repeatedly perform integration by parts, choosing u to be the power of x each time, then after a few iterations it should go away, because the third derivative of x^3 is a constant.

So, we start with Option 2: $u = x^3$, $dv = e^x dx$, $du = 3x^2 dx$, and $v = e^x$.

$$\begin{aligned} \int x^3 e^x \, dx &= \underbrace{x^3}_u \underbrace{e^x}_v - \int \underbrace{e^x}_v \cdot \underbrace{3x^2 dx}_{du} \\ &= x^3 e^x - 3 \int e^x \cdot x^2 \, dx \end{aligned}$$

Now, we take $u = x^2$ and $dv = e^x dx$, so $du = 2x dx$ and $v = e^x$. We're only using integration by parts on the actual integral—the rest of the function stays the way it is.

$$\begin{aligned}
 &= x^3 e^x - 3 \left[\underbrace{x^2 e^x}_{uv} - \int \underbrace{e^x}_v \cdot \underbrace{2x dx}_{du} \right] \\
 &= x^3 e^x - 3x^2 e^x + 6 \int x e^x dx
 \end{aligned}$$

Continuing, we take $u = x$ and $dv = e^x dx$, so $du = dx$ and $v = e^x$. This is the step where the polynomial part of the integrand finally disappears.

$$\begin{aligned}
 &= x^3 e^x - 3x^2 e^x + 6 \left[\underbrace{x e^x}_{uv} - \int \underbrace{e^x}_v \underbrace{dx}_{du} \right] \\
 &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\
 &= e^x (x^3 - 3x^2 + 6x - 6) + C
 \end{aligned}$$

Let's check that this makes sense: the derivative of $e^x (x^3 - 3x^2 + 6x - 6) + C$ should be $x^3 e^x$. We differentiate using the product rule.

$$\begin{aligned}
 \frac{d}{dx} \left\{ e^x (x^3 - 3x^2 + 6x - 6) + C \right\} &= e^x (x^3 - 3x^2 + 6x - 6) + e^x (3x^2 - 6x + 6) \\
 &= e^x (x^3 - 3x^2 + 3x^2 + 6x - 6x - 6 + 6) = x^3 e^x
 \end{aligned}$$

Remark: In order to be technically correct in our antidifferentiation, we should add the $+C$ as soon as we do the first integration by parts. However, when we are using integration by parts, we usually end up evaluating an integral at the end, and we add the $+C$ at that point. Since the $+C$ comes up eventually, it is common practice to not clutter our calculations with it until the end.

S-11: Since our integrand is two functions multiplied together, and there isn't an obvious substitution, let's try integration by parts. Here are our salient options.

Option 1:	$ \begin{array}{ll} u = x & du = 1 \, dx \\ dv = \log^3 x \, dx & v = ?? \end{array} $
Option 2:	$ \begin{array}{ll} u = \log^3 x & du = 3 \log^2 x \cdot \frac{1}{x} \, dx \\ dv = x \, dx & v = \frac{1}{2} x^2 \end{array} $

This calls for some strategizing. Using the template of Example 1.11.8 in the text, we could probably figure out the antiderivative of $\log^3 x$. Option 1 is tempting, because our

x -term goes away. So, there might be a benefit there, but on the other hand, the antiderivative of $\log^3 x$ is probably pretty complicated.

Now let's consider Option 2. When we multiply the blue functions together, we get something similar to our original integrand, but the power of logarithm is smaller. If we were to iterate this method (using integration by parts a few times, always choosing u to be the part with a logarithm) then eventually we would end up differentiating logarithm. This seems like a safer plan: let's do Option 2.

We use integration by parts with $u = \log^3 x$, $dv = x dx$, $du = \frac{3}{x} \log^2 x dx$, and $v = \frac{1}{2} x^2$.

$$\begin{aligned} \int x \log^3 x \, dx &= \underbrace{\frac{1}{2} x^2 \log^3 x}_{uv} - \int \underbrace{\frac{3}{2} x \log^2 x}_{v du} \, dx \\ &= \frac{1}{2} x^2 \log^3 x - \frac{3}{2} \int x \log^2 x \, dx \end{aligned}$$

Continuing our quest to differentiate away the logarithm, we use integration by parts with $u = \log^2 x$, $dv = x dx$, $du = \frac{2}{x} \log x dx$, and $v = \frac{1}{2} x^2$.

$$\begin{aligned} &= \frac{1}{2} x^2 \log^3 x - \frac{3}{2} \left[\underbrace{\frac{1}{2} x^2 \log^2 x}_{uv} - \int \underbrace{x \log x}_{v du} \, dx \right] \\ &= \frac{1}{2} x^2 \log^3 x - \frac{3}{4} x^2 \log^2 x + \frac{3}{2} \int x \log x \, dx \end{aligned}$$

One last integration by parts: $u = \log x$, $dv = x dx$, $du = \frac{1}{x} dx$, and $v = \frac{1}{2} x^2$.

$$\begin{aligned} &= \frac{1}{2} x^2 \log^3 x - \frac{3}{4} x^2 \log^2 x + \frac{3}{2} \left[\underbrace{\frac{1}{2} x^2 \log x}_{uv} - \int \underbrace{\frac{1}{2} x}_{v du} \, dx \right] \\ &= \frac{1}{2} x^2 \log^3 x - \frac{3}{4} x^2 \log^2 x + \frac{3}{4} x^2 \log x - \frac{3}{4} \int x \, dx \\ &= \frac{1}{2} x^2 \log^3 x - \frac{3}{4} x^2 \log^2 x + \frac{3}{4} x^2 \log x - \frac{3}{8} x^2 + C \end{aligned}$$

Once again, technically there is a $+C$ in the work after the first integration by parts, but we follow convention by conveniently suppressing it until the final integration.

S-12: The integrand is the product of two functions, without an obvious substitution, so let's see what integration by parts can do for us.

Option 1:	$u = x^2$	$du = 2x \, dx$
	$dv = \sin x \, dx$	$v = -\cos x$
Option 2:	$u = \sin x$	$du = \cos x \, dx$
	$dv = x^2 \, dx$	$v = \frac{1}{3}x^3$

Neither option gives us something immediately integrable, but Option 1 replaces our x^2 term with a *lower* power of x . If we repeatedly apply integration by parts, we can reduce this power to zero. So, we start by choosing $u = x^2$ and $dv = \sin x \, dx$, so $du = 2x \, dx$ and $v = -\cos x$.

$$\begin{aligned}\int x^2 \sin x \, dx &= \underbrace{-x^2 \cos x}_{uv} + \underbrace{\int 2x \cos x \, dx}_{-v \, du} \\ &= -x^2 \cos x + 2 \int x \cos x \, dx\end{aligned}$$

Using integration by parts again, we want to be differentiating (not antidifferentiating) x , so we choose $u = x$, $dv = \cos x \, dx$, and then $du = dx$ (x went away!), $v = \sin x$.

$$\begin{aligned}&= -x^2 \cos x + 2 \left[\underbrace{x \sin x}_{uv} - \int \underbrace{\sin x \, dx}_{v \, du} \right] \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C \\ &= (2 - x^2) \cos x + 2x \sin x + C\end{aligned}$$

S-13: This problem is similar to Questions 6 and 7: integrating a polynomial multiplied by a logarithm. Just as in these questions, if we use integration by parts with $u = \log t$, then $du = \frac{1}{t} dt$, and our new integrand will consist of powers of t —which are easy to antidifferentiate.

So, we use $u = \log t$, $dv = 3t^2 - 5t + 6$, $du = \frac{1}{t} dt$, and $v = t^3 - \frac{5}{2}t^2 + 6t$.

$$\begin{aligned}\int (3t^2 - 5t + 6) \log t \, dt &= \underbrace{\log t}_u \left(\underbrace{t^3 - \frac{5}{2}t^2 + 6t}_v \right) - \int \underbrace{\frac{1}{t} \left(t^3 - \frac{5}{2}t^2 + 6t \right) dt}_{v \, du} \\ &= \left(t^3 - \frac{5}{2}t^2 + 6t \right) \log t - \int \left(t^2 - \frac{5}{2}t + 6 \right) dt \\ &= \left(t^3 - \frac{5}{2}t^2 + 6t \right) \log t - \frac{1}{3}t^3 + \frac{5}{4}t^2 - 6t + C\end{aligned}$$

S-14: Before we jump to integration by parts, we notice that the square roots lend themselves to substitution. Let's take $w = \sqrt{s}$. Then $dw = \frac{1}{2\sqrt{s}} ds$, so $2w dw = ds$.

$$\int \sqrt{s} e^{\sqrt{s}} ds = \int w \cdot e^w \cdot 2w dw = 2 \int w^2 e^w dw$$

Now we have nearly the situation of Question 10. We can repeatedly use integration by parts, with u as the power of w , to get rid of the polynomial part. We'll start with $u = w^2$, $dv = e^w dw$, $du = 2w dw$, and $v = e^w$.

$$\begin{aligned} &= 2 \left[\underbrace{w^2 e^w}_{uv} - \int \underbrace{2we^w dw}_{v du} \right] \\ &= 2w^2 e^w - 4 \int we^w dw \end{aligned}$$

We use integration by parts again, this time with $u = w$, $dv = e^w dw$, $du = dw$, and $v = e^w$.

$$\begin{aligned} &= 2w^2 e^w - 4 \left[\underbrace{we^w}_{uv} - \int \underbrace{e^w dw}_{v du} \right] \\ &= 2w^2 e^w - 4we^w + 4e^w + C \\ &= e^w (2w^2 - 4w + 4) + C \\ &= e^{\sqrt{s}} (2s - 4\sqrt{s} + 4) + C \end{aligned}$$

S-15: Let's use integration by parts. What are our parts? We have a few options.

Solution 1: Following Example 1.11.8 in the text, we choose $u = \log^2 x$ and $dv = dx$, so that $du = \frac{2}{x} \log x dx$ and $v = x$.

$$\int \log^2 x dx = \underbrace{x \log^2 x}_{uv} - \int \underbrace{2 \log x dx}_{v du}$$

Here we can either use the antiderivative of logarithm from memory, or re-derive it. We do the latter, using integration by parts with $u = \log x$, $dv = dx$, $du = \frac{1}{x} dx$, and $v = x$.

$$\begin{aligned} &= x \log^2 x - \left[\underbrace{2x \log x}_{uv} - \int \underbrace{2 dx}_{v du} \right] \\ &= x \log^2 x - 2x \log x + 2x + C \end{aligned}$$

Solution 2: Our integrand is two functions multiplied together: $\log x$ and $\log x$. So, we will use integration by parts with $u = \log x$, $dv = \log x$, $du = \frac{1}{x}dx$, and (using the antiderivative of logarithm, found in Example 1.11.8 in the text) $v = x \log x - x$.

$$\begin{aligned}\int \log^2 x \, dx &= (\underbrace{\log x}_u)(\underbrace{x \log x - x}_v) - \int (\underbrace{x \log x - x}_v) \underbrace{\frac{1}{x}dx}_{du} \\ &= x \log^2 x - x \log x - \int (\log x - 1) \, dx \\ &= x \log^2 x - x \log x - [(x \log x - x) - x] + C \\ &= x \log^2 x - 2x \log x + 2x + C\end{aligned}$$

S-16: This is your friendly reminder that to a person with a hammer, everything looks like a nail. The integral in the problem is a classic example of an integral to solve using substitution. We have an “inside function,” $x^2 + 1$, whose derivative shows up multiplied to the rest of the integrand. We take $u = x^2 + 1$, then $du = 2x dx$, so

$$\int 2xe^{x^2+1} dx = \int e^u du = e^u + C = e^{x^2+1} + C$$

S-17: In Example 1.11.9 of the text, we saw that integration by parts was useful when the integrand has a derivative that works nicely when multiplied by x . We use the same idea here. Let $u = \arccos y$ and $dv = dy$, so that $v = y$ and $du = -\frac{dy}{\sqrt{1-y^2}}$.

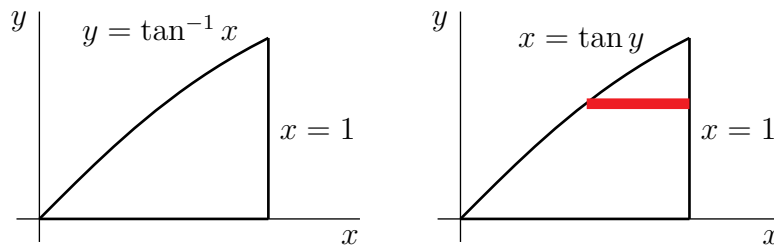
$$\int \arccos y \, dy = \underbrace{y \arccos y}_{uv} + \int \underbrace{\frac{y}{\sqrt{1-y^2}} dy}_{-v du}$$

Using the substitution $u = 1 - y^2$, $du = -2y dy$,

$$\begin{aligned}&= y \arccos y - \frac{1}{2} \int u^{-1/2} du \\ &= y \arccos y - u^{1/2} + C \\ &= y \arccos y - \sqrt{1-y^2} + C\end{aligned}$$

S-18: (a) The sketch is the figure on the left below. By integration by parts with $u = \arctan x$, $dv = dx$, $v = x$ and $du = \frac{1}{1+x^2} dx$, and then the substitution $s = 1 + x^2$,

$$\begin{aligned}A &= \int_0^1 \arctan x \, dx = \underbrace{x \arctan x}_{uv} \Big|_0^1 - \int_0^1 \underbrace{\frac{x}{1+x^2} dx}_{v du} = \arctan 1 - \frac{1}{2} \log(1+x^2) \Big|_0^1 \\ &= \frac{\pi}{4} - \frac{\log 2}{2}\end{aligned}$$



(b) We'll use horizontal washers as in Example 1.10.5 of the text.

- We cut R into thin horizontal strips of width dy as in the figure on the right above.
- When we rotate R about the y -axis, each strip sweeps out a thin washer
 - whose inner radius is $r_{in} = \tan y$ and outer radius is $r_{out} = 1$, and
 - whose thickness is dy and hence
 - whose volume $\pi(r_{out}^2 - r_{in}^2)dy = \pi(1 - \tan^2 y)dy$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \frac{\pi}{4}$ (since at the top $x = 1$ and $x = \tan y$), the total

$$\begin{aligned} \text{Volume} &= \int_0^{\pi/4} \pi(1 - \tan^2 y) dy = \int_0^{\pi/4} \pi(2 - \sec^2 y) dy = \pi[2y - \tan y]_0^{\pi/4} \\ &= \frac{\pi^2}{2} - \pi \end{aligned}$$

S-19: For a fixed value of x , if we rotate about the x -axis, we form a washer of inner radius $B(x)$ and outer radius $T(x)$ and hence of area $\pi[T(x)^2 - B(x)^2]$. We integrate this function from $x = 0$ to $x = 3$ to find the total volume V :

$$\begin{aligned} V &= \int_0^3 \pi[T(x)^2 - B(x)^2] dx \\ &= \pi \int_0^3 (\sqrt{x}e^{3x})^2 - (\sqrt{x}(1 + 2x))^2 dx \\ &= \pi \int_0^3 (xe^{6x} - (x + 4x^2 + 4x^3)) dx \\ &= \pi \int_0^3 xe^{6x} dx - \pi \left[\frac{x^2}{2} + \frac{4x^3}{3} + x^4 \right]_0^3 \\ &= \pi \int_0^3 xe^{6x} dx - \pi \left[\frac{3^2}{2} + \frac{4 \cdot 3^3}{3} + 3^4 \right] \end{aligned}$$

For the first integral, we use integration by parts with $u(x) = x$, $dv = e^{6x}dx$, so that

$du = dx$ and $v(x) = \frac{1}{6}e^{6x}$:

$$\begin{aligned}\int_0^3 x e^{6x} dx &= \underbrace{\frac{x e^{6x}}{6}}_{uv} \Big|_0^3 - \int_0^3 \underbrace{\frac{1}{6} e^{6x}}_{v du} dx \\ &= \frac{3e^{18}}{6} - 0 - \frac{1}{36} e^{6x} \Big|_0^3 = \frac{e^{18}}{2} - \left(\frac{e^{18}}{36} - \frac{1}{36} \right).\end{aligned}$$

Therefore, the total volume is

$$V = \pi \left[\frac{e^{18}}{2} - \left(\frac{e^{18}}{36} - \frac{1}{36} \right) \right] - \pi \left[\frac{3^2}{2} + \frac{4 \cdot 3^3}{3} + 3^4 \right] = \pi \left(\frac{17e^{18} - 4373}{36} \right).$$

S-20: We integrate by parts, using $u = \arctan(2y)$, $dv = 4y dy$, so that $v = 2y^2$ and $du = \frac{2 dy}{1+(2y)^2}$:

$$\int 4y \arctan(2y) dy = \underbrace{2y^2 \arctan(2y)}_{uv} - \int \underbrace{\frac{4y^2}{(2y)^2 + 1}}_{v du} dy$$

The integrand $\frac{4y^2}{(2y)^2+1}$ is a rational function. So the remaining integral can be evaluated using the method of partial fractions, starting with long division. But it is easier to just notice that $\frac{4y^2}{4y^2+1} = \frac{4y^2+1}{4y^2+1} - \frac{1}{4y^2+1}$. We therefore have:

$$\int \frac{4y^2}{4y^2+1} dy = \int \left(1 - \frac{1}{4y^2+1} \right) dy = y - \frac{1}{2} \arctan(2y) + C$$

The final answer is then

$$\int 4y \arctan(2y) dy = 2y^2 \arctan(2y) - y + \frac{1}{2} \arctan(2y) + C$$

S-21: We've got an integrand that consists of two functions multiplied together, and no obvious substitution. So, we think about integration by parts. Let's consider our options. Note in Example 1.11.9 of the text, we found that the antiderivative of arctangent is $x \arctan x - \frac{1}{2} \log(1+x^2) + C$.

Option 1:

$$u = \arctan x$$

$$dv = x^2 dx$$

$$du = \frac{1}{1+x^2} dx$$

$$v = \frac{1}{3} x^3$$

Option 2:

$$u = x^2$$

$$dv = \arctan x dx$$

$$du = 2x dx$$

$$v = x \arctan x - \frac{1}{2} \log(1+x^2)$$

Option 1 Option 1 seems likelier. Let's see how it plays out. We use integration by parts with $u = \arctan x$, $dv = x^2 dx$, $du = \frac{dx}{1+x^2}$, and $v = \frac{1}{3}x^3$.

$$\begin{aligned}\int x^2 \arctan x \, dx &= \underbrace{\frac{x^3}{3} \arctan x}_{uv} - \int \underbrace{\frac{x^3}{3(1+x^2)}}_{vdu} dx \\ &= \frac{x^3}{3} \arctan x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx\end{aligned}$$

This is starting to look like a candidate for a substitution! Let's try the denominator, $s = 1 + x^2$. Then $ds = 2x dx$, and $x^2 = s - 1$.

$$\begin{aligned}&= \frac{x^3}{3} \arctan x - \frac{1}{6} \int \frac{x^2}{1+x^2} \cdot 2x dx \\ &= \frac{x^3}{3} \arctan x - \frac{1}{6} \int \frac{s-1}{s} ds \\ &= \frac{x^3}{3} \arctan x - \frac{1}{6} \int 1 - \frac{1}{s} ds \\ &= \frac{x^3}{3} \arctan x - \frac{1}{6}s + \frac{1}{6} \log |s| + C \\ &= \frac{x^3}{3} \arctan x - \frac{1}{6}(1+x^2) + \frac{1}{6} \log(1+x^2) + C\end{aligned}$$

Option 2: What if we had tried the other option? That is, $u = x^2$, $du = 2x dx$, $dv = \arctan x$, and $v = x \arctan x - \frac{1}{2} \log(1+x^2)$. It's not always the case that both options work, but sometimes they do. (They are almost never of equal difficulty.) This solution takes advantage of two previously hard-won results: the antiderivatives of logarithm and arctangent.

$$\begin{aligned}\int x^2 \arctan x \, dx &= \underbrace{x^2}_u \underbrace{\left(x \arctan x - \frac{1}{2} \log(1+x^2)\right)}_v - \int \underbrace{\left(x \arctan x - \frac{1}{2} \log(1+x^2)\right)}_v \cdot \underbrace{2x dx}_{du} \\ &= x^3 \arctan x - \frac{x^2}{2} \log(1+x^2) - 2 \int x^2 \arctan x \, dx + \int x \log(1+x^2) dx\end{aligned}$$

Adding $2 \int x^2 \arctan x \, dx$ to both sides:

$$\begin{aligned}3 \int x^2 \arctan x \, dx &= x^3 \arctan x - \frac{x^2}{2} \log(1+x^2) + \int x \log(1+x^2) dx \\ \int x^2 \arctan x \, dx &= \frac{x^3}{3} \arctan x - \frac{x^2}{6} \log(1+x^2) + \frac{1}{3} \int x \log(1+x^2) dx\end{aligned}$$

Using the substitution $s = 1 + x^2$, $ds = 2x dx$:

$$= \frac{x^3}{3} \arctan x - \frac{x^2}{6} \log(1 + x^2) + \frac{1}{6} \int \log s ds$$

Using the antiderivative of logarithm found in Example 1.11.8 of the text,

$$\begin{aligned} &= \frac{x^3}{3} \arctan x - \frac{x^2}{6} \log(1 + x^2) + \frac{1}{6} (s \log s - s) + C \\ &= \frac{x^3}{3} \arctan x - \frac{x^2}{6} \log(1 + x^2) + \frac{1}{6} \left((1 + x^2) \log(1 + x^2) - (1 + x^2) \right) + C \\ &= \frac{x^3}{3} \arctan x + \left[-\frac{x^2}{6} + \frac{1 + x^2}{6} \right] \log(1 + x^2) - \frac{1}{6} (1 + x^2) + C \\ &= \frac{x^3}{3} \arctan x + \frac{1}{6} \log(1 + x^2) - \frac{1}{6} (1 + x^2) + C \end{aligned}$$

S-22: We begin by simplifying the integrand.

$$\int 2^{x+\log_2 x} dx = \int 2^x \cdot 2^{\log_2 x} dx = \int 2^x \cdot x dx$$

This is similar to the integral $\int x e^x dx$, which we saw in Example 1.11.1 of the text. Let's write $2 = e^{\log 2}$ to take advantage of the easy integrability of e^x .

$$= \int x \cdot e^{x \log 2} dx$$

We use integration by parts with $u = x$, $dv = e^{x \log 2} dx$; $du = dx$, $v = \frac{1}{\log 2} e^{x \log 2}$.

(Remember $\log 2$ is a constant. If you'd prefer, you can do a substitution with $s = x \log 2$ first, to have a simpler exponent of e .)

$$\begin{aligned} &= \underbrace{\frac{x}{\log 2} e^{x \log 2}}_{uv} - \int \underbrace{\frac{1}{\log 2} e^{x \log 2}}_{v du} dx \\ &= \frac{x}{\log 2} e^{x \log 2} - \frac{1}{(\log 2)^2} e^{x \log 2} + C \\ &= \frac{x}{\log 2} 2^x - \frac{1}{(\log 2)^2} 2^x + C \end{aligned}$$

S-23: It's not obvious where to start, but in general it's nice to have the arguments of our trig functions the same. So, we use the identity $\sin(2x) = 2 \sin x \cos x$.

$$\int e^{\cos x} \sin(2x) dx = 2 \int e^{\cos x} \cos x \sin x dx$$

Now we can use the substitution $w = \cos x$, $dw = -\sin x dx$.

$$= -2 \int w e^w dw$$

From here the integral should look more familiar. We can use integration by parts with $u = w$, $dv = e^w dw$, $du = dw$, and $v = e^w$.

$$\begin{aligned} &= -2 \left[\underbrace{we^w}_{uv} - \int \underbrace{e^w dw}_{v du} \right] \\ &= 2e^w [1 - w] + C \\ &= 2e^{\cos x} [1 - \cos x] + C \end{aligned}$$

S-24: We've got an integrand that consists of several functions multiplied together, and no obvious substitution. So, we think about integration by parts. We know an antiderivative for $\frac{1}{(1-x)^2}$, because we know $\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$. So let's try $dv = \frac{dx}{(1-x)^2}$ and $u = xe^{-x}$. Then $v = \frac{1}{1-x}$ and $du = (1-x)e^{-x} dx$. So, by integration by parts,

$$\begin{aligned} \int \underbrace{xe^{-x}}_u \underbrace{\frac{dx}{(1-x)^2}}_{dv} &= \underbrace{\frac{xe^{-x}}{1-x}}_{uv} - \int \underbrace{\frac{1}{1-x}}_v \underbrace{(1-x)e^{-x} dx}_{du} \\ &= \frac{xe^{-x}}{1-x} - \int e^{-x} dx \\ &= \frac{xe^{-x}}{1-x} + e^{-x} + C = \frac{e^{-x}}{1-x} + C \end{aligned}$$

S-25: (a) The "parts" in the integrand are powers of sine. Looking at the right hand side of the reduction formula, we see that it looks a little like the derivative of $\sin^{n-1} x$, although not exactly. So, let's integrate by parts with $u = \sin^{n-1} x$ and $dv = \sin x dx$, so that $du = (n-1) \sin^{n-2} x \cos x$ and $v = -\cos x$.

$$\int \sin^n x dx = \underbrace{-\sin^{n-1} x \cos x}_{uv} + \underbrace{(n-1) \int \cos^2 x \sin^{n-2} x dx}_{-\int v du}$$

Using the identity $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned} &= -\sin^{n-1} x \cos x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx \end{aligned}$$

Moving the last term on the right hand side to the left hand side gives

$$n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

Dividing across by n gives the desired reduction formula.

(b) By the reduction formula of part (a), if $n \geq 2$,

$$\int_0^{\pi/2} \sin^n(x) \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2}(x) \, dx$$

since $\sin 0 = \cos \frac{\pi}{2} = 0$. Applying this reduction formula, with $n = 8, 6, 4, 2$:

$$\begin{aligned} \int_0^{\pi/2} \sin^8(x) \, dx &= \frac{7}{8} \int_0^{\pi/2} \sin^6(x) \, dx = \frac{7}{8} \cdot \frac{5}{6} \int_0^{\pi/2} \sin^4(x) \, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \int_0^{\pi/2} \sin^2(x) \, dx \\ &= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35}{256} \pi \end{aligned}$$

Using a calculator, we see this is approximately 0.4295.

S-26: To get rid of the square root in the argument of f'' , we make the change of variables (also called “substitution”) $x = t^2$, $dx = 2t \, dt$.

$$\int_0^4 f''(\sqrt{x}) \, dx = 2 \int_0^2 t f''(t) \, dt$$

Then, to convert f'' into f' , we integrate by parts with $u = t$, $dv = f''(t) \, dt$, $v = f'(t)$.

$$\begin{aligned} \int_0^4 f''(\sqrt{x}) \, dx &= 2 \left\{ \underbrace{\left[t f'(t) \right]_0^2}_{uv} - \underbrace{\int_0^2 f'(t) \, dt}_{vdu} \right\} \\ &= 2 \left[t f'(t) - f(t) \right]_0^2 \\ &= 2 [2f'(2) - f(2) + f(0)] = 2 [2 \times 4 - 3 + 1] \\ &= 12 \end{aligned}$$

S-27: As we saw in Section 1.1 of the text, there are many different ways to interpret a limit as a Riemann sum. In the absence of instructions that restrain our choices, we go with the most convenient interpretations.

With that in mind, we choose:

- that our Riemann sum is a right Riemann sum (because we see i , not $i-1$ or $i-\frac{1}{2}$)
- $\Delta x = \frac{2}{n}$ (because it is multiplied by the rest of the integrand, and also shows up multiplied by i),
- then $x_i = a + i\Delta x = \frac{2}{n}i - 1$, which leads us to $a = -1$ and

- $f(x) = xe^x$.
- Finally, since $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and $a = -1$, we have $b = 1$.

So, the limit is equal to the definite integral

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left(\frac{2}{n}i - 1 \right) e^{\frac{2}{n}i-1} = \int_{-1}^1 xe^x dx$$

which we evaluate using integration by parts with $u = x$, $dv = e^x dx$, $du = dx$, and $v = e^x$.

$$\begin{aligned} &= \left[\underbrace{xe^x}_{uv} \right]_{-1}^1 - \int_{-1}^1 \underbrace{e^x dx}_{vdu} \\ &= \left(e + \frac{1}{e} \right) - \left(e - \frac{1}{e} \right) = \frac{2}{e} \end{aligned}$$

S-28:

Since the integrand is the product of polynomial and trigonometric functions, we suspect it might yield to integration by parts. There are a number of ways this can be accomplished.

Solution 1: Before we choose parts, let's use the identity $\sin(2x) = 2 \sin x \cos x$.

$$\int x \sin x \cos x dx = \frac{1}{2} \int x \sin(2x) dx$$

Now let $u = x$, $dv = \sin(2x) dx$; $du = dx$, and $v = -\frac{1}{2} \cos(2x)$. Using integration by parts:

$$\begin{aligned} &= \frac{1}{2} \left[-\frac{x}{2} \cos(2x) + \frac{1}{2} \int \cos(2x) dx \right] \\ &= -\frac{x}{4} \cos(2x) + \frac{1}{8} \sin(2x) + C \\ &= -\frac{x}{4} (1 - 2 \sin^2 x) + \frac{1}{4} \sin x \cos x + C \\ &= -\frac{x}{4} + \frac{x}{2} \sin^2 x + \frac{1}{4} \sin x \cos x + C \end{aligned}$$

Solution 2: If we let $u = x$, then $du = dx$, and this seems desirable for integration by parts. If $u = x$, then $dv = \sin x \cos x dx$. To find v we can use the substitution $u = \sin x$, $du = \cos x dx$.

$$v = \int \sin x \cos x dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} \sin^2 x + C$$

So, we take $v = \frac{1}{2} \sin^2 x$. Now we can apply integration by parts to our original integral.

$$\int x \sin x \cos x \, dx = \frac{x}{2} \sin^2 x - \int \frac{1}{2} \sin^2 x \, dx$$

Apply the identity $\sin^2 x = \frac{1 - \cos(2x)}{2}$.

$$\begin{aligned} &= \frac{x}{2} \sin^2 x - \frac{1}{4} \int 1 - \cos(2x) \, dx \\ &= \frac{x}{2} \sin^2 x - \frac{x}{4} + \frac{1}{8} \sin(2x) + C \\ &= \frac{x}{2} \sin^2 x - \frac{x}{4} + \frac{1}{4} \sin x \cos x + C \end{aligned}$$

Solution 3: Let $u = x \sin x$ and $dv = \cos x \, dx$; then $du = (x \cos x + \sin x) \, dx$ and $v = \sin x$.

$$\begin{aligned} \int x \sin x \cos x \, dx &= x \sin^2 x - \int \sin x (x \cos x + \sin x) \, dx \\ &= x \sin^2 x - \int x \sin x \cos x \, dx - \int \sin^2 x \, dx \end{aligned}$$

Apply the identity $\sin^2 x = \frac{1 - \cos(2x)}{2}$ to the second integral.

$$\begin{aligned} &= x \sin^2 x - \int x \sin x \cos x \, dx - \int \frac{1 - \cos(2x)}{2} \, dx \\ &= x \sin^2 x - \int x \sin x \cos x \, dx - \frac{x}{2} + \frac{1}{4} \sin(2x) + C \end{aligned}$$

So, we have the equation

$$\begin{aligned} \int x \sin x \cos x \, dx &= x \sin^2 x - \int x \sin x \cos x \, dx - \frac{x}{2} + \frac{1}{4} \sin(2x) + C \\ 2 \int x \sin x \cos x \, dx &= x \sin^2 x - \frac{x}{2} + \frac{1}{4} \sin(2x) + C \\ \int x \sin x \cos x \, dx &= \frac{x}{2} \sin^2 x - \frac{x}{4} + \frac{1}{8} \sin(2x) + \frac{C}{2} \\ &= \frac{x}{2} \sin^2 x - \frac{x}{4} + \frac{1}{4} \sin x \cos x + \frac{C}{2} \end{aligned}$$

Since C is an arbitrary constant that can take any number in $(-\infty, \infty)$, also $\frac{C}{2}$ is an arbitrary constant that can take any number in $(-\infty, \infty)$, so we're free to rename $\frac{C}{2}$ to C .

S-29: Since we have an “inside function,” we start with the substitution $s = \cos x$, so $-ds = \sin x \, dx$ and $\sin^2 x = 1 - \cos^2 x = 1 - s^2$.

$$\begin{aligned}\int \sin(\cos x) \sin^3 x \, dx &= \int \sin(\cos x) \cdot \sin^2 x \cdot \sin x \, dx \\ &= - \int \sin(s) \cdot (1 - s^2) \, ds\end{aligned}$$

We use integration by parts with $u = (1 - s^2)$, $dv = \sin s \, ds$; $du = -2s \, ds$, and $v = -\cos s$.

$$\begin{aligned}&= - \left[-(1 - s^2) \cos s - \int 2s \cos s \, ds \right] \\ &= (1 - s^2) \cos s + \int 2s \cos s \, ds\end{aligned}$$

We integrate by parts again, with $u = 2s$, $dv = \cos s \, ds$; $du = 2 \, ds$, and $v = \sin s$.

$$\begin{aligned}&= (1 - s^2) \cos s + 2s \sin s - \int 2 \sin s \, ds \\ &= (1 - s^2) \cos s + 2s \sin s + 2 \cos s + C \\ &= \sin^2 x \cdot \cos(\cos x) + 2 \cos x \cdot \sin(\cos x) + 2 \cos(\cos x) + C \\ &= (\sin^2 x + 2) \cos(\cos x) + 2 \cos x \cdot \sin(\cos x) + C\end{aligned}$$

Solutions to Exercises 1.12 — Jump to [TABLE OF CONTENTS](#)

S-1: The absolute error is the difference between the two values:

$$|1.387 - 1.5| = 0.113$$

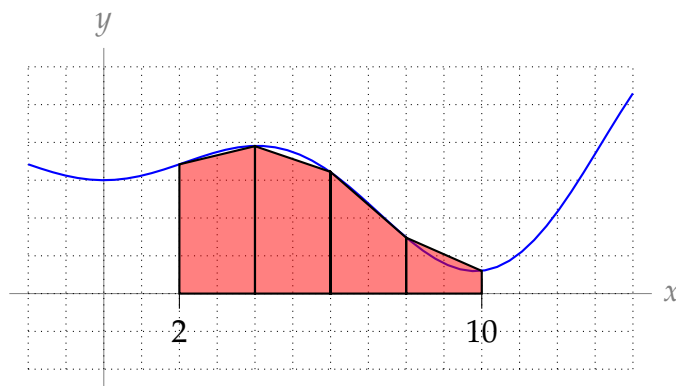
The relative error is the absolute error divided by the exact value:

$$\frac{0.113}{1.387} \approx 0.08147$$

The percent error is 100 times the relative error:

$$\approx 8.147\%$$

S-2:

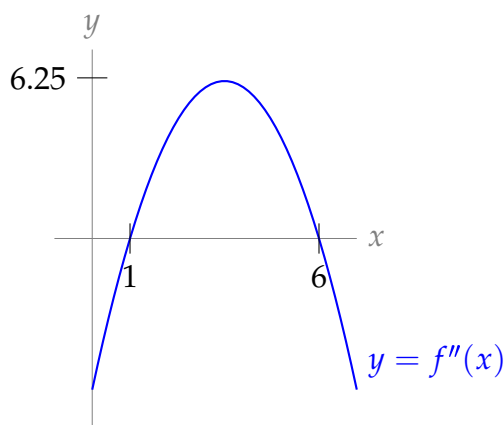


S-3:

(a) Differentiating, we find $f''(x) = -x^2 + 7x - 6$. Since $f''(x)$ is quadratic, we have a pretty good idea of what it looks like.

- It factors as $f''(x) = -(x - 6)(x - 1)$, so its two roots are at $x = 6$ and $x = 1$.
- The “flat part” of the parabola is at $x = 3.5$ (since this is exactly half way between $x = 1$ and $x = 6$; alternately, we can check that $f'''(3.5) = 0$).
- Since the coefficient of x^2 is negative, $f''(x)$ is increasing from $-\infty$ to 3.5, then decreasing from 3.5 to ∞ .

Therefore, over the interval $[1, 6]$, the largest positive value of $f''(x)$ occurs when $x = 3.5$, and this is $f''(3.5) = -(3.5 - 6)(3.5 - 1) = 6.25$.



So, we take $M = 6.25$.

(b) We differentiate further to find $f^{(4)}(x) = -2$. This is constant everywhere, so we take $L = |-2| = 2$.

S-4: Let's start by differentiating.

$$\begin{aligned} f(x) &= x \sin x + 2 \cos x \\ f'(x) &= x \cos x + \sin x - 2 \sin x = x \cos x - \sin x \\ f''(x) &= -x \sin x + \cos x - \cos x = -x \sin x \end{aligned}$$

For any value of x , $|\sin x| \leq 1$. When $-3 \leq x \leq 2$, then $|x| \leq 3$. So, it is true (and not unreasonably sloppy) that

$$f''(x) \leq 3$$

whenever x is in the interval $[-3, 2]$. So, we can take $M = 3$.

Note that $|f''(x)|$ is actually *smaller* than 3 whenever x is in the interval $[-3, 2]$, because when $x = -3$, $\sin x \neq 1$. In fact, since 3 is pretty close to π , $\sin 3$ is pretty small. (The actual maximum value of $|f''(x)|$ when $-3 \leq x \leq 2$ is about 1.8.) However, we find parameters like M for the purpose of computing error bounds. There is often not much to be gained from taking the time to find the actual maximum of a function, so we content ourselves with reasonable upper bounds.

S-5:

- (a) Let $f(x) = \cos x$. Then $f^{(4)}(x) = \cos x$, so $|f^{(4)}(x)| \leq 1$ when $-\pi \leq x \leq \pi$. So, using $L = 1$, we find the upper bound of the error using Simpson's rule with $n = 4$ is:

$$\frac{L(b-a)^5}{180n^4} = \frac{(2\pi)^5}{180 \cdot 4^4} = \frac{\pi^5}{180 \cdot 8} \approx 0.2$$

The error bound comes from Theorem 1.12.7 in the text. We used a calculator to find the approximate decimal value.

- (b) We use the general form of Simpson's rule (Equation 1.12.4 in the text) with $\Delta x = \frac{b-a}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$.

$$\begin{aligned} A &\approx \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) \\ &= \frac{\pi/2}{3} (f(-\pi) + 4f(-\frac{\pi}{2}) + 2f(0) + 4f(\frac{\pi}{2}) + f(\pi)) \\ &= \frac{\pi}{6} (-1 + 4(0) + 2(1) + 4(0) - 1) = 0 \end{aligned}$$

- (c) To find the actual error in our approximation, we compare the approximation from (b) to the exact value of A . In fact, $A = 0$: this is a fact you've probably seen before by considering the symmetry of cosine, but it's easy enough to calculate:

$$A = \int_{-\pi}^{\pi} \cos x \, dx = \sin \pi - \sin(-\pi) = 0$$

So, our approximation was exactly the same as our exact value. The absolute error is 0.

Remark: the purpose of this question was to remind you that the error bounds we calculate are not (usually) the same as the actual error. Often our approximations are better than we give them credit for. In normal circumstances, we would be approximating an integral precisely to avoid evaluating it exactly, so we wouldn't find our exact error. The bound is a quick way of ensuring that our approximation is not *too* far off.

S-6: Using Theorem 1.12.7 in the text, the error using the trapezoidal rule as described is at most

$$\frac{M(b-a)^3}{12 \cdot n^2} = \frac{M}{48} \leq \frac{3}{48} = \frac{1}{16}.$$

So, we're really being asked to find a function with the maximum possible error using the trapezoidal rule, given its second derivative.

With that in mind, our function should have the largest second derivative possible: let's set $f''(x) = 3$ for every x . Then:

$$\begin{aligned}f''(x) &= 3 \\f'(x) &= 3x + C \\f(x) &= \frac{3}{2}x^2 + Cx + D\end{aligned}$$

for some constants C and D . Now we can find the exact and approximate values of $\int_0^1 f(x) \, dx$.

$$\begin{aligned}\text{Exact:} \quad \int_0^1 f(x) \, dx &= \int_0^1 \left(\frac{3}{2}x^2 + Cx + D \right) \, dx \\&= \left[\frac{1}{2}x^3 + \frac{C}{2}x^2 + Dx \right]_0^1 \\&= \frac{1}{2} + \frac{C}{2} + D\end{aligned}$$

$$\begin{aligned}\text{Approximate:} \quad \int_0^1 f(x) \, dx &\approx \Delta x \left[\frac{1}{2}f(0) + f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) \right] \\&= \frac{1}{2} \left[\frac{1}{2}(D) + \left(\frac{3}{8} + \frac{C}{2} + D \right) + \frac{1}{2} \left(\frac{3}{2} + C + D \right) \right] \\&= \frac{1}{2} \left[\frac{9}{8} + C + 2D \right] \\&= \frac{9}{16} + \frac{C}{2} + D\end{aligned}$$

So, the absolute error associated with the trapezoidal approximation is:

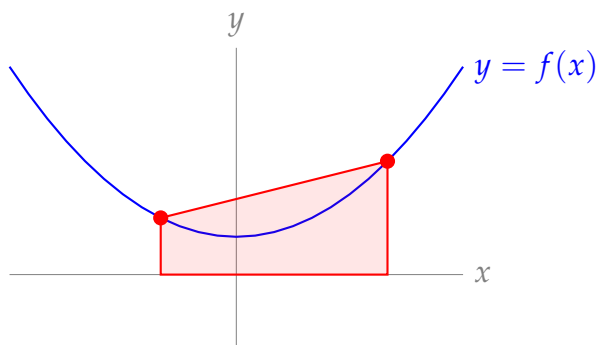
$$\left| \left(\frac{1}{2} + \frac{C}{2} + D \right) - \left(\frac{9}{16} + \frac{C}{2} + D \right) \right| = \frac{1}{16}$$

So, for any constants C and D , $f(x) = \frac{3}{2}x^2 + Cx + D$ has the desired error.

Remark: contrast this question with Question 5. In this problem, our absolute error was exactly as bad as the bound predicted, but sometimes it is much better. The thing to remember is that, in general, we *don't know* our absolute error. We only guarantee that it's not any worse than some worst-case-scenario bound.

S-7: Under any reasonable assumptions⁵, my mother is older than I am.

S-8: True. Because $f(x)$ is positive and concave up, the graph of $f(x)$ is always below the top edges of the trapezoids used in the trapezoidal rule.



S-9: According to Theorem 1.12.7 in the text, the error associated with the Simpson's rule approximation is no more than $\frac{L}{180} \frac{(b-a)^5}{n^4}$, where L is a constant such that $|f^{(4)}(x)| \leq L$ for all x in $[a, b]$. If $L = 0$, then the error is no more than 0 regardless of a , b , or n —that is, the approximation is exact.

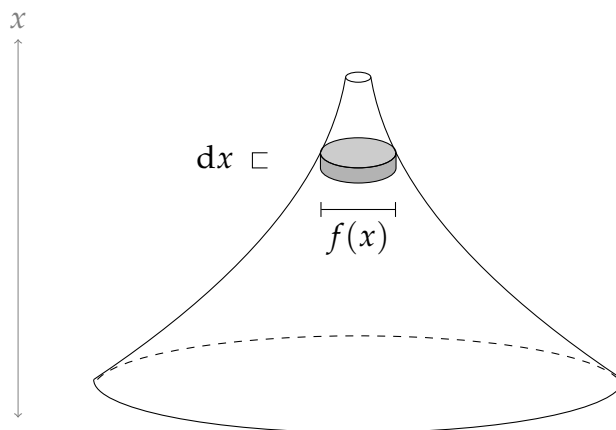
Any polynomial $f(x)$ of degree at most 3 has $f^{(4)}(x) = 0$ for all x . So, any polynomial of degree at most 3 is an acceptable answer. For example, $f(x) = 5x^3 - 27$, or $f(x) = x^2$.

S-10:

- (a) D holds the product of the entries in B and C, so we write `=B2*C2` in cell D2.
- (b) The first and last function values are multiplied by 1. The function value $f(x_0)$ is stored in row 2, so we put a 1 in cell C2. The function value $f(x_n)$ is stored in row $n + 2$, so we put a 1 in cell C($n+2$).
- (c) We want the values 4,2,4,2,4,... in column C, starting in cell C3. That is, the values are 3 ± 1 , where we *add* one when the row is odd, and *subtract* one when the row is even. So, we write `=3+(-1)^(ROW()+1)` in cell C3. (Note the parity [evenness/oddness] of our rows is opposite of the example in Subsection 1.12.4, which is why the formula we write is different. This is because we used Row 1 for labels.)

S-11: Let $f(x)$ denote the diameter at height x . As in Example 1.10.6 of the text, we slice V into thin horizontal “pancakes”, which in this case are circular.

5 Anyone caught trying to come up with a scenario in which I am older than my mother will be sent to maximum security grad school.



- We are told that the pancake at height x is a circular disk of diameter $f(x)$ and so
- has cross-sectional area $\pi\left(\frac{f(x)}{2}\right)^2$ and thickness dx and hence
- has volume $\pi\left(\frac{f(x)}{2}\right)^2 dx$.

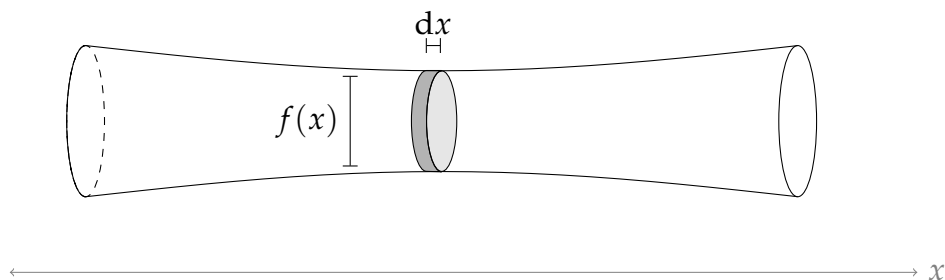
Hence the volume of V is

$$\begin{aligned} \int_0^{40} \pi \left[\frac{f(x)}{2} \right]^2 dx &\approx \frac{\pi}{4} 10 \left[\frac{1}{2} f(0)^2 + f(10)^2 + f(20)^2 + f(30)^2 + \frac{1}{2} f(40)^2 \right] \\ &= \frac{\pi}{4} 10 \left[\frac{1}{2} 24^2 + 16^2 + 10^2 + 6^2 + \frac{1}{2} 4^2 \right] \\ &= 688 \times 2.5\pi = 1720\pi \approx 5403.5 \end{aligned}$$

where we have approximated the integral using the trapezoidal rule with $\Delta x = 10$, and used a calculator to get a decimal approximation.

S-12: Let $f(x)$ be the diameter a distance x from the left end of the log. If we slice our log into thin disks, the disks x metres from the left end of the log has

- radius $\frac{f(x)}{2}$,
- width dx , and so
- volume $\pi \left(\frac{f(x)}{2} \right)^2 dx = \frac{\pi}{4} f(x)^2 dx$.

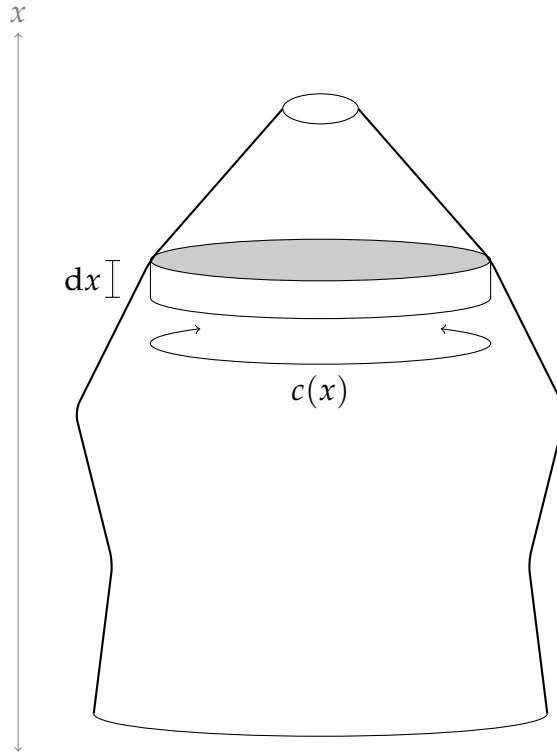


Using Simpson's Rule with $\Delta x = 1$, the volume of the log is:

$$\begin{aligned} V &= \int_0^6 \frac{\pi}{4} f(x)^2 dx \approx \frac{\pi}{4} \frac{1}{3} \left[f(0)^2 + 4f(1)^2 + 2f(2)^2 + 4f(3)^2 + 2f(4)^2 + 4f(5)^2 + f(6)^2 \right] \\ &= \frac{\pi}{12} \left[1.2^2 + 4(1)^2 + 2(0.8)^2 + 4(0.8)^2 + 2(1)^2 + 4(1)^2 + 1.2^2 \right] \\ &= \frac{\pi}{12} (16.72) \\ &\approx 4.377 \text{ m}^3 \end{aligned}$$

where we used a calculator to approximate the decimal value.

S-13: At height x metres, let the circumference of the tree be $c(x)$. The corresponding radius is $\frac{c(x)}{2\pi}$, so the corresponding cross-sectional area is $\pi \left(\frac{c(x)}{2\pi} \right)^2 = \frac{c(x)^2}{4\pi}$.



The height of a very thin cross-sectional disk is dx , so the volume of a cross-sectional disk is $\frac{c(x)^2}{4\pi} dx$. Therefore, total volume of the tree is:

$$\begin{aligned} \int_0^8 \frac{c(x)^2}{4\pi} dx &\approx \frac{1}{4\pi} \frac{2}{3} \left[c(0)^2 + 4c(2)^2 + 2c(4)^2 + 4c(6)^2 + c(8)^2 \right] \\ &= \frac{1}{6\pi} \left[1.2^2 + 4(1.1)^2 + 2(1.3)^2 + 4(0.9)^2 + 0.2^2 \right] \\ &= \frac{12.94}{6\pi} \approx 0.6865 \end{aligned}$$

where we used Simpson's rule with $\Delta x = 2$ and $n = 4$ to approximate the value of the integral based on the values of $c(x)$ given in the table.

S-14: For both approximations, $\Delta x = 10$ and $n = 6$.

(a) The Trapezoidal Rule gives

$$\begin{aligned} V &= \int_0^{60} A(h) \, dh \approx 10 \left[\frac{1}{2}A(0) + A(10) + A(20) + A(30) + A(40) + A(50) + \frac{1}{2}A(60) \right] \\ &= 363,500 \end{aligned}$$

(b) Simpson's Rule gives

$$\begin{aligned} V &= \int_0^{60} A(h) \, dh \approx \frac{10}{3} \left[A(0) + 4A(10) + 2A(20) + 4A(30) + 2A(40) + 4A(50) + A(60) \right] \\ &= 367,000 \end{aligned}$$

S-15: Call the curve in the graph $y = f(x)$. It looks like

$$f(2) = 3 \quad f(3) = 8 \quad f(4) = 7 \quad f(5) = 6 \quad f(6) = 4$$

We're estimating $\int_2^6 f(x) \, dx$ with $n = 4$, so $\Delta x = \frac{6-2}{4} = 1$.

(a) The trapezoidal rule gives

$$T_4 = \left[\frac{3}{2} + 8 + 7 + 6 + \frac{4}{2} \right] \times 1 = \frac{49}{2}$$

(b) Simpson's rule gives

$$S_4 = \frac{1}{3} [3 + 4 \times 8 + 2 \times 7 + 4 \times 6 + 4] \times 1 = \frac{77}{3}$$

S-16: In general the error in approximating $\int_a^b f(x) \, dx$ using Simpson's rule with n steps is bounded by $\frac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $L \geq |f^{(4)}(x)|$ for all $a \leq x \leq b$. In this case, $a = 1$, $b = 5$, $n = 4$ and $f(x) = \frac{1}{x}$. We need to find L , so we differentiate.

$$f'(x) = -\frac{1}{x^2} \quad f''(x) = \frac{2}{x^3} \quad f^{(3)}(x) = -\frac{6}{x^4} \quad f^{(4)}(x) = \frac{24}{x^5}$$

and

$$|f^{(4)}(x)| \leq 24 \text{ for all } x \geq 1$$

So we may take $L = 24$ and $\Delta x = \frac{5-1}{4} = 1$, which leads to

$$|\text{Error}| \leq \frac{24(5-1)}{180}(1)^4 = \frac{24}{45} = \frac{8}{15}$$

S-17: In general, the error in approximating $\int_a^b f(x) dx$ using Simpson's rule with n steps is bounded by $\frac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $L \geq |f^{(4)}(x)|$ for all $a \leq x \leq b$. In this case, $a = 0$, $b = 1$, $n = 6$ and $f(x) = e^{-2x} + 3x^3$. We need to find L , so we differentiate.

$$f'(x) = -2e^{-2x} + 9x^2 \quad f''(x) = 4e^{-2x} + 18x \quad f^{(3)}(x) = -8e^{-2x} + 18 \quad f^{(4)}(x) = 16e^{-2x}$$

Since $e^{-2x} = \frac{1}{e^{2x}}$, we see $f^{(4)}(x)$ is a positive, decreasing function. So, its maximum occurs when x is as small as possible. In the interval $[0, 1]$, that means $x = 0$.

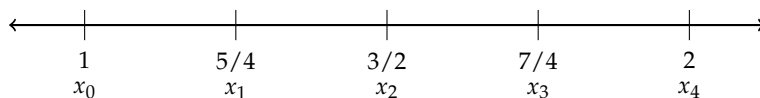
$$|f^{(4)}(x)| \leq f(0) = 16 \text{ for all } x \geq 0$$

So, we take $L = 16$ and $\Delta x = \frac{1-0}{6} = \frac{1}{6}$.

$$|\text{Error}| \leq \frac{L(b-a)}{180}(\Delta x)^4 = \frac{16(1-0)}{180}(1/6)^4 = \frac{16}{180 \times 6^4} = \frac{1}{180 \times 3^4} = \frac{1}{14580}$$

S-18: For both approximations, $a = 1$, $b = 2$, $n = 4$, $f(x) = \frac{1}{x}$ and $\Delta x = \frac{b-a}{n} = \frac{1}{4}$.

Then $x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4}$, and $x_4 = 2$.



(a)

$$\begin{aligned} T_4 &= \Delta x \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4) \right] \\ &= \Delta x \left[\frac{1}{2}f(1) + f(5/4) + f(3/2) + f(7/4) + \frac{1}{2}f(2) \right] \\ &= \frac{1}{4} \left[\left(\frac{1}{2} \times 1 \right) + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \left(\frac{1}{2} \times \frac{1}{2} \right) \right] \end{aligned}$$

(b)

$$\begin{aligned} S_4 &= \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \\ &= \frac{\Delta x}{3} [f(1) + 4f(5/4) + 2f(3/2) + 4f(7/4) + f(2)] \\ &= \frac{1}{12} \left[1 + \left(4 \times \frac{4}{5} \right) + \left(2 \times \frac{2}{3} \right) + \left(4 \times \frac{4}{7} \right) + \frac{1}{2} \right] \end{aligned}$$

(c) In this case, $a = 1$, $b = 2$, $n = 4$ and $f(x) = \frac{1}{x}$. We need to find L , so we differentiate.

$$f'(x) = -\frac{1}{x^2} \quad f''(x) = \frac{2}{x^3} \quad f^{(3)}(x) = -\frac{6}{x^4} \quad f^{(4)}(x) = \frac{24}{x^5}$$

So,

$$|f^{(4)}(x)| \leq 24 \text{ for all } x \text{ in the interval } [1, 2]$$

We take $L = 24$.

$$|\text{Error}| \leq \frac{L(b-a)^5}{180 \times n^4} \leq \frac{24(2-1)^5}{180 \times 4^4} = \frac{24}{180 \times 4^4} = \frac{1}{1920}$$

S-19: Set $a = 0$ and $b = 8$. Since we have information about $s(x)$ when x is 0, 2, 4, 6, and 8, we set $\Delta x = \frac{b-a}{n} = 2$, so $n = 4$. (Recall with the trapezoid rule and Simpson's rule, $n = 4$ intervals actually uses the value of the function at 5 points.)

We could perform the trapezoidal approximations with fewer intervals, for example $n = 2$, but this would involve ignoring some of the points we're given. Since the question asks for the best estimation we can give, we use $n = 4$ intervals and no fewer.

(a)

$$\begin{aligned} T_4 &= \Delta x \left[\frac{1}{2}s(0) + s(2) + s(4) + s(6) + \frac{1}{2}s(8) \right] \\ &= 2 \left[\frac{1.00664}{2} + 1.00543 + 1.00435 + 1.00331 + \frac{1.00233}{2} \right] \\ &= 8.03515 \\ S_4 &= \frac{\Delta x}{3} [s(0) + 4s(2) + 2s(4) + 4s(6) + s(8)] \\ &= \frac{2}{3} [1.00664 + 4 \times 1.00543 + 2 \times 1.00435 + 4 \times 1.00331 + 1.00233] \\ &\approx 8.03509 \end{aligned}$$

(b) The information $|s^{(k)}(x)| \leq \frac{k}{1000}$, with $k = 2$, tells us $|s''(x)| \leq \frac{2}{1000}$ for all x in the interval $[0, 8]$. So, we take K_2 (also called M in your text) to be $\frac{2}{1000}$.

Then the absolute error associated with our trapezoid rule approximation is at most

$$\left| \int_a^b f(x) \, dx - T_n \right| \leq \frac{K_2(b-a)^3}{12n^2} \leq \frac{2}{1000} \cdot \frac{8^3}{12(4)^2} \leq 0.00533$$

For $k = 4$, we see $|s^{(4)}(x)| \leq \frac{4}{1000}$ for all x in the interval $[0, 8]$. So, we take K_4 (also called L in your text) to be $\frac{4}{1000}$.

Then the absolute error associated with our Simpson's rule approximation is at most

$$\left| \int_a^b f(x) \, dx - S_n \right| \leq \frac{K_4(b-a)^5}{180n^4} \leq \frac{4}{1000} \cdot \frac{8^5}{180(4)^4} \leq 0.00284$$

S-20: In this case, $a = 1, b = 4$. Since $-2 \leq f''(x) \leq 0$ over the relevant interval, we take $M = 2$. (Remember M is an upper bound on $|f''(x)|$, not $f''(x)$.) So we need n to obey

$$\frac{2(4-1)^3}{12n^2} \leq 0.001 \iff n^2 \geq \frac{2(3)^3}{12} 1000 = \frac{27000}{6} = \frac{9000}{2} = 4500$$

One obvious allowed n is 100. Since $\sqrt{4500} \approx 67.01$, and n has to be a whole number, any $n \geq 68$ works.

S-21: $\Delta = \frac{2\pi/3 - \pi/2}{50} = \frac{\pi}{300}$, so our x -values (which we'll store in column A), are $x_i = \frac{\pi}{2} + \frac{\pi}{300}i$, from $i = 1$ to $i = 50$.

So, we'll set up the spreadsheet as below, where x_i values are in column A and the function values are in column B.

	A	B
1	=PI () / 2 + PI () / 300	=SIN (A1)
2	=A1 + PI () / 300	↓
3	↓	↓

We copy the contents down through row 50.

In a blank cell, we write

$$=PI () / 300 * SUM (B1 : B50)$$

to find

$$\int_{\pi/2}^{2\pi/3} \sin x \, dx \approx 0.499293941 .$$

S-22: $\Delta x = \frac{1-1/2}{100} = \frac{1}{200}$. We'll store x_i values in column A, and function values in column B.

	A	B
1	=1 / 2	=1 / A1
2	=A1 + 1 / 200	↓
3	↓	↓

We copy the cells down through row 101. In a free cell, we write

$$= ((B1 + B101) / 2 + SUM (B2 : B100)) / 200$$

and find

$$\int_{\frac{1}{2}}^1 \frac{1}{x} dx \approx 0.69315343 .$$

S-23: $\Delta x = \frac{1}{500}$. We'll store x_i values in column A and function values in column B, as in questions 21 and 22, and copy them down through row 501. We'll store coefficients in column C. To that end, we enter 1 into columns C1 and C 501. Into C2, we write

$=3+(-1)^{\wedge \text{ROW}()}$, and copy that entry down through row 500. Finally, in D1, we write $=B1 * C1$.

	A	B	C	D
1	0	$=\text{EXP}(A1 * A1)$	1	$=B1 * C1$
2	$=A1 + 1 / 500$	↓	$=3 + (-1)^{\wedge \text{ROW}()}$	↓
3	↓	↓	↓	↓
⋮	⋮	⋮	⋮	⋮
50	↓	↓	↓	↓
51	↓	↓	1	↓

Into a free cell, we write

$$=\text{SUM}(D1:D501) / 1500$$

(since $\frac{\Delta x}{3} = \frac{1}{1500}$) and find $\int_0^1 e^{x^2} dx \approx 1.462651746$.

S-24: Denote by $f(x)$ the width of the pool x feet from the left-hand end. From the sketch, $f(0) = 0$, $f(2) = 10$, $f(4) = 12$, $f(6) = 10$, $f(8) = 8$, $f(10) = 6$, $f(12) = 8$, $f(14) = 10$ and $f(16) = 0$.

A cross-section of the pool x feet from the left end is half of a circular disk with diameter $f(x)$ (so, radius $\frac{f(x)}{2}$) and thickness dx . So, the volume of the part of the pool with x -coordinate running from x to $(x + dx)$ is $\frac{1}{2}\pi\left(\frac{f(x)}{2}\right)^2 dx = \frac{\pi}{8}[f(x)]^2 dx$.

The total volume is given by the following integral.

$$\begin{aligned}
 V &= \frac{\pi}{8} \int_0^{16} f(x)^2 dx \\
 &\approx \frac{\pi}{8} \cdot \frac{\Delta x}{3} \left[f(0)^2 + 4f(2)^2 + 2f(4)^2 + 4f(6)^2 + 2f(8)^2 + 4f(10)^2 + 2f(12)^2 + 4f(14)^2 + f(16)^2 \right] \\
 &= \frac{\pi}{8} \cdot \frac{2}{3} \left[0 + 4(10)^2 + 2(12)^2 + 4(10)^2 + 2(8)^2 + 4(6)^2 + 2(8)^2 + 4(10)^2 + 0 \right] \\
 &= \frac{472}{3}\pi \approx 494 \text{ ft}^3
 \end{aligned}$$

S-25: (a) The Trapezoidal Rule with $n = 4$, $a = 0$, $b = 1$, and $\Delta x = \frac{1}{4}$ gives:

$$\begin{aligned}
 W &= 2\pi 10^{-6} \int_0^1 r g(r) dr \approx 2\pi 10^{-6} \Delta x \left[\frac{1}{2} x_0 g(x_0) + x_1 g(x_1) + x_2 g(x_2) + x_3 g(x_3) + \frac{1}{2} x_4 g(x_4) \right] \\
 &= 2\pi 10^{-6} \frac{1}{4} \left[\frac{1}{2} 0 g(0) + \frac{1}{4} g\left(\frac{1}{4}\right) + \frac{1}{2} g\left(\frac{1}{2}\right) + \frac{3}{4} g\left(\frac{3}{4}\right) + \frac{1}{2} g(1) \right] \\
 &= \pi 10^{-6} \frac{1}{2} \left[\frac{8100}{4} + \frac{8144}{2} + \frac{3 \cdot 8170}{4} + \frac{8190}{2} \right] \\
 &= \frac{32639\pi}{4 \cdot 10^6} \approx 0.025635
 \end{aligned}$$

(b) Using the product rule, the integrand $f(r) = 2\pi 10^{-6} r g(r)$ obeys

$$f''(r) = 2\pi 10^{-6} \frac{d}{dr} [g(r) + r g'(r)] = 2\pi 10^{-6} [2g'(r) + r g''(r)]$$

and hence, for $0 \leq r \leq 1$,

$$|f''(r)| \leq 2\pi 10^{-6} [2 \times 200 + 1 \times 150] = 1.1\pi 10^{-3}$$

So,

$$|\text{Error}| \leq \frac{1.1\pi 10^{-3} (1-0)^3}{12(4)^2} \leq 1.8 \times 10^{-5}$$

S-26: (a) Let $f(x) = \frac{1}{x}$, $a = 1$, $b = 2$ and $\Delta x = \frac{b-a}{6} = \frac{1}{6}$. Using Simpson's rule:

$$\begin{aligned} \int_1^2 \frac{1}{x} dx &\approx \frac{\Delta x}{3} \left[f(1) + 4f\left(\frac{7}{6}\right) + 2f\left(\frac{8}{6}\right) + 4f\left(\frac{9}{6}\right) + 2f\left(\frac{10}{6}\right) + 4f\left(\frac{11}{6}\right) + f(2) \right] \\ &= \frac{1}{18} \left[1 + \frac{24}{7} + \frac{12}{8} + \frac{24}{9} + \frac{12}{10} + \frac{24}{11} + \frac{1}{2} \right] \approx 0.6931698 \end{aligned}$$

(b) The integrand is $f(x) = \frac{1}{x}$. The first four derivatives of $f(x)$ are:

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f^{(3)}(x) = -\frac{6}{x^4}, \quad f^{(4)}(x) = \frac{24}{x^5}$$

On the interval $1 \leq x \leq 2$, the fourth derivative is never bigger in magnitude than $L = 24$.

$$|E_n| \leq \frac{L(b-a)^5}{180n^4} = \frac{24(2-1)^5}{180n^4} = \frac{4}{30n^4}$$

So, we want an even number n such that

$$\begin{aligned} \frac{4}{30n^4} &\leq 0.00001 = \frac{1}{10^5} \\ n^4 &\geq \frac{40000}{3} \\ n &\geq \sqrt[4]{\frac{40000}{3}} \approx 10.7 \end{aligned}$$

So, any even number greater than or equal to 12 will do.

S-27: (a) From the figure, we see that the magnitude of $|f'''(x)|$ never exceeds 310 for $0 \leq x \leq 2$. So, the absolute error is bounded by

$$\frac{310(2-0)^5}{180 \times 8^4} \leq 0.01345$$

(b) We want to choose n such that:

$$\begin{aligned}\frac{310(2-0)^5}{180 \times n^4} &\leq 10^{-4} \\ n^4 &\geq \frac{310 \times 2^5}{180} 10^4 \\ n &\geq 10 \sqrt[4]{\frac{310 \times 32}{180}} \approx 27.2\end{aligned}$$

For Simpson's rule, n must be even, so any even integer obeying $n \geq 28$ will guarantee us the requisite accuracy.

S-28: Let $g(x) = \int_0^x \sin(\sqrt{t}) \, dt$. By the Fundamental Theorem of Calculus Part 1, $g'(x) = \sin(\sqrt{x})$. By its definition, $f(x) = g(x^2)$, so we use the chain rule to differentiate $f(x)$.

$$f'(x) = 2xg'(x^2) = 2x \sin x \qquad f''(x) = 2 \sin x + 2x \cos x$$

Since $|\sin x|, |\cos x| \leq 1$, we have $|f''(x)| \leq 2 + 2|x|$ and, for $0 \leq t \leq 1$, $|f''(t)| \leq 4$. When the trapezoidal rule with n subintervals is applied, the resulting error E_n obeys

$$E_n \leq \frac{4(1-0)^3}{12n^2} = \frac{1}{3n^2}$$

We want an integer n such that

$$\begin{aligned}\frac{1}{3n^2} &\leq 0.000005 \\ n^2 &\geq \frac{4}{12 \times 0.000005} \\ n &\geq \sqrt{\frac{1}{3 \times 0.000005}} \approx 258.2\end{aligned}$$

Any integer $n \geq 259$ will do.

S-29: Before we can take our Simpson's rule approximation of $\int_1^x \frac{1}{t} \, dt$, we need to know how many intervals to use. That means we need to bound our error, which means we need to bound $\frac{d^4}{dt^4} \left\{ \frac{1}{t} \right\}$.

$$\begin{aligned}\frac{d}{dt} \left\{ \frac{1}{t} \right\} &= -\frac{1}{t^2} & \frac{d^2}{dt^2} \left\{ \frac{1}{t} \right\} &= \frac{2}{t^3} \\ \frac{d^3}{dt^3} \left\{ \frac{1}{t} \right\} &= -\frac{6}{t^4} & \frac{d^4}{dt^4} \left\{ \frac{1}{t} \right\} &= \frac{24}{t^5}\end{aligned}$$

So, over the interval $[1, 3]$, $\left| \frac{d^4}{dt^4} \left\{ \frac{1}{t} \right\} \right| \leq 24$.

Now, we can find an appropriate n to ensure our error will be less than 0.1 for any x in $[1, 3]$:

$$\begin{aligned}\frac{L(b-a)^5}{180n^4} &< 0.1 \\ \frac{24(x-1)^5}{180n^4} &< \frac{1}{10} \\ n^4 &> \frac{24 \cdot (x-1)^5}{18}\end{aligned}$$

Because $x - 1 \leq 2$ for every x in $[1, 3]$, if $n^4 > \frac{24 \cdot 2^5}{18}$, then $n^4 > \frac{24 \cdot (x-1)^5}{18}$ for every allowed x .

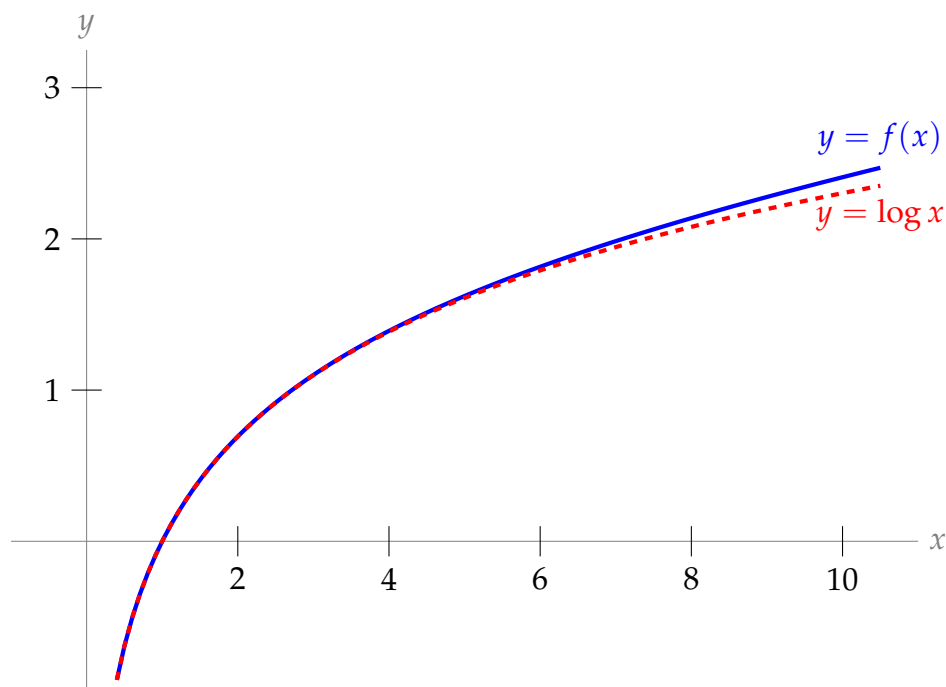
$$\begin{aligned}n^4 &> \frac{24 \cdot 2^5}{18} = \frac{128}{3} \\ n &> \sqrt[4]{\frac{128}{3}} \approx 2.6\end{aligned}$$

Since n must be even, $n = 4$ is enough intervals to guarantee our error is not too high for any x in $[1, 3]$. Now we find our Simpson's rule approximation with $n = 4$, $a = 1$, $b = x$, and $\Delta x = \frac{x-1}{4}$. The points where we evaluate $\frac{1}{t}$ are:

$$\begin{array}{ccccccccc}x_0 = 1 & x_1 = 1 + \frac{x-1}{4} & x_2 = 1 + 2\frac{x-1}{4} & x_3 = 1 + 3\frac{x-1}{4} & x_4 = 1 + 4\frac{x-1}{4} & & & & \\ & = \frac{x+3}{4} & = \frac{x+1}{2} & = \frac{3x+1}{4} & = x & & & & \\ \hline \leftarrow & | & | & | & | & | & & & \rightarrow \\ & 1 & \frac{x+3}{4} & \frac{x+1}{2} & \frac{3x+1}{4} & x & & & \\ & x_0 & x_1 & x_2 & x_3 & x_4 & & & \end{array}$$

$$\begin{aligned}\log x &= \int_1^x \frac{1}{t} dt \approx \frac{\Delta x}{3} \left[\frac{1}{x_0} + \frac{4}{x_1} + \frac{2}{x_2} + \frac{4}{x_3} + \frac{1}{x_4} \right] \\ &= \frac{x-1}{12} \left[1 + \frac{16}{x+3} + \frac{4}{x+1} + \frac{16}{3x+1} + \frac{1}{x} \right] \\ &= f(x)\end{aligned}$$

Below is a graph of our approximation $f(x)$ and natural logarithm on the same axes. The natural logarithm function is shown red and dashed, while our approximating function is solid blue. Our approximation appears to be quite accurate for small, positive values of x .



S-30: First, we want a strategy for approximating $\arctan 2$. Our hints are that involves integrating $\frac{1}{1+x^2}$, which is the antiderivative of arctangent, and the number $\frac{\pi}{4}$, which is the same as $\arctan(1)$. With that in mind:

$$\int_1^2 \frac{1}{1+x^2} dx = \arctan(2) - \arctan(1) = \arctan(2) - \frac{\pi}{4}$$

So, $\arctan(2) = \frac{\pi}{4} + \int_1^2 \frac{1}{1+x^2} dx$ (*)

We won't know the value of the integral exactly, but we'll have an approximation A bounded by some positive error bound ε . Then,

$$-\varepsilon \leq \left(\int_1^2 \frac{1}{1+x^2} dx - A \right) \leq \varepsilon$$

$$A - \varepsilon \leq \left(\int_1^2 \frac{1}{1+x^2} dx \right) \leq A + \varepsilon$$

So, from (*), $\frac{\pi}{4} + A - \varepsilon \leq \arctan(2) \leq \frac{\pi}{4} + A + \varepsilon$

Which approximation should we use? We're given the fourth derivative of $\frac{1}{1+x^2}$, which is the derivative we need for Simpson's rule. Simpson's rule is also usually quite efficient, and we're very interested in not adding up dozens of terms, so we choose Simpson's rule.

Now that we've chosen Simpson's rule, we should decide how many intervals to use. In order to bound our error, we need to find a bound for the fourth derivative. To that end, define $N(x) = 24(5x^4 - 10x^2 + 1)$. Then $N'(x) = 24(20x^3 - 20x) = 480x(x^2 - 1)$, which

is positive over the interval $[1, 2]$. So, $N(x) \leq N(2) = 24(5 \cdot 2^4 - 10 \cdot 2^2 + 1) = 984$ when $1 \leq x \leq 2$. Furthermore, let $D(x) = (x^2 + 1)^5$. If $1 \leq x \leq 2$, then $D(x) \geq 2^5$. Now we can find a reasonable value of L :

$$|f^{(4)}(x)| = \left| \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5} \right| = \left| \frac{N(x)}{D(x)} \right| \leq \frac{984}{2^5} = \frac{123}{4} = 30.75$$

So, we take $L = 30.75$.

We want $\left[\frac{\pi}{4} + A - \varepsilon, \frac{\pi}{4} + A + \varepsilon \right]$ to look something like $\left[\frac{\pi}{4} + 0.321, \frac{\pi}{4} + 0.323 \right]$. Note ε is half the length of the first interval. Half the length of the second interval is $0.001 = \frac{1}{1000}$. So, we want a value of ε that is no larger than this. Now we can find our n :

$$\begin{aligned} \frac{L(b-a)^5}{180 \cdot n^4} &\leq \frac{1}{1000} \\ \frac{30.75}{180 \cdot n^4} &\leq \frac{1}{1000} \\ n^4 &\geq \frac{30.75 \times 1000}{180} \\ n &\geq \sqrt[4]{\frac{30750}{180}} \approx 3.62 \end{aligned}$$

So, we choose $n = 4$, and are guaranteed that the absolute error in our approximation will be no more than $\frac{30.75}{180 \cdot 4^4} < 0.00067$.

Since $n = 4$, then $\Delta x = \frac{b-a}{n} = \frac{1}{4}$, so:

$$x_0 = 1 \quad x_1 = \frac{5}{4} \quad x_2 = \frac{3}{2} \quad x_3 = \frac{7}{4} \quad x_4 = 2$$

Now we can find our Simpson's rule approximation A :

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &\approx \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \\ &= \frac{1/4}{3} [f(1) + 4f(5/4) + 2f(3/2) + 4f(7/4) + f(2)] \\ &= \frac{1}{12} \left[\frac{1}{1+1} + \frac{4}{25/16+1} + \frac{2}{9/4+1} + \frac{4}{49/16+1} + \frac{1}{4+1} \right] \\ &= \frac{1}{12} \left[\frac{1}{2} + \frac{4 \cdot 16}{25+16} + \frac{2 \cdot 4}{9+4} + \frac{4 \cdot 16}{49+16} + \frac{1}{5} \right] \\ &= \frac{1}{12} \left[\frac{1}{2} + \frac{64}{41} + \frac{8}{13} + \frac{64}{65} + \frac{1}{5} \right] \\ &\approx 0.321748 = A \end{aligned}$$

As we saw before, the error associated with this approximation is at most

$$\frac{30.75}{180 \cdot 4^4} < 0.00067 = \varepsilon. \text{ So,}$$

$$\begin{aligned} A - \varepsilon &\leq \int_1^2 \frac{1}{1+x^2} dx \leq A + \varepsilon \\ \Rightarrow 0.321748 - 0.00067 &\leq \int_1^2 \frac{1}{1+x^2} dx \leq 0.321748 + 0.00067 \\ \Rightarrow 0.321078 &\leq \int_1^2 \frac{1}{1+x^2} dx \leq 0.322418 \\ \Rightarrow 0.321 &\leq \int_1^2 \frac{1}{1+x^2} dx \leq 0.323 \\ \Rightarrow \frac{\pi}{4} + 0.321 &\leq \int_1^2 \frac{1}{1+x^2} dx + \frac{\pi}{4} \leq \frac{\pi}{4} + 0.323 \\ \Rightarrow \frac{\pi}{4} + 0.321 &\leq \arctan(2) \leq \frac{\pi}{4} + 0.323 \end{aligned}$$

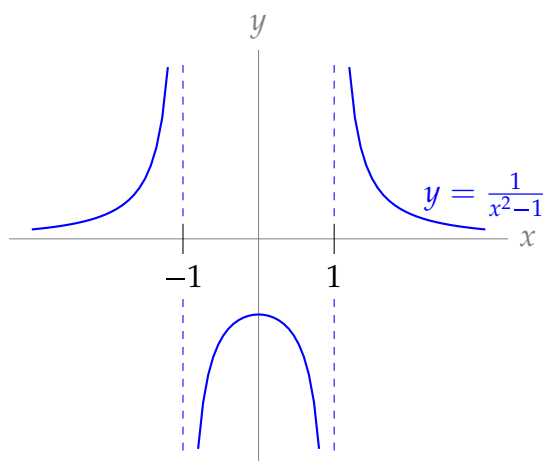
This is precisely what we wanted to show.

Solutions to Exercises 1.13 — [Jump to TABLE OF CONTENTS](#)

S-1: If $b = \pm\infty$, then our integral is improper because one limit is not a real number.

Furthermore, our integral will be improper if its domain of integration contains either of its infinite discontinuities, $x = 1$ and $x = -1$. Since one limit of integration is 0, the integral is improper if $b \geq 1$ or if $b \leq -1$.

Below, we've graphed $\frac{1}{x^2-1}$ to make it clearer why values of b in $(-1, 1)$ are the only values that *don't* result in an improper integral when the other limit of integration is $a = 0$.



S-2: Since the integrand is continuous for all real x , the only kind of impropriety available to us is to set $b = \pm\infty$.

S-3: For large values of x , $|\text{red function}| \leq (\text{blue function})$ and $0 \leq (\text{blue function})$. If the blue function's integral converged, then the red function's integral would as well (by the comparison test, Theorem 1.13.17 in the text). Since one integral converges and the other diverges, the blue function is $g(x)$ and the red function is $f(x)$.

S-4: False. The inequality goes the “wrong way” for Theorem 1.13.17 in the text: the area under the curve $f(x)$ is finite, but the area under $g(x)$ could be much larger, even infinitely larger.

For example, if $f(x) = e^{-x}$ and $g(x) = 1$, then $0 \leq f(x) \leq g(x)$ and $\int_1^\infty f(x) \, dx$ converges, but $\int_1^\infty g(x) \, dx$ diverges.

S-5:

(a) Not enough information to decide. For example, consider $h(x) = 0$ versus $h(x) = -1$. In both cases, $h(x) \leq f(x)$. However, $\int_0^\infty 0 \, dx$ converges to 0, while $\int_0^\infty -1 \, dx$ diverges.

Note: if we had also specified $0 \leq h(x)$, then we would be able to conclude that $\int_0^\infty h(x) \, dx$ converges by the comparison test.

(b) Not enough information to decide. For example, consider $h(x) = f(x)$ versus $h(x) = g(x)$. In both cases, $f(x) \leq h(x) \leq g(x)$.

(c) $\int_0^\infty h(x) \, dx$ converges.

- From the given information, $|h(x)| \leq 2f(x)$.

- We claim $\int_0^\infty 2f(x) \, dx$ converges.

- We can see this by writing $\int_0^\infty 2f(x) \, dx = 2 \int_0^\infty f(x) \, dx$ and noting that the second integral converges.

- Alternately, we can use the limiting comparison test, Theorem 1.13.22 in the text. Since $f(x) \geq 0$, $\int_0^\infty f(x) \, dx$ converges, and $\lim_{x \rightarrow \infty} \frac{2f(x)}{f(x)} = 2$ (the limit exists), we conclude $\int_0^\infty 2f(x) \, dx$ converges.

- So, comparing $h(x)$ to $2f(x)$, by the comparison test (Theorem 1.13.17 in the text) $\int_0^\infty h(x) \, dx$ converges.

S-6: The denominator is zero when $x = 1$, but the numerator is not, so the integrand has a singularity (infinite discontinuity) at $x = 1$. Let's replace the limit $x = 1$ with a variable that creeps toward 1.

$$\int_0^1 \frac{x^4}{x^5 - 1} \, dx = \lim_{t \rightarrow 1^-} \int_0^t \frac{x^4}{x^5 - 1} \, dx$$

To evaluate this integral we use the substitution $u = x^5$, $du = 5x^4 dx$. When $x = 0$ we have $u = 0$, and when $x = t$ we have $u = t^5$, so

$$\begin{aligned} \int_0^1 \frac{x^4}{x^5 - 1} \, dx &= \lim_{t \rightarrow 1^-} \int_0^t \frac{x^4}{x^5 - 1} \, dx = \lim_{t \rightarrow 1^-} \int_{u=0}^{u=t^5} \frac{1}{5(u - 1)} \, du \\ &= \lim_{t \rightarrow 1^-} \left(\left[\frac{1}{5} \log |u - 1| \right]_0^{t^5} \right) = \lim_{t \rightarrow 1^-} \frac{1}{5} \log |t^5 - 1| = -\infty \end{aligned}$$

The limit does not exist, so the integral diverges.

S-7: The denominator of the integrand is zero when $x = -1$, but the numerator is not. So, the integrand has a singularity (infinite discontinuity) at $x = -1$. This is the only "source of impropriety" in this integral, so we only need to make one break in the domain of integration.

$$\int_{-2}^2 \frac{1}{(x+1)^{4/3}} \, dx = \lim_{t \rightarrow -1^-} \int_{-2}^t \frac{1}{(x+1)^{4/3}} \, dx + \lim_{t \rightarrow -1^+} \int_t^2 \frac{1}{(x+1)^{4/3}} \, dx$$

Let's start by considering the left limit.

$$\begin{aligned} \lim_{t \rightarrow -1^-} \int_{-2}^t \frac{1}{(x+1)^{4/3}} \, dx &= \lim_{t \rightarrow -1^-} \left(\left[-\frac{3}{(x+1)^{1/3}} \right]_{-2}^t \right) \\ &= \lim_{t \rightarrow -1^-} \left(-\frac{3}{(t+1)^{1/3}} + \frac{3}{(-1)^{1/3}} \right) = \infty \end{aligned}$$

Since this limit diverges, the integral diverges. (A similar argument shows that the second integral diverges. Either one of them diverging is enough to conclude that the original integral diverges.)

S-8: First, let's identify all "sources of impropriety." The integrand has a singularity when $4x^2 - x = 0$, that is, when $x(4x - 1) = 0$, so at $x = 0$ and $x = \frac{1}{4}$. Neither of these are in our domain of integration, so the only "source of impropriety" is the unbounded domain of integration.

We could antidifferentiate this function (it looks like a nice candidate for a trig substitution), but it seems easier to use a comparison. For large values of x , the term x^2

will be much larger than x , so we might guess that our integral behaves similarly to $\int_1^\infty \frac{1}{\sqrt{4x^2}} dx = \int_1^\infty \frac{1}{2x} dx$.

For all $x \geq 1$, $\sqrt{4x^2 - x} \leq \sqrt{4x^2} = 2x$. So, $\frac{1}{\sqrt{4x^2 - x}} \geq \frac{1}{2x}$. Note $\int_1^\infty \frac{1}{2x} dx$ diverges:

$$\lim_{t \rightarrow \infty} \int_1^t \frac{1}{2x} dx = \lim_{t \rightarrow \infty} \left(\frac{1}{2} [\log x]_1^t \right) = \lim_{t \rightarrow \infty} \frac{1}{2} \log t = \infty$$

So:

- $\frac{1}{2x}$ and $\frac{1}{\sqrt{4x^2 - x}}$ are defined and continuous for all $x \geq 1$,
- $\frac{1}{2x} \geq 0$ for all $x \geq 1$,
- $\frac{1}{\sqrt{4x^2 - x}} \geq \frac{1}{\sqrt{4x^2}} = \frac{1}{2x}$ for all $x \geq 1$, and
- $\int_1^\infty \frac{1}{2x} dx$ diverges.

By the comparison test, Theorem 1.13.17 in the text, the integral does not converge.

S-9: The integrand is positive everywhere. So, either the integral converges to some finite number, or it is infinite. We want to generate a guess as to which it is.

When x is small, $\sqrt{x} > x^2$, so we might guess that our integral behaves like the integral of $\frac{1}{\sqrt{x}}$ when x is near to 0. On the other hand, when x is large, $\sqrt{x} < x^2$, so we might guess that our integral behaves like the integral of $\frac{1}{x^2}$ as x goes to infinity. This is the hunch that drives the following work:

$$0 \leq \frac{1}{x^2 + \sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{and the integral } \int_0^1 \frac{dx}{\sqrt{x}} \text{ converges by Example 1.13.9 in the text}$$

$$0 \leq \frac{1}{x^2 + \sqrt{x}} \leq \frac{1}{x^2} \quad \text{and the integral } \int_1^\infty \frac{dx}{x^2} \text{ converges by Example 1.13.8 in the text}$$

Note that $\frac{1}{x^2 + \sqrt{x}}$ is defined and continuous for all $x > 0$, $\frac{1}{\sqrt{x}}$ is defined and continuous for $x > 0$, and $\frac{1}{x^2}$ is defined and continuous for $x \geq 1$. So, the integral converges by the comparison test, Theorems 1.13.17 and 1.13.24 in the text.

S-10: There are two “sources of impropriety”: the two (infinite) limits of integration. So, we break our integral into two pieces.

$$\begin{aligned} \int_{-\infty}^{\infty} \cos x \, dx &= \int_{-\infty}^0 \cos x \, dx + \int_0^{\infty} \cos x \, dx \\ &= \lim_{a \rightarrow -\infty} \left[\int_{-a}^0 \cos x \, dx \right] + \lim_{b \rightarrow \infty} \left[\int_0^b \cos x \, dx \right] \end{aligned}$$

These are easy enough to antdifferentiate.

$$\begin{aligned} &= \lim_{a \rightarrow -\infty} [\sin 0 - \sin(-a)] + \lim_{b \rightarrow \infty} [\sin b - \sin 0] \\ &= \text{DNE} \end{aligned}$$

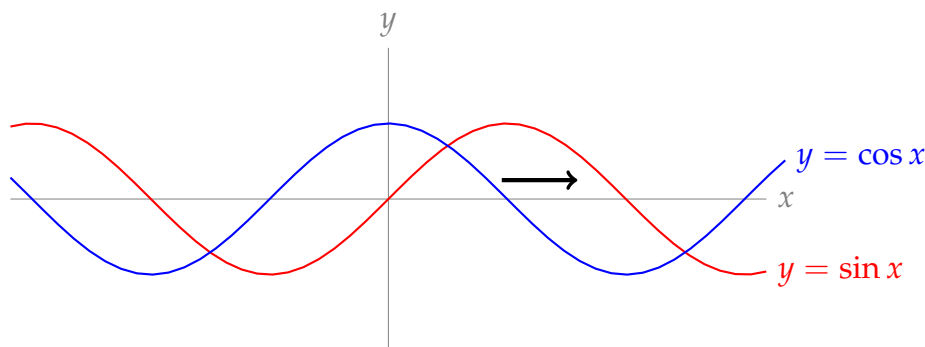
Since the limits don't exist, the integral diverges. (It happens that both limits don't exist; even if only one failed to exist, the integral would still diverge.)

S-11: There are two “sources of impropriety”: the $-\infty$ and the $+\infty$. So, we break our integral into two pieces.

$$\begin{aligned}\int_{-\infty}^{\infty} \sin x \, dx &= \int_{-\infty}^0 \sin x \, dx + \int_0^{\infty} \sin x \, dx \\ &= \lim_{a \rightarrow \infty} \left[\int_{-a}^0 \sin x \, dx \right] + \lim_{b \rightarrow \infty} \left[\int_0^b \sin x \, dx \right] \\ &= \lim_{a \rightarrow \infty} [-\cos 0 + \cos(-a)] + \lim_{b \rightarrow \infty} [-\cos b + \cos 0] \\ &= \text{DNE}\end{aligned}$$

Since the limits don't exist, the integral diverges. (It happens that both limits don't exist; even if only one failed to exist, the integral would diverge.)

Remark: it's very tempting to think that this integral should converge, because as an odd function the area to the right of the x -axis “cancels out” the area to the left when the limits of integration are symmetric. One justification for not using this intuition is given in Example 1.13.11 in the text. Here's another: In Question 10 we saw that $\int_{-\infty}^{\infty} \cos x \, dx$ diverges. Since $\sin x = \cos(x - \pi/2)$, the area bounded by sine and the area bounded by cosine over an infinite region seem to be the same—only shifted by $\pi/2$. So if $\int_{-\infty}^{\infty} \sin x \, dx = 0$, then we ought to also have $\int_{-\infty}^{\infty} \cos x \, dx = 0$, but we saw in Question 10 this is not the case.



S-12: First, we check that the integrand has no singularities. The denominator is always positive when $x \geq 10$, so our only “source of impropriety” is the infinite limit of integration.

We further note that, for large values of x , the integrand resembles $\frac{x^4}{x^5} = \frac{1}{x}$. So, we have a two-part hunch: that the integral diverges, and that we can show it diverges by comparing it to $\int_{10}^{\infty} \frac{1}{x} \, dx$.

In order to use the comparison test, we'd need to show that $\frac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8} \geq \frac{1}{x}$. If this

is true, it will be difficult to prove—and it's not at all clear that it's true. So, we will use the limiting comparison test instead, Theorem 1.13.22 in the text, with $g(x) = \frac{1}{x}$,

$$f(x) = \frac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8}, \text{ and } a = 10.$$

- Both $f(x)$ and $g(x)$ are defined and continuous for all $x > 0$, so in particular they are defined and continuous for $x \geq 10$.
- $g(x) \geq 0$ for all $x \geq 10$
- $\int_{10}^{\infty} g(x) \, dx$ diverges.
- Using l'Hôpital's rule (5 times!), or simply dividing both the numerator and denominator by x^5 (the common leading term), tells us:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{\frac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} x \cdot \frac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8} \\ &= \lim_{x \rightarrow \infty} \frac{x^5 - 5x^4 + 2x^2 - 7x}{x^5 + 3x + 8} = 1 \end{aligned}$$

That is, the limit exists and is nonzero.

By the limiting comparison test, we conclude $\int_{10}^{\infty} f(x) \, dx$ diverges.

S-13: Our domain of integration is finite, so the only potential “sources of impropriety” are infinite discontinuities in the integrand. To find these, we factor.

$$\int_0^{10} \frac{x-1}{x^2 - 11x + 10} \, dx = \int_0^{10} \frac{x-1}{(x-1)(x-10)} \, dx$$

A removable discontinuity doesn't affect the integral.

$$= \int_0^{10} \frac{1}{x-10} \, dx$$

Use the substitution $u = x - 10$, $du = dx$. When $x = 0$, $u = -10$, and when $x = 10$, $u = 0$.

$$= \int_{-10}^0 \frac{1}{u} \, du$$

This is a p -integral with $p = 1$. From Example 1.13.9 and Theorem 1.13.20 in the text, we know it diverges.

S-14: You might think that, because the integrand is odd, the integral converges to 0. This is a common mistake—see Example 1.13.11 in the text, or Question 11 in this section. In the absence of such a shortcut, we use our standard procedure: identifying problem spots over the domain of integration, and replacing them with limits.

There are two “sources of impropriety,” namely $x \rightarrow +\infty$ and $x \rightarrow -\infty$. So, we split the integral in two, and treat the two halves separately. The integrals below can be evaluated with the substitution $u = x^2 + 1$, $\frac{1}{2}du = xdx$.

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x}{x^2+1} dx &= \int_{-\infty}^0 \frac{x}{x^2+1} dx + \int_0^{+\infty} \frac{x}{x^2+1} dx \\ \int_{-\infty}^0 \frac{x}{x^2+1} dx &= \lim_{R \rightarrow \infty} \int_{-R}^0 \frac{x}{x^2+1} dx = \lim_{R \rightarrow \infty} \frac{1}{2} \log(x^2+1) \Big|_{-R}^0 \\ &= \lim_{R \rightarrow \infty} \frac{1}{2} [\log 1 - \log(R^2+1)] = \lim_{R \rightarrow \infty} -\frac{1}{2} \log(R^2+1) = -\infty \\ \int_0^{+\infty} \frac{x}{x^2+1} dx &= \lim_{R \rightarrow \infty} \int_0^R \frac{x}{x^2+1} dx = \lim_{R \rightarrow \infty} \frac{1}{2} \log(x^2+1) \Big|_0^R \\ &= \lim_{R \rightarrow \infty} \frac{1}{2} [\log(R^2+1) - \log 1] = \lim_{R \rightarrow \infty} \frac{1}{2} \log(R^2+1) = +\infty\end{aligned}$$

Both halves diverge, so the whole integral diverges.

Once again: after we found that one of the limits diverged, we could have stopped and concluded that the original integrand diverges. Don’t make the mistake of thinking that $\infty - \infty = 0$. That can get you into *big* trouble. ∞ is not a normal number. For example $2\infty = \infty$. So if ∞ were a normal number we would have both $\infty - \infty = 0$ and $\infty - \infty = 2\infty - \infty = \infty$.

S-15: We don’t want to antidifferentiate this integrand, so let’s use a comparison. Note the integrand is positive when $x > 0$.

For any x , $|\sin x| \leq 1$, so $\frac{|\sin x|}{x^{3/2} + x^{1/2}} \leq \frac{1}{x^{3/2} + x^{1/2}}$.

Since $x = 0$ and $x \rightarrow \infty$ both cause the integral to be improper, we need to break it into two pieces. Since both terms in the denominator give positive numbers when x is positive, $\frac{1}{x^{3/2} + x^{1/2}} \leq \frac{1}{x^{3/2}}$ and $\frac{1}{x^{3/2} + x^{1/2}} \leq \frac{1}{x^{1/2}}$. That gives us two options for comparison.

When x is positive and close to zero, $x^{1/2} \geq x^{3/2}$, so we guess that we should compare our integrand to $\frac{1}{x^{1/2}}$ near the limit $x = 0$. In contrast, when x is very large, $x^{1/2} \leq x^{3/2}$, so we guess that we should compare our integrand to $\frac{1}{x^{3/2}}$ as x goes to infinity.

$\frac{|\sin x|}{x^{3/2} + x^{1/2}} \leq \frac{1}{x^{1/2}}$ and the integral $\int_0^1 \frac{dx}{x^{1/2}}$ converges by the p -test, Example 1.13.9 in the text

$\frac{|\sin x|}{x^{3/2} + x^{1/2}} \leq \frac{1}{x^{3/2}}$ and the integral $\int_1^\infty \frac{dx}{x^{3/2}}$ converges by the p -test, Example 1.13.8 in the text

Now we have all the data we need to apply the comparison test, Theorems 1.13.17 and 1.13.24 in the text.

-
- $\frac{|\sin x|}{x^{3/2} + x^{1/2}}$, $\frac{1}{x^{1/2}}$, and $\frac{1}{x^{3/2}}$ are defined and continuous for $x > 0$
 - $\frac{1}{x^{1/2}}$ and $\frac{1}{x^{3/2}}$ are nonnegative for $x \geq 0$
 - $\frac{|\sin x|}{x^{3/2} + x^{1/2}} \leq \frac{1}{x^{1/2}}$ for all $x > 0$ and $\int_0^1 \frac{1}{x^{1/2}} dx$ converges, so $\int_0^1 \frac{|\sin x|}{x^{3/2} + x^{1/2}} dx$ converges.
 - $\frac{|\sin x|}{x^{3/2} + x^{1/2}} \leq \frac{1}{x^{3/2}}$ for all $x \geq 1$ and $\int_1^\infty \frac{1}{x^{3/2}} dx$ converges, so $\int_1^\infty \frac{|\sin x|}{x^{3/2} + x^{1/2}} dx$ converges.

Therefore, our integral $\int_0^\infty \frac{|\sin x|}{x^{3/2} + x^{1/2}} dx$ converges.

S-16: The integrand is positive everywhere, so either the integral converges to some finite number or it is infinite. There are two potential “sources of impropriety” — a possible singularity at $x = 0$ and the fact that the domain of integration extends to ∞ . So we split up the integral.

$$\int_0^\infty \frac{x+1}{x^{1/3}(x^2+x+1)} dx = \int_0^1 \frac{x+1}{x^{1/3}(x^2+x+1)} dx + \int_1^\infty \frac{x+1}{x^{1/3}(x^2+x+1)} dx$$

Let’s develop a hunch about whether the integral converges or diverges. When $x \approx 0$, x^2 and x are both a lot smaller than 1, so we guess we should compare the integrand to $\frac{1}{x^{1/3}}$.

$$\frac{x+1}{x^{1/3}(x^2+x+1)} \approx \frac{1}{x^{1/3}(1)} = \frac{1}{x^{1/3}}$$

Note $\int_0^1 \frac{1}{x^{1/3}} dx$ converges by Example 1.13.9 in the text (it’s a p -type integral), so we guess $\int_0^1 \frac{x+1}{x^{1/3}(x^2+x+1)} dx$ converges as well.

When x is very large, x^2 is much bigger than x , which is much bigger than 1, so we guess we should compare the integrand to $\frac{1}{x^{4/3}}$.

$$\frac{x+1}{x^{1/3}(x^2+x+1)} \approx \frac{x}{x^{1/3}(x^2)} = \frac{1}{x^{4/3}}$$

Note $\int_1^\infty \frac{1}{x^{4/3}} dx$ converges by Example 1.13.8 in the text (it’s a p -type integral), so we guess $\int_1^\infty \frac{x+1}{x^{1/3}(x^2+x+1)} dx$ converges as well.

Now it’s time to verify our guesses with the limiting comparison test, Theorems 1.13.22 and 1.13.25 in the text. Be careful: our “ $\approx$ ” signs are not strong enough to use either the limiting comparison test or the comparison test, they are only enough to suggest a reasonable function to compare to.

- $\frac{x+1}{x^{1/3}(x^2+x+1)}$, $\frac{1}{x^{1/3}}$, and $\frac{1}{x^{4/3}}$ are defined and continuous for all $x > 0$
- $\frac{1}{x^{1/3}}$ and $\frac{1}{x^{4/3}}$ are positive for all $x > 0$

-
- $\int_0^1 \frac{1}{x^{1/3}} dx$ and $\int_1^\infty \frac{1}{x^{4/3}} dx$ both converge
 - $\lim_{x \rightarrow 0} \frac{\frac{x+1}{x^{1/3}(x^2+x+1)}}{\frac{1}{x^{1/3}}} = \lim_{x \rightarrow 0} \frac{x+1}{x^2+x+1} = \frac{0+1}{0+0+1} = 1$; in particular, this limit exists.
 - Using the limiting comparison test (Theorem 1.13.25 in the text), $\int_0^1 \frac{x+1}{x^{1/3}(x^2+x+1)} dx$ converges.
 - $\lim_{x \rightarrow \infty} \frac{\frac{x+1}{x^{1/3}(x^2+x+1)}}{\frac{1}{x^{4/3}}} = \lim_{x \rightarrow \infty} \frac{x(x+1)}{x^2+x+1} = 1$; in particular, this limit exists.
 - Using the limiting comparison test (Theorem 1.13.22 in the text), $\int_1^\infty \frac{x+1}{x^{1/3}(x^2+x+1)} dx$ converges.

We conclude $\int_0^\infty \frac{x+1}{x^{1/3}(x^2+x+1)} dx$ converges.

S-17: To find the volume of the solid, we cut it into horizontal slices, which are thin circular disks. At height y , the disk has radius $x = \frac{1}{y}$ and thickness dy , so its volume is $\frac{\pi}{y^2} dy$. The base of the solid is at height $y = 1$, and its top is at height $y = \frac{1}{a}$. So, the volume of the entire solid is:

$$\int_1^{1/a} \frac{\pi}{y^2} dy = \left[-\frac{\pi}{y} \right]_1^{1/a} = \pi(1 - a)$$

If we imagine sliding a closer and closer to 0, the volume increases, getting closer and closer to π units, but never quite reaching it.

So, the statement is false. For example, if we set $M = 4$, no matter which a we choose our solid has volume strictly less than M .

Remark: we've seen before that $\int_0^1 \frac{1}{x} dx$ diverges. If we imagine the solid that would result from choosing $a = 0$, it would have a scant volume of π cubic units, but a silhouette (side view) of infinite area.

S-18: Our goal is to decide when this integral diverges, and where it converges. We will leave q as a variable, and antidifferentiate. In order to antidifferentiate without knowing q , we'll need different cases. The integrand is x^{-5q} , so when $-5q \neq -1$, we use the power

rule (that is, $\int x^n dx = \frac{x^{n+1}}{n+1}$) to antidifferentiate. Note $x^{(-5q)+1} = x^{1-5q} = \frac{1}{x^{5q-1}}$.

$$\begin{aligned} \int_1^t \frac{1}{x^{5q}} dx &= \begin{cases} \left[\frac{x^{1-5q}}{1-5q} \right]_1^t \text{ with } 1-5q > 0 & \text{if } q < \frac{1}{5} \\ \left[\log x \right]_1^t & \text{if } q = \frac{1}{5} \\ \left[\frac{1}{(1-5q)x^{5q-1}} \right]_1^t \text{ with } 5q-1 > 0 & \text{if } q > \frac{1}{5} \end{cases} \\ &= \begin{cases} \frac{1}{1-5q} (t^{1-5q} - 1) \text{ with } 1-5q > 0 & \text{if } q < \frac{1}{5} \\ \log t & \text{if } q = \frac{1}{5} \\ \frac{1}{5q-1} (1 - \frac{1}{t^{5q-1}}) \text{ with } 5q-1 > 0 & \text{if } q > \frac{1}{5}. \end{cases} \end{aligned}$$

Therefore,

$$\int_1^\infty \frac{1}{x^{5q}} dx = \lim_{t \rightarrow \infty} \left(\int_1^t \frac{1}{x^{5q}} dx \right) = \begin{cases} \frac{1}{1-5q} \left(\lim_{t \rightarrow \infty} t^{1-5q} - 1 \right) = \infty & \text{if } q < \frac{1}{5} \\ \lim_{t \rightarrow \infty} \log t = \infty & \text{if } q = \frac{1}{5} \\ \frac{1}{5q-1} \left(1 - \lim_{t \rightarrow \infty} \frac{1}{t^{5q-1}} \right) = \frac{1}{5q-1} & \text{if } q > \frac{1}{5}. \end{cases}$$

The first two cases are divergent, and so the largest such value is $q = \frac{1}{5}$. (Alternatively, we might recognize this as a “ p -integral” with $p = 5q$, and recall that the p -integral diverges precisely when $p \leq 1$.)

S-19: This integrand is a nice candidate for the substitution $u = x^2 + 1$, $\frac{1}{2}du = xdx$. Remember when we use substitution on a definite integral, we also need to adjust the limits of integration.

$$\begin{aligned} \int_0^\infty \frac{x}{(x^2 + 1)^p} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{x}{(x^2 + 1)^p} dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_1^{t^2+1} \frac{1}{u^p} du \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_1^{t^2+1} u^{-p} du \\ &= \begin{cases} \frac{1}{2} \lim_{t \rightarrow \infty} \left[\frac{u^{1-p}}{1-p} \right]_1^{t^2+1} & \text{if } p \neq 1 \\ \frac{1}{2} \lim_{t \rightarrow \infty} \left[\log |u| \right]_1^{t^2+1} & \text{if } p = 1 \end{cases} \\ &= \begin{cases} \frac{1}{2} \lim_{t \rightarrow \infty} \frac{1}{1-p} \left[(t^2 + 1)^{1-p} - 1 \right] & \text{if } p \neq 1 \\ \frac{1}{2} \lim_{t \rightarrow \infty} \left[\log(t^2 + 1) \right] = \infty & \text{if } p = 1 \end{cases} \end{aligned}$$

At this point, we can see that the integral diverges when $p = 1$. When $p \neq 1$, we have the limit

$$\lim_{t \rightarrow \infty} \frac{1/2}{1-p} \left[(t^2 + 1)^{1-p} - 1 \right] = \frac{1/2}{1-p} \left[\lim_{t \rightarrow \infty} (t^2 + 1)^{1-p} \right] - \frac{1/2}{1-p}$$

Since $t^2 + 1 \rightarrow \infty$, this limit converges exactly when the exponent $1 - p$ is negative; that is, it converges when $p > 1$, and diverges when $p < 1$.

So, the integral in the question converges when $p > 1$.

S-20:

- First, we notice there is only one “source of impropriety:” the domain of integration is infinite. (The integrand has a singularity at $t = 1$, but this is not in the domain of integration, so it’s not a problem for us.)
- We should try to get some intuition about whether the integral converges or diverges. When $t \rightarrow \infty$, notice the integrand “looks like” the function $\frac{1}{t^4}$. We know $\int_1^\infty \frac{1}{t^4} dt$ converges, because it’s a p -integral with $p = 4 > 1$ (see Example 1.13.8 in the text). So, our integral probably converges as well. If we were only asked show it converges, we could use a comparison test, but we’re asked more than that.
- Since we guess the integral converges, we’ll need to evaluate it. The integrand is a rational function, and there’s no obvious substitution, so we use partial fractions.

$$\frac{1}{t^4 - 1} = \frac{1}{(t^2 + 1)(t^2 - 1)} = \frac{1}{(t^2 + 1)(t + 1)(t - 1)} = \frac{At + B}{t^2 + 1} + \frac{C}{t + 1} + \frac{D}{t - 1}$$

Multiply by the original denominator.

$$1 = (At + B)(t + 1)(t - 1) + C(t^2 + 1)(t - 1) + D(t^2 + 1)(t + 1) \quad (*)$$

Set $t = 1$.

$$1 = 0 + 0 + D(2)(2) \quad \Rightarrow \quad D = \frac{1}{4}$$

Set $t = -1$.

$$1 = 0 + C(2)(-2) + 0 \quad \Rightarrow \quad C = -\frac{1}{4}$$

Simplify (*) using $D = \frac{1}{4}$ and $C = -\frac{1}{4}$.

$$\begin{aligned} 1 &= (At + B)(t + 1)(t - 1) - \frac{1}{4}(t^2 + 1)(t - 1) + \frac{1}{4}(t^2 + 1)(t + 1) \\ &= (At + B)(t + 1)(t - 1) + \frac{1}{2}(t^2 + 1) \\ &= At^3 + \left(B + \frac{1}{2}\right)t^2 - At + \left(\frac{1}{2} - B\right) \end{aligned}$$

By matching up coefficients of corresponding powers of t , we find $A = 0$ and $B = -\frac{1}{2}$.

$$\begin{aligned}\int_2^\infty \frac{1}{t^4 - 1} dt &= \int_2^\infty \left(\frac{-1/2}{t^2 + 1} - \frac{1/4}{t + 1} + \frac{1/4}{t - 1} \right) dt \\&= \lim_{R \rightarrow \infty} \int_2^R \left(\frac{-1/2}{t^2 + 1} - \frac{1/4}{t + 1} + \frac{1/4}{t - 1} \right) dt \\&= \lim_{R \rightarrow \infty} \left[-\frac{1}{2} \arctan t - \frac{1}{4} \log |t + 1| + \frac{1}{4} \log |t - 1| \right]_2^R \\&= \lim_{R \rightarrow \infty} \left[-\frac{1}{2} \arctan t + \frac{1}{4} \log \left| \frac{t - 1}{t + 1} \right| \right]_2^R \\&= \lim_{R \rightarrow \infty} \left(-\frac{1}{2} \arctan R + \frac{1}{2} \arctan 2 + \frac{1}{4} \log \left| \frac{R - 1}{R + 1} \right| - \frac{1}{4} \log \left| \frac{2 - 1}{2 + 1} \right| \right)\end{aligned}$$

We can use l'Hôpital's rule to see $\lim_{R \rightarrow \infty} \frac{R - 1}{R + 1} = 1$. Also note $-\log(1/3) = \log 3$.

$$\begin{aligned}&= -\frac{1}{2} \left(\frac{\pi}{2} \right) + \frac{1}{2} \arctan 2 + \frac{1}{4} \log 1 + \frac{1}{4} \log 3 \\&= \frac{\log 3 - \pi}{4} + \frac{1}{2} \arctan 2\end{aligned}$$

S-21: There are three singularities in the integrand: $x = 0$, $x = 1$, and $x = 2$. We'll need to break up the integral at each of these places.

$$\begin{aligned}&\int_{-5}^5 \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x - 1|}} + \frac{1}{\sqrt{|x - 2|}} \right) dx \\&= \int_{-5}^0 \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x - 1|}} + \frac{1}{\sqrt{|x - 2|}} \right) dx + \int_0^1 \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x - 1|}} + \frac{1}{\sqrt{|x - 2|}} \right) dx \\&\quad + \int_1^2 \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x - 1|}} + \frac{1}{\sqrt{|x - 2|}} \right) dx + \int_2^5 \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x - 1|}} + \frac{1}{\sqrt{|x - 2|}} \right) dx\end{aligned}$$

This looks rather unfortunate. Let's think again. If all of the integrals below converge, then we can write:

$$\int_{-5}^5 \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x - 1|}} + \frac{1}{\sqrt{|x - 2|}} \right) dx = \int_{-5}^5 \frac{1}{\sqrt{|x|}} dx + \int_{-5}^5 \frac{1}{\sqrt{|x - 1|}} dx + \int_{-5}^5 \frac{1}{\sqrt{|x - 2|}} dx$$

That looks a lot better. Also, we have a good reason to guess these integrals converge—they look like p -integrals with $p = \frac{1}{2}$. Let's take a closer look at each one.

$$\begin{aligned}\int_{-5}^5 \frac{1}{\sqrt{|x|}} dx &= \int_{-5}^0 \frac{1}{\sqrt{|x|}} dx + \int_0^5 \frac{1}{\sqrt{|x|}} dx \\&= 2 \int_0^5 \frac{1}{\sqrt{x}} dx \quad (\text{even function}) \\&= 2 \int_0^5 \frac{1}{\sqrt{x}} dx\end{aligned}$$

This is a p -integral, with $p = \frac{1}{2}$. By Example 1.13.9 in the text (and Theorem 1.13.20, since the upper limit of integration is not 1), it converges. The other two pieces behave similarly.

$$\int_{-5}^5 \frac{1}{\sqrt{|x-1|}} dx = \int_{-5}^1 \frac{1}{\sqrt{|x-1|}} dx + \int_1^5 \frac{1}{\sqrt{|x-1|}} dx$$

Use $u = x - 1$, $du = dx$

$$\begin{aligned} &= \int_{-6}^0 \frac{1}{\sqrt{|u|}} du + \int_0^4 \frac{1}{\sqrt{|u|}} du \\ &= \int_0^6 \frac{1}{\sqrt{u}} du + \int_0^4 \frac{1}{\sqrt{u}} du \end{aligned}$$

Since our function is even, we use the reasoning of Example 1.2.10 in the text to consider the area under the curve when $x \geq 0$, rather than when $x \leq 0$. Again, these are p -integrals with $p = \frac{1}{2}$, so they both converge. Finally:

$$\int_{-5}^5 \frac{1}{\sqrt{|x-2|}} dx = \int_{-5}^2 \frac{1}{\sqrt{|x-2|}} dx + \int_2^5 \frac{1}{\sqrt{|x-2|}} dx$$

Use $u = x - 2$, $du = dx$.

$$\begin{aligned} &= \int_{-7}^0 \frac{1}{\sqrt{|u|}} du + \int_0^3 \frac{1}{\sqrt{|u|}} du \\ &= \int_0^7 \frac{1}{\sqrt{u}} du + \int_0^3 \frac{1}{\sqrt{u}} du \end{aligned}$$

Since $p = \frac{1}{2}$, so they both converge.

We conclude our original integral, as the sum of convergent integrals, converges.

S-22: The integrand is positive everywhere. So either the integral converges to some finite number or it is infinite. There are two potential “sources of impropriety” — a possible singularity at $x = 0$ and the fact that the domain of integration extends to ∞ . So, we split up the integral.

$$\int_0^\infty \frac{\sin^4 x}{x^2} dx = \int_0^1 \frac{\sin^4 x}{x^2} dx + \int_1^\infty \frac{\sin^4 x}{x^2} dx$$

Let's consider the first integral. Recall $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Consequently,

$$\lim_{x \rightarrow 0} \frac{\sin^4 x}{x^2} = \left(\lim_{x \rightarrow 0} \sin^2 x \right) \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) = 0 \times 1 \times 1 = 0$$

and the first integral is not even improper.

Now for the second integral. Since $|\sin x| \leq 1$, we'll compare it to $\int_1^\infty \frac{1}{x^2}$.

- $\frac{\sin^4 x}{x^2}$ and $\frac{1}{x^2}$ are defined and continuous for every $x \geq 1$
- $0 \leq \frac{\sin^4 x}{x^2} \leq \frac{1^4}{x^2} = \frac{1}{x^2}$ for every $x \geq 1$
- $\int_1^\infty \frac{1}{x^2} dx$ converges by Example 1.13.8 in the text (it's a p -type integral with $p > 1$)

By the comparison test, Theorem 1.13.17 in the text, $\int_1^\infty \frac{\sin^4 x}{x^2} dx$ converges.

Since $\int_0^1 \frac{\sin^4 x}{x^2} dx$ and $\int_1^\infty \frac{\sin^4 x}{x^2} dx$ both converge, we conclude $\int_0^\infty \frac{\sin^4 x}{x^2} dx$ converges as well.

S-23: Since the denominator is positive for all $x \geq 0$, the integrand is continuous over $[0, \infty)$. So, the only “source of impropriety” is the infinite domain of integration.

Solution 1: Let's try to use a direct comparison. Note $\frac{x}{e^x + \sqrt{x}} \geq 0$ whenever $x \geq 0$. Also note that, for large values of x , e^x is much larger than $\sqrt{x}$. That leads us to consider the following inequality:

$$0 \leq \frac{x}{e^x + \sqrt{x}} \leq \frac{x}{e^x}$$

If $\int_0^\infty \frac{x}{e^x} dx$ converges, we're in business. Let's figure it out. The integrand looks like a candidate for integration by parts: take $u = x$, $dv = e^{-x} dx$, so $du = dx$ and $v = -e^{-x}$.

$$\begin{aligned} \int_0^\infty \frac{x}{e^x} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{x}{e^x} dx = \lim_{b \rightarrow \infty} \left(\left[-\frac{x}{e^x} \right]_0^b + \int_0^b e^{-x} dx \right) \\ &= \lim_{b \rightarrow \infty} \left(-\frac{b}{e^b} + \left[-e^{-x} \right]_0^b \right) = \lim_{b \rightarrow \infty} \left(-\frac{b}{e^b} - \frac{1}{e^b} + 1 \right) \\ &= \lim_{b \rightarrow \infty} \left(1 - \underbrace{\frac{b+1}{e^b}}_{\substack{\text{num} \rightarrow \infty \\ \text{den} \rightarrow \infty}} \right) = \lim_{b \rightarrow \infty} \left(1 - \frac{1}{e^b} \right) = 1 \end{aligned}$$

Using l'Hôpital's rule, we see $\int_0^\infty \frac{x}{e^x} dx$ converges. All together:

- $\frac{x}{e^x}$ and $\frac{x}{e^x + \sqrt{x}}$ are defined and continuous for all $x \geq 0$,
- $\left| \frac{x}{e^x + \sqrt{x}} \right| \leq \frac{x}{e^x}$, and
- $\int_0^\infty \frac{x}{e^x} dx$ converges.

So, by Theorem 1.13.17 in the text, our integral $\int_0^\infty \frac{x}{e^x + \sqrt{x}} dx$ converges.

Solution 2: Let's try to use a different direct comparison from Solution 1, and avoid integration by parts. We'd like to compare to something like $\frac{1}{e^x}$, but the inequality goes the wrong way. So, we make a slight modification: we consider $2e^{-x/2}$. To that

end, we claim $x < 2e^{x/2}$ for all $x \geq 0$. We can prove this by noting the following two facts:

- $0 < 2 = 2e^{0/2}$, and
- $\frac{d}{dx}\{x\} = 1 \leq e^{x/2} = \frac{d}{dx}\{2e^{x/2}\}$.

So, when $x = 0$, $x < 2e^{x/2}$, and then as x increases, $2e^{x/2}$ grows faster than x .

Now we can make the following comparison:

$$0 \leq \frac{x}{e^x + \sqrt{x}} \leq \frac{x}{e^x} < \frac{2e^{x/2}}{e^x} = \frac{2}{e^{x/2}}$$

We have a hunch that $\int_0^\infty \frac{2}{e^{x/2}} dx$ converges, just like $\int_0^\infty \frac{1}{e^x} dx$. This is easy enough to prove. We can guess an antiderivative, or use the substitution $u = x/2$.

$$\begin{aligned} \int_0^\infty \frac{2}{e^{x/2}} dx &= \lim_{R \rightarrow \infty} \int_0^R \frac{2}{e^{x/2}} dx = \lim_{R \rightarrow \infty} \left[-\frac{4}{e^{x/2}} \right]_0^R \\ &= \lim_{R \rightarrow \infty} \left[\frac{4}{e^0} - \frac{4}{e^{R/2}} \right]_0^R = 4 \end{aligned}$$

Now we know:

- $0 \leq \frac{x}{e^x + \sqrt{x}} \leq \frac{2}{e^{x/2}}$, and
- $\int_0^\infty \frac{2}{e^{x/2}} dx$ converges.
- Furthermore, $\frac{x}{e^x + \sqrt{x}}$ and $\frac{2}{e^{x/2}}$ are defined and continuous for all $x \geq 0$.

By the comparison test (Theorem 1.13.17) in the text, we conclude the integral converges.

S-24:

(a) Since $f(x)$ is odd, using the reasoning of Example 1.2.11 in the text,

$$\int_{-\infty}^{-1} f(x) dx = \lim_{t \rightarrow \infty} \int_{-t}^{-1} f(x) dx = \lim_{t \rightarrow \infty} - \int_1^t f(x) dx = - \lim_{t \rightarrow \infty} \int_1^t f(x) dx$$

Since $\int_1^\infty f(x) dx$ converges, the last limit above converges. Therefore, $\int_{-\infty}^{-1} f(x) dx$ converges.

(b) Since $f(x)$ is even, using the reasoning of Example 1.2.10 in the text,

$$\int_{-\infty}^{-1} f(x) dx = \lim_{t \rightarrow \infty} \int_{-t}^{-1} f(x) dx = \lim_{t \rightarrow \infty} \int_1^t f(x) dx = \lim_{t \rightarrow \infty} \int_1^t f(x) dx$$

Since $\int_1^\infty f(x) \, dx$ converges, the last limit above converges. Since $f(x)$ is continuous everywhere, by Theorem 1.13.20 in the text, $\int_{-1}^\infty f(x) \, dx$ converges (note the adjusted lower limit). Then, since

$$\int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^{-1} f(x) \, dx + \int_{-1}^\infty f(x) \, dx$$

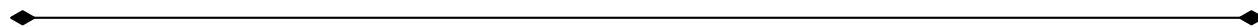
and both terms converge, our original integral converges as well.

S-25: Define $F(x) = \int_0^x \frac{1}{e^t} dt$.

$$F(x) = \int_0^x \frac{1}{e^t} dt = \left[-\frac{1}{e^t} \right]_0^x = \frac{1}{e^0} - \frac{1}{e^x} < \frac{1}{e^0} = 1$$

So, the statement is false: there is no x such that $F(x) = 1$. For every real x , $F(x) < \frac{1}{e^0} = 1$.

We note here that $\lim_{x \rightarrow \infty} \int_0^x \frac{1}{e^t} dt = 1$. So, as x grows larger, the gap between $F(x)$ and 1 grows infinitesimally small. But there is no real value of x where $F(x)$ is exactly equal to 1.



Solutions to Exercises 1.14 — Jump to [TABLE OF CONTENTS](#)

S-1: (A) Note $\int f'(x)f(x) \, dx = \int u \, du$ if we substitute $u = f(x)$. This is the kind of integrand described in (I). It's quite possible that a $u = f(x)$ substitution would work on the others, as well, but (I) is the most reliable kind of integrand for a $u = f(x)$ substitution.

(B) A trigonometric substitution usually allows us to cancel out a square root containing a quadratic function, as in (IV).

(C) We can often antidifferentiate the product of a polynomial with an exponential function using integration by parts: see Examples 1.11.1, 1.11.6 in the text. If we let u be the polynomial function and dv be the exponential, as long as we can antidifferentiate dv , we can repeatedly apply integration by parts until the polynomial function goes away. So, we go with (II)

(D) We apply partial fractions to rational functions, (III).

Note: without knowing more about the functions, there's no guarantee that the methods we chose will be the best methods, or even that they will work (with the exception of (I)). With practice, you gain intuition about likely methods for different integrals. Luckily for you, there's lots of practice below.

S-2:

The integrand is a product of powers of sine and cosine. Since cosine has an odd power, we want to substitute $u = \sin x$, $du = \cos x \, dx$. Therefore, we should:

- reserve one cosine for the derivative of sine in our substitution, and

- change the rest of the cosines to sines using the identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}
 \int_0^{\pi/2} \sin^4 x \cos^5 x dx &= \int_0^{\pi/2} \sin^4 x (\cos^2 x)^2 \cos x dx \\
 &= \int_0^{\pi/2} \sin^4 x (1 - \sin^2 x)^2 \underbrace{\cos x dx}_{du} \\
 &= \int_{\sin(0)}^{\sin(\pi/2)} u^4 (1 - u^2)^2 du \\
 &= \int_0^1 u^4 (1 - 2u^2 + u^4) du \\
 &= \int_0^1 (u^4 - 2u^6 + u^8) du \\
 &= \left[\frac{1}{5} u^5 - \frac{2}{7} u^7 + \frac{1}{9} u^9 \right]_{u=0}^{u=1} \\
 &= \left(\frac{1}{5} - \frac{2}{7} + \frac{1}{9} \right) - 0 \\
 &= \frac{8}{315}
 \end{aligned}$$

S-3:

We notice that there is a quadratic equation under the square root. If that equation were a perfect square, we could get rid of the square root: so we'll mould it into a perfect square using a trig substitution.

Our candidates will use one of the following identities:

$$1 - \sin^2 \theta = \cos^2 \theta \quad \tan^2 \theta + 1 = \sec^2 \theta \quad \sec^2 \theta - 1 = \tan^2 \theta$$

We'll be substituting $x = (\text{something})$, so we notice that $3 - 5x^2$ has the general form of (constant)–(function), as does $1 - \sin^2 \theta$. In order to get the constant right, we multiply through by three:

$$3 - 3 \sin^2 \theta = 3 \cos^2 \theta$$

Our goal is to get $3 - 5x^2 = 3 - 3 \sin^2 \theta$; so we solve this equation for x and decide on the substitution

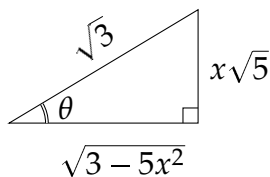
$$x = \sqrt{\frac{3}{5}} \sin \theta, \quad dx = \sqrt{\frac{3}{5}} \cos \theta d\theta$$

Now we evaluate our integral.

$$\begin{aligned}
 \int \sqrt{3-5x^2} dx &= \int \sqrt{3-5\left(\sqrt{\frac{3}{5}} \sin \theta\right)^2} \sqrt{\frac{3}{5}} \cos \theta d\theta \\
 &= \int \sqrt{3-3\sin^2 \theta} \sqrt{3/5} \cos \theta d\theta \\
 &= \int \sqrt{3\cos^2 \theta} \sqrt{3/5} \cos \theta d\theta \\
 &= \int \sqrt{3} \cos \theta \sqrt{3/5} \cos \theta d\theta \\
 &= \frac{3}{\sqrt{5}} \int \cos^2 \theta d\theta \\
 &= \frac{3}{\sqrt{5}} \int \frac{1+\cos 2\theta}{2} d\theta \\
 &= \frac{3}{2\sqrt{5}} \int (1+\cos 2\theta) d\theta \\
 &= \frac{3}{2\sqrt{5}} \left[\theta + \frac{1}{2} \sin(2\theta) \right] + C \\
 &= \frac{3}{2\sqrt{5}} [\theta + \sin \theta \cos \theta] + C
 \end{aligned}$$

From our substitution $x = \sqrt{3/5} \sin \theta$, we glean $\sin \theta = x\sqrt{5/3}$, and $\theta = \arcsin(x\sqrt{5/3})$.

To figure out $\cos \theta$, we draw a right triangle. Let θ be one angle, and since $\sin \theta = \frac{x\sqrt{5}}{\sqrt{3}}$, we let the hypotenuse be $\sqrt{3}$ and the side opposite θ be $x\sqrt{5}$. By Pythagoras, the missing side (adjacent to θ) has length $\sqrt{3-5x^2}$.



Therefore, $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3-5x^2}}{\sqrt{3}}$. So our integral evaluates to:

$$\begin{aligned}
 \frac{3}{2\sqrt{5}} [\theta + \sin \theta \cos \theta] + C &= \frac{3}{2\sqrt{5}} \left[\arcsin(x\sqrt{5/3}) + x\sqrt{5/3} \cdot \frac{\sqrt{3-5x^2}}{\sqrt{3}} \right] + C \\
 &= \frac{3}{2\sqrt{5}} \arcsin(x\sqrt{5/3}) + \frac{x}{2} \cdot \sqrt{3-5x^2} + C
 \end{aligned}$$

S-4: First, we note the integral is improper. So, we'll need to replace the top bound with a variable, and take a limit. Second, we're going to have to antidifferentiate. The

integrand is the product of an exponential function, e^{-x} , with a polynomial function, $x - 1$, so we use integration by parts with $u = x - 1$, $dv = e^{-x} du$, $du = dx$, and $v = -e^{-x}$.

$$\begin{aligned}\int \frac{x-1}{e^x} dx &= -(x-1)e^{-x} + \int e^{-x} dx \\ &= -(x-1)e^{-x} - e^{-x} + C = -xe^{-x} + C\end{aligned}$$

So,

$$\begin{aligned}\int_0^\infty \frac{x-1}{e^x} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{x-1}{e^x} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{x}{e^x} \right]_0^b = \lim_{b \rightarrow \infty} \left[-\underbrace{\frac{b}{e^b}}_{\substack{\text{num} \rightarrow \infty \\ \text{den} \rightarrow \infty}} \right] \\ &\stackrel{(*)}{=} \lim_{b \rightarrow \infty} -\frac{1}{e^b} = 0\end{aligned}$$

(In the equality marked (*), we used l'Hôpital's rule.)

$$\text{So, } \int_0^\infty \frac{x-1}{e^x} dx = 0.$$

Remark: this shows that, interestingly, $\int_0^\infty \frac{x}{e^x} dx = \int_0^\infty \frac{1}{e^x} dx$.

S-5:

Solution 1: Notice the denominator factors as $(x+1)(3x+1)$. Since the integrand is a rational function (the quotient of two polynomials), we can use partial fraction decomposition.

$$\begin{aligned}\frac{-2}{3x^2 + 4x + 1} &= \frac{-2}{(x+1)(3x+1)} \\ &= \frac{A}{x+1} + \frac{B}{3x+1} \\ &= \frac{A(3x+1) + B(x+1)}{(x+1)(3x+1)} \\ &= \frac{(3A+B)x + (A+B)}{(x+1)(3x+1)}\end{aligned}$$

So:

$$\begin{aligned}-2 &= (3A+B)x + (A+B) \\ 0 &= 3A+B \text{ and } -2 = A+B \\ B &= -3A \text{ and hence } -2 = A + (-3A) \\ A &= 1 \text{ so then } B = -3\end{aligned}$$

So now:

$$\begin{aligned}\frac{-2}{3x^2 + 4x + 1} &= \frac{1}{x+1} - \frac{3}{3x+1} \\ \int \frac{-2}{3x^2 + 4x + 1} dx &= \int \left(\frac{1}{x+1} - \frac{3}{3x+1} \right) dx \\ &= \log|x+1| - \log|3x+1| + C \\ &= \log \left| \frac{x+1}{3x+1} \right| + C\end{aligned}$$

Solution 2: The previous solution is probably the nicest. However, for the foolhardy or the brave, this integral can also be evaluated using trigonometric substitution.

We start by completing the square on the denominator.

$$\begin{aligned}3x^2 + 4x + 1 &= 3 \left(x^2 + \frac{4}{3}x + \frac{1}{3} \right) \\ &= 3 \left(x^2 + 2 \cdot \frac{2}{3}x + \frac{4}{9} - \frac{4}{9} + \frac{1}{3} \right) \\ &= 3 \left(\left(x + \frac{2}{3} \right)^2 - \frac{4}{9} + \frac{3}{9} \right) \\ &= 3 \left(\left(x + \frac{2}{3} \right)^2 - \frac{1}{9} \right) \\ &= 3 \left(x + \frac{2}{3} \right)^2 - \frac{1}{3}\end{aligned}$$

This has the form of a function minus a constant, which matches the trigonometric identity $\sec^2 \theta - 1 = \tan^2 \theta$. Multiplying through by $\frac{1}{3}$, we see we can use the identity $\frac{1}{3} \sec^2 \theta - \frac{1}{3} = \frac{1}{3} \tan^2 \theta$. So, to get the substitution right, we want to choose a substitution that makes the following true:

$$\begin{aligned}3 \left(x + \frac{2}{3} \right)^2 - \frac{1}{3} &= \frac{1}{3} \sec^2 \theta - \frac{1}{3} \\ 3 \left(x + \frac{2}{3} \right)^2 &= \frac{1}{3} \sec^2 \theta \\ 9 \left(x + \frac{2}{3} \right)^2 &= \sec^2 \theta \\ \color{red}{3x + 2} &= \color{blue}{\sec \theta}\end{aligned}$$

And, accordingly:

$$\color{red}{3}dx = \color{blue}{\sec \theta \tan \theta} d\theta$$

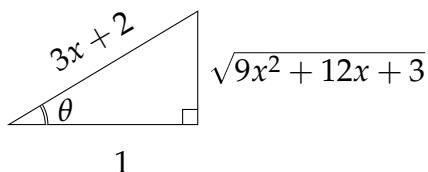
Now, let's simplify a little and use this substitution on our integral:

$$\begin{aligned}\int \frac{-2}{3x^2 + 4x + 1} dx &= \int \frac{-2}{3 \left(x + \frac{2}{3}\right)^2 - \frac{1}{3}} dx \\&= \int \frac{-2}{9 \left(x + \frac{2}{3}\right)^2 - 1} 3dx \\&= \int \frac{-2}{(3x + 2)^2 - 1} 3dx \\&= \int \frac{-2}{(\sec \theta)^2 - 1} \sec \theta \tan \theta d\theta \\&= \int \frac{-2}{\tan^2 \theta} \sec \theta \tan \theta d\theta \\&= \int -2 \frac{\sec \theta}{\tan \theta} d\theta \\&= \int -2 \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} d\theta \\&= \int -2 \frac{1}{\sin \theta} d\theta \\&= \int -2 \csc \theta d\theta\end{aligned}$$

Using the result of Question 16 of Section 1.9,

$$= 2 \log |\csc \theta + \cot \theta| + C$$

Our final task is to translate this back from θ to x . Recall we used the substitution $3x + 2 = \sec \theta$. Using this information, and $\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$, we can fill in two sides of a right triangle with angle θ . The Pythagorean theorem tells us the third side (opposite to θ) has measure $\sqrt{(3x + 2)^2 - 1} = \sqrt{9x^2 - 12x + 3}$.



$$\begin{aligned}
 2 \log |\csc \theta + \cot \theta| + C &= 2 \log \left| \frac{3x+2}{\sqrt{9x^2+12x+3}} + \frac{1}{\sqrt{9x^2+12x+3}} \right| + C \\
 &= 2 \log \left| \frac{3x+3}{\sqrt{9x^2+12x+3}} \right| + C \\
 &= \log \left| \frac{(3x+3)^2}{\sqrt{9x^2+12x+3}^2} \right| + C \\
 &= \log \left| \frac{(3x+3)^2}{9x^2+12x+3} \right| + C \\
 &= \log \left| \frac{9(x+1)^2}{3(3x+1)(x+1)} \right| + C \\
 &= \log \left| \frac{3(x+1)^2}{(3x+1)(x+1)} \right| + C \\
 &= \log \left| \frac{3(x+1)}{3x+1} \right| + C \\
 &= \log \left| \frac{x+1}{3x+1} \right| + \log 3 + C
 \end{aligned}$$

Since C is an arbitrary constant, we can write our final answer as

$$\log \left| \frac{x+1}{3x+1} \right| + C$$

S-6:

We see that we have two functions multiplied, but they don't simplify nicely with each other. However, if we differentiate logarithm, and integrate x^2 , we'll get a polynomial. So, let's use integration by parts.

$$u = \log x \quad dv = x^2 dx$$

$$du = (1/x)dx \quad v = x^3/3$$

First, let's antidifferentiate. We'll deal with the limits of integration later.

$$\begin{aligned}
 \int x^2 \log x \, dx &= \underbrace{(\log x)}_u \underbrace{(x^3/3)}_v - \int \underbrace{(x^3/3)}_v \underbrace{(1/x)}_{du} dx \\
 &= \frac{1}{3} x^3 \log x - \frac{1}{3} \int x^2 dx \\
 &= \frac{1}{3} x^3 \log x - \frac{1}{3} \cdot \frac{1}{3} x^3 + C \\
 &= \frac{1}{3} x^3 \log x - \frac{1}{9} x^3 + C
 \end{aligned}$$

We use the Fundamental Theorem of Calculus Part 2 to evaluate the definite integral.

$$\begin{aligned}
 \int_1^2 x^3 \log x \, dx &= \left[\frac{1}{3} x^3 \log x - \frac{1}{9} x^3 \right]_1^2 \\
 &= \left[\frac{1}{3} 2^3 \log 2 - \frac{1}{9} 2^3 \right] - \left[\frac{1}{3} 1^3 \log 1 - \frac{1}{9} 1^3 \right] \\
 &= \frac{8 \log 2}{3} - \frac{8}{9} - 0 + \frac{1}{9} \\
 &= \frac{8}{3} \log 2 - \frac{7}{9}
 \end{aligned}$$

S-7: The derivative of the denominator shows up in the numerator, only differing by a constant, so we perform a substitution. Specifically, substitute $u = x^2 - 3$, $du = 2x \, dx$. This gives

$$\int \frac{x}{x^2 - 3} \, dx = \int \frac{du/2}{u} = \frac{1}{2} \log |u| + C = \frac{1}{2} \log |x^2 - 3| + C$$

S-8: (a) Although a quadratic under a square root often suggests trigonometric substitution, in this case we have an easier substitution. Specifically, let $y = 9 + x^2$. Then $dy = 2x \, dx$, $x \, dx = \frac{dy}{2}$, $y(0) = 9$, and $y(4) = 25$.

$$\int_0^4 \frac{x}{\sqrt{9 + x^2}} \, dx = \int_9^{25} \frac{1}{\sqrt{y}} \frac{dy}{2} = \frac{1}{2} \cdot \frac{\sqrt{y}}{1/2} \Big|_9^{25} = 5 - 3 = 2$$

(b) The power of cosine is odd, so we can reserve one cosine for the differential and change the rest to sines. Substituting $y = \sin x$, $dy = \cos x \, dx$, $y(0) = 0$, $y(\pi/2) = 1$, $\cos^2 x = 1 - y^2$:

$$\begin{aligned}
 \int_0^{\pi/2} \cos^3 x \sin^2 x \, dx &= \int_0^{\pi/2} \cos^2 x \sin^2 x \cos x \, dx = \int_0^1 (1 - y^2) y^2 \, dy = \int_0^1 (y^2 - y^4) \, dy \\
 &= \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_0^1 = \frac{1}{3} - \frac{1}{5} \\
 &= \frac{2}{15}
 \end{aligned}$$

(c) The integrand is the product of two different kinds of functions, with no obvious substitution or simplification. If we differentiate $\log x$, it will match better with the polynomial nature of the rest of the integrand. So, integrate by parts with $u(x) = \log x$ and $dv = x^3 dx$, then $du = \frac{1}{x} dx$ and $v = x^4/4$.

$$\begin{aligned}\int_1^e x^3 \log x \, dx &= \left. \frac{x^4}{4} \log x \right|_1^e - \int_1^e \frac{x^4}{4} \cdot \frac{1}{x} \, dx = \frac{e^4}{4} - \int_1^e \frac{x^3}{4} \, dx = \left. \frac{e^4}{4} - \frac{x^4}{16} \right|_1^e \\ &= \frac{3e^4}{16} + \frac{1}{16}\end{aligned}$$

S-9: (a) Integrate by parts with $u = x$ and $dv = \sin x \, dx$ so that $du = dx$ and $v = -\cos x$.

$$\begin{aligned}\int x \sin x \, dx &= -x \cos x - \int (-\cos x) \, dx = -x \cos x + \sin x + C \\ \text{So, } \int_0^{\pi/2} x \sin x \, dx &= \left[-x \cos x + \sin x \right]_0^{\pi/2} = 1\end{aligned}$$

(b) The power of cosine is odd, so we can reserve one cosine for du and change the rest into sines. Make the substitution $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned}\int_0^{\pi/2} \cos^5 x \, dx &= \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x \, dx = \int_0^1 (1 - u^2)^2 \, du = \int_0^1 (1 - 2u^2 + u^4) \, du \\ &= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}\end{aligned}$$

S-10: (a) This is a classic integration-by-parts example. If we integrate e^x , it doesn't change, and if we differentiate x it becomes a constant. So, let $u = x$ and $dv = e^x \, dx$, so that $du = dx$ and $v = e^x$.

$$\int_0^2 x e^x \, dx = \left[x e^x \right]_0^2 - \int_0^2 e^x \, dx = 2e^2 - \left[e^x \right]_0^2 = e^2 + 1$$

(b) We have a quadratic function underneath a square root. In the absence of an easier substitution, we can get rid of the square root with a trigonometric substitution.

Substitute $x = \tan y$, $dx = \sec^2 y \, dy$. When $x = 0$, $\tan y = 0$ so $y = 0$. When $x = 1$, $\tan y = 1$ so $y = \frac{\pi}{4}$. Also $\sqrt{1+x^2} = \sqrt{1+\tan^2 y} = \sqrt{\sec^2 y} = \sec y$, since $\sec y \geq 0$ for all $0 \leq y \leq \frac{\pi}{4}$.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1+x^2}} \, dx &= \int_0^{\pi/4} \frac{\sec^2 y \, dy}{\sec y} = \int_0^{\pi/4} \sec y \, dy = \left[\log |\sec y + \tan y| \right]_0^{\pi/4} \\ &= \log \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \log |\sec 0 + \tan 0| \\ &= \log |\sqrt{2} + 1| - \log |1 + 0| = \log(\sqrt{2} + 1)\end{aligned}$$

$$\text{So, } \int_0^1 \frac{1}{\sqrt{1+x^2}} \, dx = \log(\sqrt{2} + 1)$$

(c) The integral is a rational function. In the absence of an obvious substitution, we use partial fractions.

$$\frac{4x}{(x^2-1)(x^2+1)} = \frac{4x}{(x-1)(x+1)(x^2+1)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}$$

Multiplying by the denominator,

$$4x = a(x+1)(x^2+1) + b(x-1)(x^2+1) + (cx+d)(x-1)(x+1) \quad (*)$$

Setting $x = 1$ gives $4a = 4$, so $a = 1$. Setting $x = -1$ gives $-4b = -4$, so $b = 1$.

Substituting in $a = b = 1$ in $(*)$ gives:

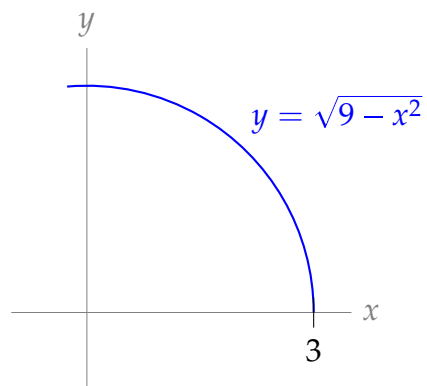
$$\begin{aligned} 4x &= (x+1)(x^2+1) + (x-1)(x^2+1) + (cx+d)(x-1)(x+1) \\ 4x &= 2x(x^2+1) + (cx+d)(x-1)(x+1) \\ 4x - 2x(x^2+1) &= (cx+d)(x-1)(x+1) \\ -2x(x^2-1) &= (cx+d)(x^2-1) \\ -2x &= cx+d \\ c &= -2, d = 0 \end{aligned}$$

So,

$$\begin{aligned} \int_3^5 \frac{4x}{(x^2-1)(x^2+1)} dx &= \int_3^5 \left(\frac{1}{x-1} + \frac{1}{x+1} - \frac{2x}{x^2+1} \right) dx \\ &= \left[\log|x-1| + \log|x+1| - \log(x^2+1) \right]_3^5 \\ &= \log 4 + \log 6 - \log 26 - \log 2 - \log 4 + \log 10 \\ &= \log \frac{6 \times 10}{26 \times 2} = \log \frac{15}{13} \approx 0.1431 \end{aligned}$$

S-11: (a) $\int_0^3 \sqrt{9-x^2} dx$ is the area of the portion of the disk $x^2 + y^2 \leq 9$ that lies in the first quadrant. It is $\frac{1}{4}\pi 3^2 = \frac{9}{4}\pi$. Alternatively, you could also evaluate this integral using the substitution $x = 3 \sin y$, $dx = 3 \cos y dy$.

$$\begin{aligned} \int_0^3 \sqrt{9-x^2} dx &= \int_0^{\pi/2} \sqrt{9-9\sin^2 y} (3 \cos y) dy = 9 \int_0^{\pi/2} \cos^2 y dy \\ &= \frac{9}{2} \int_0^{\pi/2} [1 + \cos(2y)] dy = \frac{9}{2} \left[y + \frac{\sin(2y)}{2} \right]_0^{\pi/2} \\ &= \frac{9}{4}\pi \end{aligned}$$



(b) It's not immediately obvious what to do with this one, but remember we found $\int \log x dx$ using integration by parts with $u = \log x$ and $dv = dx$. Let's hope a similar trick works here. Integrate by parts, using $u = \log(1 + x^2)$ and $dv = dx$, so that $du = \frac{2x}{1+x^2} dx$, $v = x$.

$$\begin{aligned} \int_0^1 \log(1 + x^2) dx &= \left[x \log(1 + x^2) \right]_0^1 - \int_0^1 x \frac{2x}{1 + x^2} dx = \log 2 - 2 \int_0^1 \frac{x^2}{1 + x^2} dx \\ &= \log 2 - 2 \int_0^1 \left(1 - \frac{1}{1 + x^2} \right) dx = \log 2 - 2 \left[x - \arctan x \right]_0^1 \\ &= \log 2 - 2 + \frac{\pi}{2} \approx 0.264 \end{aligned}$$

(c) The integrand is a rational function with no obvious substitution, so we use partial fractions.

$$\frac{x}{(x-1)^2(x-2)} = \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c}{x-2} = \frac{a(x-2) + b(x-1)(x-2) + c(x-1)^2}{(x-1)^2(x-2)}$$

Multiply by the denominator.

$$x = a(x-2) + b(x-1)(x-2) + c(x-1)^2$$

Setting $x = 1$ gives $a = -1$. Setting $x = 2$ gives $c = 2$. Substituting in $a = -1$ and $c = 2$ gives

$$\begin{aligned} b(x-1)(x-2) &= x + (x-2) - 2(x-1)^2 = -2x^2 + 6x - 4 = -2(x-1)(x-2) \\ \implies b &= -2 \end{aligned}$$

Hence

$$\begin{aligned}
\int_3^\infty \frac{x}{(x-1)^2(x-2)} dx &= \lim_{M \rightarrow \infty} \int_3^M \left(-\frac{1}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{x-2} \right) dx \\
&= \lim_{M \rightarrow \infty} \left[\frac{1}{x-1} - 2 \log |x-1| + 2 \log |x-2| \right]_3^M \\
&= \lim_{M \rightarrow \infty} \left[\frac{1}{x-1} + 2 \log \left| \frac{x-2}{x-1} \right| \right]_3^M \\
&= \lim_{M \rightarrow \infty} \left[\frac{1}{M-1} + 2 \log \left| \frac{M-2}{M-1} \right| \right] - \left[\frac{1}{3-1} + 2 \log \left| \frac{3-2}{3-1} \right| \right] \\
&= 2 \log 2 - \frac{1}{2} \approx 0.886
\end{aligned}$$

since

$$\begin{aligned}
\lim_{M \rightarrow \infty} \log \frac{M-2}{M-1} &= \lim_{M \rightarrow \infty} \log \frac{1-2/M}{1-1/M} = \log 1 = 0 \\
\text{and} \quad \log \frac{1}{2} &= -\log 2
\end{aligned}$$

S-12: This looks quite a lot like a rational function, but with variable $\sin \theta$ instead of x . So, we use the substitution $x = \sin \theta$, $dx = \cos \theta d\theta$.

$$\int \frac{\sin^4 \theta - 5 \sin^3 \theta + 4 \sin^2 \theta + 10 \sin \theta}{\sin^2 \theta - 5 \sin \theta + 6} \cos \theta d\theta = \int \frac{x^4 - 5x^3 + 4x^2 + 10x}{x^2 - 5x + 6} dx$$

Since the numerator does not have smaller degree than the denominator, we need to do some long division before we can set up our partial fractions decomposition.

$$\begin{array}{r}
 \overline{x^2 - 2} \\
x^2 - 5x + 6 \overline{x^4 - 5x^3 + 4x^2 + 10x} \\
 \underline{-x^4 + 5x^3 - 6x^2} \\
 -2x^2 + 10x \\
 \underline{2x^2 - 10x + 12} \\
 12
\end{array}$$

That is,

$$\frac{x^4 - 5x^3 + 4x^2 + 10x}{x^2 - 5x + 6} = x^2 - 2 + \frac{12}{x^2 - 5x + 6} = x^2 - 2 + \frac{12}{(x-2)(x-3)}$$

We use partial fractions decomposition on the rightmost term.

$$\begin{aligned}
\frac{12}{(x-2)(x-3)} &= \frac{A}{x-2} + \frac{B}{x-3} \\
12 &= A(x-3) + B(x-2)
\end{aligned}$$

Setting $x = 3$ and $x = 2$ gives us

$$B = 12, \quad A = -12$$

Now we can evaluate our integral.

$$\begin{aligned} \int \frac{\sin^4 \theta - 5 \sin^3 \theta + 4 \sin^2 \theta + 10 \sin \theta}{\sin^2 \theta - 5 \sin \theta + 6} \cos \theta d\theta &= \int \frac{x^4 - 5x^3 + 4x^2 + 10x}{x^2 - 5x + 6} dx \\ &= \int \left(x^2 - 2 + \frac{12}{(x-2)(x-3)} \right) dx \\ &= \int \left(x^2 - 2 - \frac{12}{x-2} + \frac{12}{x-3} \right) dx \\ &= \frac{1}{3}x^3 - 2x - 12 \log |x-2| + 12 \log |x-3| + C \\ &= \frac{1}{3}x^3 - 2x + 12 \log \left| \frac{x-3}{x-2} \right| + C \\ &= \frac{1}{3} \sin^3 \theta - 2 \sin \theta + 12 \log \left| \frac{\sin \theta - 3}{\sin \theta - 2} \right| + C \end{aligned}$$

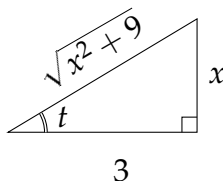
S-13: (a) It doesn't matter to us right now that the arguments of sine and cosine are $2x$ rather than x . This is still the integral of powers of products of sines and cosines. Since cosine has an odd power, we make the substitution $u = \sin(2x)$, $du = 2 \cos(2x) dx$.

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin^2(2x) \cos^3(2x) dx &= \int_0^{\frac{\pi}{4}} \sin^2(2x) [1 - \sin^2(2x)] \cos(2x) dx = \frac{1}{2} \int_0^1 u^2 [1 - u^2] du \\ &= \frac{1}{2} \int_0^1 (u^2 - u^4) du = \frac{1}{2} \left[\frac{1}{3} u^3 - \frac{1}{5} u^5 \right]_0^1 = \frac{1}{15} \end{aligned}$$

(b) Make the substitution $x = 3 \tan t$, $dx = 3 \sec^2 t dt$ and use the trig identity $9 + 9 \tan^2 t = 9 \sec^2 t$.

$$\begin{aligned} \int (9 + x^2)^{-\frac{3}{2}} dx &= \int (9 + 9 \tan^2 t)^{-\frac{3}{2}} 3 \sec^2 t dt = \int (3 \sec t)^{-3} 3 \sec^2 t dt \\ &= \frac{1}{9} \int \cos t dt = \frac{1}{9} \sin t + C = \frac{1}{9} \frac{x}{\sqrt{x^2 + 9}} + C \end{aligned}$$

To convert back to x , in the last step, we used the triangle below, which is rigged to have $\tan t = \frac{x}{3}$.



(c) Seeing a rational function with no obvious substitutions, we use partial fractions.

$$\frac{1}{(x-1)(x^2+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+1} = \frac{a(x^2+1) + (bx+c)(x-1)}{(x-1)(x^2+1)}$$

Multiply by the original denominator.

$$1 = a(x^2+1) + (bx+c)(x-1) \quad (*)$$

Setting $x = 1$ gives $2a = 1$ or $a = \frac{1}{2}$. Substituting in $a = \frac{1}{2}$ in $(*)$ gives

$$\begin{aligned} \frac{1}{2}(x^2+1) + (bx+c)(x-1) &= 1 \\ \Leftrightarrow (bx+c)(x-1) &= \frac{1}{2}(1-x^2) = -\frac{1}{2}(x-1)(x+1) \\ \Leftrightarrow (bx+c) &= -\frac{1}{2}(x+1) \\ \Leftrightarrow b=c &= -\frac{1}{2} \end{aligned}$$

So,

$$\int \frac{dx}{(x-1)(x^2+1)} = \int \left[\frac{1/2}{x-1} - \frac{\frac{1}{2}(x+1)}{x^2+1} \right] dx$$

To antidifferentiate the second piece, we split it into two integrals: one that can be handled with the substitution $u = x^2 + 1$, and another that looks like the derivative of arctangent.

$$\begin{aligned} &= \int \left(\frac{1/2}{x-1} - \frac{x/2}{x^2+1} - \frac{1/2}{x^2+1} \right) dx \\ &= \int \left(\frac{1/2}{x-1} - \frac{1}{4} \cdot \frac{2x}{x^2+1} - \frac{1/2}{x^2+1} \right) dx \\ &= \frac{1}{2} \log|x-1| - \frac{1}{4} \log(x^2+1) - \frac{1}{2} \arctan x + C \end{aligned}$$

(d) We know the derivative of arctangent, and it would integrate nicely if multiplied to the antiderivative of x . So, we integrate by parts with $u = \arctan x$ and $dv = x dx$ so that $du = \frac{1}{1+x^2} dx$ and $v = \frac{1}{2}x^2$. Then

$$\begin{aligned} \int x \arctan x dx &= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ &= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} dx \\ &= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\ &= \frac{1}{2} [x^2 \arctan x - x + \arctan x] + C \end{aligned}$$

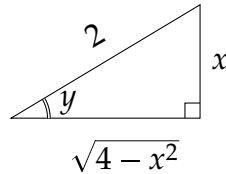
S-14: (a) We substitute $y = \sin(2x)$, $dy = 2 \cos(2x) dx$. Note $\sin(2 \cdot 0) = 0$ and $\sin(2 \cdot \frac{\pi}{4}) = 1$.

$$\int_0^{\pi/4} \sin^5(2x) \cos(2x) dx = \int_0^1 y^5 \frac{dy}{2} = \frac{1}{12} \left[y^6 \right]_0^1 = \frac{1}{12}$$

(b) We can get rid of the square root with a trig substitution. Substituting $x = 2 \sin y$, $dx = 2 \cos y dy$,

$$\begin{aligned} \int \sqrt{4-x^2} dx &= \int \sqrt{4-4\sin^2 y} 2 \cos y dy = 4 \int \cos^2 y dy = 2 \int [1 + \cos(2y)] dy \\ &= 2y + \sin(2y) + C = 2y + 2 \sin y \cos y + C \\ &= 2 \sin^{-1} \frac{x}{2} + x \sqrt{1 - \frac{x^2}{4}} + C \end{aligned}$$

since $\sin y = \frac{x}{2}$ and $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{x^2}{4}}$. Alternately, we can draw a triangle with $\sin y = \frac{x}{2}$, and use the Pythagorean theorem to find the adjacent side.



(c) Seeing a rational function with no obvious substitution, we use the method of partial fractions. The denominator is already completely factored.

$$\begin{aligned} \frac{x+1}{x^2(x-1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \\ x+1 &= Ax(x-1) + B(x-1) + Cx^2 \end{aligned}$$

Setting $x = 1$ gives us $C = 2$. Setting $x = 0$ gives us $B = -1$. Furthermore, the coefficient of x^2 on the left hand side (after collecting like terms), namely $A + C$, must be the same as the coefficient of x^2 on the right hand side, namely 0. So $A + C = 0$ and $A = -2$.

Checking,

$$-2x(x-1) - (x-1) + 2x^2 = -2x^2 + 2x - x + 1 + 2x^2 = x + 1$$

as desired. Thus,

$$\int \frac{x+1}{x^2(x-1)} dx = \int \left[-\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1} \right] dx = -2 \log |x| + \frac{1}{x} + 2 \log |x-1| + C$$

S-15:

(a) We can cancel out the square root if we use a trig substitution. Substitute $x = \sqrt{2} \tan y$, $dx = \sqrt{2} \sec^2 y dy$.

$$\int_0^{\sqrt{2}} \frac{1}{(2+x^2)^{3/2}} dx = \sqrt{2} \int_0^{\pi/4} \frac{\sec^2 y}{(2+2 \tan^2 y)^{3/2}} dy = \frac{1}{2} \int_0^{\pi/4} \cos y dy = \frac{1}{2} \left[\sin y \right]_0^{\pi/4} = \frac{1}{2\sqrt{2}}$$

(b)

Solution 1: Integrate by parts, using $u = \log(1 + x^2)$ and $dv = x \, dx$, so that $du = \frac{2x}{1+x^2}$, $v = \frac{x^2}{2}$.

$$\begin{aligned}\int_0^1 x \log(1 + x^2) \, dx &= \left[\frac{1}{2} x^2 \log(1 + x^2) \right]_0^1 - \int_0^1 \frac{x^3}{1 + x^2} \, dy = \frac{1}{2} \log 2 - \int_0^1 \left[x - \frac{x}{1 + x^2} \right] \, dx \\ &= \frac{1}{2} \log 2 - \left[\frac{x^2}{2} - \frac{1}{2} \log(1 + x^2) \right]_0^1 = \log 2 - \frac{1}{2} \approx 0.193\end{aligned}$$

Solution 2: First substitute $y = 1 + x^2$, $dy = 2x \, dx$.

$$\int_0^1 x \log(1 + x^2) \, dx = \frac{1}{2} \int_1^2 \log y \, dy$$

Then integrate by parts, using $u = \log y$ and $dv = dy$, so that $du = \frac{1}{y}$, $v = y$.

$$\int_0^1 x \log(1 + x^2) \, dx = \frac{1}{2} \int_1^2 \log y \, dy = \left[\frac{1}{2} y \log y \right]_1^2 - \frac{1}{2} \int_1^2 y \frac{1}{y} \, dy = \log 2 - \frac{1}{2} \approx 0.193$$

(c) Seeing a rational function with no obvious substitution, we use partial fractions.

$$\begin{aligned}\frac{1}{(x-1)^2(x-2)} &= \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c}{x-2} \\ 1 &= a(x-2) + b(x-1)(x-2) + c(x-1)^2\end{aligned}\quad (*)$$

Setting $x = 1$ gives $a = -1$. Setting $x = 2$ gives $c = 1$. Substituting in $a = -1$ and $c = 1$ to (*) gives

$$\begin{aligned}b(x-1)(x-2) &= 1 + (x-2) - (x-1)^2 \\ &= -x^2 + 3x - 2 \\ &= -(x-1)(x-2) \\ \implies \quad b &= -1\end{aligned}$$

Hence:

$$\begin{aligned}\int_3^\infty \frac{1}{(x-1)^2(x-2)} \, dx &= \lim_{M \rightarrow \infty} \int_3^M \left(-\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x-2} \right) \, dx \\ &= \lim_{M \rightarrow \infty} \left[\frac{1}{x-1} - \log(x-1) + \log(x-2) \right]_3^M \\ &= \lim_{M \rightarrow \infty} \left[\frac{1}{M-1} + \log \frac{M-2}{M-1} \right] - \left[\frac{1}{3-1} + \log \frac{3-2}{3-1} \right] \\ &= \log 2 - \frac{1}{2} \approx 0.193\end{aligned}$$

since

$$\lim_{M \rightarrow \infty} \log \frac{M-2}{M-1} = \lim_{M \rightarrow \infty} \log \frac{1-2/M}{1-1/M} = \log 1 = 0$$

S-16: (a) Integrate by parts with $u = \log x$ and $dv = x \, dx$, so that $du = \frac{dx}{x}$ and $v = \frac{1}{2}x^2$.

$$\int x \log x \, dx = \frac{1}{2}x^2 \log x - \frac{1}{2} \int x^2 \cdot \frac{1}{x} \, dx = \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C$$

(b) The denominator is an irreducible quadratic, so partial fractions can't get us any further. To integrate a function whose denominator is quadratic, we split the numerator up so that one piece can be evaluated with a u -substitution, and the other piece looks like arctangent.

$$\begin{aligned} \int \frac{(x-1)dx}{x^2+4x+5} &= \int \frac{x+2-3}{x^2+4x+5} dx \\ &= \frac{1}{2} \int \frac{2x+4}{x^2+4x+5} dx - \int \frac{3}{x^2+4x+5} dx \\ &= \frac{1}{2} \int \frac{2x+4}{x^2+4x+5} dx - 3 \int \frac{1}{(x+2)^2+1} dx \\ &= \frac{1}{2} \log[x^2+4x+5] - 3 \arctan(x+2) + C \end{aligned}$$

For the last step, you can guess the antiderivative, or use the substitutions $u_1 = x^2 + 4x + 5$ and $u_2 = x + 2$, respectively, for the two integrals.

(c) We use partial fractions.

$$\begin{aligned} \frac{1}{x^2-4x+3} &= \frac{1}{(x-3)(x-1)} = \frac{a}{x-3} + \frac{b}{x-1} \\ 1 &= a(x-1) + b(x-3) \end{aligned}$$

Setting $x = 3$ gives $a = \frac{1}{2}$. Setting $x = 1$ gives $b = -\frac{1}{2}$. So,

$$\int \frac{dx}{x^2-4x+3} = \int \left(\frac{1/2}{x-3} - \frac{1/2}{x-1} \right) dx = \frac{1}{2} \log|x-3| - \frac{1}{2} \log|x-1| + C$$

(d) Substitute $y = x^3$, $dy = 3x^2 \, dx$.

$$\int \frac{x^2 \, dx}{1+x^6} = \frac{1}{3} \int \frac{dy}{1+y^2} = \frac{1}{3} \arctan y + C = \frac{1}{3} \arctan x^3 + C$$

S-17: (a) Integrate by parts with $u = \arctan x$, $dv = dx$, $du = \frac{dx}{1+x^2}$ and $v = x$. This gives

$$\int_0^1 \arctan x \, dx = [x \arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2} dx = \arctan 1 - \left[\frac{1}{2} \log(1+x^2) \right]_0^1 = \frac{\pi}{4} - \frac{1}{2} \log 2$$

(b) Note that the derivative of the denominator is $2x - 2$, which differs from the numerator only by 1.

$$\begin{aligned} \int \frac{2x-1}{x^2-2x+5} dx &= \int \frac{2x-2}{x^2-2x+5} dx + \int \frac{1}{x^2-2x+5} dx \\ &= \int \frac{2x-2}{x^2-2x+5} dx + \int \frac{1}{(x-1)^2+4} dx \\ &= \log|x^2-2x+5| + \frac{1}{2} \arctan \frac{x-1}{2} + C \end{aligned}$$

In the last step, you can guess the antiderivative, or use the substitutions $u_1 = x^2 - 2x + 5$ and $u_2 = (x - 1)/2$, respectively.

S-18: (a) Substituting $u = x^3 + 1$, $du = 3x^2 dx$

$$\int \frac{x^2}{(x^3 + 1)^{101}} dx = \int \frac{1}{u^{101}} \cdot \frac{du}{3} = \frac{u^{-100}}{-100} \cdot \frac{1}{3} + C = -\frac{1}{300(x^3 + 1)^{100}} + C$$

(b) Substituting $u = \sin x$, $du = \cos x dx$, $\cos^2 x = 1 - \sin^2 x = 1 - u^2$,

$$\begin{aligned} \int \cos^3 x \sin^4 x dx &= \int \cos^2 x \sin^4 x \cos x dx = \int (1 - u^2)u^4 du \\ &= \int (u^4 - u^6) du = \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C \end{aligned}$$

S-19: First, we note that the integral is improper, because $\sin \pi = 0$. So, we'll have to use a limit.

Second, we need to antidifferentiate. The substitution $u = \sin x$, $du = \cos x dx$ fits just right.

$$\begin{aligned} \int_{\pi/2}^{\pi} \frac{\cos x}{\sqrt{\sin x}} dx &= \lim_{b \rightarrow \pi^-} \int_{\pi/2}^b \frac{\cos x}{\sqrt{\sin x}} dx = \lim_{b \rightarrow \pi^-} \int_1^{\sin b} \frac{1}{\sqrt{u}} du \\ &= \lim_{b \rightarrow \pi^-} \left[2\sqrt{u} \right]_1^{\sin b} = 2\sqrt{0} - 2\sqrt{1} = -2 \end{aligned}$$

S-20: (a) If the integrand had x 's instead of e^x 's it would be a rational function, ripe for the application of partial fractions. So let's start by making the substitution $u = e^x$, $du = e^x dx$:

$$\int \frac{e^x}{(e^x + 1)(e^x - 3)} dx = \int \frac{du}{(u + 1)(u - 3)}$$

Now, we follow the partial fractions protocol, starting with expressing

$$\frac{1}{(u + 1)(u - 3)} = \frac{A}{u + 1} + \frac{B}{u - 3}$$

To find A and B , the sneaky way, we cross multiply by the denominator

$$1 = A(u - 3) + B(u + 1)$$

and find A and B by evaluating at $u = -1$ and $u = 3$, respectively.

$$1 = A(-1 - 3) + B(-1 + 1) \iff A = -\frac{1}{4}$$

$$1 = A(3 - 3) + B(3 + 1) \iff B = \frac{1}{4}$$

Finally, we can do the integral:

$$\begin{aligned} \int \frac{e^x}{(e^x + 1)(e^x - 3)} dx &= \int \frac{du}{(u + 1)(u - 3)} = \int \left(\frac{-1/4}{u + 1} + \frac{1/4}{u - 3} \right) du \\ &= -\frac{1}{4} \log |u + 1| + \frac{1}{4} \log |u - 3| + C \\ &= -\frac{1}{4} \log |e^x + 1| + \frac{1}{4} \log |e^x - 3| + C \end{aligned}$$

(b) The argument of the square root is

$$12 + 4x - x^2 = 12 - (x - 2)^2 + 4 = 16 - (x - 2)^2$$

Hmmm. The numerator is $x^2 - 4x + 4 = (x - 2)^2$. So let's make the integral look somewhat simpler by substituting $u = x - 2$, $du = dx$. When $x = 2$ we have $u = 0$, and when $x = 4$ we have $u = 2$, so:

$$\int_{x=2}^{x=4} \frac{x^2 - 4x + 4}{\sqrt{12 + 4x - x^2}} dx = \int_{u=0}^{u=2} \frac{u^2}{\sqrt{16 - u^2}} du$$

This is perfect for the trig substitution $u = 4 \sin \theta$, $du = 4 \cos(\theta) d\theta$. When $u = 0$ we have $4 \sin \theta = 0$ and hence $\theta = 0$. When $u = 2$ we have $4 \sin \theta = 2$ and hence $\theta = \frac{\pi}{6}$. So

$$\begin{aligned} \int_{u=0}^{u=2} \frac{u^2}{\sqrt{16 - u^2}} du &= \int_{\theta=0}^{\theta=\pi/6} \frac{16 \sin^2 \theta}{\sqrt{16 - 16 \sin^2 \theta}} 4 \cos \theta d\theta \\ &= 16 \int_0^{\pi/6} \sin^2 \theta d\theta \\ &= 8 \int_0^{\pi/6} (1 - \cos(2\theta)) d\theta \\ &= 8 \left[\theta - \frac{1}{2} \sin(2\theta) \right]_0^{\pi/6} = 8 \left[\frac{\pi}{6} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right] \\ &= \frac{4\pi}{3} - 2\sqrt{3} \end{aligned}$$

S-21: (a) Substituting $y = \cos x$, $dy = -\sin x dx$, $\sin^2 x = 1 - \cos^2 x = 1 - y^2$

$$\begin{aligned} \int \frac{\sin^3 x}{\cos^3 x} dx &= \int \frac{\sin^2 x}{\cos^3 x} \sin x dx = \int \frac{1 - y^2}{y^3} (-dy) = - \int (y^{-3} - y^{-1}) dy \\ &= -\frac{y^{-2}}{-2} + \log |y| + C = \frac{1}{2} \sec^2 x + \log |\cos x| + C \end{aligned}$$

(b) The integrand is an even function, and the limits of integration are symmetric. So, we can slightly simplify the integral by replacing the lower limit with 0, and doubling the integral.

We'd rather not use partial fractions here, because it would be pretty complicated. Instead, notice that the numerator is only off by a constant from the derivative of x^5 . Substituting $x^5 = 4y$, $5x^4 dx = 4 dy$, and using that $x = 2 \implies 2^5 = 4y \implies y = 8$,

$$\begin{aligned}\int_{-2}^2 \frac{x^4}{x^{10} + 16} dx &= 2 \int_0^2 \frac{x^4}{x^{10} + 16} dx = 2 \cdot \frac{4}{5} \int_0^8 \frac{1}{16y^2 + 16} dy = \frac{1}{10} \int_0^8 \frac{1}{y^2 + 1} dy \\ &= \frac{1}{10} \arctan 8 \approx 0.1446\end{aligned}$$

S-22:

Solution 1: Let's use the substitution $u = x - 1$, $du = dx$.

$$\begin{aligned}\int x\sqrt{x-1} dx &= \int (u+1)\sqrt{u} du \\ &= \int (u^{3/2} + u^{1/2}) du \\ &= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C\end{aligned}$$

Solution 2: We have an integrand with x multiplied by something integrable. So, if we use integration by parts with $u = x$ and $dv = \sqrt{x-1} dx$, then $du = dx$ (that is, the x goes away) and $v = \frac{2}{3}(x-1)^{3/2}$.

$$\begin{aligned}\int x\sqrt{x-1} dx &= \frac{2}{3}x\sqrt{x-1}^3 - \frac{2}{3} \int (x-1)^{3/2} dx \\ &= \frac{2}{3}x\sqrt{x-1}^3 - \frac{2}{3} \left(\frac{2}{5}(x-1)^{5/2} \right) + C \\ &= \frac{2}{3}\sqrt{x-1} \left(x(x-1) - \frac{2}{5}(x-1)^2 \right) + C \\ &= \frac{2}{15}\sqrt{x-1} \cdot (3x^2 - x - 2) + C \\ &= \frac{2}{15}\sqrt{x-1} \cdot (3(x^2 - 2x + 1) + 5x - 5) + C \\ &= \frac{2}{15}\sqrt{x-1} \cdot (3(x-1)^2 + 5(x-1)) + C \\ &= \frac{2}{15} \cdot 3\sqrt{x-1}^5 + \frac{2}{15} \cdot 5\sqrt{x-1}^3 + C \\ &= \frac{2}{5}\sqrt{x-1}^5 + \frac{2}{3}\sqrt{x-1}^3 + C\end{aligned}$$

We have just seen two solutions. There are other solutions too. For example, one could use the substitution $u = \sqrt{x-1}$. Or, as another example, one could write

$$x\sqrt{x-1} = (x-1)\sqrt{x-1} + \sqrt{x-1} = (x-1)^{3/2} + (x-1)^{1/2}$$

and just guess a function whose derivative is $(x-1)^{3/2}$ and a function whose derivative is $(x-1)^{1/2}$.

S-23: We can use partial fraction decomposition to break this into chunks that we can deal with. The denominator has a repeated linear factor, so it can be decomposed as the sum of constants divided by powers of that factor.

$$\begin{aligned}\frac{3x^2 + 4x + 6}{(x+1)^3} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \\ &= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} \\ \Rightarrow 3x^2 + 4x + 6 &= A(x+1)^2 + B(x+1) + C \\ &= Ax^2 + (2A+B)x + (A+B+C)\end{aligned}$$

So, by matching coefficients:

$$\begin{aligned}A &= 3, 2A + B = 4, \text{ and } A + B + C = 6 \\ A &= 3, B = -2, C = 5\end{aligned}$$

Therefore:

$$\frac{3x^2 + 4x + 6}{(x+1)^3} = \frac{3}{x+1} + \frac{-2}{(x+1)^2} + \frac{5}{(x+1)^3}$$

Now, the integration is easy, with a substitution of $u = x+1$ and $du = dx$:

$$\begin{aligned}\int \frac{3x^2 + 4x + 6}{(x+1)^3} dx &= \int \left(\frac{3}{x+1} + \frac{-2}{(x+1)^2} + \frac{5}{(x+1)^3} \right) dx \\ &= \int (3u^{-1} - 2u^{-2} + 5u^{-3}) du \\ &= 3 \log |u| + 2u^{-1} - \frac{5}{2}u^{-2} + C \\ &= 3 \log |x+1| + \frac{2}{x+1} - \frac{5}{2(x+1)^2} + C\end{aligned}$$

S-24:

If the denominator were $x^2 + 1$, the antiderivative would be arctangent. So, by completing the square, let's aim for the fraction to look like $\frac{1}{u^2 + 1}$, for some u . This is a good strategy for integrating an irreducible quadratic under a constant.

First: complete the square

$$\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{x^2 + x + \frac{1}{4} + \frac{3}{4}} dx = \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

Second: get the denominator in the form $u^2 + 1$. To do this, we need to fix the constant

$$\begin{aligned} &= \int \left(\frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \right) \left(\frac{\frac{4}{3}}{\frac{4}{3}} \right) dx \\ &= \frac{4}{3} \int \frac{1}{\frac{4}{3} \cdot \left(x + \frac{1}{2}\right)^2 + 1} dx \end{aligned}$$

Now a quick wiggle to make that first part of the denominator into something squared again:

$$= \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} dx$$

Now we see that $u = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$, $du = \frac{2}{\sqrt{3}}dx$ will do the job

$$\begin{aligned} &= \frac{4}{3} \int \frac{1}{u^2 + 1} \cdot \frac{\sqrt{3}}{2} du = \frac{2}{\sqrt{3}} \int \frac{1}{u^2 + 1} du \\ &= \frac{2}{\sqrt{3}} \arctan u + C \\ &= \frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \right) + C \end{aligned}$$

S-25: Since $\tan x = \frac{\sin x}{\cos x}$,

$$\begin{aligned} \int \sin x \cos x \tan x \, dx &= \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos(2x)) \, dx \\ &= \frac{1}{2} \left(x - \frac{1}{2} \sin(2x) \right) + C \\ &= \frac{1}{2} (x - \sin x \cos x) + C \end{aligned}$$

S-26: We have the integral of a rational function with no obvious substitution, so we use partial fractions. That means we need to factor the denominator. We see that $x = -1$ is a root of the denominator, so $x + 1$ is a factor. You might be able to figure out the rest of the factorization by inspection, or from having seen this common expression before; alternately, we can use long division.

$$\begin{array}{r}
 x^2 - x + 1 \\
 x+1 \overline{) \begin{array}{r} x^3 + 1 \\ -x^3 - x^2 \\ \hline -x^2 \\ x^2 + x \\ \hline x + 1 \\ -x - 1 \\ \hline 0 \end{array}}
 \end{array}$$

Note $x^2 - x + 1$ is an irreducible quadratic.

$$\begin{aligned}
 \frac{1}{x^3 + 1} &= \frac{1}{(x+1)(x^2 - x + 1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 - x + 1} \\
 1 &= A(x^2 - x + 1) + (Bx + C)(x+1)
 \end{aligned} \tag{*}$$

When $x = -1$, we see $1 = 3A$, so $\frac{1}{3} = A$. We plug this into (*).

$$\begin{aligned}
 1 &= \frac{1}{3}(x^2 - x + 1) + (Bx + C)(x+1) \\
 -\frac{1}{3}x^2 + \frac{1}{3}x + \frac{2}{3} &= Bx^2 + (B+C)x + C
 \end{aligned}$$

Matching up coefficients of corresponding power of x , we see $B = -\frac{1}{3}$ and $C = \frac{2}{3}$.

$$\int \frac{1}{x^3 + 1} dx = \int \left(\frac{1/3}{x+1} - \frac{\frac{1}{3}x - \frac{2}{3}}{x^2 - x + 1} \right) dx$$

To integrate the second fraction, we break it up into two pieces: one we can integrate using the substitution $u = x^2 - x + 1$, the other will look like the derivative of arctangent.

$$\begin{aligned}
 &= \frac{1}{3} \log|x+1| - \int \frac{\frac{1}{3}x - \frac{1}{6} - \frac{1}{2}}{x^2 - x + 1} dx \\
 &= \frac{1}{3} \log|x+1| - \frac{1}{6} \int \frac{2x-1}{x^2 - x + 1} dx + \frac{1}{2} \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx \\
 &= \frac{1}{3} \log|x+1| - \frac{1}{6} \log|x^2 - x + 1| + \frac{1}{2} \int \frac{1}{\frac{3}{4} \left(\left(\frac{2x-1}{\sqrt{3}} \right)^2 + 1 \right)} dx \\
 &= \frac{1}{3} \log|x+1| - \frac{1}{6} \log|x^2 - x + 1| + \frac{2}{3} \int \frac{1}{\left(\frac{2x-1}{\sqrt{3}} \right)^2 + 1} dx
 \end{aligned}$$

Let $u = \frac{2x-1}{\sqrt{3}}$, $du = \frac{2}{\sqrt{3}} dx$.

$$\begin{aligned}
 &= \frac{1}{3} \log|x+1| - \frac{1}{6} \log|x^2 - x + 1| + \frac{1}{\sqrt{3}} \int \frac{1}{u^2 + 1} dx \\
 &= \frac{1}{3} \log|x+1| - \frac{1}{6} \log|x^2 - x + 1| + \frac{1}{\sqrt{3}} \arctan \left(\frac{2x-1}{\sqrt{3}} \right) + C
 \end{aligned}$$

S-27: By process of elimination, we decide to use integration by parts. We won't get anything better by antidifferentiating arcsine, so let's plan on differentiating it:

$$\begin{aligned} u &= \arcsin x & dv &= (3x)^2 dx \\ du &= \frac{1}{\sqrt{1-x^2}} dx & v &= 3x^3 \end{aligned}$$

$$\begin{aligned} \int (3x)^2 \arcsin x dx &= \underbrace{\arcsin x}_u \cdot \underbrace{3x^3}_v - \int \underbrace{3x^3}_v \cdot \underbrace{\frac{1}{\sqrt{1-x^2}}}_{du} dx \\ &= 3x^3 \arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}} dx \end{aligned}$$

So: we've gotten rid of the ugly pairing of arcsine with a polynomial, but now we're in another pickle. From here, two options present themselves. We could use the substitution $u = 1 - x^2$, or we could use a trig substitution.

Option 1: Let $u = 1 - x^2$. Then $-\frac{1}{2}du = dx$, and $x^2 = 1 - u$.

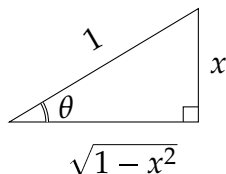
$$\begin{aligned} \int (3x)^2 \arcsin x dx &= 3x^3 \arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}} dx \\ &= 3x^3 \arcsin x - 3 \int \frac{x^2}{\sqrt{1-x^2}} \cdot x dx \\ &= 3x^3 \arcsin x + \frac{3}{2} \int \frac{1-u}{\sqrt{u}} du \\ &= 3x^3 \arcsin x + \frac{3}{2} \int (u^{-1/2} - u^{1/2}) du \\ &= 3x^3 \arcsin x + \frac{3}{2} \left(2u^{1/2} - \frac{2}{3}u^{3/2} \right) + C \\ &= 3x^3 \arcsin x + 3\sqrt{1-x^2} - \sqrt{1-x^2}^3 + C \end{aligned}$$

Option 2: If we let $x = \sin \theta$, then $\sqrt{1-x^2} = \sqrt{\cos^2 \theta} = \cos \theta$. So let's use the substitution $x = \sin \theta$, $dx = \cos \theta d\theta$.

$$\begin{aligned} \int (3x)^2 \arcsin x dx &= 3x^3 \arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}} dx \\ &= 3x^3 \arcsin x - \int \frac{3 \sin^3 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta \\ &= 3x^3 \arcsin x - \int 3 \sin^3 \theta d\theta \end{aligned}$$

And now: a substitution from Section 1.7.1 of the text, $u = \cos x$ and $du = -\sin x dx$

$$\begin{aligned}
 3x^3 \arcsin x - \int 3 \sin^3 \theta d\theta &= 3x^3 \arcsin x - 3 \int \sin^2 \theta \sin \theta d\theta \\
 &= 3x^3 \arcsin x - 3 \int (1 - \cos^2 \theta) \sin \theta d\theta \\
 &= 3x^3 \arcsin x + 3 \int (1 - u^2) du \\
 &= 3x^3 \arcsin x + 3 \left(u - \frac{1}{3} u^3 \right) + C \\
 &= 3x^3 \arcsin x + 3u - u^3 + C \\
 &= 3x^3 \arcsin x + 3 \cos \theta - \cos^3 \theta + C
 \end{aligned}$$



Recall $x = \sin \theta$; so we draw a triangle with angle θ , opposite side x , hypotenuse 1. Then by Pythagoras, adjacent side is $\sqrt{1-x^2}$, so $\cos \theta = \sqrt{1-x^2}$.

$$= 3x^3 \arcsin x + 3\sqrt{1-x^2} - (1-x^2)^{3/2} + C$$

S-28: We would like to not have that square root there. Luckily, there's a way of turning cosine into cosine squared: the identity $\cos(2x) = 2\cos^2 x - 1$. If we take $2x = t$, then $\cos t = 2\cos^2(t/2) - 1$.

$$\int_0^{\pi/2} \sqrt{\cos t + 1} dt = \int_0^{\pi/2} \sqrt{2\cos^2(t/2)} dt = \sqrt{2} \int_0^{\pi/2} |\cos(t/2)| dt$$

Over the interval $[0, \frac{\pi}{2}]$, $\cos(t/2) > 0$, so we can drop the absolute values.

$$\begin{aligned}
 &= \sqrt{2} \int_0^{\pi/2} \cos(t/2) dt = \sqrt{2} \left[2 \sin\left(\frac{t}{2}\right) \right]_0^{\pi/2} \\
 &= 2\sqrt{2} \sin\left(\frac{\pi}{4}\right) = 2
 \end{aligned}$$

S-29:

Solution 1: Using logarithm rules, $\log \sqrt{x} = \log(x^{1/2}) = \frac{1}{2} \log x$, so we can simplify:

$$\int_1^e \frac{\log \sqrt{x}}{x} dx = \int_1^e \frac{\log x}{2x} dx$$

We use the substitution $u = \log x$, $du = \frac{1}{x} dx$:

$$\begin{aligned}\int_1^e \frac{\log x}{2x} dx &= \frac{1}{2} \int_1^e \underbrace{\log(x)}_u \cdot \underbrace{\frac{1}{x} dx}_{du} \\&= \frac{1}{2} \int_{\log(1)}^{\log(e)} u \, du \\&= \frac{1}{2} \int_0^1 u \, du \\&= \frac{1}{2} \left[\frac{1}{2} u^2 \right]_0^1 \\&= \frac{1}{2} \left[\frac{1}{2} - 0 \right] = \frac{1}{4}\end{aligned}$$

Solution 2: We use the substitution $u = \log \sqrt{x}$. Then $\frac{du}{dx} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$, hence

$2du = \frac{1}{x} dx$. This fits our integral nicely!

$$\begin{aligned}\int_1^e \frac{\log \sqrt{x}}{x} dx &= \int_{\log \sqrt{1}}^{\log \sqrt{e}} u \cdot 2du \\&= \left[u^2 \right]_0^{1/2} \\&= \left(\frac{1}{2} \right)^2 - 0^2 = \frac{1}{4}\end{aligned}$$

S-30:

$$\int_{0.1}^{0.2} \frac{\tan x}{\log(\cos x)} dx$$

It might not be immediately obvious how to proceed on this one, so this is another example of an integral where you should not be discouraged by finding methods that don't work. One thing that's worked for us in the past is to use a u -substitution with the denominator. With that in mind, let's find the derivative of the denominator.

$$\frac{d}{dx} \{\log(\cos x)\} = \frac{1}{\cos x} \cdot (-\sin x) = \frac{-\sin x}{\cos x} = -\tan x$$

So, if we let $u = \log(\cos x)$, we see $-du = \tan x dx$, which will work for a substitution.

$$\begin{aligned}
\int_{0.1}^{0.2} \frac{\tan x}{\log(\cos x)} dx &= \int_{\log(\cos(0.1))}^{\log(\cos(0.2))} \frac{-du}{u} \\
&= \left[-\log |u| \right]_{\log(\cos(0.1))}^{\log(\cos(0.2))} \\
&= -\log |\log(\cos 0.2)| + \log |\log(\cos 0.1)| \\
&= \log \left| \frac{\log(\cos(0.1))}{\log(\cos(0.2))} \right| \\
&= \log \left(\frac{\log(\cos(0.1))}{\log(\cos(0.2))} \right)
\end{aligned}$$

Things to notice: the integrand is only defined when $\log(\cos x)$ exists AND is nonzero. So, for instance, it is not defined when $x = 0$, because then $\log \cos x = \log 1 = 0$, and we can't divide by zero.

In the final simplification, since 0.1 and 0.2 are between 0 and $\pi/2$, the cosine term is positive but less than one, so $\log(\cos 0.1)$ and $\log(\cos 0.2)$ are both negative; then their quotient is positive, so we can drop the absolute value signs.

Using the base change formula, we can also write the final answer as $\log \left(\log_{\cos(0.2)} \cos(0.1) \right)$.

S-31:

The integrand is of the form $N(x)/D(x)$ with $N(x)$ of lower degree than $D(x)$. So we factor $D(x) = (x-2)(x-3)$ and look for a partial fractions decomposition:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Multiplying through by the denominator yields

$$1 = A(x-3) + B(x-2)$$

Setting $x = 2$ we find:

$$1 = A(2-3) + 0 \implies A = -1$$

Setting $x = 3$ we find:

$$1 = 0 + B(3-2) \implies B = 1$$

So we have found that $A = -1$ and $B = 1$. Therefore

$$\begin{aligned}
\int \frac{1}{(x-2)(x-3)} dx &= \int \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx \\
&= \log |x-3| - \log |x-2| + C
\end{aligned}$$

and the definite integral

$$\begin{aligned}\int_0^1 \frac{1}{(x-2)(x-3)} dx &= \left[\log|x-3| - \log|x-2| \right]_0^1 \\ &= [\log 2 - \log 1] - [\log 3 - \log 2] \\ &= 2\log 2 - \log 3 = \log \frac{4}{3}\end{aligned}$$

S-32: (a) If we expand the integrand, one part of it is quite familiar—a portion of a circle. So, we split the specified integral in two.

$$\int_0^3 (x+1)\sqrt{9-x^2} dx = \int_0^3 \sqrt{9-x^2} dx + \int_0^3 x\sqrt{9-x^2} dx$$

The first piece represents the area above the x -axis and below the curve $y = \sqrt{9-x^2}$, i.e. $x^2 + y^2 = 9$, with $0 \leq x \leq 3$. That's the area of one quadrant of a disk of radius 3. So

$$\int_0^3 \sqrt{9-x^2} dx = \frac{1}{4}(\pi \cdot 3^2) = \frac{9}{4}\pi$$

For the second part, we substitute $u = 9 - x^2$, $du = -2x dx$. Note $u(0) = 9$ and $u(3) = 0$. So,

$$\int_0^3 x\sqrt{9-x^2} dx = \int_9^0 \sqrt{u} \frac{du}{-2} = -\frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_9^0 = -\frac{1}{2} \left[-\frac{27}{3/2} \right] = 9$$

All together,

$$\int_0^3 (x+1)\sqrt{9-x^2} dx = \frac{9}{4}\pi + 9$$

(b) The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. We immediately look for a partial fractions decomposition:

$$\frac{4x+8}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}.$$

Multiplying through by the denominator yields

$$4x+8 = A(x^2+4) + (Bx+C)(x-2) \quad (*)$$

Setting $x = 2$ we find:

$$8+8 = A(4+4) + 0 \implies 16 = 8A \implies A = 2$$

Substituting $A = 2$ in $(*)$ gives

$$\begin{aligned}4x+8 &= A(x^2+4) + (Bx+C)(x-2) \\ \implies -2x^2+4x &= (x-2)(Bx+C) \\ \implies (-2x)(x-2) &= (Bx+C)(x-2) \\ \implies B &= -2, C = 0\end{aligned}$$

So we have found that $A = 2$, $B = -2$, and $C = 0$. Therefore

$$\begin{aligned}\int \frac{4x+8}{(x-2)(x^2+4)} dx &= \int \left(\frac{2}{x-2} - \frac{2x}{x^2+4} \right) dx \\ &= 2 \log|x-2| - \log(x^2+4) + C\end{aligned}$$

Here the second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2 + 4$, $du = 2x dx$.

(c) The given integral is improper, but only because of its infinite limits of integration. (The integrand is continuous for all real numbers.) So, we'll have to take two limits. Before we do that, though, let's find the antiderivative. We would like to use the substitution $u = e^x$, $du = e^x dx$. That is, $\frac{1}{u} du = dx$.

$$\begin{aligned}\int \frac{1}{e^x + e^{-x}} dx &= \int \frac{1}{u(u + \frac{1}{u})} du = \int \frac{1}{u^2 + 1} = \arctan u + C \\ &= \arctan(e^x) + C\end{aligned}$$

Now we can deal with the limits of integration.

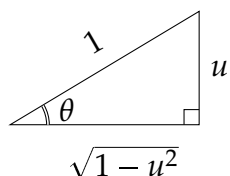
$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx &= \int_{-\infty}^0 \frac{1}{e^x + e^{-x}} dx + \int_0^{\infty} \frac{1}{e^x + e^{-x}} dx \\ &= \lim_{a \rightarrow -\infty} \left[\int_a^0 \frac{1}{e^x + e^{-x}} dx \right] + \lim_{b \rightarrow \infty} \left[\int_0^b \frac{1}{e^x + e^{-x}} dx \right] \\ &= \lim_{a \rightarrow -\infty} [\arctan(e^x)]_a^0 + \lim_{b \rightarrow \infty} [\arctan(e^x)]_0^b \\ &= \lim_{a \rightarrow -\infty} [\arctan(e^0) - \arctan(e^a)] + \lim_{b \rightarrow \infty} [\arctan(e^b) - \arctan(e^0)] \\ &= \lim_{a \rightarrow -\infty} [-\arctan(e^a)] + \lim_{b \rightarrow \infty} [\arctan(e^b)] \\ &= -\arctan(0) + \frac{\pi}{2} = \frac{\pi}{2}\end{aligned}$$

S-33: It's not immediately clear where to start, but a common method we've seen is to use the denominator in a u -substitution, especially when square roots are involved.

Let $u = \sqrt{1-x}$, $du = -\frac{1}{2\sqrt{1-x}} dx$. Then $u^2 = 1-x$, so $x = 1-u^2$.

$$\int \sqrt{\frac{x}{1-x}} dx = 2 \int \frac{\sqrt{x}}{2\sqrt{1-x}} dx = -2 \int \sqrt{1-u^2} du$$

Now we're back in familiar territory. Let $u = \sin \theta$, $du = \cos \theta \, d\theta$.



$$\begin{aligned}
 &= -2 \int \sqrt{1 - \sin^2 \theta} \cos \theta \, d\theta \\
 &= -2 \int \cos^2 \theta \, d\theta \\
 &= - \int (1 + \cos(2\theta)) \, d\theta \\
 &= -\theta - \frac{1}{2} \sin(2\theta) + C \\
 &= -\theta - \sin \theta \cos \theta + C \\
 &= -\arcsin u - u\sqrt{1-u^2} + C \\
 &= -\arcsin(\sqrt{1-x}) - \sqrt{1-x}\sqrt{x} + C
 \end{aligned} \tag{*}$$

In (*), to convert from θ to u , our substitution $u = \sin \theta$ tells us $\theta = \arcsin u$. To find $\cos \theta$, we can either trace our work backwards to see that we already simplified $\sqrt{1-u^2}$ into $\cos \theta$, or we can draw a right triangle with angle θ and $\sin \theta = u$, then use the Pythagorean theorem to find the length of the adjacent side of the triangle and $\cos \theta$.

S-34: Let's use the substitution $u = e^x$. There are a few reasons to think this is a good choice. It's an "inside function," in that if we let $f(x) = e^x$, then $f(e^x) = e^{e^x}$, which is a piece of our integrand. Also its derivative, e^x , is multiplied by the rest of the integrand, since $e^{2x} = e^x \cdot e^x$.

Let $u = e^x$, $du = e^x dx$. When $x = 0$, $u = 1$, and when $x = 1$, $u = e$.

$$\int_0^1 e^{2x} e^{e^x} dx = \int_1^e e^x e^{e^x} dx = \int_1^e u e^u du$$

This is more familiar. We use integration by parts with $dv = e^u du$, $v = e^u$. Conveniently, the " u " we brought in with the substitution is what we want to use for the " u " in integration by parts, so we don't have to change the names of our variables.

$$\begin{aligned}
 &= [ue^u]_1^e - \int_1^e e^u du \\
 &= e \cdot e^e - e - e^e + e = e^e(e-1)
 \end{aligned}$$

S-35: The substitution $u = x + 1$ looks promising at first, but doesn't result in something easily integrable. We can't use partial fractions because our integration isn't rational. This doesn't look like something from the trig-substitution family. So, let's think about integration by parts. There's a lot of different ways we could break up the integrand into two parts. For example, we could view it as $\left(\frac{x}{(x+1)^2}\right)\left(e^x\right)$, or we could view it as $\left(\frac{x}{x+1}\right)\left(\frac{e^x}{x+1}\right)$. After some trial and error, we settle on $u = xe^x$ and $dv = (x+1)^{-2} dx$.

Then $du = e^x(x+1)$ and $v = \frac{-1}{x+1}$.

$$\begin{aligned}\int \frac{xe^x}{(x+1)^2} dx &= -\frac{xe^x}{x+1} + \int \frac{e^x(x+1)}{x+1} dx \\ &= -\frac{xe^x}{x+1} + \int e^x dx \\ &= -\frac{xe^x}{x+1} + e^x + C \\ &= \frac{e^x}{x+1} + C\end{aligned}$$

S-36: It would be nice to use integration by parts with $u = x$, because then we would integrate $\int v du$, and $du = dx$. That is, the x would go away, and we'd be left with a pure trig integral. If we use $u = x$, then $dv = \frac{\sin x}{\cos^2 x}$. We need to find v :

$$v = \int \frac{\sin x}{\cos^2 x} dx = \int \tan x \sec x dx = \sec x$$

Now we use integration by parts.

$$\int \frac{x \sin x}{\cos^2 x} dx = x \sec x - \int \sec x dx = x \sec x - \log |\sec x + \tan x| + C$$

S-37: If the unknown exponent gives you the jitters, think about what this looks like in easier cases. If n is a whole number, the integrand is a polynomial. Not so scary, right? However, it's a little complicated to expand. (You can do it using the very handy binomial theorem.) Let's think of an easier way.

If we had simply the variable x raised to the power n , rather than the binomial $x+a$, that might be nicer. So, let's use the substitution $u = x+a$, $du = dx$. Note $x = u-a$.

$$\int x(x+a)^n dx = \int (u-a)u^n dx = \int (u^{n+1} - au^n) du$$

Now, if $n \neq -1$ and $n \neq -2$, we can just use the power rule:

$$\begin{aligned}&= \frac{u^{n+2}}{n+2} - a \frac{u^{n+1}}{n+1} + C \\ &= \frac{(x+a)^{n+2}}{n+2} - a \frac{(x+a)^{n+1}}{n+1} + C\end{aligned}$$

If $n = -1$, then

$$\begin{aligned}\int x(x+a)^n dx &= \int (u^{n+1} - au^n) du = \int \left(1 - \frac{a}{u}\right) du \\ &= u - a \log |u| + C = (x+a) - a \log |x+a| + C\end{aligned}$$

If $n = -2$, then

$$\begin{aligned}\int x(x+a)^n dx &= \int (u^{n+1} - au^n) du = \int \left(\frac{1}{u} - au^{-2} \right) du \\ &= \log |u| + \frac{a}{u} + C = \log |x+a| + \frac{a}{x+a} + C\end{aligned}$$

All together,

$$\int x(x+a)^n dx = \begin{cases} \frac{(x+a)^{n+2}}{n+2} - a \frac{(x+a)^{n+1}}{n+1} + C & \text{if } n \neq -1, -2 \\ (x+a) - a \log |a+x| + C & \text{if } n = -1 \\ \log |x+a| + \frac{a}{x+a} + C & \text{if } n = -2 \end{cases}$$

Solutions to Exercises 2.1 — Jump to [TABLE OF CONTENTS](#)

S-1: $Pr(X = 5) = 0.1$

S-2: The outcomes $X \leq 5$ and $X > 5$ are disjoint, so

$$Pr(X \leq 5 \text{ or } X > 5) = Pr(X \leq 5) + Pr(X > 5)$$

Since “ $X \leq 5$ or $X > 5$ ” describes all real numbers, $Pr(X \leq 5 \text{ or } X > 5) = 1$. So, it follows that

$$Pr(X > 5) = Pr(X \leq 5 \text{ or } X > 5) - Pr(X \leq 5) = 1 - \frac{99}{100} = \frac{1}{100}$$

S-3: The outcomes $X = x$ and $X \neq x$ are disjoint, so

$$Pr(X = x \text{ or } X \neq x) = Pr(X = x) + Pr(X \neq x)$$

Since “ $X = x$ or $X \neq x$ ” describes all real numbers, $Pr(X = x \text{ or } X \neq x) = 1$. So, it follows that

$$Pr(X \neq 5) = 1 - \frac{78}{93} = \frac{15}{93}$$

Solutions to Exercises 2.2 — Jump to [TABLE OF CONTENTS](#)

S-1:

(a) First, let's think about what $f(x) = 4x(x-1)(x-2)$ looks like on the interval $[0, 1]$.

- $f(x)$ is cubic function with roots at $x = 0$, $x = 1$, and $x = 2$
- $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$. Knowing the basic shapes cubics take, we know that $f(x)$ will be positive on the interval $(0, 1)$.

So $f(0) = 0$, then $f(x)$ increases and decreases, returning to the x -axis at $f(1) = 0$. That means we will have fewer marks close to $x = 0$ and $x = 1$, and more marks in the middle. (If we wanted to be really precise, we could find that $f(x)$ has a local maximum at $x = 1 - \frac{1}{\sqrt{3}}$. Density diagrams aren't very precise, so there isn't much point to going that far.)

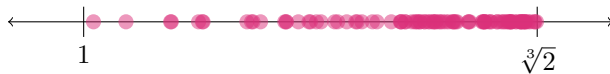


$$(b) f(x) = \begin{cases} \frac{1}{2} & 0 < x < 1 \\ \frac{1}{3} & 1 < x < 2 \\ \frac{1}{6} & 2 < x < 3 \end{cases} = \begin{cases} \frac{3}{6} & 0 < x < 1 \\ \frac{2}{6} & 1 < x < 2 \\ \frac{1}{6} & 2 < x < 3 \end{cases}$$

Comparing the intervals $(0, 1)$, $(1, 2)$, and $(1, 3)$: we expect $(1, 2)$ to be twice as dense as $(2, 3)$, and we expect $(0, 1)$ to be three times as dense as $(2, 3)$.



(c) $f(x) = x^2$ increases on the interval $[1, \sqrt[3]{2}]$, so the marks will be sparse near $x = 1$ and dense near $x = \sqrt[3]{2}$.



S-2: By Corollary 2.2.5,

$$Pr(4 \leq W \leq 17) = \int_4^{17} f(x) dx = \int_4^{17} \frac{10/\pi}{1 + 100x^2} dx$$

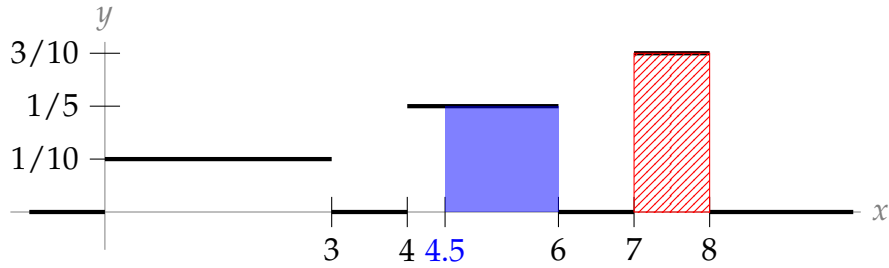
The integrand looks like the derivative of arctangent, so we use the substitution $u = 10x$, $du = 10dx$.

$$\begin{aligned} &= \frac{1}{\pi} \int_4^{17} \frac{1}{1 + 100x^2} \cdot 10dx = \frac{1}{\pi} \int_{40}^{170} \frac{1}{1 + u^2} du \\ &= \frac{1}{\pi} [\arctan u]_{40}^{170} = \frac{\arctan(170) - \arctan(40)}{\pi} \approx 0.006 \end{aligned}$$

S-3: By Corollary 2.2.5,

$$Pr(Q \geq 4.5) = Pr(4.5 \leq Q < \infty) = \int_{4.5}^{\infty} f(x) dx$$

If we wanted to find this area using the Fundamental Theorem of Calculus, we'd need to chop the interval $[4.5, \infty)$ up into subintervals $(4.5, 6)$, $(6, 7)$, $(7, 8)$, and $(8, \infty)$. It's easier to use geometry.



The blue solid rectangle has area $(1.5)(\frac{1}{5}) = 0.3$, and the pink striped rectangle has area $(1)(\frac{3}{10}) = 0.3$, so all together $Pr(Q \geq 4.5) = 0.6$.

S-4: Because the two events are disjoint,

$$Pr(0 < M < 1 \text{ OR } 9 < M < 10) = Pr(0 < M < 1) + Pr(9 < M < 10)$$

Since M is a continuous variable, $Pr(a < M < b) = Pr(a \leq M \leq b)$.

$$\begin{aligned} &= \int_0^1 f(x)dx + \int_9^{10} f(x)dx = \int_0^1 \frac{x}{50}dx + \int_9^{10} \frac{x}{50}dx \\ &= \left[\frac{x^2}{100} \right]_0^1 + \left[\frac{x^2}{100} \right]_9^{10} = \left[\frac{1}{100} - 0 \right] + \left[\frac{100}{100} - \frac{81}{100} \right] = \frac{2}{10} \end{aligned}$$

S-5: Properties of a PDF are given in Corollary 2.2.5 in the text. In particular, the area under the entire function is 1.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= 0 + \int_{-1}^b e^x dx \\ &= e^x \Big|_{-1}^b \\ &= e^b - e^{-1} \\ 1 + \frac{1}{e} &= e^b \\ \ln \left(1 + \frac{1}{e} \right) &= b \end{aligned}$$

S-6: Properties of a PDF are given in Corollary 2.2.5 in the text. In particular, the area under the entire function is 1. **Solution 1:**

$$\begin{aligned}
 1 &= \int_{-\infty}^{\infty} f(x) dx \\
 &= \int_{-\infty}^{\infty} \frac{A}{x^2 + 4} dx
 \end{aligned}$$

Let $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$. Note $\theta = \arctan\left(\frac{x}{2}\right)$ and $\lim_{x \rightarrow \pm\infty} \arctan\left(\frac{x}{2}\right) = \pm\frac{\pi}{2}$

$$\begin{aligned}
 1 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{A}{4 \tan^2 \theta + 4} \cdot 2 \sec^2 \theta d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2A}{4 \sec^2 \theta} \cdot \sec^2 \theta d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{A}{2} d\theta \\
 &= \frac{A}{2} \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= \frac{A}{2} (\pi) \\
 \frac{2}{\pi} &= A
 \end{aligned}$$

Solution 2:

$$\begin{aligned}
 1 &= \int_{-\infty}^{\infty} f(x) dx \\
 &= \int_{-\infty}^{\infty} \frac{A}{x^2 + 4} dx \\
 &= \int_{-\infty}^{\infty} \frac{A}{4 \left(\frac{x}{2}\right)^2 + 4} dx \\
 &= \frac{A}{4} \int_{-\infty}^{\infty} \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx
 \end{aligned}$$

Let $u = \frac{x}{2}$, $du = \frac{1}{2} dx$.

$$\begin{aligned}
 1 &= \frac{A}{4} \int_{-\infty}^{\infty} \frac{1}{u^2 + 1} \cdot 2 du \\
 &= \frac{A}{2} \left[\lim_{a \rightarrow -\infty} \arctan u \Big|_a^0 + \lim_{b \rightarrow \infty} \arctan u \Big|_0^b \right] \\
 &= \frac{A}{2} \left[0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 \right] \\
 &= \frac{\pi}{2} A \\
 \frac{2}{\pi} &= A
 \end{aligned}$$

S-7: We know $Pr(0 \leq X \leq 1) = \int_0^1 f(x)dx$. Equation 1.12.4 gives us Simpson's Rule:

$$\int_a^b f(x) dx \approx \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \right. \\ \left. \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right] \frac{\Delta x}{3}$$

where $\Delta x = \frac{b-a}{n}$. In this case, $\Delta x = \frac{1}{4}$. So,

$$\begin{aligned} Pr(0 \leq X \leq 1) &= \int_0^1 f(x)dx \\ &\approx \left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right] \frac{1/4}{3} \\ &= \left[\sqrt{\frac{2}{\pi}} + 4 \left(\sqrt{\frac{2}{\pi}} e^{-1/8} \right) + 2 \left(\sqrt{\frac{2}{\pi}} e^{-1/2} \right) + 4 \left(\sqrt{\frac{2}{\pi}} e^{-9/8} \right) + \left(\sqrt{\frac{2}{\pi}} e^{-2} \right) \right] \frac{1}{12} \end{aligned}$$

Solutions to Exercises 2.3 — [Jump to TABLE OF CONTENTS](#)

S-1: False.

Indeed, it is often the case that $\mathbb{E}(X)$ is not even in the sample space of X . For example, imagine X results from a fair coin flip, where $X = 0$ when the flip is tails and $X = 1$ when the flip is heads. Then $\mathbb{E}(X) = \frac{1}{2}$, which is *never* a value of X .

S-2: Z does not have to be uniformly distributed. For any probability density function (PDF) $f(x)$ that has *even symmetry*, the function $xf(x)$ has odd symmetry, so

$$\mathbb{E}(Z) = \int_{-1}^1 xf(x)dx = 0$$

If $f(x)$ is a constant function on the interval $[-1, 1]$, then Z is uniformly distributed, but this isn't the only function with even symmetry. For example, we could have $f(x) = \frac{3}{2}x^2$.

S-3: Using Definition 2.3.1,

$$\begin{aligned} \mathbb{E}(M) &= \int_{-\infty}^{\infty} x \cdot f(x)dx = 0 + \int_0^{100} x \cdot \frac{x}{5000}dx \\ &= \left[\frac{x^3}{15000} \right]_0^{100} = \frac{10^6}{15000} = \frac{1000}{15} = 66 + \frac{2}{3} \end{aligned}$$

The sample space of M is the interval $\mathcal{S} = [0, 100]$, so by Theorem 2.3.5, $0 \leq \mathbb{E}(X) \leq 100$. The PDF of M is increasing on its sample space, so by Theorem 2.3.6, $\mathbb{E}(X) > 50$. All together, we must have $50 < \mathbb{E}(M) \leq 100$, and indeed this is the case.

S-4: Using Definition 2.3.1,

$$\mathbb{E}(N) = \int_{-\infty}^{\infty} x \cdot f(x) dx = 0 + \int_{-1}^1 x \cdot \frac{2/\pi}{x^2 + 1} dx$$

The function $c \frac{x}{x^2+1}$ has odd symmetry, so

$$\int_{-1}^1 x \cdot \frac{2/\pi}{x^2 + 1} dx = 0$$

So, $\mathbb{E}(N) = 0$.

Note $\mathcal{S} = [-1, 1]$, so by Theorem 2.3.5, $-1 \leq \mathbb{E}(N) \leq 1$, which is true of 0. Since $f(x)$ is not increasing, nor is it decreasing, we cannot use Theorem 2.3.6.

S-5: Using Definition 2.3.1,

$$\begin{aligned} \mathbb{E}(P) &= \int_{-\infty}^{\infty} x \cdot f(x) dx = 0 + \int_0^1 x \cdot \frac{2}{3} x(1-x) dx + \int_2^3 x \cdot \frac{2}{3} (x-2) dx \\ &= \frac{2}{3} \int_0^1 (x^2 - x^3) dx + \frac{2}{3} \int_2^3 (x^2 - 2x) dx \\ &= \frac{2}{3} \left[\frac{1}{3} x^3 - \frac{1}{4} x^4 \right]_0^1 + \frac{2}{3} \left[\frac{1}{3} x^3 - x^2 \right]_2^3 = \frac{17}{18} \end{aligned}$$

Since $f(x)$ is neither increasing nor decreasing, we can't apply Theorem 2.3.6. However, by Theorem 2.3.5, we need $0 \leq \mathbb{E}(P) \leq 3$, so our answer passes that check.

S-6: Using Definition 2.3.1,

$$\mathbb{E}(Q) = \int_1^e x \cdot \left(\frac{e}{e-2} \right) \cdot \frac{\ln x}{x^2} dx = \frac{e}{e-2} \int_1^e \ln x \cdot \frac{1}{x} dx$$

Let $u = \ln x$, $du = \frac{1}{x} dx$

$$\begin{aligned} &= \frac{e}{e-2} \int_{\ln 1}^{\ln e} u du = \frac{e}{e-2} \int_0^1 u du = \frac{e}{e-2} \left[\frac{1}{2} u^2 \right]_0^1 \\ &= \frac{e}{e-2} \left(\frac{1}{2} \right) \end{aligned}$$

Q takes on values from the interval $[1, e]$, so we expect $1 \leq \mathbb{E}(Q) \leq e$.

Using a calculator: we expect $1 \leq \mathbb{E}(Q) < 2.72$ and $\mathbb{E}(Q) \approx 1.9$.

By hand:

$$\begin{aligned} &e < 4 \\ \implies &-4 < -e \\ \implies &2e - 4 < 2e - e = e \\ \implies &1 < \frac{e}{2e - 4}, \text{ as desired} \end{aligned}$$

S-7: Since Y is uniformly distributed over $[a, b]$, its PDF is a constant on that interval: $f(x) = c, a \leq x \leq b$. In order for $f(x)$ to be a PDF, we must have the area underneath it equal to 1, so

$$1 = c(b - a) \quad \implies \quad c = \frac{1}{b - a}$$

Now, we can compute $\mathbb{E}(Y)$:

$$\begin{aligned} \mathbb{E}(Y) &= \int_a^b x \cdot \frac{1}{b - a} dx = \frac{1}{2(b - a)} x^2 \Big|_a^b = \frac{b^2 - a^2}{2(b - a)} \\ &= \frac{(b + a)(b - a)}{2(b - a)} = \frac{b + a}{2} \end{aligned}$$

It makes intuitive sense that, if Y is uniformly distributed on $[a, b]$, then its long-term average will be exactly in the middle of that interval.

S-8:

(a) Corollary 2.2.5 has properties of PDFs.

- $f_p(x) \geq 0$ for all real x in the domain of $f_p(x)$: this is true for any positive p
- We need $\int_1^\infty f_p(x) dx = 1$; in particular, that means $\int_1^\infty \frac{1}{x^p} dx$ has to converge. From Example 1.13.10 in the text, this happens when $p > 1$.

So, this setup only makes sense when $p > 1$.

(b) For $p > 1$:

$$\begin{aligned} 1 &= \int_1^\infty f_p(x) dx = a_p \int_1^\infty x^{-p} dx = a_p \lim_{b \rightarrow \infty} \left[\frac{1}{1 - p} x^{1-p} \right]_1^b \\ &= \frac{a_p}{1 - p} \lim_{b \rightarrow \infty} [b^{1-p} - 1] = \frac{a_p}{1 - p} (-1) = \frac{a_p}{p - 1} \\ a_p &= p - 1 \end{aligned}$$

(c) For $p > 1$:

$$\mathbb{E}(X_p) = \int_1^\infty x \cdot \frac{a_p}{x^p} dx = (p - 1) \int_1^\infty \frac{1}{x^{p-1}} dx$$

Again from Example 1.13.10 in the text, this integral converges when $p - 1 > 1$, i.e. when $p > 2$.

S-9:

$$\mathbb{E}(A) = \int_0^1 x^2 e^x dx$$

Use integration by parts with $u = x^2$, $dv = e^x dx$; $du = 2x dx$, $v = e^x$

$$= \left[x^2 e^x \right]_0^1 - \int_0^1 2x e^x dx = e - 2 \int_0^1 x e^x dx$$

Use integration by parts with $u = x$, $dv = e^x dx$; $du = dx$, $v = e^x$

$$\begin{aligned} &= e - 2 \left(\left[x e^x \right]_0^1 - \int_0^1 e^x dx \right) = e - 2 \left(e - \left[e^1 - e^0 \right] \right) \\ &= e - 2 \end{aligned}$$

Note: $0 \leq e - 2 \leq 1$, so it passes our first check.

S-10:

$$\mathbb{E}(B) = \int_1^2 x \cdot \frac{2}{\ln^2 2} \left(\frac{\ln x}{x} \right) dx = \frac{2}{\ln^2 2} \int_1^2 \ln x dx$$

We use integration by parts to evaluate $\int \ln x dx$, as in Example 1.11.8 in the text. Let $u = \ln x$, $dv = dx$; $du = \frac{1}{x} dx$, $v = x$

$$\begin{aligned} &= \frac{2}{\ln^2 2} \left(x \ln x \Big|_1^2 - \int_1^2 1 dx \right) = \frac{2}{\ln^2 2} \left(2 \ln 2 - \left[x \right]_1^2 \right) \\ &= \frac{2}{\ln^2 2} (2 \ln 2 - 1) \end{aligned}$$

S-11:

$$\mathbb{E}(C) = \int_2^b \frac{x^2}{x^2 - 1} dx = \int_2^b \frac{x^2 - 1 + 1}{x^2 - 1} dx = \int_2^b \left(1 + \frac{1}{x^2 - 1} \right) dx$$

To use integration by partial fractions, we factor

$$= \int_2^b \left(1 + \frac{1}{(x+1)(x-1)} \right) dx$$

Intermediate calculation:

$$\begin{aligned} \frac{1}{(x+1)(x-1)} &= \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)} \\ x = 1 &\implies B = \frac{1}{2} \quad x = -1 \implies A = -\frac{1}{2} \end{aligned}$$

Returning to the integral:

$$\begin{aligned}\mathbb{E}(C) &= \int_2^b \left(1 + \frac{1/2}{x-1} - \frac{1/2}{x+1}\right) dx = \left[x + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| \right]_2^b \\ &= \left[x + \frac{1}{2} \ln\left(\frac{x-1}{x+1}\right) \right]_2^b \\ &= (b-2) + \frac{1}{2} \ln\left(\frac{b-1}{b+1}\right) - \frac{1}{2} \ln\left(\frac{1}{3}\right) \\ &= (b-2) + \frac{1}{2} \ln\left(3 \frac{b-1}{b+1}\right)\end{aligned}$$

S-12:

$$\begin{aligned}\mathbb{E}(D) &= \frac{4}{4-\pi} \int_0^{\pi/4} x \cdot \tan^2 x dx = \frac{4}{4-\pi} \left(\int_0^{\pi/4} x(\sec^2 x - 1) dx \right) \\ &= \frac{4}{4-\pi} \left(\int_0^{\pi/4} x \cdot \sec^2 x dx - \int_0^{\pi/4} x dx \right)\end{aligned}$$

Use integration by parts with $u = x$, $dv = \sec^2 x dx$; $du = dx$, $v = \tan x$

$$\begin{aligned}&= \frac{4}{4-\pi} \left(\left[x \tan x \right]_0^{\pi/4} - \int_0^{\pi/4} \tan x dx - \left[\frac{1}{2} x^2 \right]_0^{\pi/4} \right) \\ &= \frac{4}{4-\pi} \left(\frac{\pi}{4} - \int_0^{\pi/4} \tan x dx - \left[\frac{\pi^2}{32} \right] \right) \\ &= \frac{4}{4-\pi} \left(\frac{\pi}{4} - \left[\ln|\sec x| \right]_0^{\pi/4} - \left[\frac{\pi^2}{32} \right] \right) \\ &= \frac{4}{4-\pi} \left(\frac{\pi}{4} - \left[\ln \sqrt{2} \right] - \left[\frac{\pi^2}{32} \right] \right) \\ &= \frac{1}{4-\pi} \left(\pi - 2 \ln 2 - \frac{\pi^2}{8} \right)\end{aligned}$$

S-13: Using the definition of expectation,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_{-\infty}^{\infty} c x e^{bx^2} dx$$

The integrand has odd symmetry, so since the integral does converge, it will be equal to 0. That's the easiest way to find $\mathbb{E}(X)$.

$$= 0$$

However, we could also evaluate the integral with the substitution $u = bx^2$, $du = 2bxdx$. To avoid carrying limits throughout our work, let's first evaluate the indefinite integral.

$$\begin{aligned}\int cxe^{bx^2} dx &= \int \frac{c}{2b} e^u du = \frac{c}{2b} e^u + C = \frac{c}{2b} e^{bx^2} + C \\ \int_{-\infty}^{\infty} cxe^{bx^2} dx &= \lim_{A \rightarrow -\infty} \left[\int_A^0 cxe^{bx^2} dx \right] + \lim_{B \rightarrow \infty} \left[\int_0^B cxe^{bx^2} dx \right] \\ &= \lim_{A \rightarrow -\infty} \left[\frac{c}{2b} e^{bx^2} \Big|_A^0 \right] + \lim_{B \rightarrow \infty} \left[\frac{c}{2b} e^{bx^2} \Big|_0^B \right] \\ &= \frac{c}{2b} \lim_{A \rightarrow -\infty} [1 - e^{bA^2}] + \frac{c}{2b} \lim_{B \rightarrow \infty} [e^{bB^2} - 1]\end{aligned}$$

To evaluate the limits, remember b is *negative*, so $\lim_{x \rightarrow \pm\infty} bx^2 = -\infty$. Hence $\lim_{x \rightarrow \pm\infty} e^{bx^2} = 0$.

$$= \frac{c}{2b} [1 - 0] + \frac{c}{2b} [0 - 1] = 0$$

Solutions to Exercises 2.4 — Jump to [TABLE OF CONTENTS](#)

S-1:

A - 2: we introduced expected value in Section 2.3.1 precisely as a long-term average.

B - 4: Section 2.4.1 motivated the study of “average difference from the average;” in defining variance, we replaced “difference from the average” with “squared difference from the average.”

C - 3: because variance introduces squares, to get our numbers to have the right kind of size, standard deviation undoes the squaring, trying to get back to the original motivation of “average difference from the average” (but with functions that are differentiable).

D - 1: The probability density function (PDF) differs from the probability mass function (PMF) because in a continuous function, the probability of any one value happening is 0. However, it serves a similar purpose, helping us decide which regions are more or less likely.

S-2: If X is at most one standard deviation from the mean, that means $|X - 10| \leq 1.5$, so $8.5 \leq X \leq 11.5$. This tells us that bundles of 8.5 to 11.5 kg are typical, while bundles less than 8.5 kg or more than 11.5 kg are atypical.

S-3: A small standard deviation implies that most values are close to the mean, and a large standard deviation implies that the values do not cluster in that way. From the second paragraph, we interpret Brand B to have the larger standard deviation.

The values in Paragraph A are not necessary for solving this question, but they are somewhat reasonable: in order to make sure you have a minimum mass (what it says on the label) nearly all the time, you either need to have very accurate measurements, or you need to add in extra.

S-4: A high standard deviation means X takes values that are very spread out. So, we let X have sample space $\mathbb{S} = \{-100, 100\}$ let it be equally likely to take either value.

$$\begin{aligned}\mathbb{E}(X) &= \sum_x x \Pr(X = x) = -100(1/2) + 100(1/2) = 0 \\ \mathbb{E}(X^2) &= \sum_x x^2 \Pr(X = x) = 100^2(1/2) + 100^2(1/2) = 100^2\end{aligned}$$

Using Corollary 2.4.4:

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 100^2 \\ \sigma(X) &= \sqrt{\text{Var}(X)} = 100\end{aligned}$$

We note that Corollary 2.4.6 tells us $\sigma(X) \leq \frac{100 - (-100)}{2} = 100$, so we've given X the highest possible standard deviation, given its restricted range.

S-5:

- (a) Since X is uniformly distributed, its probability density function (PDF) is constant on the interval $[a, b]$:

$$f(x) = c, \quad a \leq x \leq b$$

If $f(x)$ is a probability density function (PDF), the area underneath it must be one. Conveniently, this area is a rectangle.

$$1 = c(b - a) \quad \implies \quad c = \frac{1}{b - a}$$

So, we have our probability density function (PDF): $f(x) = \frac{1}{b-a}$, $a \leq x \leq b$. Now we can apply our definitions.

$$\begin{aligned}\mathbb{E}(X) &= \int_a^b \frac{x}{b-a} dx = \frac{x^2}{2(b-a)} \Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2} \\ \mathbb{E}(X^2) &= \int_a^b \frac{x^2}{b-a} dx = \frac{x^3}{3(b-a)} \Big|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{1}{3}(b^2 + ab + a^2) \\ \text{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{3}(b^2 + ab + a^2) - \frac{1}{4}(b+a)^2 = \frac{(b-a)^2}{12} \\ \sigma(X) &= \sqrt{\text{Var}(X)} = \frac{b-a}{2\sqrt{3}}\end{aligned}$$

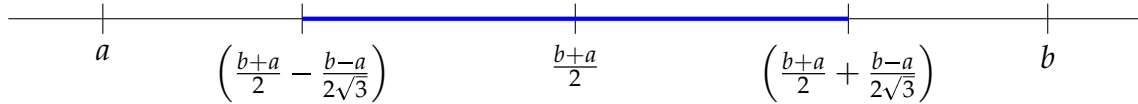
Note: had we wanted to calculate $\text{Var}(X)$ the other way, it would go like this:

$$\begin{aligned}
 \text{Var}(X) &= \int_a^b (x - \mathbb{E}(X)) f(x) dx = \frac{1}{b-a} \int_a^b \left(x - \frac{b+a}{2}\right)^2 \\
 &= \frac{1}{b-a} \int_a^b \left(x^2 - (b+a)x + \frac{(b+a)^2}{4}\right) \\
 &= \frac{1}{b-a} \left[\frac{1}{3}x^3 - \frac{b+a}{2}x^2 + \frac{b^2+2ab+a^2}{4}x \right]_a^b \\
 &= \frac{1}{b-a} \left[\frac{1}{3}b^3 - \frac{b+a}{2}b^2 + \frac{b^2+2ab+a^2}{4}b - \frac{1}{3}a^3 + \frac{b+a}{2}a^2 - \frac{b^2+2ab+a^2}{4}a \right] \\
 &= \frac{1}{b-a} \left[\frac{b^3 - 3ab^2 + 3a^2b - a^3}{12} \right] = \frac{(b-a)^3}{12(b-a)} = \frac{(b-a)^2}{12}
 \end{aligned}$$

- (b) The values that fall within one standard deviation of $\mathbb{E}(X)$ are those on the interval $[\mathbb{E}(X) - \sigma(X), \mathbb{E}(X) + \sigma(X)]$. In this case,

$$2\sigma(X) = \frac{b-a}{\sqrt{3}}$$

So the interval in question is the middle part of the interval $[a, b]$, with length $\frac{1}{\sqrt{3}}$ times the length of the whole interval. That's a little over half the entire interval $[a, b]$ included in the interval $[\mathbb{E}(X) - \sigma(X), \mathbb{E}(X) + \sigma(X)]$.



S-6: If $f(x)$ has even symmetry, then $xf(x)$ has odd symmetry, so $\mathbb{E}(X) = \int_{-a}^a xf(x)dx = 0$.

This works intuitively: the values taken by X are evenly distributed between positive and negative values, so these should cancel each other out in the long-term average.

S-7:

- (a) Before we compute $\text{Var}(X)$, we need to know $\mathbb{E}(X)$.

$$\mathbb{E}(X) = \int_0^{\frac{\pi}{2}} x \cdot \sin x dx$$

We use integration by parts with $u = x$, $dv = \sin x dx$; $du = dx$, $v = -\cos x$

$$\begin{aligned}
 &= \left[-x \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x dx = \int_0^{\frac{\pi}{2}} \cos x dx \\
 &= \sin \frac{\pi}{2} - \sin 0 = 1
 \end{aligned}$$

Now we can find $\text{Var}(X)$

$$\mathbb{E}(X^2) = \int_0^{\frac{\pi}{2}} x^2 \sin x dx$$

We use integration by parts with $u = x^2$, $dv = \sin x dx$; $du = 2x dx$, $v = -\cos x$

$$= \underbrace{\left[-x^2 \cos x \right]_0^{\frac{\pi}{2}}}_0 + 2 \int_0^{\frac{\pi}{2}} x \cos x dx$$

We use integration by parts with $u = x$, $dv = \cos x dx$; $du = dx$, $v = \sin x$

$$\begin{aligned} &= 2 \left[\underbrace{x \sin x}_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} -\sin x dx \right] \\ &= 2 \left[\frac{\pi}{2} + (\cos(\pi/2) - \cos 0) \right] = 2 \left[\frac{\pi}{2} - 1 \right] = \pi - 2 \end{aligned}$$

$$\text{So, } \text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = (\pi - 2) - 1 = \pi - 3$$

Note: we can also use the other method to compute $\text{Var}(X)$:

$$\begin{aligned} \text{Var}(X) &= \int_{-\infty}^{\infty} f(x) \cdot (x - \mathbb{E}(X))^2 dx = \int_0^{\frac{\pi}{2}} \sin x \cdot (x - 1)^2 dx \\ &= \int_0^{\frac{\pi}{2}} (x^2 \sin x - 2x \sin x + \sin x) dx \\ &= \int_0^{\frac{\pi}{2}} x^2 \sin x dx - 2 \int_0^{\frac{\pi}{2}} x \sin x dx + \int_0^{\frac{\pi}{2}} \sin x dx \end{aligned}$$

The second integral is $\mathbb{E}(X)$; the third integral is equal to $\int_{-\infty}^{\infty} f(x) dx$, which is 1 by Corollary 2.2.5 part 1.

$$= \int_0^{\frac{\pi}{2}} x^2 \sin x dx - 2 + 1$$

Use integration by parts with $u = x^2$, $dv = \sin x dx$; $du = 2x dx$, $v = -\cos x$

$$= \underbrace{-x^2 \cos x}_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} x \cos x dx - 1$$

Use integration by parts with $u = x$, $dv = \cos x dx$; $du = dx$, $v = \sin x$

$$\begin{aligned} &= 2 \left[\underbrace{x \sin x}_0^{\frac{\pi}{2}} - \underbrace{\int_0^{\frac{\pi}{2}} \sin x dx}_1 \right] - 1 \\ &= 2 \left[\frac{\pi}{2} - 1 \right] - 1 = \pi - 3 \end{aligned}$$

(b) $\sigma(X) = \sqrt{\pi - 3}$

S-8: To find $\mathbb{E}(X)$, it's easiest to note that $f(x)$ is an even function, so $xf(x)$ is an odd function, so $\mathbb{E}(X) = \int_{-1}^1 xf(x)dx = 0$. But we can also compute $\mathbb{E}(X)$ using the (piecewise) definition of the absolute value function.

$$\begin{aligned} |x| &= \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases} \\ \mathbb{E}(X) &= \int_{-1}^1 xf(x)dx = \int_{-1}^0 x(1+x)dx + \int_0^1 x(1-x)dx \\ &= \int_{-1}^0 (x+x^2)dx + \int_0^1 (x-x^2)dx = \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-1}^0 + \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \\ &= \left[0 - \left(\frac{1}{2} - \frac{1}{3} \right) \right] + \left[\left(\frac{1}{2} - \frac{1}{3} \right) - 0 \right] = 0 \end{aligned}$$

To compute variance, we'll now find $\mathbb{E}(X^2)$.

$$\begin{aligned} \mathbb{E}(X^2) &= \int_{-1}^1 x^2 f(x)dx = \int_{-1}^0 x^2(1+x)dx + \int_0^1 x^2(1-x)dx \\ &= \int_{-1}^0 (x^2+x^3)dx + \int_0^1 (x^2-x^3)dx = \left[\frac{x^3}{3} + \frac{1}{4}x^4 \right]_{-1}^0 + \left[\frac{x^3}{3} - \frac{1}{4}x^4 \right]_0^1 \\ &= \left[0 - \left(\frac{-1}{3} + \frac{1}{4} \right) \right] + \left[\left(\frac{1}{3} - \frac{1}{4} \right) - 0 \right] = \frac{1}{6} \\ \text{So, } \text{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{6} - 0 = \frac{1}{6} \end{aligned}$$

Alternately, we can compute variance the other way.

$$\begin{aligned} \text{Var}(X) &= \int_{-1}^1 (x - \mathbb{E}(X))^2 f(x)dx = \int_{-1}^0 x^2(1+x)dx + \int_0^1 x^2(1-x)dx \\ &= \left[\frac{x^3}{3} + \frac{1}{4}x^4 \right]_{-1}^0 + \left[\frac{x^3}{3} - \frac{1}{4}x^4 \right]_0^1 = \frac{1}{6} \end{aligned}$$

Now, $\sigma(X) = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}}$.

Solutions to Exercises 3.1 — Jump to [TABLE OF CONTENTS](#)

S-1: (a) The values of the sequence seem to be getting closer and closer to -2, so we guess the limit of this sequence is -2.

(b) Overall, the values of the sequence seem to be getting extremely close to 0, so we approximate the limit of this sequence as 0. It doesn't matter that the sequence changes signs, or that the numbers are sometimes farther from 0, sometimes closer.

(c) This limit does not exist. The sequence is sometimes 0, sometimes -2, and not consistently staying extremely near to either one.

S-2: True. We consider the end behaviour of the sequences, which does not depend on any finite number of terms at their beginning.

S-3: (a) We follow the arithmetic of limits, Theorem 3.1.8 in the text: $\frac{A - B}{C}$

(b) Since $\lim_{n \rightarrow \infty} c_n$ is some real number, and n grows without bound, $\lim_{n \rightarrow \infty} \frac{c_n}{n} = 0$.

(c) We note $\lim_{n \rightarrow \infty} a_{2n+5} = \lim_{n \rightarrow \infty} a_n$, so $\frac{a_{2n+5}}{b_n} = \frac{A}{B}$.

S-4: There are many possible answers. One is:

$$a_n = \begin{cases} 3000 - n & \text{if } n \leq 1000 \\ -2 + \frac{1}{n} & \text{if } n > 1000 \end{cases}$$

where we have a series that looks different before and after its thousandth term. Note every term is smaller than the term preceding it.

Another sequence with the desired properties is:

$$a_n = \frac{1,002,001}{n} - 2$$

When $n \leq 1000$, $a_n \geq \frac{1,002,001}{1000} - 2 > \frac{1,002,000}{1,000} - 2 = 1000$. That is, $a_n > 1000$ when $n \leq 1000$. As n gets larger, a_n gets smaller, so $a_{n+1} < a_n$ for all n . Finally, $\lim_{n \rightarrow \infty} a_n = 0 - 2 = -2$.

S-5: One possible answer is $a_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, -1, \dots\}$.

Another is $a_n = n(-1)^n = \{-1, 2, -3, 4, -5, 6, -7, \dots\}$.

S-6: If the terms of a sequence are alternating sign, but the limit of the sequence exists, the limit must be zero. (If it were a positive number, the negative terms would not get very close to it; if it were a negative number, the positive terms would not get very close to it.)

This gives us the idea to modify an answer from Question 5. One possible sequence:

$$a_n = \frac{(-1)^n}{n} = \left\{ -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots \right\}$$

S-7:

(a) • Note $a_n = b_n$ since $n = |n|$ for all $n \geq 1$. Then $a_n = b_n = 1 + \frac{1}{n} = \frac{n+1}{n}$. So, whenever n is a whole number, a_n and b_n are the same as $h(n)$ and $i(n)$. (Be careful here: $h(x) \neq i(x)$ when x is not a whole number.)

- $c_n = e^{-n} = \frac{1}{e^n} = j(n)$
- For any integer n , $\cos(\pi n) = (-1)^n$. So, $d_n = f(n)$.
- Similarly, $e_n = g(n)$.

(b) According to Theorem 3.1.6 in the text, if any of the functions on the right have limits that exist as $x \rightarrow \infty$, then these limits match the limits of their corresponding sequences. So, we only have to be suspicious of $f(x)$ and $i(x)$, since these do not converge.

The limit $\lim_{x \rightarrow \infty} f(x)$ does not exist, and $f(n) = d_n$; the limit $\lim_{n \rightarrow \infty} d_n$ also does not exist. (We generally don't write equality for two things that don't exist: equality refers to numerical value, and these have none.⁶)

The limit $\lim_{x \rightarrow \infty} i(x)$ does not exist, because $i(x) = 0$ when x is not a whole number, while $i(x)$ approaches 1 when x is a whole number. However, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 1$.

So, using our answers from part (a), we match the following:

- $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \lim_{x \rightarrow \infty} h(x) = 1$
- $\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} e_n = \lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} j(x) = 0$
- $\lim_{n \rightarrow \infty} d_n$, $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow \infty} i(x)$ do not exist.

S-8: (a) We want to find odd multiples of π that are close to integers.

Solution 1: One way to do that is to remember that π is somewhat close to $\frac{22}{7}$. Then when we multiply π by a multiple of 7, we should get something close to an integer. In particular, 7π , 21π , and 35π should be reasonably close to $7 \left(\frac{22}{7} \right) = 22$, $21 \left(\frac{22}{7} \right) = 66$, and $35 \left(\frac{22}{7} \right) = 110$, respectively. We check whether they are close enough:

$$7\pi \approx 21.99 \quad 21\pi \approx 65.97 \quad 35\pi \approx 109.96$$

So indeed, 22, 66, and 110 are all within 0.1 of some odd multiple of π .

Since the cosine of an odd multiple of π is -1 , we expect all of the sequence values to be close to -1 . Using a calculator:

$$\begin{aligned} a_{22} &= \cos(22) \approx -0.99996, \\ a_{66} &= \cos(66) \approx -0.99965, \\ a_{110} &= \cos(110) \approx -0.99902 \end{aligned}$$

6 The idea "two things that both don't exist are equal" is also rejected because it can lead to contradictions. For example, in the real numbers $\sqrt{-1}$ and $\sqrt{-2}$ don't exist; if we write $\sqrt{-1} = \sqrt{-2}$, then squaring both sides yields the inanity $-1 = -2$.

Solution 2: Alternately, we could have just listed odd multiple of π until we found three that are close to integers.

$2k + 1$	$(2k + 1)\pi$
1	3.14
3	9.42
5	15.71
7	21.99
9	28.27
11	34.56
13	40.84
15	47.12
17	53.41
19	59.69
21	65.97
23	72.26
25	78.54
27	84.82
29	91.11
31	97.39
33	103.67
35	109.96

Some earlier odd multiples of π (like 15π and 29π) get fairly close to integers, but not within 0.1.

(b) If $x = \frac{2k+1}{2}\pi$ for some integer k (that is, x is an odd multiple of $\pi/2$), then $\cos x = 0$. So, we can either list out the first few terms of a_n until we find three that are very close to 0, or we can use our approximation $\pi \approx \frac{22}{7}$ to choose values of n that are close to $\frac{2k+1}{2}\pi$.

Solution 1:

$$\frac{2k+1}{2}\pi \approx \frac{(2k+1) \times 22}{2 \times 7} = 11 \times \frac{2k+1}{7}$$

So, we expect our values to be close to integers when $2k+1$ is a multiple of 7. For example, $2k+1 = 7$, $2k+1 = 21$, and $2k+1 = 35$.

We check:

x	n	a_n
$7 \times \frac{\pi}{2} \approx 10.99557$	11	$a_{11} \approx 0.0044$
$21 \times \frac{\pi}{2} \approx 32.98672$	33	$a_{33} \approx -0.0133$
$35 \times \frac{\pi}{2} \approx 54.97787$	55	$a_{55} \approx 0.0221$

These seem like values of a_n that are all pretty close to 0.

Solution 2: We could have listed the first several values of a_n , and looked for some that are close to 0.

n	a_n
1	0.54
2	-0.42
3	-0.99
4	-0.65
5	0.28
6	0.96
7	0.75
8	-0.15
9	-0.91
10	-0.84

Oof. Nothing very close yet. Maybe a better way is to list values of $\frac{2k+1}{2}\pi$, and see which ones are close to integers.

$2k+1$	$\frac{2k+1}{2}\pi$
1	1.57
3	4.71
5	7.85
7	10.996
9	14.14
11	17.28
13	20.42
15	23.56
17	26.70
19	29.85
21	32.99
23	36.13
25	39.27
27	42.41
29	45.55
31	48.69
33	51.84
35	54.98

We find roughly the same candidates we did in Solution 1, depending on what we're ready to accept as "close".

(c) One can use the same strategies as we did for parts (a) and (b). But since we already know some n 's with $\cos(n)$ close to -1 , it's easier to use the trig identity

$$\cos(2m) = 2\cos^2(m) - 1$$

This identity shows that if $\cos(m)$ is close to -1 , then $\cos(2m)$ is close to $+1$. So let's try

$$n = 2m = 2 \times 22 = 44, 2 \times 66 = 132 \text{ and } 2 \times 110 = 220.$$

n	a_n
44	0.9998
132	0.9986
220	0.9961

They do the trick.

Remark: it is possible to turn the ideas of this question into a rigorous proof that

$\lim_{n \rightarrow \infty} \cos n$ is undefined.

- Let, for each integer $k \geq 1$, n_k be the integer that is closest to $2k\pi$. Then $2k\pi - \frac{1}{2} \leq n_k \leq 2k\pi + \frac{1}{2}$ so that $\cos(n_k) \geq \cos \frac{1}{2} \geq 0.8$. Consequently, if $\lim_{n \rightarrow \infty} \cos n = c$ exists, we must have $c \geq 0.8$.
- Let, for each integer $k \geq 1$, n'_k be the integer that is closest to $(2k+1)\pi$. Then $(2k+1)\pi - \frac{1}{2} \leq n'_k \leq (2k+1)\pi + \frac{1}{2}$ so that $\cos(n'_k) \leq -\cos \frac{1}{2} \leq -0.8$. Consequently, if $\lim_{n \rightarrow \infty} \cos n = c$ exists, we must have $c \leq -0.8$.
- It is impossible to have both $c \geq 0.8$ and $c \leq -0.8$, so $\lim_{n \rightarrow \infty} \cos n$ does not exist.

Remark: This question also hints at a property of the set of all numbers $\cos(n)$, with n running over the integers. Mathematicians say that this set is “dense in the interval $[-1, 1]$ ”. This means that if you pick any number $-1 \leq r \leq 1$, there is an integer n such that $\cos(n)$, while not necessarily being exactly r , is as close to r as you like. If you feed “ $\cos(n)$ dense” into your favourite search engine, you can find out more.

S-9: When determining the end behaviour of rational functions, recall from last semester that we can either cancel out the highest power of n from the numerator and denominator, or skip this step and compare the highest powers of the numerator and denominator.

- (a) Since the numerator has a higher degree than the denominator, this sequence will diverge to positive or negative infinity; since its terms are positive for large n , its limit is (positive) infinity. (You can imagine that the numerator is growing much, much faster than the denominator, leading the terms to have a very, very large absolute value.)

Calculating the longer way:

$$a_n = \frac{3n^2 - 2n + 5}{4n + 3} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) = \frac{3n - 2 + \frac{5}{n}}{4 + \frac{3}{n}}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3n - 2 + \frac{5}{n}}{4 + \frac{3}{n}} = \lim_{n \rightarrow \infty} \frac{3n - 2 + 0}{4 + 0} = \infty$$

- (b) Since the numerator has the same degree as the denominator, as n goes to infinity, this sequence will converge to the ratio of their leading coefficients: $\frac{3}{4}$. (You can

imagine that the numerator is growing at roughly the same rate as the denominator, so the terms settle into an almost-constant ratio.)

Calculating the longer way:

$$b_n = \frac{3n^2 - 2n + 5}{4n^2 + 3} \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right) = \frac{3 - \frac{2}{n} + \frac{5}{n^2}}{4 + \frac{3}{n^2}}$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{3 - \frac{2}{n} + \frac{5}{n^2}}{4 + \frac{3}{n^2}} = \frac{3 - 0 + 0}{4 + 0} = \frac{3}{4}$$

- (c) Since the numerator has a lower degree than the denominator, this sequence will converge to 0 as n goes to infinity. (You can imagine that the denominator is growing much, much faster than the numerator, leading the terms to be very, very small.)

Calculating the longer way:

$$c_n = \frac{3n^2 - 2n + 5}{4n^3 + 3} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right) = \frac{\frac{3}{n} - \frac{2}{n^2} + \frac{5}{n^3}}{4 + \frac{3}{n^3}}$$

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} \frac{\frac{3}{n} - \frac{2}{n^2} + \frac{5}{n^3}}{4 + \frac{3}{n^3}} = \frac{0 - 0 + 0}{4 + 0} = 0$$

S-10: At first glance, we see both the numerator and denominator grow huge as n increases, so we'll need to think a little further to find the limit.

We don't have a rational function, but we can still divide the top and bottom by n^e to get a clearer picture.

$$a_n = \frac{4n^3 - 21}{n^e + \frac{1}{n}} \left(\frac{\frac{1}{n^e}}{\frac{1}{n^e}} \right) = \frac{4n^{3-e} - \frac{21}{n^e}}{1 + \frac{1}{n^{e+1}}}$$

Since $e < 3$, we see $3 - e$ is positive, so $\lim_{n \rightarrow \infty} n^{3-e} = \infty$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{4n^{3-e} - \frac{21}{n^e}}{1 + \frac{1}{n^{e+1}}} = \lim_{n \rightarrow \infty} \frac{4n^{3-e} - 0}{1 + 0} = \infty$$

S-11: This isn't a rational sequence, but factoring out $\sqrt{n}$ from the top and bottom will still clear things up.

$$b_n = \frac{\sqrt[4]{n} + 1}{\sqrt{9n} + 3} \left(\frac{\frac{1}{\sqrt[4]{n}}}}{\frac{1}{\sqrt[4]{n}}}} \right) = \frac{\frac{1}{\sqrt[4]{n}} + \frac{1}{\sqrt{n}}}}{\sqrt{9 + \frac{3}{n}}}}$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt[4]{n}} + \frac{1}{\sqrt{n}}}}{\sqrt{9 + \frac{3}{n}}}} = \frac{0 + 0}{\sqrt{9 + 0}} = 0$$

S-12:

$$d_n = e^{-1/n} = \frac{1}{e^{1/n}}$$
$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} \frac{1}{e^{1/n}} = \frac{1}{e^0} = \frac{1}{1} = 1$$

S-13: First, we note that both numerator and denominator grow without bound. So, we have to decide whether one outstrips the other, or whether they reach a stable ratio.

Solution 1: Let's try dividing the numerator and denominator by 2^n (the dominant term in the denominator; this is the same idea behind factoring out the leading term in rational expressions).

$$b_n = \frac{e^n}{2^n + n^2} \left(\frac{\frac{1}{2^n}}{\frac{1}{2^n}} \right) = \frac{\left(\frac{e}{2}\right)^n}{1 + \frac{n^2}{2^n}}$$

Since $e > 2$, we see $\frac{e}{2} > 1$, and so $\lim_{n \rightarrow \infty} \left(\frac{e}{2}\right)^n = \infty$. Since exponential functions grow much, much faster than polynomial functions, we also see $\lim_{n \rightarrow \infty} \frac{n^2}{2^n} = 0$. So,

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\left(\frac{e}{2}\right)^n}{1 + \frac{n^2}{2^n}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{e}{2}\right)^n}{1 + 0} = \infty$$

Solution 2: Since the numerator and denominator both increase without bound, we apply l'Hôpital's rule. Recall $\frac{d}{dx}\{2^x\} = 2^x \log 2$.

$$\begin{aligned} \lim_{n \rightarrow \infty} b_n &= \lim_{n \rightarrow \infty} \frac{e^n}{\underbrace{2^n + n^2}_{\substack{\text{num} \rightarrow \infty \\ \text{den} \rightarrow \infty}}} \\ &= \lim_{n \rightarrow \infty} \frac{e^n}{\underbrace{2^n \log 2 + 2n}_{\substack{\text{num} \rightarrow \infty \\ \text{den} \rightarrow \infty}}} \\ &= \lim_{n \rightarrow \infty} \frac{e^n}{\underbrace{2^n (\log 2)^2 + 2}_{\substack{\text{num} \rightarrow \infty \\ \text{den} \rightarrow \infty}}} \\ &= \lim_{n \rightarrow \infty} \frac{e^n}{2^n (\log 2)^3} \\ &= \frac{1}{(\log 2)^3} \lim_{n \rightarrow \infty} \left(\frac{e}{2}\right)^n \\ &= \infty \end{aligned}$$

Since $e > 2$, we see $\frac{e}{2} > 1$, and so $\lim_{n \rightarrow \infty} \left(\frac{e}{2}\right)^n = \infty$.

S-14: First, we note that $\lim_{n \rightarrow \infty} \frac{6n^2 + 5n}{n^2 + 1} = 6$. We see this either by comparing the leading terms in the numerator and denominator, or by factoring out n^2 from the top and the bottom.

Second, since $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$, we see $\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n^2}\right) = \cos 0 = 1$.

Using arithmetic of limits, Theorem 3.1.8 in the text, we conclude

$$\lim_{n \rightarrow \infty} \left[\frac{6n^2 + 5n}{n^2 + 1} + 3 \cos(1/n^2) \right] = 6 + 3(1) = 9.$$

S-15: Let's take stock: $\sin(1/n) \rightarrow \sin(0) = 0$ as $n \rightarrow \infty$, so $\log(\sin(1/n)) \rightarrow -\infty$. However, $\log(2n) \rightarrow \infty$. So, we have some tension here: the two pieces behave in ways that pull the terms of the sequence in different directions. (Recall we *cannot* conclude anything like " $-\infty + \infty = 0$.")

We try using logarithm rules to get a clearer picture.

$$\log\left(\sin \frac{1}{n}\right) + \log(2n) = \log\left(2n \sin\left(\frac{1}{n}\right)\right)$$

Still, we have indeterminate behaviour: $2n \sin(1/n)$ is the product of $2n$, which grows without bound, and $\sin(1/n)$, which approaches zero. In the past, we learned that we can handle the indeterminate form $0 \cdot \infty$ with l'Hôpital's rule (after a little algebra), but there's a slicker way. Note $1/n \rightarrow 0$ as $n \rightarrow \infty$. If we write $\frac{1}{n} = x$, then this piece of our limit resembles something familiar.

$$2n \sin\left(\frac{1}{n}\right) = 2 \left(\frac{\sin x}{x} \right)$$

If $n \rightarrow \infty$, then $x = \frac{1}{n} \rightarrow 0$.

$$\lim_{n \rightarrow \infty} 2n \sin\left(\frac{1}{n}\right) = 2 \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

That limit is familiar:

$$= 2(1) = 2$$

Then:

$$\lim_{n \rightarrow \infty} \log\left(2n \sin\left(\frac{1}{n}\right)\right) = \log 2$$

Note: if you have forgotten that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, you can also evaluate this limit using l'Hôpital's rule:

$$\underbrace{\lim_{x \rightarrow 0} \frac{\sin x}{x}}_{\substack{\text{num} \rightarrow 0 \\ \text{den} \rightarrow 0}} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1$$

S-16: First, although this sequence is not defined for some small values of n , it is defined as long as $n \geq 5$, so it's not a problem to take the limit as $n \rightarrow \infty$. Second, we notice that our limit has the indeterminate form $\infty - \infty$. Since this form is indeterminate, more work is needed to find our limit, if it exists.

A standard trick we saw last semester with functions of this form was to multiply and divide by the conjugate of the expression, $\sqrt{n^2 + 5n} + \sqrt{n^2 - 5n}$. Then the denominator will be the *sum* of two similar things, rather than their difference. See the work below to find out why that is helpful.

$$\begin{aligned}\sqrt{n^2 + 5n} - \sqrt{n^2 - 5n} &= (\sqrt{n^2 + 5n} - \sqrt{n^2 - 5n}) \left(\frac{\sqrt{n^2 + 5n} + \sqrt{n^2 - 5n}}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 5n}} \right) \\ &= \frac{(n^2 + 5n) - (n^2 - 5n)}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 5n}} \\ &= \frac{10n}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 5n}}\end{aligned}$$

Now, we'll cancel out n from the top and the bottom. Note $n = \sqrt{n^2}$.

$$\begin{aligned}&= \frac{10n}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 5n}} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \\ &= \frac{10n}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 5n}} \left(\frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2}}} \right) \\ &= \frac{10}{\sqrt{1 + \frac{5}{n}} + \sqrt{1 - \frac{5}{n}}}\end{aligned}$$

Now, the limit is clear.

$$\lim_{n \rightarrow \infty} \frac{10}{\sqrt{1 + \frac{5}{n}} + \sqrt{1 - \frac{5}{n}}} = \frac{10}{\sqrt{1 + 0} + \sqrt{1 + 0}} = \frac{10}{1 + 1} = 5$$

S-17: First, although this sequence is not defined for some small values of n , it is defined as long as $n \geq \sqrt{2.5}$, so it's not a problem to take the limit as $n \rightarrow \infty$. Second, we notice that our limit has the indeterminate form $\infty - \infty$. Since this form is indeterminate, more work is needed to find our limit, if it exists.

In Question 16, we saw a similar limit, and made use of the conjugate. However, in this

case, there's an easier path: let's factor out n from each term.

$$\begin{aligned}\sqrt{n^2 + 5n} - \sqrt{2n^2 - 5} &= \sqrt{n^2 \left(1 + \frac{5}{n}\right)} - \sqrt{n^2 \left(2 - \frac{5}{n^2}\right)} \\ &= n\sqrt{1 + \frac{5}{n}} - n\sqrt{2 - \frac{5}{n^2}} \\ &= n \left(\sqrt{1 + \frac{5}{n}} - \sqrt{2 - \frac{5}{n^2}} \right)\end{aligned}$$

Now, the limit is clear.

$$\begin{aligned}\lim_{n \rightarrow \infty} \left[\sqrt{n^2 + 5n} - \sqrt{2n^2 - 5} \right] &= \lim_{n \rightarrow \infty} \left[n \left(\sqrt{1 + \frac{5}{n}} - \sqrt{2 - \frac{5}{n^2}} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[n \left(\sqrt{1 + 0} - \sqrt{2 - 0} \right) \right] \\ &= \lim_{n \rightarrow \infty} [n(-1)] = -\infty\end{aligned}$$

Remark: check Question 16 to see whether a similar trick would work there. Why or why not?

S-18: First, we note that we have in indeterminate form: as n grows, $2 + \frac{1}{n} \rightarrow 2$, so $n \left[\left(2 + \frac{1}{n}\right)^{100} - 2^{100} \right]$ has the form $\infty \cdot 0$. To overcome this difficulty, we could use some algebra and l'Hôpital's rule, but there's a slicker way. If we let $h = \frac{1}{n}$, then $h \rightarrow 0$ as $n \rightarrow \infty$, and our limit looks like:

$$\lim_{n \rightarrow \infty} n \left[\left(2 + \frac{1}{n}\right)^{100} - 2^{100} \right] = \lim_{h \rightarrow 0} \frac{(2 + h)^{100} - 2^{100}}{h}$$

This reminds us of the definition of a derivative.

$$\frac{d}{dx} \{x^{100}\} = \lim_{h \rightarrow 0} \frac{(x + h)^{100} - x^{100}}{h}$$

So, if we set $f(x) = x^{100}$, our limit is simply $f'(2)$. That is, $[100x^{99}]_{x=2} = 100 \cdot 2^{99}$.

S-19: Using the definition of a derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

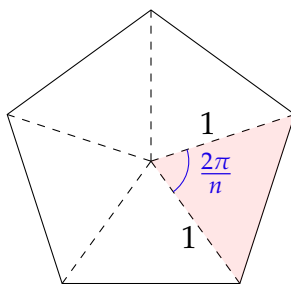
We want $n \rightarrow \infty$, so we set $h = \frac{1}{n}$.

$$\begin{aligned}&= \lim_{\frac{1}{n} \rightarrow 0} \frac{f\left(a + \frac{1}{n}\right) - f(a)}{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} n \left[f\left(a + \frac{1}{n}\right) - f(a) \right]\end{aligned}$$

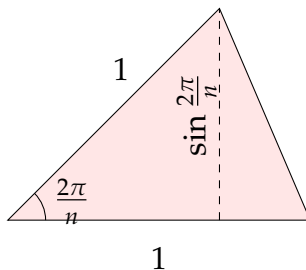
We also could have chosen $h = -\frac{1}{n}$, which leads to the following:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{-\frac{1}{n} \rightarrow 0} \frac{f\left(a - \frac{1}{n}\right) - f(a)}{-1/n} \\ &= \lim_{n \rightarrow \infty} -n \left(f\left(a - \frac{1}{n}\right) - f(a) \right) \\ &= \lim_{n \rightarrow \infty} n \left(f(a) - f\left(a - \frac{1}{n}\right) \right)\end{aligned}$$

S-20: (a) To find the area A_n , note that the figure with n sides can be divided up into n isosceles triangles, each with two sides of length 1 and angle between them of $\frac{2\pi}{n}$:



Each of these triangles has area $\frac{1}{2} \sin\left(\frac{2\pi}{n}\right)$:



All together, the area of the n -sided figure is $A_n = \frac{n}{2} \sin\left(\frac{2\pi}{n}\right)$.

(b) We will discuss two ways to find $\lim_{n \rightarrow \infty} A_n$, which has the indeterminate form $\infty \times 0$.

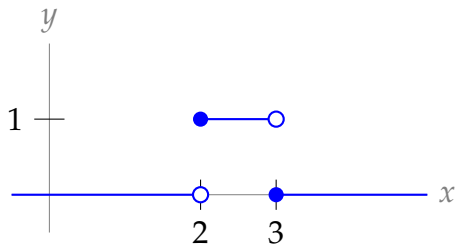
First, note that as $n \rightarrow \infty$, our figures look more and more like a circle of radius 1. So, we see A_n is approaching the area of a circle of radius 1. That is, $\lim_{n \rightarrow \infty} A_n = \pi$.

Alternately, we can make use of the limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Let $x = \frac{2\pi}{n}$. Note if $n \rightarrow \infty$, then $x \rightarrow 0$.

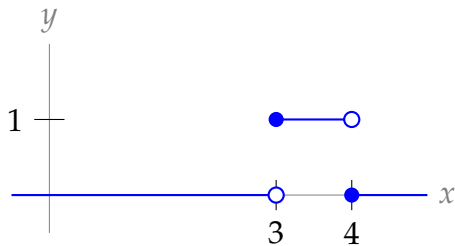
$$\begin{aligned}\lim_{n \rightarrow \infty} A_n &= \lim_{n \rightarrow \infty} \frac{n}{2} \sin\left(\frac{2\pi}{n}\right) = \lim_{n \rightarrow \infty} \frac{\pi}{\frac{2\pi}{n}} \sin\left(\frac{2\pi}{n}\right) \\ &= \lim_{x \rightarrow 0} \pi \frac{\sin x}{x} = \pi \times 1 = \pi\end{aligned}$$

S-21:

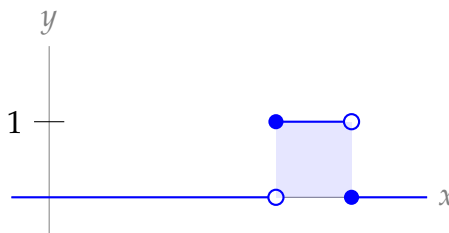
(a) $f_2(x) = \begin{cases} 1 & 2 \leq x < 3 \\ 0 & \text{else} \end{cases}$



(b) $f_3(x) = \begin{cases} 1 & 3 \leq x < 4 \\ 0 & \text{else} \end{cases}$



- (c) For any n , $f_n(x) = 1$ for an interval of length 1, and $f_n(x) = 0$ for all other x . So, the area under the curve is a square of side length one.



Then $A_n = \int_0^\infty f_n(x) dx = 1$ for all n . That is, the sequence $\{A_n\}$ is simply $\{1, 1, \dots, 1\}$, a sequence of all 1s.

- (d) Given the description above, $\lim_{n \rightarrow \infty} A_n = 1$.

- (e) For any fixed x , recall $\{f_n(x)\} = \{0, \dots, 0, 1, 0, \dots, 0, 0, 0, \dots\}$. In particular, there are infinitely many zeroes at its end. So, $\lim_{n \rightarrow \infty} f_n(x) = 0$. Then $g(x) = 0$ for every x .

- (f) Given the description above, $\int_0^\infty g(x) dx = \int_0^\infty 0 dx = 0$.

Remark: what we've shown here is that, for this particular $f_n(x)$,

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx \neq \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx$$

That is, we can't necessarily swap a limit with an integral (which is, in this case, another

limit, since the integral is improper). The interested reader can look up “uniform convergence” to learn about the conditions under which these *can* be swapped.

S-22: If we naively try to find the limit, we run up against the indeterminate form 1^∞ . We’d like to use l’Hôpital’s rule, but we don’t have the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ —we’ll need to use a logarithm. Additionally, l’Hôpital’s rule applies to differentiable functions defined for real numbers—so we’ll consider a function, rather than the sequence.

Note the terms of the sequence are all positive.

Solution 1: Define $x = \frac{1}{n}$, and $f(x) = (1 + 3x + 5x^2)^{1/x}$. Then $b_n = f\left(\frac{1}{n}\right) = f(x)$, and

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}\right) = \lim_{x \rightarrow 0^+} f(x).$$

If this limit exists, it is equal to $\lim_{n \rightarrow \infty} b_n$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (1 + 3x + 5x^2)^{1/x} \\ \lim_{x \rightarrow 0^+} \log[f(x)] &= \lim_{x \rightarrow 0^+} \log \left[(1 + 3x + 5x^2)^{1/x} \right] = \lim_{x \rightarrow 0^+} \underbrace{\frac{\log [1 + 3x + 5x^2]}{x}}_{\substack{\text{num} \rightarrow 0 \\ \text{den} \rightarrow 0}} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{3+10x}{1+3x+5x^2}}{1} = 3 \\ \lim_{x \rightarrow 0^+} f(x) &= e^3 \end{aligned}$$

Since the limit exists, $\lim_{n \rightarrow \infty} b_n = e^3$.

Solution 2: If we didn’t see the nice simplifying trick of letting $x = \frac{1}{n}$, we can still solve the problem using $g(x) = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$:

$$\begin{aligned} g(x) &= \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x \\ \log[g(x)] &= x \log \left[1 + \frac{3}{x} + \frac{5}{x^2}\right] = \underbrace{\frac{\log \left[1 + \frac{3}{x} + \frac{5}{x^2}\right]}{1/x}}_{\substack{\text{num} \rightarrow 0 \\ \text{den} \rightarrow 0}} \\ \lim_{x \rightarrow \infty} \log[g(x)] &= \lim_{x \rightarrow \infty} \frac{\frac{-\frac{3}{x^2} - \frac{10}{x^3}}{1 + \frac{3}{x} + \frac{5}{x^2}}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} x^2 \frac{\frac{3}{x^2} + \frac{10}{x^3}}{1 + \frac{3}{x} + \frac{5}{x^2}} = \lim_{x \rightarrow \infty} \frac{3 + \frac{10}{x}}{1 + \frac{3}{x} + \frac{5}{x^2}} = \frac{3 + 0}{1 + 0 + 0} = 3 \\ \lim_{x \rightarrow \infty} f(x) &= e^3 \end{aligned}$$

Since the limit exists, $\lim_{n \rightarrow \infty} b_n = e^3$

S-23:

- (a) When $a_1 = 4$, we see $a_2 = \frac{4+8}{3} = 4$, and so on. That is, $a_n = 4$ for every n . So,
 $\lim_{n \rightarrow \infty} a_n = 4$.
- (b) Cross-multiplying, we see $3x = x + 8$, hence $x = 4$.
- (c) In order for our sequence to converge to 4, the terms should be getting infinitely close to 4. So, we find the relationship between $a_{n+1} - 4$ and $a_n - 4$.

$$a_{n+1} = \frac{a_n + 8}{3}$$
$$a_{n+1} - 4 = \frac{a_n + 8}{3} - 4 = \frac{a_n - 4}{3}$$

So, the distance between our sequence terms and the number 4 is decreasing by a factor of 3 each term. This implies that the terms get infinitely close to 4 as n grows. That is, $\lim_{n \rightarrow \infty} a_n = 4$.

S-24:

- (a) Since w_1 has the highest frequency, w_2 has the next-highest frequency, and so on, we know f_1 is larger than the other members of its sequence, f_2 is the next largest, etc. So, $\{f_n\}$ is a decreasing sequence.
- (b) The most-used word in a language is w_1 , while the n -th most used word in a language is w_n . So, we re-state the law as:

$$f_1 = n f_n$$

Then we can rewrite this formula a little more naturally as $f_n = \frac{1}{n} f_1$.

- (c) Then $f_3 = \frac{1}{3} f_1$. In this case, we expect the third-most used word to account for $\frac{1}{3}(6\%) = 2\%$ of all words.
- (d) From (b), we know $f_{10} = \frac{1}{10} f_1$. Note $f_1 = 6 f_6 = 6(0.3\%)$. Then:

$$f_{10} = \frac{1}{10} f_1 = \frac{1}{10} 6 f_6 = \frac{1}{10} (6)(0.3\%) = \frac{1.8}{10}\% = 0.18\%$$

So, f_{10} should be 0.18% of all words.

- (e) The use of the word “frequency” in the statement of Zipf’s law implies $f_n = \frac{\text{uses of } w_n}{\text{total number of words}}$. The question asks for the total uses of w_n . If we call this quantity t_n , and the total number of all words is T , then Zipf’s law tells us $\frac{t_n}{T} = \frac{1}{n} \frac{t_1}{T}$, hence $t_n = \frac{1}{n} t_1$.

With this notation, the problem states $t_1 = 22,038,615$, $w_1 = \text{the}$, $w_2 = \text{be}$, and $w_3 = \text{and}$.

Following Zipf's law, $t_n = \frac{1}{n}t_1$. So, we expect $t_2 = \frac{t_1}{2} = 11,019,307.5$; since this isn't an integer, let's say we expect $t_2 \approx 11,019,308$. Similarly, we expect $t_3 = \frac{t_1}{3} = 7,346,205$.

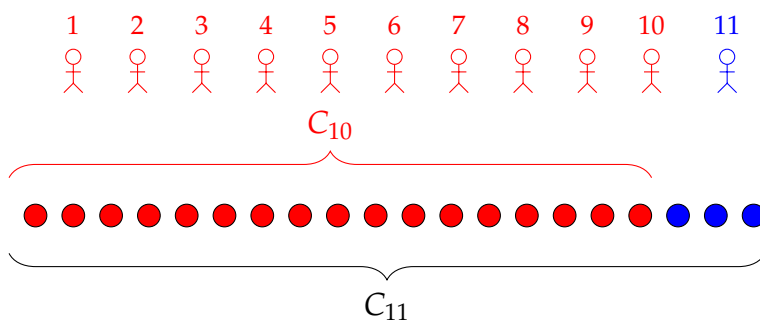
Remark: The 450-million-word source material that used "the" 22,038,615 times also contained 12,545,825 instances of "be," and 10,741,073 instances of "and." While Zipf's Law might be a nice model for our data overall, in these few instances it does not appear to be extremely accurate.

Solutions to Exercises 3.2 — Jump to [TABLE OF CONTENTS](#)

S-1: The N th term of the sequence of partial sums, S_N , is the sum of the first N terms of the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

N	S_N
1	1
2	$1 + \frac{1}{2}$
3	$1 + \frac{1}{2} + \frac{1}{3}$
4	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
5	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

S-2: If there were a total of 17 cookies before Student 11 came, and 20 cookies after, then Student 11 brought 3 cookies.



S-3:

- (a) We find $\{a_n\}$ from $\{S_N\}$ using the same logic as Question 2. S_N is the sum of the first N terms of $\{a_n\}$, and S_{N-1} is the sum of all the same terms *except* a_N . So, $a_N = S_N - S_{N-1}$ when $N \geq 2$. Written another way:

$$S_N = a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1} + a_N$$

$$S_{N-1} = a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1}$$

So,

$$\begin{aligned} S_N - S_{N-1} &= \left[a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1} + a_N \right] \\ &\quad - \left[a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1} \right] \\ &= a_N \end{aligned}$$

So, we calculate

$$\begin{aligned} a_N &= S_N - S_{N-1} = \left(\frac{N}{N+1} \right) - \left(\frac{N-1}{N-1+1} \right) \\ &= \frac{N^2}{N(N+1)} - \frac{N^2-1}{N(N+1)} \\ &= \frac{1}{N(N+1)} \end{aligned}$$

Therefore,

$$a_n = \frac{1}{n(n+1)}$$

Remark: the formula given for S_N has $S_0 = 0$, which makes sense: the sum of no terms at all should be 0. However, it is common for a sequence of partial sums to start at $N = 1$. (This fits our definition of a partial sum—we don't really define the "sum of no terms.") In this case, a_1 must be calculated separately from the other terms of $\{a_n\}$. To find a_1 , we simply set $a_1 = S_1$, which (to reiterate) might not be the same as $S_1 - S_0$.

(b)

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n(n+1)} = 0.$$

That is, the terms we're adding up are getting very, very small as we go along.

(c) By Definition 3.2.3 in the text,

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \frac{N}{N+1} = 1$$

That is, as we add more and more terms of our series, our cumulative sum gets very, very close to 1.

S-4: As in Question 3,

$$\begin{aligned} a_N &= S_N - S_{N-1} = \left[(-1)^N + \frac{1}{N} \right] - \left[(-1)^{N-1} + \frac{1}{N-1} \right] \\ &= (-1)^N - (-1)^{N-1} + \frac{1}{N} - \frac{1}{N-1} \\ &= (-1)^N + (-1)^N + \frac{N-1}{N(N-1)} - \frac{N}{N(N-1)} \\ &= 2(-1)^N - \frac{1}{N(N-1)} \end{aligned}$$

Note, however, that a_N is only the same as $S_N - S_{N-1}$ when $N \geq 2$: otherwise, we're trying to calculate $S_1 - S_0$, but S_0 is not defined. So, we find a_1 separately:

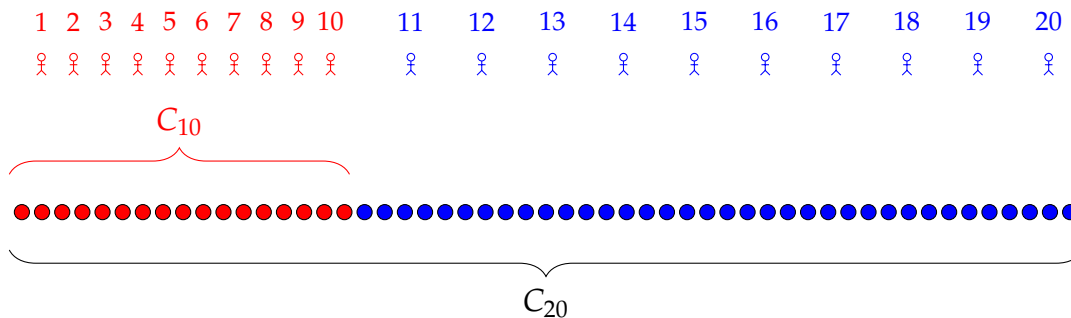
$$a_1 = S_1 = (-1)^1 + \frac{1}{1} = 0$$

All together:

$$a_n = \begin{cases} 0 & \text{if } n = 1 \\ 2(-1)^n - \frac{1}{n(n-1)} & \text{else} \end{cases}$$

S-5: If $f'(N) < 0$, that means $f(N)$ is decreasing. So, adding more terms makes for a smaller sum. That means the terms we're adding are negative. That is, $a_n < 0$ for all $n \geq 2$.

S-6: After twenty students have brought their cookies, the pile numbers 53 cookies. 17 of these cookies were brought by students one through ten. So, the remainder ($53 - 17 = 36$) is the number of cookies brought by students 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20, together.



S-7: Using arithmetic of series, Theorem 3.2.5 in the text, we see

$$\sum_{n=1}^{\infty} (a_n + b_n + c_{n+1}) = A + B + \sum_{n=1}^{\infty} c_{n+1}$$

The question remaining is what do to with the last series. If we write out the terms, we see the difference between $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} c_{n+1}$ is simply that the latter is missing c_1 :

$$\begin{aligned} \sum_{n=1}^{\infty} c_{n+1} &= c_2 + c_3 + c_4 + c_5 + \cdots \\ &= -c_1 + c_1 + c_2 + c_3 + c_4 + c_5 + \cdots \\ &= -c_1 + \sum_{n=1}^{\infty} c_n \end{aligned}$$

So,

$$\sum_{n=1}^{\infty} (a_n + b_n + c_{n+1}) = A + B + C - c_1$$

S-8: Theorem 3.2.5 in the text, arithmetic of series, doesn't mention division, because in general it doesn't work the way the question suggests. For example, let $\{a_n\} = \{b_n\} = \frac{1}{2^n}$. Then:

- $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \frac{1}{1-\frac{1}{2}} = 2$, while
- $\sum_{n=0}^{\infty} \frac{a_n}{b_n} = \sum_{n=0}^{\infty} 1 = \infty$.

For the statement in the question, we can take $\{a_n\} = \{b_n\} = \frac{1}{2^n}$, $A = B = 2$, $\{c_n\} = \{0, 0, 0, \dots\}$, and $C = 0$. We see the statement is false in this case.

So, in general, the statement given is false.

S-9:

- Cell A1 holds $\frac{1}{10}$; cell A2 holds $\frac{1/10}{10} = \frac{1}{10^2}$; cell A3 holds $\frac{1/10^2}{10} = \frac{1}{10^3}$, etc. So $a_n = \frac{1}{10^n}$.
- Bn holds the n th partial sum, $\sum_{i=1}^n a_i$. The dollar signs fix the starting index at 1. Without them, it would just be $=\text{SUM}(A_n : A_n)$, i.e. the same as entry A_n .
- As we go down column A, we approximate $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{10^n} = 0$. So, we expect those entries to get closer and closer to 0.
- As we go down column B, we approximate $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i = 0.1 + 0.01 + 0.001 + 0.0001 + \dots = 0.\overline{11} = \frac{1}{9}$. So, we expect those entries to get closer and closer to $\frac{1}{9}$.

S-10: We'll store the a_n values in column A, and the partial sums in column B. Since our index starts at $n = 2$, we allow ourselves Row 1 for labels.

	A	B
1	an	An
2	=1 / (ROW() * ROW() + ROW())	=SUM(\$A\$2 : A2)
3	↓	↓

As we copy the cells down the columns, column B entries start looking a lot like 0.5.

S-11: (a) As in Question 2, since

$$s_{n-1} = a_1 + a_2 + \dots + a_{n-1}$$

$$s_n = a_1 + a_2 + \dots + a_{n-1} + a_n$$

we can find a_n by subtracting:

$$\begin{aligned}
 a_n &= s_n - s_{n-1} \\
 &= \frac{1+3n}{5+4n} - \frac{1+3(n-1)}{5+4(n-1)} = \frac{3n+1}{4n+5} - \frac{3n-2}{4n+1} \\
 &= \frac{(3n+1)(4n+1) - (3n-2)(4n+5)}{(4n+1)(4n+5)} \\
 &= \frac{11}{16n^2 + 24n + 5}
 \end{aligned}$$

(b) Using Definition 3.2.3 in the text,

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1+3n}{5+4n} = \lim_{n \rightarrow \infty} \frac{1/n+3}{5/n+4} = \frac{0+3}{0+4} = \frac{3}{4}$$

The series converges to $\frac{3}{4}$.

S-12: Since $\{\mathcal{S}_M\}$ is the sequence of partial sums of $\sum_{N=1}^{\infty} S_N$, we can find $\{S_N\}$ from $\{\mathcal{S}_M\}$ as in Question 3:

$$\begin{aligned}
 S_N &= \mathcal{S}_N - \mathcal{S}_{N-1} = \frac{N+1}{N} - \frac{N}{N-1} = -\frac{1}{N(N-1)} \quad \text{if } N \geq 2, \\
 S_1 &= \mathcal{S}_1 = 2
 \end{aligned}$$

Similarly, we find $\{a_n\}$ from $\{S_N\}$. Do be careful: S_N only follows the formula we found above when $N \geq 2$. In the next line, we use an expression containing S_{n-1} ; in order for the subscript to be at least two (so the formula fits), we need $n \geq 3$.

$$\begin{aligned}
 a_n &= S_n - S_{n-1} = -\frac{1}{n(n-1)} - \frac{-1}{(n-1)(n-2)} \\
 &= \frac{2}{n(n-1)(n-2)} \quad \text{if } n \geq 3, \\
 a_2 &= S_2 - S_1 = -\frac{1}{2(2-1)} - 2 = -\frac{5}{2}, \\
 a_1 &= S_1 = 2
 \end{aligned}$$

All together,

$$a_n = \begin{cases} \frac{2}{n(n-1)(n-2)} & \text{if } n \geq 3, \\ -\frac{5}{2} & \text{if } n = 2, \\ 2 & \text{if } n = 1 \end{cases}$$

Solutions to Exercises 3.3 — Jump to [TABLE OF CONTENTS](#)

S-1:

- (A) $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, so the divergence test is inconclusive. It's true that this series diverges, but we can't show it using the divergence test.
- (B) $\lim_{n \rightarrow \infty} \frac{n^2}{n+1} = \infty$, which is not zero, so the divergence test tells us this series diverges.
- (C) We'll show below that $\lim_{n \rightarrow \infty} \sin n$ does not exist at all. In particular, it is not zero. Therefore, the divergence test tells us this series diverges.

Now we'll show that $\lim_{n \rightarrow \infty} \sin n$ does not exist. Suppose that it does exist and takes the value S . We will now see that this assumption leads to a contradiction. Add together the two trig identities

$$\begin{aligned}\sin(n+1) &= \sin(n)\cos(1) + \cos(n)\sin(1) \\ \sin(n-1) &= \sin(n)\cos(1) - \cos(n)\sin(1)\end{aligned}$$

This gives

$$\sin(n+1) + \sin(n-1) = 2\sin(n)\cos(1)$$

Taking the limit $n \rightarrow \infty$ gives $2S = 2S\cos(1)$. Since $\cos(1) \neq 1$, this forces $S = 0$. Now the first trig identity above gives

$$\cos(n) = \frac{\sin(n+1) - \sin(n)\cos(1)}{\sin(1)}$$

Taking the limit as $n \rightarrow \infty$ of that gives

$$\lim_{n \rightarrow \infty} \cos(n) = \frac{S - S\cos(1)}{\sin(1)} = 0$$

But that provides the contradiction. Because $\sin^2(n) + \cos^2(n) = 1$, we can't have both $\sin(n)$ and $\cos(n)$ converging to zero. So $\lim_{n \rightarrow \infty} \sin n$ does not exist.

- (D) For all whole numbers n , $\sin(\pi n) = 0$, so $\lim_{n \rightarrow \infty} \sin(\pi n) = 0$ and the divergence test is inconclusive.

S-2: Let $f(x)$ be a function with $f(n) = a_n$ for all whole numbers n . In order to apply the integral test (Theorem 3.3.6 in the text) we need $f(x)$ to be positive and decreasing for all sufficiently large values of n .

- (A) $f(x) = \frac{1}{x}$, which is positive and decreasing for all $x \geq 1$, so the integral test does apply here.
- (B) $f(x) = \frac{x^2}{x+1}$, which is not decreasing—in fact, it goes to infinity. So, the integral test does not apply here. (The divergence test tells us the series diverges, though.)
- (C) $f(x) = \sin x$, which is neither consistently positive nor consistently decreasing, so the integral test does not apply. (The divergence test tells us the series diverges, though.)

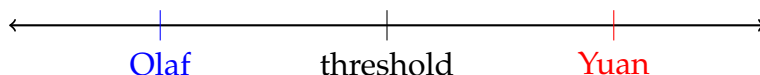
(D) $f(x) = \frac{\sin x + 1}{x^2}$ is positive for all whole numbers n . To determine whether it is decreasing, we consider its derivative.

$$f'(x) = \frac{x^2(\cos x) - (\sin x + 1)(2x)}{x^4} = \frac{x \cos x - 2 \sin x - 2}{x^3}$$

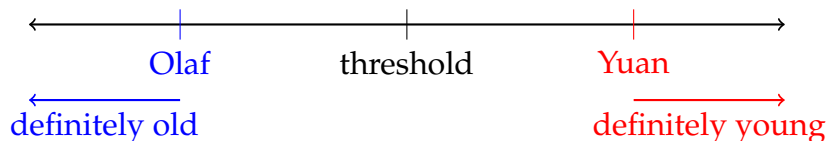
This is sometimes positive, and sometimes negative. (For example, if $x = 100\pi$, $f'(x) = \frac{100\pi - 0 - 2}{(100\pi)^3} > 0$, but if $x = 101\pi$ then $f'(x) = \frac{101\pi(-1) - 0 - 2}{(101\pi)^3} < 0$.) Then $f(x)$ is not a decreasing function, so the integral test does not apply.

- S-3: (a) If Olaf is old, and I am *even older*, then I am old as well.
 (b) If Olaf is old, and I am not as old, then perhaps I am old as well (just slightly less so), or perhaps I am young. There is not enough information to tell.
 (c) If Yuan is young, and I am older, then perhaps I am much older and I am old, or perhaps I am only a little older, and I am young. There is not enough information to tell.
 (d) If Yuan is young, and I am *even younger*, then I must also be young.

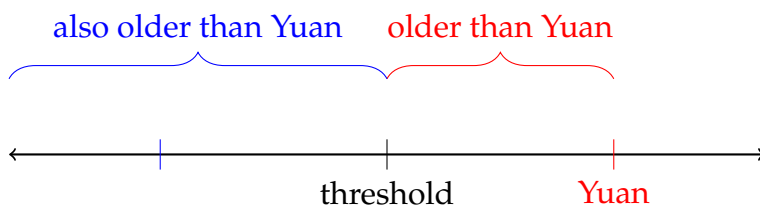
Another way to think about this is with a timeline of birthdates. People born before the threshold are old, and people born after it are young.



If I'm born before (older than) Olaf, I'm born before the threshold, so I'm old.
 If I'm born after (younger than) Yuan I'm born after the threshold, so I'm young.



If I'm born after Olaf or before Yuan, I don't know which side of the threshold I'm on. I could be old or I could be young.



S-4: The comparison test is Theorem 3.3.11 in the text. However, rather than trying to memorize which way the inequalities go in all cases, we use the same reasoning as Question 3.

If a sequence has positive terms, it either converges, or it diverges to infinity, with the partial sums increasing and increasing without bound. If one sequence diverges, and the other sequence is larger, then the other sequence diverges—just like being older than an old person makes you old.

If $\sum a_n$ converges, and $\{a_n\}$ is the red (larger) series, then $\sum b_n$ converges: it's smaller than a sequence that doesn't add up to infinity, so it too does not add up to infinity.

If $\sum a_n$ diverges, and $\{a_n\}$ is the blue (smaller) series, then $\sum b_n$ diverges: it's larger than a sequence that adds up to infinity, so it too adds up to infinity.

In the other cases, we can't say anything. If $\{a_n\}$ is the red (larger) series, and $\sum a_n$ diverges, then perhaps $\{b_n\}$ behaves similarly to $\{a_n\}$ and $\sum b_n$ diverges, or perhaps $\{b_n\}$ is much, much smaller than $\{a_n\}$ and $\sum b_n$ converges.

Similarly, if $\{a_n\}$ is the blue (smaller) series, and $\sum a_n$ converges, then perhaps $\{b_n\}$ behaves similarly to $\{a_n\}$ and $\sum b_n$ converges, or perhaps $\{b_n\}$ is much, much bigger than $\{a_n\}$ and $\sum b_n$ diverges.

	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$ CONVERGES	inconclusive
and if $\{a_n\}$ is the blue series	inconclusive	then $\sum b_n$ DIVERGES

S-5: (a) Since $\sum \frac{1}{n}$ is divergent, we can only use it to prove series with *larger* terms are divergent. This is the case here, since $\frac{1}{n-1} > \frac{1}{n}$. So, the direct comparison test is valid.

For the limit comparison test, we calculate:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n-1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{n}} = 1$$

Since the limit exists and is not zero, the limit comparison test is also valid.

(b) Since the series $\sum \frac{1}{n^2}$ converges, we can only use the direct comparison test to show the convergence of a series if its terms have *smaller* absolute values. Indeed,

$$\left| \frac{\sin n}{n^2 + 1} \right| = \frac{|\sin n|}{n^2 + 1} < \frac{1}{n^2}$$

so the series are set for a direct comparison.

To check whether a limit comparison will work, we compute:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\sin n}{n^2 + 1}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} \sin n = \lim_{n \rightarrow \infty} (1) \sin n$$

The limit does not exist, so the limit comparison test is not a valid test to compare these two series.

(c) Since the series $\sum \frac{1}{n^3}$ converges, we can only use the direct comparison test to conclude something about a series with *smaller* terms. However,

$$\frac{n^3 + 5n + 1}{n^6 - 2} > \frac{n^3}{n^6} = \frac{1}{n^3}.$$

Therefore the direct comparison test does not apply to this pair of series.

For the limit comparison test, we calculate:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{n^3+5n+1}{n^6-2}}{\frac{1}{n^3}} = \lim_{n \rightarrow \infty} \frac{n^3+5n+1}{n^3-\frac{2}{n^3}} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1+\frac{5}{n^2}+\frac{1}{n^3}}{1-\frac{2}{n^6}} = 1\end{aligned}$$

Since the limit is a nonzero real number, we can use the limit comparison test to compare this pair of series.

(d) Since the series $\sum \frac{1}{\sqrt[4]{n}}$ diverges, we can only use the direct comparison test to show that a series with *larger* terms diverges. However,

$$\frac{1}{\sqrt{n}} < \frac{1}{\sqrt[4]{n}}$$

so the direct comparison test isn't valid with this pair of series.

For the limit comparison test, we calculate:

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt[4]{n}}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[4]{n}} = 0$$

Since the series $\sum \frac{1}{\sqrt[4]{n}}$ diverges, it is part (b) of the limit comparison test, Theorem 3.3.14 in the text, that is appropriate here. As the limit $L = 0$, the limit comparison test doesn't apply.

S-6: It diverges by the divergence test, because $\lim_{n \rightarrow \infty} a_n \neq 0$.

S-7: The integral test does not apply because $f(x)$ is not decreasing.

S-8: The inequality goes the wrong way, so the direct comparison test (with this comparison series) is inconclusive.

S-9: Although the terms of (A) are sometimes negative and sometimes positive, they are not strictly alternating in a positive-negative-positive-negative pattern. For instance, $\sin 1$ and $\sin 2$ are both positive. So, (A) is not an alternating series.

When n is a whole number, $\cos(\pi n) = (-1)^n$, so (B) is alternating.

Since the exponent of $(-n)$ in (C) is even, the terms are always positive. Therefore (C) is not alternating.

(D) is an alternating series.

S-10: One possible answer: $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This series converges (it's a p -series with $p = 2 > 1$), but if we take the ratio of consecutive terms:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1$$

The limit of the ratio is 1, so the ratio test is inconclusive.

S-11: By the divergence test, for a series $\sum a_n$ to *converge*, we need $\lim_{n \rightarrow \infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller. If $\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| < 1$ or (equivalently) $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $|a_{n+1}| > |a_n|$ for sufficiently large n , so the terms are actually *growing* in magnitude. That means the series diverges, by the divergence test.

S-12: The terms of the series only see a small portion of the domain of the integral. We can try to think of a function $f(x)$ that behaves “nicely” when x is a whole number (that is, it produces a sequence whose sum converges), but is more unruly when x is not a whole number.

For example, suppose $f(x) = \sin(\pi x)$. Then $f(x) = 0$ for every integer x , but this is not representative of the function as a whole. Indeed, our corresponding series has terms $\{a_n\} = \{0, 0, 0, \dots\}$.

- $\int_1^{\infty} \sin(\pi x) \, dx = \lim_{R \rightarrow \infty} \left[-\frac{1}{\pi} \cos(\pi x) \right]_1^R = \lim_{R \rightarrow \infty} \left[-\cos(\pi R) \right] - \frac{1}{\pi}$
Since the limit does not exist, the integral diverges.

- $\sum_{n=1}^{\infty} \sin(\pi n) = \sum_{n=1}^{\infty} 0 = 0$. The series converges.

S-13: When n is very large, the term 2^n dominates the numerator, and the term 3^n dominates the denominator. So when n is very large $a_n \approx \frac{2^n}{3^n}$. Therefore we should take $b_n = \frac{2^n}{3^n}$. Note that, with this choice of b_n ,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2^n + n}{3^n + 1} \frac{3^n}{2^n} = \lim_{n \rightarrow \infty} \frac{1 + n/2^n}{1 + 1/3^n} = 1$$

as desired.

S-14: (a) In general false. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p -test with $p = 1$.

(b) Be careful. You were not told that the a_n 's are positive. So this is false in general. If $a_n = (-1)^n \frac{1}{n}$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is again the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

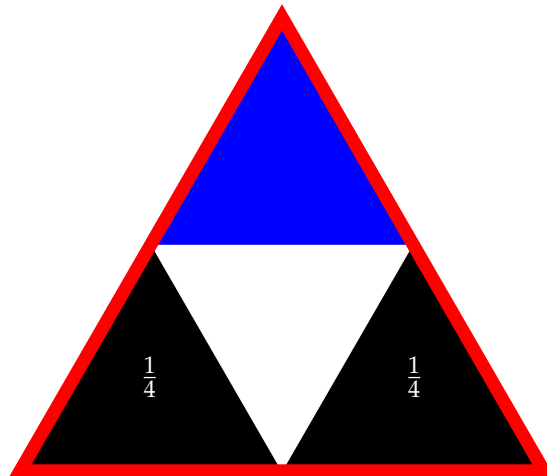
(c) In general false. Take, for example, $a_n = 0$ and $b_n = 1$.

S-15: (a) To generate the pattern, we repeat the following steps:

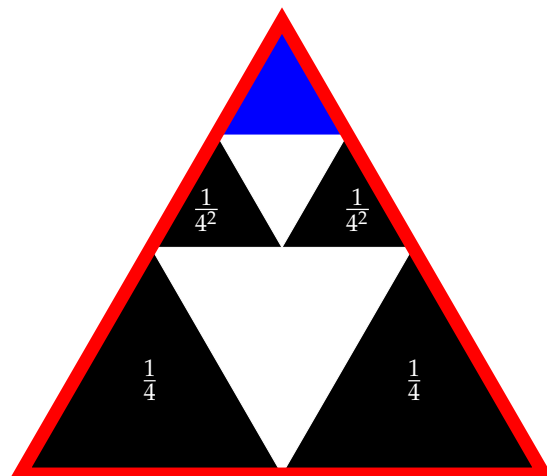
- divide the top triangle into four triangles of equal area,
- colour the bottom two of them black, and
- leave the middle one white.

Every time we repeat this sequence, we divide up a triangle with an area one-quarter the size of our previous triangle, and take two of the four resulting pieces. So, our area should end up as a geometric sum with common ratio $r = \frac{1}{4}$, and coefficient $a = 2$. This is shown more explicitly below.

Since the entire triangle (outlined in red) has area 1, the four smaller triangles below each have area $\frac{1}{4}$. The two black triangles will be added to our total black area; the blue triangle will be subdivided.



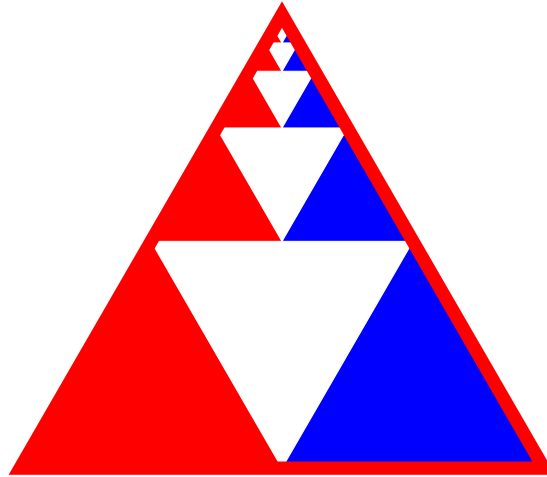
The blue triangle had area $\frac{1}{4}$, so each of the small black triangles below has area $\left(\frac{1}{4}\right) \left(\frac{1}{4}\right) = \frac{1}{4^2}$.



Each time we make another subdivision, we add two black triangles, each with $\frac{1}{4}$ the area of the previous black triangles. So, our total black area is:

$$2 \left(\frac{1}{4} \right) + 2 \left(\frac{1}{4^2} \right) + 2 \left(\frac{1}{4^3} \right) + 2 \left(\frac{1}{4^4} \right) + \cdots = \sum_{n=1}^{\infty} \frac{2}{4^n}$$

(b) To evaluate the series, we imagine gathering up all our little triangles and sorting them into three identical piles: the bottom three triangles go in three different piles, the three triangles directly above them go in three different piles, etc. (In the picture below, different colours correspond to different piles.)



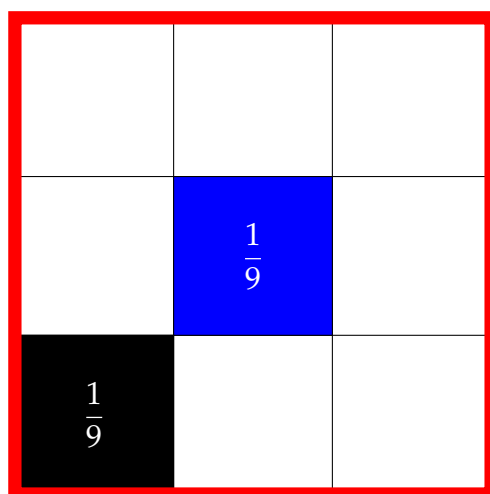
Since the piles all have equal area, each pile has a total area of $\frac{1}{3}$. The black area shaded in the problem corresponds to *two* piles (red and blue above), so

$$\sum_{n=1}^{\infty} \frac{2}{4^n} = \frac{2}{3}$$

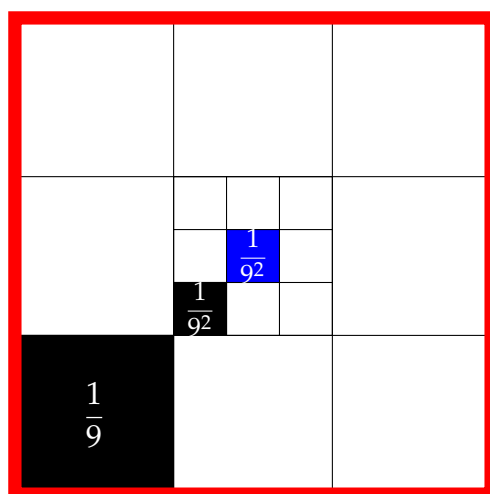
S-16: (a) The pattern can be described as follows: divide the innermost square into 9 equal parts (a 3×3 grid), choose one square to be black, and another square to subdivide.

The area of the red (outermost) square is 1, so the area of the largest black square is $\frac{1}{9}$.

The area of the central, blue square below is also $\frac{1}{9}$.



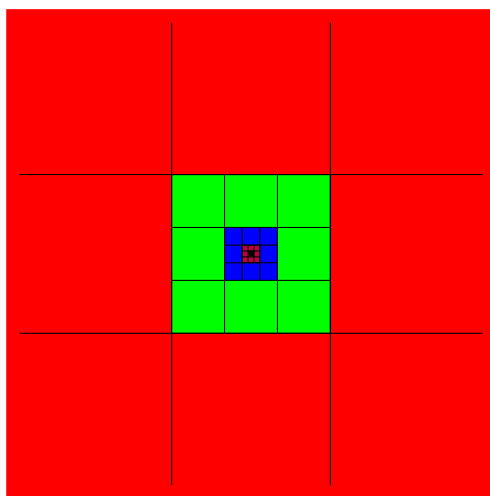
When we subdivide the blue square, the subdivisions each have one-ninth its area, or $\frac{1}{9^2}$.



We continue taking squares that are one-ninth the area of the previous square. So, our total black area is

$$\frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9^n}$$

(b) If we cut up this square along the marks, we can easily share it equally among 8 friends: there are eight squares of area $\frac{1}{9}$ along the outer ring, eight squares of area $\frac{1}{9^2}$ along the next ring in, and so on.



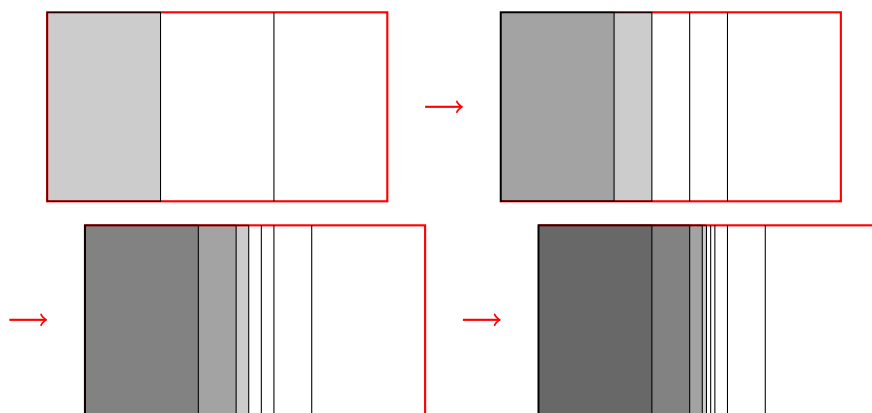
Since the eight friends all get the same total area, the area each friend gets is $\frac{1}{8}$. The area shaded in black in the question corresponds to the pile given to one friend. So,

$$\sum_{n=1}^{\infty} \frac{1}{9^n} = \frac{1}{8}$$

S-17:

If we start with a shape of area 1, and iteratively divide it into thirds, taking one of the three newly created pieces each time, then the area we take will be equal to the desired series, $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

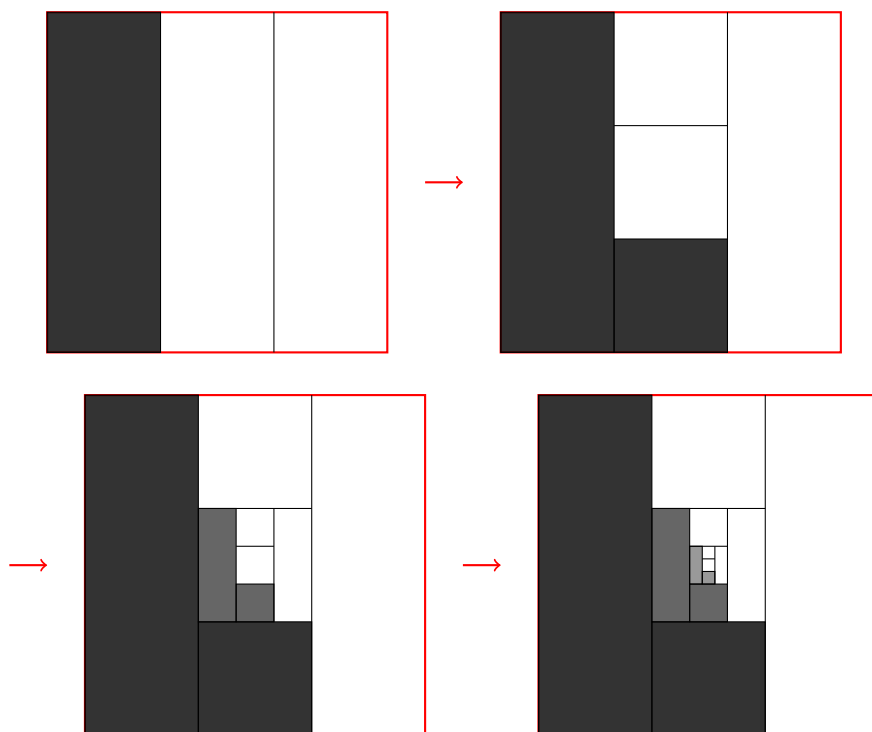
One way to do this is to start with a rectangle, make three vertical strips, then keep the left strip and subdivide the middle strip.



We see that the total area we take approaches one-half the total area of the figure, so

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

Alternately, instead of always taking vertical strips, we could alternate vertical and horizontal slices.



In this setup, we notice that our strips come in pairs: two large vertical strips, two smaller horizontal strips, two smaller vertical strips, etc. We shaded exactly one of each, so the shaded area is one-half the total area: $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$.

Other solutions are possible, as well.

S-18: Equation 3.3.1 in the text tells us $\sum_{n=0}^N ar^n = a \frac{1-r^{N+1}}{1-r}$, for $r \neq 1$. Our geometric sum has $a = 1$, $r = \frac{1}{5}$, and $N = 100$. So:

$$\sum_{n=0}^{100} \frac{1}{5^n} = \frac{1 - \frac{1}{5^{101}}}{1 - \frac{1}{5}} = \frac{5^{101} - 1}{4 \cdot 5^{100}}$$

S-19:

Solution 1: Using the ideas of Question 6, we see:

$$\sum_{n=50}^{100} \frac{1}{5^n} = \sum_{n=0}^{100} \frac{1}{5^n} - \sum_{n=0}^{49} \frac{1}{5^n}$$

That is, we want start with the sum of all the terms up to $\frac{1}{5^{100}}$, and then subtract off the ones we actually don't want, which is everything up to $\frac{1}{5^{49}}$. Now, both series are in a form appropriate for Equation 3.3.1 in the text.

$$\begin{aligned}\sum_{n=0}^{100} \frac{1}{5^n} - \sum_{n=0}^{49} \frac{1}{5^n} &= \frac{1 - \frac{1}{5^{101}}}{1 - \frac{1}{5}} - \frac{1 - \frac{1}{5^{50}}}{1 - \frac{1}{5}} \\ &= \frac{5^{101} - 1}{4 \cdot 5^{100}} - \frac{5^{50} - 1}{4 \cdot 5^{49}} \left(\frac{5^{51}}{5^{51}} \right) \\ &= \frac{5^{101} - 1}{4 \cdot 5^{100}} - \frac{5^{101} - 5^{51}}{4 \cdot 5^{100}} \\ &= \frac{5^{51} - 1}{4 \cdot 5^{100}}\end{aligned}$$

Solution 2: If we write out the first few terms of our series, we see we can factor out a constant to change the starting index.

$$\begin{aligned}\sum_{n=50}^{100} \frac{1}{5^n} &= \frac{1}{5^{50}} + \frac{1}{5^{51}} + \frac{1}{5^{52}} + \frac{1}{5^{53}} + \cdots + \frac{1}{5^{100}} \\ &= \frac{1}{5^{50}} \left(\frac{1}{5^0} + \frac{1}{5^1} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots + \frac{1}{5^{50}} \right) \\ &= \sum_{n=0}^{50} \frac{1}{5^{50}} \cdot \frac{1}{5^n}\end{aligned}$$

Now, our sum is in the form of Equation 3.3.1 in the text with $a = \frac{1}{5^{50}}$, $r = \frac{1}{5}$, and $N = 50$.

$$\begin{aligned}\sum_{n=0}^{50} \frac{1}{5^{50}} \cdot \frac{1}{5^n} &= \frac{1}{5^{50}} \cdot \frac{1 - \frac{1}{5^{51}}}{1 - \frac{1}{5}} = \frac{1 - \frac{1}{5^{51}}}{4 \cdot 5^{49}} \left(\frac{5^{51}}{5^{51}} \right) \\ &= \frac{5^{51} - 1}{4 \cdot 5^{100}}\end{aligned}$$

S-20: First, we'll check the divergence test. It doesn't always work, but if it does, it's likely the easiest path.

$$\lim_{n \rightarrow \infty} \frac{n^2}{3n^2 + \sqrt{n}} \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right) = \lim_{n \rightarrow \infty} \frac{1}{3 + \frac{1}{n\sqrt{n}}} = \frac{1}{3} \neq 0$$

Since the limit of the terms being added is not zero, the series diverges by the divergence test.

S-21: This precise question was asked on a 2014 final exam. Note that the n^{th} term in the series is $a_n = \frac{5^k}{4^k + 3^k}$ and does *not* depend on n ! There are two possibilities. Either this was intentional (and the instructor was being particularly nasty) or it was a typo and the intention was to have $a_n = \frac{5^n}{4^n + 3^n}$. In both cases, the limit

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{5^k}{4^k + 3^k} = \frac{5^k}{4^k + 3^k} \neq 0 \\ \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{5^n}{4^n + 3^n} = \lim_{n \rightarrow \infty} \frac{(5/4)^n}{1 + (3/4)^n} = +\infty \neq 0\end{aligned}$$

is nonzero, so the series diverges by the divergence test.

S-22: We recognize that this is a geometric series:

$$\begin{aligned}1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots &= \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{3^n}\end{aligned}$$

Using Equation 3.3.2 in the text with $r = \frac{1}{3}$ and $a = 1$,

$$= \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}.$$

S-23: This is a geometric series, with ratio $r = \frac{1}{8}$. However, it doesn't start at $k = 0$, which is what we're used to.

Solution 1: We write out the first few terms of the series to figure out a convenient constant to factor out.

$$\begin{aligned}\sum_{k=7}^{\infty} \frac{1}{8^k} &= \frac{1}{8^7} + \frac{1}{8^8} + \frac{1}{8^9} + \cdots \\ &= \frac{1}{8^7} \left(\frac{1}{8^0} + \frac{1}{8^1} + \frac{1}{8^2} + \cdots \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{8^7} \cdot \frac{1}{8^n}\end{aligned}$$

We now evaluate the series using Equation 3.3.2 in the text with $r = \frac{1}{8}$, $a = \frac{1}{8^7}$.

$$= \frac{1}{8^7} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{1}{7 \times 8^6}$$

Solution 2: Using the idea of Question 6, we express the series we're interested in as the difference of two series that we can easily evaluate.

$$\sum_{k=7}^{\infty} \frac{1}{8^k} = \sum_{k=0}^{\infty} \frac{1}{8^k} - \sum_{k=0}^6 \frac{1}{8^k}$$

Using Equations 3.3.2 and 3.3.1 in the text,

$$\begin{aligned} &= \frac{1}{1 - \frac{1}{8}} - \frac{1 - \frac{1}{8^7}}{1 - \frac{1}{8}} \\ &= \frac{1}{7 \times 8^6} \end{aligned}$$

S-24: What we have is a geometric series, but we need to get it into the proper form before we can evaluate it.

$$\sum_{n=2}^{\infty} \frac{3 \cdot 4^{n+1}}{8 \cdot 5^n} = \sum_{n=2}^{\infty} \frac{3 \cdot 4 \cdot 4^n}{8 \cdot 5^n} = \frac{3}{2} \sum_{n=2}^{\infty} \left(\frac{4}{5}\right)^n$$

Solution 1: If we factor our $\left(\frac{4}{5}\right)^2$, we can change our index to something more convenient.

$$\begin{aligned} \frac{3}{2} \sum_{n=2}^{\infty} \left(\frac{4}{5}\right)^n &= \frac{3}{2} \sum_{n=2}^{\infty} \left(\frac{4}{5}\right)^2 \left(\frac{4}{5}\right)^{n-2} \\ &= \frac{3}{2} \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^2 \left(\frac{4}{5}\right)^n \end{aligned}$$

We use Equation 3.3.2 in the text with $r = \frac{4}{5}$.

$$= \frac{3}{2} \left(\frac{4}{5}\right)^2 \cdot \frac{1}{1 - \frac{4}{5}} = \frac{24}{5}$$

Solution 2: Using the idea of Question 6, we view our series as a more convenient series, minus a few initial terms.

$$\begin{aligned} \frac{3}{2} \sum_{n=2}^{\infty} \left(\frac{4}{5}\right)^n &= \frac{3}{2} \left(\left[\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n \right] - \left(\frac{4}{5}\right)^1 - \left(\frac{4}{5}\right)^0 \right) \\ &= \frac{3}{2} \left(\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n - \frac{9}{5} \right) \end{aligned}$$

We use Equation 3.3.2 in the text with $r = \frac{4}{5}$.

$$= \frac{3}{2} \left(\frac{1}{1 - \frac{4}{5}} - \frac{9}{5} \right) = \frac{24}{5}$$

S-25: The number is:

$$\begin{aligned} 0.2 + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots &= \frac{1}{5} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots \\ &= \frac{1}{5} + \frac{3}{10^2} \left(\frac{1}{10^0} + \frac{1}{10^1} + \frac{1}{10^2} + \cdots \right) \\ &= \frac{1}{5} + \frac{3}{10^2} \sum_{n=0}^{\infty} \frac{1}{10^n} \end{aligned}$$

We use Equation 3.3.2 in the text with $r = \frac{1}{10}$.

$$\begin{aligned} &= \frac{1}{5} + \frac{3}{10^2} \cdot \frac{1}{1 - \frac{1}{10}} \\ &= \frac{1}{5} + \frac{1}{30} = \frac{7}{30} \end{aligned}$$

S-26: The number is:

$$\begin{aligned} 2 + \frac{65}{100} + \frac{65}{10000} + \frac{65}{1000000} + \cdots &= 2 + \frac{65}{100} + \frac{65}{100^2} + \frac{65}{100^3} + \cdots \\ &= 2 + \frac{65}{100} \sum_{n=0}^{\infty} \frac{1}{100^n} \end{aligned}$$

We use Equation 3.3.2 in the text with $r = \frac{1}{100}$.

$$\begin{aligned} &= 2 + \frac{65}{100} \cdot \frac{1}{1 - \frac{1}{100}} \\ &= 2 + \frac{65}{99} = \frac{263}{99} \end{aligned}$$

S-27: The number is:

$$\begin{aligned} 0.\overline{321} &= 0.321321321 \dots \\ &= \frac{321}{1000} + \frac{321}{10^6} + \frac{321}{10^9} + \cdots \\ &= \frac{321}{1000} \left(\left(\frac{1}{10^3} \right)^0 + \left(\frac{1}{10^3} \right)^1 + \left(\frac{1}{10^3} \right)^2 + \cdots \right) \\ &= \frac{321}{1000} \sum_{n=0}^{\infty} \left(\frac{1}{10^3} \right)^n \end{aligned}$$

We use Equation 3.3.2 in the text with $r = \frac{1}{10^3}$.

$$= \frac{321}{1000} \cdot \frac{1}{1 - \frac{1}{10^3}} = \frac{321}{999} = \frac{107}{333}$$

S-28: We split the sum into two parts.

$$\sum_{n=1}^{\infty} \left[\left(\frac{1}{3} \right)^n + \left(-\frac{2}{5} \right)^{n-1} \right] = \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n + \sum_{n=1}^{\infty} \left(-\frac{2}{5} \right)^{n-1}$$

Both are geometric series.

$$\begin{aligned} &= \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^{n+1} + \sum_{n=0}^{\infty} \left(-\frac{2}{5} \right)^n \\ &= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n + \sum_{n=0}^{\infty} \left(-\frac{2}{5} \right)^n \end{aligned}$$

We use Equation 3.3.2 in the text with $a_1 = \frac{1}{3}$ and $r_1 = \frac{1}{3}$, then with $a_2 = 1$ and $r_2 = -\frac{2}{5}$.

$$\begin{aligned} &= \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}} + \frac{1}{1 + \frac{2}{5}} \\ &= \frac{1}{2} + \frac{5}{7} = \frac{17}{14} \end{aligned}$$

S-29: We split the sum into two parts.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1 + 3^{n+1}}{4^n} &= \sum_{n=0}^{\infty} \frac{1}{4^n} + \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} \\ &= \sum_{n=0}^{\infty} \frac{1}{4^n} + 3 \sum_{n=0}^{\infty} \left(\frac{3}{4} \right)^n \end{aligned}$$

Using Equation 3.3.2 in the text,

$$\begin{aligned} &= \frac{1}{1 - \frac{1}{4}} + \frac{3}{1 - \frac{3}{4}} \\ &= \frac{4}{3} + 12 = \frac{40}{3} \end{aligned}$$

S-30: We usually check the divergence test first, to look for low-hanging fruit. The limit of the terms being added is zero:

$$\lim_{n \rightarrow \infty} \frac{1}{n + \frac{1}{2}} = 0$$

so the divergence test is inconclusive. That is, we need to look harder.

Next, we might consider a comparison test—these can also provide us (if we're lucky) with an easy path. The terms we're adding look somewhat like $\frac{1}{n}$, but our terms are *smaller* than these terms, which form the terms of the *divergent* harmonic series. So, a direct comparison seems unlikely. Now we search for more exotic tests.

Let $f(x) = \frac{1}{x + \frac{1}{2}}$. Note $f(x)$ is positive and decreases as x increases. So, by the integral test, which is Theorem 3.3.6 in the text, the given series converges if and only if the integral $\int_0^{\infty} \frac{1}{x + \frac{1}{2}} dx$ converges. Since

$$\begin{aligned} \int_0^{\infty} \frac{1}{x + \frac{1}{2}} dx &= \lim_{R \rightarrow \infty} \int_0^R \frac{1}{x + \frac{1}{2}} dx = \lim_{R \rightarrow \infty} \left[\log \left(x + \frac{1}{2} \right) \right]_{x=0}^{x=R} \\ &= \lim_{R \rightarrow \infty} \left[\log \left(R + \frac{1}{2} \right) - \log \frac{1}{2} \right] \end{aligned}$$

diverges, the series diverges.

S-31: The terms of the series tend to 0, so we can't use the divergence test.

To generate a guess about its convergence, we do the following:

$$\sum \frac{1}{\sqrt{k}\sqrt{k+1}} = \sum \frac{1}{\sqrt{k^2+k}} \approx \sum \frac{1}{\sqrt{k^2}} = \sum \frac{1}{k}$$

We guess that our series behaves like the harmonic series, and the harmonic series diverges (which can be demonstrated by p -test or integral test). So, we guess that our series diverges. However, in order to directly compare our series to the harmonic series and show our series diverges, our terms would have to be *bigger* than the terms in the harmonic series, and this is not the case. So, we use limit comparison.

$$\begin{aligned} \frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2+k}}} &= \frac{\sqrt{k^2+k}}{k} = \frac{\sqrt{k^2+k}}{\sqrt{k^2}} = \sqrt{\frac{k^2+k}{k^2}} = \sqrt{1 + \frac{1}{k}}, \text{ so} \\ \lim_{k \rightarrow \infty} \frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2+k}}} &= \lim_{k \rightarrow \infty} \sqrt{1 + \frac{1}{k}} = 1 \end{aligned}$$

Since 1 is a real number greater than 0, by the Limit Comparison Test, $\sum \frac{1}{\sqrt{k}\sqrt{k+1}}$ diverges, like $\sum \frac{1}{k}$.

S-32: This is a geometric series with $r = 1.001$. Since $|r| > 1$, it is divergent.

S-33: This is a geometric series with $r = \frac{-1}{5}$. Since $|r| < 1$, it is convergent.

We want to use the formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, but our series does not start at 0, so we re-write it:

$$\begin{aligned}\sum_{n=3}^{\infty} \left(\frac{-1}{5}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n - \sum_{n=0}^2 \left(\frac{-1}{5}\right)^n = \frac{1}{1 - (-1/5)} - \left(1 - \frac{1}{5} + \frac{1}{25}\right) \\ &= \frac{1}{6/5} - 1 + \frac{1}{5} - \frac{1}{25} = \frac{5}{6} + \frac{-25 + 5 - 1}{25} = -\frac{1}{150}\end{aligned}$$

S-34: For any integer n , $\sin(\pi n) = 0$, so $\sum \sin(\pi n) = \sum 0 = 0$. So, this series converges.

S-35: For any integer n , $\cos(\pi n) = \pm 1$, so $\lim_{n \rightarrow \infty} \cos(\pi n) \neq 0$.

By the divergence test, this series diverges.

S-36: Factorials grow super fast. Like, wow, really fast. Even faster than exponentials. So the terms are going to zero, and the divergence test won't help us. Let's use ratio—it's a good go-to test with factorials.

$$\frac{a_{k+1}}{a_k} = \frac{\frac{e^{k+1}}{(k+1)!}}{\frac{e^k}{k!}} = \frac{e^{k+1}}{e^k} \cdot \frac{k!}{(k+1)!} = e \cdot \frac{k(k-1) \cdots (1)}{(k+1)(k)(k-1) \cdots (1)} = e \cdot \frac{1}{k+1} = \frac{e}{k+1}$$

Since e is a constant,

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{e}{k+1} = 0$$

Since $0 < 1$, by the ratio test, the series converges.

S-37: This is close to being in the form of a geometric series. First, we should have our powers be k , not $k+2$, but we notice $3^{k+2} = 3^k 3^2 = 9 \cdot 3^k$, so:

$$\sum_{k=0}^{\infty} \frac{2^k}{3^{k+2}} = \sum_{k=0}^{\infty} \frac{2^k}{9 \cdot 3^k} = \frac{1}{9} \sum_{k=0}^{\infty} \frac{2^k}{3^k} = \frac{1}{9} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

Now it looks like a geometric series with $r = \frac{2}{3}$

$$= \frac{1}{9} \left(\frac{1}{1 - (2/3)} \right) = \frac{1}{3}$$

In conclusion: this (geometric) series is convergent, and its sum is $\frac{1}{3}$.

S-38: Usually with factorials, we want to use the divergence test or the ratio test. Since the terms are indeed tending towards zero, we are left with the ratio test.

$$\begin{aligned}
 \frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1)!(n+1)!}{(2n+2)!}}{\frac{n!n!}{(2n)!}} = \frac{(n+1)!(n+1)!}{n!n!} \cdot \frac{(2n)!}{(2n+2)!} \\
 &= \frac{(n+1)(n)(n-1)\cdots(1)}{n(n-1)\cdots(1)} \cdot \frac{(n+1)(n)(n-1)\cdots(1)}{n(n-1)\cdots(1)} \cdot \frac{(2n)(2n-1)(2n-2)\cdots(1)}{(2n+2)(2n+1)(2n)(2n-1)(2n-2)\cdots(1)} \\
 &= (n+1)(n+1) \cdot \frac{1}{(2n+2)(2n+1)}, \text{ so} \\
 \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)}{2(n+1)(2n+1)} = \lim_{n \rightarrow \infty} \frac{n+1}{4n+2} = \frac{1}{4}
 \end{aligned}$$

Since the limit is a number less than 1, the series converges by the ratio test.

S-39: We want to make an estimation, when n gets big:

$$\frac{n^2 + 1}{2n^4 + n} \approx \frac{n^2}{2n^4} = \frac{1}{2n^2}$$

Since $\sum \frac{1}{2n^2}$ is a convergent series (by p -test, or integral test), we guess that our series is convergent as well. If we wanted to use comparison test, we should have to show $\frac{n^2+1}{2n^4+n} < \frac{1}{2n^2}$, which seems unpleasant, so let's use limit comparison.

$$\lim_{n \rightarrow \infty} \frac{\frac{n^2+1}{2n^4+n}}{\frac{1}{2n^2}} = \lim_{n \rightarrow \infty} \frac{(n^2+1)2n^2}{2n^4+n} = \lim_{n \rightarrow \infty} \frac{2n^4+2n^2}{2n^4+n} \left(\frac{1/n^4}{1/n^4} \right) = \lim_{n \rightarrow \infty} \frac{2 + \frac{2}{n^2}}{2 + \frac{1}{n^3}} = 1$$

Since the limit is a positive finite number, by the Limit Comparison Test, $\sum \frac{n^2+1}{2n^4+n}$ does the same thing $\sum \frac{1}{2n^2}$ does: it converges.

S-40: First, we rule out some of the easier tests. The limit of the terms being added is zero, so the divergence test is inconclusive. The terms being added are smaller than the terms of the (divergent) harmonic series, $\sum \frac{1}{n}$, so we can't directly compare these two series, and there isn't another obvious series to compare ours to. However, the terms being added seem like a function we could integrate.

Let $f(x) = \frac{5}{x(\log x)^{3/2}}$. Then $f(x)$ is positive and decreases as x increases. So the sum

$\sum_3^\infty f(n)$ and the integral $\int_3^\infty f(x) dx$ either both converge or both diverge, by the integral test, which is Theorem 3.3.6 in the text. For the integral, we use the substitution $u = \log x$, $du = \frac{dx}{x}$ to get

$$\int_3^\infty \frac{5 dx}{x(\log x)^{3/2}} = \int_{\log 3}^\infty \frac{5 du}{u^{3/2}}$$

which converges by the p -test (which is Example 1.13.8 in the text) with $p = \frac{3}{2} > 1$.

S-41: Let $f(x) = \frac{1}{x(\log x)^p}$. Then $f(x)$ is positive for $n \geq 3$, and $f(x)$ decreases as x increases. So, we can use the integral test, Theorem 3.3.6 in the text.

$$\int_2^{\infty} \frac{1}{x(\log x)^p} dx = \lim_{R \rightarrow \infty} \int_2^R \frac{1}{(\log x)^p} \frac{dx}{x} = \lim_{R \rightarrow \infty} \int_{\log 2}^{\log R} \frac{1}{u^p} du \quad \text{with } u = \log x, du = \frac{dx}{x}$$

Using the results about p -series, Example 3.3.7 in the text, we know this integral converges if and only if $p > 1$, so the same is true for the series by the integral test.

S-42: As usual, let's see whether the "easy" tests work. The terms we're adding converge to zero:

$$\lim_{n \rightarrow \infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}e^{\sqrt{n}}} = 0$$

so the divergence test is inconclusive. Our series isn't geometric, and it doesn't seem obvious how to compare it to a geometric series. However, the terms we're adding seem like they would make an integrable function.

Set $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$. For $x \geq 1$, this function is positive and decreasing (since it is the product of the two positive decreasing functions $e^{-\sqrt{x}}$ and $\frac{1}{\sqrt{x}}$). We use the integral test with this function. Using the substitution $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}} dx$, we see that

$$\begin{aligned} \int_1^{\infty} f(x) dx &= \lim_{R \rightarrow \infty} \int_1^R \left(\frac{e^{-\sqrt{x}}}{\sqrt{x}} dx \right) \\ &= \lim_{R \rightarrow \infty} \left(\int_1^{\sqrt{R}} e^{-u} \cdot 2 du \right) \\ &= \lim_{R \rightarrow \infty} \left(-2e^{-u} \Big|_1^{\sqrt{R}} \right) \\ &= \lim_{R \rightarrow \infty} \left(-2e^{-\sqrt{R}} + 2e^{-\sqrt{1}} \right) = 0 + 2e^{-1}, \end{aligned}$$

and so this improper integral converges. By the integral test, the given series also converges.

S-43: We first develop some intuition. For very large n , $3n^2$ dominates 7 so that

$$\frac{\sqrt{3n^2 - 7}}{n^3} \approx \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2}$$

The series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by the p -test with $p = 2$, so we expect the given series to converge too.

To verify that our intuition is correct, it suffices to observe that

$$0 < a_n = \frac{\sqrt{3n^2 - 7}}{n^3} < \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2} = c_n$$

for all $n \geq 2$. As the series $\sum_{n=2}^{\infty} c_n$ converges, the comparison test says that $\sum_{n=2}^{\infty} a_n$ converges too.

S-44: We first develop some intuition. For very large k , k^4 dominates 1 so that the numerator $\sqrt[3]{k^4 + 1} \approx \sqrt[3]{k^4} = k^{4/3}$, and k^5 dominates 9 so that the denominator $\sqrt{k^5 + 9} \approx \sqrt{k^5} = k^{5/2}$ and the summand

$$\frac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}} \approx \frac{k^{4/3}}{k^{5/2}} = \frac{1}{k^{7/6}}$$

The series $\sum_{n=1}^{\infty} \frac{1}{k^{7/6}}$ converges by the p -test with $p = \frac{7}{6} > 1$, so we expect the given series to converge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_k = \frac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}} \quad \text{and} \quad b_k = \frac{1}{k^{7/6}} = \frac{k^{4/3}}{k^{5/2}}$$

which is valid since

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\sqrt[3]{k^4 + 1}/k^{4/3}}{\sqrt{k^5 + 9}/k^{5/2}} = \lim_{k \rightarrow \infty} \frac{\sqrt[3]{1 + 1/k^4}}{\sqrt{1 + 9/k^5}} = 1$$

exists. Since the series $\sum_{k=1}^{\infty} b_k$ is a convergent p -series (with ratio $p = \frac{7}{6} > 1$), the given series converges.

Note: to apply the direct comparison test with our chosen comparison series, we would need to show that

$$\frac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}} \leq \frac{1}{k^{7/6}}$$

for all k sufficiently large. However, this is not true: the opposite inequality holds when k is large.

S-45:

Solution 1: Let's see whether the divergence test works here.

$$\lim_{n \rightarrow \infty} \frac{n^4 2^{n/3}}{(2n + 7)^4} \left(\frac{\frac{1}{n^4}}{\frac{1}{n^4}} \right) = \lim_{n \rightarrow \infty} \frac{2^{n/3}}{(2 + 7/n)^4} = \lim_{n \rightarrow \infty} \frac{2^{n/3}}{(2 + 0)^4} = \infty$$

The summands of our series do not converge to zero. By the divergence test, the series diverges.

Solution 2: Let's develop some intuition for a comparison. For very large n , $2n$ dominates 7 so that

$$\frac{n^4 2^{n/3}}{(2n+7)^4} \approx \frac{n^4 2^{n/3}}{(2n)^4} = \frac{1}{16} 2^{n/3}$$

The series $\sum_{n=1}^{\infty} 2^{n/3}$ is a geometric series with ratio $r = 2^{1/3} > 1$ and so diverges. (It also fails the divergence test.) We expect the given series to diverge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_n = \frac{n^4 2^{n/3}}{(2n+7)^4} \quad \text{and} \quad b_n = 2^{n/3}$$

which is valid since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^4}{(2n+7)^4} = \lim_{n \rightarrow \infty} \frac{1}{(2+7/n)^4} = \frac{1}{2^4}$$

exists and is nonzero. Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent geometric series (with ratio $r = 2^{1/3} > 1$), the given series diverges.

(It is possible to use the plain comparison test as well. One needs to show something like $a_n = \frac{n^4 2^{n/3}}{(2n+7)^4} \geq \frac{n^4 2^{n/3}}{(2n+7n)^4} = \frac{1}{9^4} b_n$.)

Solution 3: Alternately, one can apply the ratio test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^4 2^{(n+1)/3} / (2(n+1)+7)^4}{n^4 2^{n/3} / (2n+7)^4} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^4 (2n+7)^4}{n^4 (2n+9)^4} \cdot \frac{2^{(n+1)/3}}{2^{n/3}} \\ &= \lim_{n \rightarrow \infty} \frac{(1+1/n)^4 (2+7/n)^4}{(2+9/n)^4} \cdot 2^{1/3} = 1 \cdot 2^{1/3} > 1. \end{aligned}$$

Since the ratio of consecutive terms is greater than one, by the ratio test, the series diverges.

S-46: (a) For large n , $n^2 \gg 1$ and so $\sqrt{n^2+1} \approx \sqrt{n^2} = n$. This suggests that we apply the limit comparison test with $a_n = \frac{1}{\sqrt{n^2+1}}$ and $b_n = \frac{1}{n}$. Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+1/n^2}} = 1$$

and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the given series diverges.

(b) Since $\cos(n\pi) = (-1)^n$, the given series converges by the alternating series test. To check that $a_n = \frac{n}{2^n}$ decreases to 0 as n tends to infinity, note that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)2^{-(n+1)}}{n2^{-n}} = \left(1 + \frac{1}{n}\right) \frac{1}{2}$$

is smaller than 1 (so that $a_{n+1} \leq a_n$) for all $n \geq 1$, and is smaller than $\frac{3}{4}$ (so $a_{n+1} \leq \frac{3}{4}a_n$) for all $n \geq 2$.

S-47: For large k , $k^4 \gg 2k^3 - 2$ and $k^5 \gg k^2 + k$ so

$$\frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k} \approx \frac{k^4}{k^5} = \frac{1}{k}.$$

This suggests that we apply the limit comparison test with $a_k = \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k}$ and $b_k = \frac{1}{k}$. Since

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_k}{b_k} &= \lim_{k \rightarrow \infty} \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k} \cdot \frac{k}{1} = \lim_{k \rightarrow \infty} \frac{k^5 - 2k^4 + 2k}{k^5 + k^2 + k} = \lim_{k \rightarrow \infty} \frac{1 - 2/k + 2/k^4}{1 + 1/k^3 + 1/k^4} \\ &= 1 \end{aligned}$$

and since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (by the p -test with $p = 1$), the given series diverges.

S-48: (a) For large n , $n^2 \gg n + 1$ and so the numerator $n^2 + n + 1 \approx n^2$. For large n , $n^5 \gg n$ and so the denominator $n^5 - n \approx n^5$. So, for large n ,

$$\frac{n^2 + n + 1}{n^5 - n} \approx \frac{n^2}{n^5} = \frac{1}{n^3}.$$

This suggests that we apply the limit comparison test with $a_n = \frac{n^2 + n + 1}{n^5 - n}$ and $b_n = \frac{1}{n^3}$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{(n^2 + n + 1)/(n^5 - n)}{1/n^3} = \lim_{n \rightarrow \infty} \frac{n^5 + n^4 + n^3}{n^5 - n} = \lim_{n \rightarrow \infty} \frac{1 + 1/n + 1/n^2}{1 - 1/n^4} \\ &= 1 \end{aligned}$$

exists and is nonzero, and since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (by the p -test with $p = 3 > 1$), the given series converges.

(b) For large m , $3m \gg |\sin \sqrt{m}|$ and so

$$\frac{3m + \sin \sqrt{m}}{m^2} \approx \frac{3m}{m^2} = \frac{3}{m}.$$

This suggests that we apply the limit comparison test with $a_m = \frac{3m + \sin \sqrt{m}}{m^2}$ and $b_m = \frac{1}{m}$. (We could also use $b_m = \frac{3}{m}$.) Since

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{a_m}{b_m} &= \lim_{m \rightarrow \infty} \frac{(3m + \sin \sqrt{m})/m^2}{1/m} = \lim_{m \rightarrow \infty} \frac{3m + \sin \sqrt{m}}{m} = \lim_{m \rightarrow \infty} 3 + \frac{\sin \sqrt{m}}{m} \\ &= 3 \end{aligned}$$

exists and is nonzero, and since $\sum_{m=1}^{\infty} \frac{1}{m}$ diverges (by the p -test with $p = 1$), the given series diverges.

S-49:

$$\begin{aligned}
 \sum_{n=5}^{\infty} \frac{1}{e^n} &= \sum_{n=5}^{\infty} \left(\frac{1}{e}\right)^n \\
 &= \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n - \sum_{n=0}^4 \left(\frac{1}{e}\right)^n \\
 &= \frac{1}{1 - \frac{1}{e}} - \frac{1 - \left(\frac{1}{e}\right)^5}{1 - \frac{1}{e}} \\
 &= \frac{\left(\frac{1}{e}\right)^5}{1 - \frac{1}{e}} = \frac{1}{e^5 \left(1 - \frac{1}{e}\right)} \\
 &= \frac{1}{e^5 - e^4}
 \end{aligned}$$

S-50: This is a geometric series.

$$\sum_{n=2}^{\infty} \frac{6}{7^n} = \sum_{n=0}^{\infty} \frac{6}{7^{n+2}} = \sum_{n=0}^{\infty} \frac{6}{7^2} \cdot \frac{1}{7^n}$$

We use Equation 3.3.2 in the text with $a = \frac{6}{7^2}$ and $r = \frac{1}{7}$.

$$= \frac{6}{7^2} \cdot \frac{1}{1 - \frac{1}{7}} = \frac{6}{42} = \frac{1}{7}$$

S-51: (a)

Solution 1: The given series is

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} a_n \text{ with } a_n = \frac{1}{2n-1}$$

First we'll develop some intuition by observing that, for very large n , $a_n \approx \frac{1}{2n}$. We know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p -test with $p = 1$. So let's apply the limit comparison test with $b_n = \frac{1}{n}$. Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}$$

the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges. So the given series diverges.

Solution 2: The series

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots &\geq \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots \\ &= \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \right) \end{aligned}$$

The series in the brackets is the harmonic series which we know diverges, by the p -test with $p = 1$. So the series on the right hand side diverges. By the direct comparison test, the series on the left hand side diverges too.

(b) We'll use the ratio test with $a_n = \frac{(2n+1)}{2^{2n+1}}$. Since

$$\frac{a_{n+1}}{a_n} = \frac{(2n+3)}{2^{2n+3}} \cdot \frac{2^{2n+1}}{(2n+1)} = \frac{1}{4} \frac{(2n+3)}{(2n+1)} = \frac{1}{4} \frac{(2+3/n)}{(2+1/n)} \rightarrow \frac{1}{4} < 1 \text{ as } n \rightarrow \infty$$

the series converges.

S-52: (a) For very large k , $k \ll k^2$ so that

$$a_n = \frac{\sqrt[3]{k}}{k^2 - k} \approx \frac{\sqrt[3]{k}}{k^2} = \frac{1}{k^{5/3}}.$$

We apply the limit comparison test with $b_k = \frac{1}{k^{5/3}}$. Since

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\sqrt[3]{k}/(k^2 - k)}{1/k^{5/3}} = \lim_{k \rightarrow \infty} \frac{k^2}{k^2 - k} = \lim_{k \rightarrow \infty} \frac{1}{1 - 1/k} = 1$$

exists and is nonzero, and $\sum_{k=1}^{\infty} \frac{1}{k^{5/3}}$ converges (by the p -test with $p = \frac{5}{3} > 1$), the given series converges by the limit comparison test.

(b) The k^{th} term in this series is $a_k = \frac{k^{10} 10^k (k!)^2}{(2k)!}$. Factorials often work well with the ratio test, because they simplify so nicely in quotients.

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{(k+1)^{10} 10^{k+1} ((k+1)!)^2}{(2k+2)!} \cdot \frac{(2k)!}{k^{10} 10^k (k!)^2} = 10 \left(\frac{k+1}{k} \right)^{10} \frac{(k+1)^2}{(2k+2)(2k+1)} \\ &= 10 \left(1 + \frac{1}{k} \right)^{10} \frac{(1 + 1/k)^2}{(2 + 2/k)(2 + 1/k)} \end{aligned}$$

As k tends to ∞ , this converges to $10 \times 1 \times \frac{1}{2 \times 2} > 1$. So the series diverges by the ratio test.

(c) We'll use the integral test. The k^{th} term in the series is $a_k = \frac{1}{k(\log k)(\log \log k)} = f(k)$ with $f(x) = \frac{1}{x(\log x)(\log \log x)}$, which is continuous, positive and decreasing for $x \geq 3$.

$$\begin{aligned} \int_3^\infty f(x) \, dx &= \int_3^\infty \frac{dx}{x(\log x)(\log \log x)} = \lim_{R \rightarrow \infty} \int_3^R \frac{dx}{x(\log x)(\log \log x)} \\ &= \lim_{R \rightarrow \infty} \int_{\log 3}^{\log R} \frac{dy}{y \log y} \quad \text{with } y = \log x, \, dy = \frac{dx}{x} \\ &= \lim_{R \rightarrow \infty} \int_{\log \log 3}^{\log \log R} \frac{dt}{t} \quad \text{with } t = \log y, \, dt = \frac{dy}{y} \\ &= \lim_{R \rightarrow \infty} \left[\log t \right]_{\log \log 3}^{\log \log R} = \infty \end{aligned}$$

Since the integral is divergent, the series is divergent as well by the integral test.

S-53: For large n , the numerator $n^3 - 4 \approx n^3$ and the denominator $2n^5 - 6n \approx 2n^5$, so the n th term is approximately $\frac{n^3}{2n^5} = \frac{1}{2n^2}$. So we apply the limit comparison test with $a_n = \frac{n^3 - 4}{2n^5 - 6n}$ and $b_n = \frac{1}{n^2}$. Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(n^3 - 4)/(2n^5 - 6n)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1 - \frac{4}{n^3}}{2 - \frac{6}{n^4}} = \frac{1}{2}$$

exists and is nonzero, the given series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges. Since the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p -series (with $p = 2$), both series converge.

S-54: By the alternating series test, the error introduced when we approximate the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 10^n} \text{ by } \sum_{n=1}^N \frac{(-1)^n}{n \cdot 10^n} \text{ is at most the magnitude of the first omitted term,}$$

$\frac{1}{(N+1)10^{(N+1)}}$. By trial and error, we find that this expression becomes smaller than 10^{-6} when $N+1 \geq 6$. So the smallest allowable value is $N = 5$.

S-55: The sequence $\{\frac{1}{n^2}\}$ decreases to zero as n increases to infinity. So, by the alternating series error bound, which is given in Theorem 3.3.17 in the text, $\frac{\pi^2}{12} - S_N$ lies between zero and the first omitted term, $\frac{(-1)^N}{(N+1)^2}$. We therefore need $\frac{1}{(N+1)^2} \leq 10^{-6}$, which is equivalent to $N+1 \geq 10^3$ and $N \geq 999$.

S-56: The error introduced when we approximate S by the N^{th} partial sum

$$S_N = \sum_{n=1}^N \frac{(-1)^{n+1}}{(2n+1)^2} \text{ lies between 0 and the first term dropped, which is}$$

$\left. \frac{(-1)^{n+1}}{(2n+1)^2} \right|_{n=N+1} = \frac{(-1)^{N+2}}{(2N+3)^2}$. So we need the smallest positive integer N obeying

$$\begin{aligned} \frac{1}{(2N+3)^2} &\leq \frac{1}{100} \\ (2N+3)^2 &\geq 100 \\ 2N+3 &\geq 10 \\ N &\geq \frac{7}{2} \end{aligned}$$

So we need $N = 4$ and then

$$S_4 = \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2}$$

S-57: The volume of a sphere of radius $\frac{1}{\pi^n}$ is

$$v_n = \frac{4}{3}\pi \left(\frac{1}{\pi^n} \right)^3 = \frac{4\pi}{3} \left(\frac{1}{\pi^3} \right)^n$$

So, the volume of all the spheres together is:

$$\begin{aligned} \sum_{n=1}^{\infty} v_n &= \sum_{n=1}^{\infty} \frac{4\pi}{3} \left(\frac{1}{\pi^3} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{4\pi}{3} \left(\frac{1}{\pi^3} \right)^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{4}{3\pi^2} \left(\frac{1}{\pi^3} \right)^n \end{aligned}$$

We use Equation 3.3.2 in the text with $a = \frac{4}{3\pi^2}$ and $r = \frac{1}{\pi^3}$.

$$= \frac{4}{3\pi^2} \cdot \frac{1}{1 - \frac{1}{\pi^3}} = \frac{4\pi}{3(\pi^3 - 1)}$$

S-58: (a) There are plenty of powers/factorials. So let's try the ratio test with $a_n = \frac{n^n}{9^n n!}$.

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{9^{n+1}(n+1)!} \frac{9^n n!}{n^n} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{n^n 9(n+1)} = \lim_{n \rightarrow \infty} \frac{(1 + 1/n)^n}{9} = \frac{e}{9}$$

Here we have used that $\lim_{n \rightarrow \infty} (1 + 1/n)^n = e$. As $e < 9$, our series converges.

(b) We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the p -test with $p = 2$, and also that $\log n \geq 2$ for all $n \geq e^2$. So let's use the limit comparison test with $a_n = \frac{1}{n^{\log n}}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{n^{\log n}} \cdot \frac{n^2}{1} = \lim_{n \rightarrow \infty} \frac{1}{n^{\log n - 2}} = 0$$

So our series converges, by the limit comparison test.

S-59: (a)

Solution 1: • Our first task is to identify the potential sources of impropriety for this integral.

- The domain of integration extends to $+\infty$. On the domain of integration the denominator is never zero so the integrand is continuous. Thus the only problem is at $+\infty$.
- Our second task is to develop some intuition about the behavior of the integrand for very large x . When x is very large:
 - $|\sin x| \leq 1 \ll x$, so that the numerator $x + \sin x \approx x$, and
 - $1 \ll x^2$, so that denominator $1 + x^2 \approx x^2$, and
 - the integrand $\frac{x + \sin x}{1 + x^2} \approx \frac{x}{x^2} = \frac{1}{x}$
- Now, since $\int_2^\infty \frac{dx}{x}$ diverges, we would expect $\int_2^\infty \frac{x + \sin x}{1 + x^2} dx$ to diverge too.
- Our final task is to verify that our intuition is correct. To do so, we set

$$f(x) = \frac{x + \sin x}{1 + x^2} \qquad g(x) = \frac{1}{x}$$

and compute

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{x + \sin x}{1 + x^2} \div \frac{1}{x} \\ &= \lim_{x \rightarrow \infty} \frac{(1 + \sin x/x)x}{(1/x^2 + 1)x^2} \times x \\ &= \lim_{x \rightarrow \infty} \frac{1 + \sin x/x}{1/x^2 + 1} \\ &= 1 \end{aligned}$$

- Since $\int_2^\infty g(x) dx = \int_2^\infty \frac{dx}{x}$ diverges, by Example 1.13.8 in the text⁷, with $p = 1$, Theorem 1.13.22(b) in the text now tells us that $\int_2^\infty f(x) dx = \int_2^\infty \frac{x + \sin x}{1 + x^2} dx$ diverges too.

Solution 2: Let's break up the integrand as $\frac{x + \sin x}{1 + x^2} = \frac{x}{1 + x^2} + \frac{\sin x}{1 + x^2}$. First, we consider the integral $\int_2^\infty \frac{\sin x}{1 + x^2} dx$.

7 To change the lower limit of integration from 1 to 2, just apply Theorem 1.13.20 in the text.

- $\frac{|\sin x|}{1+x^2} \leq \frac{1}{1+x^2}$, so if we can show $\int \frac{1}{1+x^2} dx$ converges, we can conclude that $\int \frac{|\sin x|}{1+x^2} dx$ converges as well by the comparison test.
- $\int_2^\infty \frac{1}{1+x^2} dx \leq \int_2^\infty \frac{1}{x^2} dx$
- $\int_2^\infty \frac{1}{x^2} dx$ converges (by the p -test with $p = 2$)
- So the integral $\int_2^\infty \frac{\sin x}{1+x^2} dx$ converges by the comparison test, and hence
- $\int_2^\infty \frac{\sin x}{1+x^2} dx$ converges as well.

Therefore, $\int_2^\infty \frac{x + \sin x}{1+x^2} dx$ converges if and only if $\int_2^\infty \frac{x}{1+x^2} dx$ converges. But

$$\int_2^\infty \frac{x}{1+x^2} dx = \lim_{r \rightarrow \infty} \int_2^r \frac{x}{1+x^2} dx = \lim_{r \rightarrow \infty} \left[\frac{1}{2} \log(1+x^2) \right]_2^r = \infty$$

diverges, so $\int_2^\infty \frac{x + \sin x}{1+x^2} dx$ diverges.

(b) The problem is that $f(x) = \frac{x + \sin x}{1+x^2}$ is not a decreasing function. To see this, compute the derivative:

$$f'(x) = \frac{(1 + \cos x)(1 + x^2) - (x + \sin x)(2x)}{(1 + x^2)^2} = \frac{(\cos x - 1)x^2 - 2x \sin x + 1 + \cos x}{(1 + x^2)^2}$$

If $x = 2m\pi$, the numerator is $0 - 0 + 1 + 1 > 0$.

Therefore, the integral test does not apply.

(c)

Solution 1: Set $a_n = \frac{n + \sin n}{1 + n^2}$. We first try to develop some intuition about the behaviour of a_n for large n and then we confirm that our intuition was correct.

- *Step 1: Develop intuition.* When $n \gg 1$, the numerator $n + \sin n \approx n$, and the denominator $1 + n^2 \approx n^2$ so that $a_n \approx \frac{n}{n^2} = \frac{1}{n}$ and it looks like our series should diverge by the p -test (Example 3.3.7 in the text) with $p = 1$.
- *Step 2: Verify intuition.* To confirm our intuition we set $b_n = \frac{1}{n}$ and compute the limit

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n + \sin n}{1 + n^2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n[n + \sin n]}{1 + n^2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{\sin n}{n}}{\frac{1}{n^2} + 1} = 1$$

We already know that the series $\sum_{n=1}^\infty b_n = \sum_{n=1}^\infty \frac{1}{n}$ diverges by the p -test with $p = 1$. So our series diverges by the limit comparison test, Theorem 3.3.14 in the text.

Solution 2: Since $|\frac{\sin n}{1+n^2}| \leq \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p -test with $p = 2$, the series $\sum_{n=1}^{\infty} \frac{\sin n}{1+n^2}$ converges. Hence $\sum_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ converges if and only if the series $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ converges. Now $f(x) = \frac{x}{1+x^2}$ is a continuous, positive, decreasing function on $[1, \infty)$ since

$$f'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

is negative for all $x > 1$. We saw in part (a) that the integral $\int_2^{\infty} \frac{x}{1+x^2} dx$ diverges. So the integral $\int_1^{\infty} \frac{x}{1+x^2} dx$ diverges too and the sum $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ diverges by the integral test. So $\sum_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges.

S-60: Note that $\frac{e^{-\sqrt{x}}}{\sqrt{x}} = \frac{1}{\sqrt{x}e^{\sqrt{x}}}$ decreases as x increases. Hence, for every $n \geq 1$,

$$\begin{aligned} \frac{e^{-\sqrt{x}}}{\sqrt{x}} &\geq \frac{e^{-\sqrt{n}}}{\sqrt{n}} && \text{for } x \text{ in the interval } [n-1, n] \\ \text{So, } \int_{n-1}^n \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx &\geq \int_{n-1}^n \frac{e^{-\sqrt{n}}}{\sqrt{n}} dx \\ &= \left[\frac{e^{-\sqrt{n}}}{\sqrt{n}} x \right]_{x=n-1}^{x=n} \\ &= \frac{e^{-\sqrt{n}}}{\sqrt{n}} \end{aligned}$$

Then, for every $N \geq 1$,

$$\begin{aligned} E_N &= \sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \leq \sum_{n=N+1}^{\infty} \int_{n-1}^n \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx \\ &= \int_N^{N+1} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx + \int_{N+1}^{N+2} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx + \cdots \\ &= \int_N^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx \end{aligned}$$

Substituting $y = \sqrt{x}$, $dy = \frac{1}{2} \frac{dx}{\sqrt{x}}$,

$$\int_N^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = 2 \int_{\sqrt{N}}^{\infty} e^{-y} dy = -2e^{-y} \Big|_{\sqrt{N}}^{\infty} = 2e^{-\sqrt{N}}$$

This shows that $\sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges and is between 0 and $2e^{-\sqrt{N}}$. Since $E_{14} = 2e^{-\sqrt{14}} = 0.047$, we may truncate the series at $n = 14$.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} &= \sum_{n=1}^{14} \frac{e^{-\sqrt{n}}}{\sqrt{n}} + E_{14} \\ &= 0.3679 + 0.1719 + 0.1021 + 0.0677 + 0.0478 \\ &\quad + 0.0352 + 0.0268 + 0.0209 + 0.0166 + 0.0134 \\ &\quad + 0.0109 + 0.0090 + 0.0075 + 0.0063 + E_{14} \\ &= 0.9042 + E_{14} \end{aligned}$$

The sum is between 0.9035 and 0.9535. (This even allows for a roundoff error of 0.00005 in each term as we were calculating the partial sum.)

S-61: Let's get some intuition to guide us through a proof. Since $\sum_{n=1}^{\infty} a_n$ converges a_n must converge to zero as $n \rightarrow \infty$. So, when n is quite large, $\frac{a_n}{1-a_n} \approx \frac{a_n}{1-0} = \frac{a_n}{1}$, and we know $\sum a_n$ converges. So, we want to separate the "large" indices from a finite number of smaller ones.

Since $\lim_{n \rightarrow \infty} a_n = 0$, there must be⁸ some integer N such that $\frac{1}{2} > a_n \geq 0$ for all $n > N$. Then, for $n > N$,

$$\frac{a_n}{1-a_n} \leq \frac{a_n}{1-1/2} = 2a_n$$

From the information in the problem statement, we know

$$\sum_{n=N+1}^{\infty} 2a_n = 2 \sum_{n=N+1}^{\infty} a_n \quad \text{converges.}$$

So, by the direct comparison test,

$$\sum_{n=N+1}^{\infty} \frac{a_n}{1-a_n} \quad \text{converges as well.}$$

Since the convergence of a series is not affected by its first N terms, as long as N is finite, we conclude

$$\sum_{n=1}^{\infty} \frac{a_n}{1-a_n} \quad \text{converges.}$$

⁸ We could have chosen any positive number strictly less than 1, not only $\frac{1}{2}$.

S-62: By the divergence test, the fact that $\sum_{n=0}^{\infty} (1 - a_n)$ converges guarantees that $\lim_{n \rightarrow \infty} (1 - a_n) = 0$, or equivalently, that $\lim_{n \rightarrow \infty} a_n = 1$. So, by the divergence test, a second time, the fact that

$$\lim_{n \rightarrow \infty} 2^n a_n = +\infty$$

guarantees that $\sum_{n=0}^{\infty} 2^n a_n$ diverges too.

S-63: We are told that $\sum_{n=1}^{\infty} a_n$ converges. Thus we must have that $\lim_{n \rightarrow \infty} a_n = 0$. In particular, there is an index N such that $0 \leq a_n \leq 1$ for all $n \geq N$. Then:

$$0 \leq a_n^2 \leq a_n \quad \text{for } n > N$$

By the direct comparison test,

$$\sum_{n=N+1}^{\infty} a_n^2 \quad \text{converges.}$$

Since convergence doesn't depend on the first N terms of a series for any finite N ,

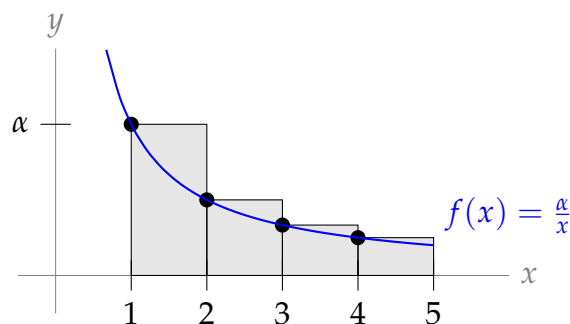
$$\sum_{n=1}^{\infty} a_n^2 \quad \text{converges as well.}$$

S-64: The most-commonly used word makes up α percent of all the words. So, we want to find α .

If we add together the frequencies of all the words, they should amount to 100%. That is,

$$\sum_{n=1}^{20,000} \frac{\alpha}{n} = 100$$

We can approximate the sum (with α left as a parameter) using the ideas behind the integral test. (See Example 3.3.5.)



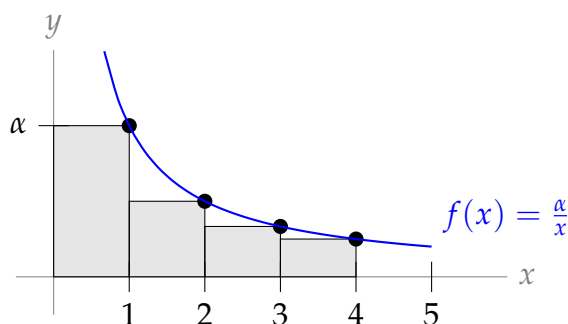
As we see in the diagram above, $\sum_{n=1}^N \frac{\alpha}{n}$ (which is the sum of the areas of the rectangles) is *greater* than $\int_1^{N+1} \frac{\alpha}{x} dx$ (the area under the curve). That is,

$$\int_1^{N+1} \frac{\alpha}{x} dx < \sum_{n=1}^N \frac{\alpha}{n}.$$

Using the fact that our language's 20,000 words make up 100% of the words used, we can find a lower bound for α .

$$\begin{aligned} 100 &= \sum_{n=1}^{20,000} \frac{\alpha}{n} > \int_1^{20,001} \frac{\alpha}{x} dx = \left[\alpha \log(x) \right]_1^{20,001} = \alpha \log(20,001) \\ \alpha &< \frac{100}{\log(20,001)} \end{aligned}$$

We can find an upper bound for α in a similar manner.



From the diagram, we see $\sum_{n=2}^N \frac{\alpha}{n}$ (which is the sum of the areas of the rectangles, excluding the first) is *less* than $\int_1^N \frac{\alpha}{x} dx$. (The reason for excluding the first rectangle is to avoid comparing our series to an integral that diverges.) That is,

$$\sum_{n=2}^N \frac{\alpha}{n} < \int_1^N \frac{\alpha}{x} dx.$$

Therefore,

$$\begin{aligned} 100 &= \sum_{n=1}^{20,000} \frac{\alpha}{n} = \alpha + \sum_{n=2}^{20,000} \frac{\alpha}{n} \\ &< \alpha + \int_1^{20,000} \frac{\alpha}{x} dx = \alpha + \alpha \log(20,000) = \alpha [1 + \log(20,000)] \\ \alpha &> \frac{100}{1 + \log(20,000)} \end{aligned}$$

Using a calculator, we see

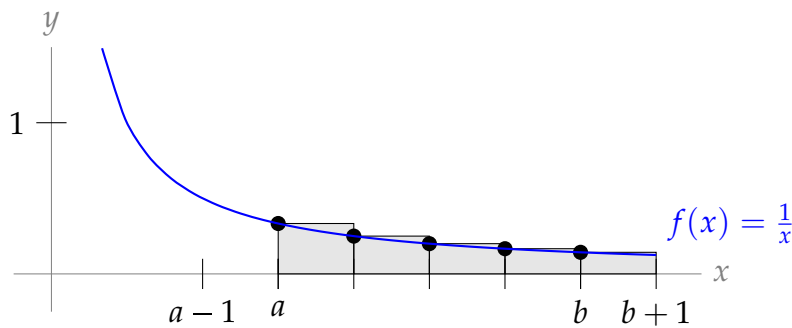
$$9.17 < \alpha < 10.01$$

So, the most-commonly used word makes up about 9-10 percent of the total words.

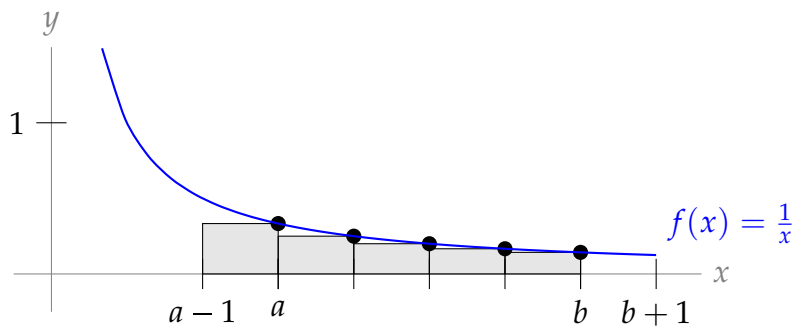
S-65: Generalizing our work in Question 64, we find the approximations:

$$\int_a^{b+1} \frac{1}{x} dx < \sum_{n=a}^b \frac{1}{n} < \int_{a-1}^b \frac{1}{x} dx$$

when $a \geq 2$. The inequality $\int_a^{b+1} \frac{1}{x} dx < \sum_{n=a}^b \frac{1}{n}$ can be read off of the sketch



and the inequality $\sum_{n=a}^b \frac{1}{n} < \int_{a-1}^b \frac{1}{x} dx$ can be read off of the sketch



We will evaluate the total population by writing

$$\sum_{n=1}^{2 \times 10^6} \frac{2 \times 10^6}{n} = \sum_{n=1}^{a-1} \frac{2 \times 10^6}{n} + \sum_{n=a}^{2 \times 10^6} \frac{2 \times 10^6}{n}$$

and applying the above integral approximations to the second sum. We want our error to be less than one million, so we need to choose a value of a such that:

$$\begin{aligned}
& \underbrace{2 \times 10^6 \int_{a-1}^{2 \times 10^6} \frac{1}{x} dx}_{\text{upper bound}} - \underbrace{2 \times 10^6 \int_a^{2 \times 10^6 + 1} \frac{1}{x} dx}_{\text{lower bound}} < 10^6 \\
& \int_{a-1}^{2 \times 10^6} \frac{1}{x} dx - \int_a^{2 \times 10^6 + 1} \frac{1}{x} dx < \frac{1}{2} \\
& \left[\log(2 \times 10^6) - \log(a-1) \right] - \left[\log(2 \times 10^6 + 1) - \log(a) \right] < \frac{1}{2} \\
& \left[\log(2 \times 10^6) - \log(2 \times 10^6 + 1) \right] + [\log(a) - \log(a-1)] < \frac{1}{2} \\
& \log\left(\frac{2 \times 10^6}{2 \times 10^6 + 1}\right) + \log\left(\frac{a}{a-1}\right) < \frac{1}{2}
\end{aligned}$$

The first term is extremely close to 0, so we ignore it.

$$\begin{aligned}
\log\left(\frac{a}{a-1}\right) &< \frac{1}{2} \\
\frac{a}{a-1} &< e^{1/2} = \sqrt{e} \\
a &< a\sqrt{e} - \sqrt{e} \\
\sqrt{e} &< a(\sqrt{e} - 1) \\
\frac{\sqrt{e}}{\sqrt{e} - 1} &< a
\end{aligned}$$

Since $\frac{\sqrt{e}}{\sqrt{e} - 1} \approx 2.5$, we use $a = 3$. That is, we will approximate the value of $\sum_{n=3}^{2 \times 10^6} \frac{1}{n}$ using an integral. Then, we will use that approximation to estimate our total population.

$$\begin{aligned}
\int_3^{2 \times 10^6 + 1} \frac{1}{x} dx &< \sum_{n=3}^{2 \times 10^6} \frac{1}{n} &< \int_2^{2 \times 10^6} \frac{1}{x} dx \\
\log(2 \times 10^6 + 1) - \log(3) &< \sum_{n=3}^{2 \times 10^6} \frac{1}{n} &< \log(2 \times 10^6) - \log(2) \\
1 + \frac{1}{2} + \log(2 \times 10^6 + 1) - \log(3) &< \sum_{n=1}^{2 \times 10^6} \frac{1}{n} &< 1 + \frac{1}{2} + \log(2 \times 10^6) - \log(2) \\
\frac{3}{2} + \log\left(\frac{2 \times 10^6 + 1}{3}\right) &< \sum_{n=1}^{2 \times 10^6} \frac{1}{n} &< \frac{3}{2} + 6 \log(10) \\
2 \times 10^6 \left(\frac{3}{2} + \log\left(\frac{2}{3} \times 10^6 + \frac{1}{3}\right) \right) &< \sum_{n=1}^{2 \times 10^6} \frac{2 \times 10^6}{n} &< 2 \times 10^6 \left(\frac{3}{2} + 6 \log(10) \right) \\
29,820,091 &< \text{population} &< 30,631,021
\end{aligned}$$

S-66: We consider a circle of radius R , with an “inner ring” from $\frac{R}{3}$ to $\frac{2R}{3}$ and an “outer ring” from $\frac{2R}{3}$ to R .

The area of the outer ring is:

$$\pi R^2 - \pi \left(\frac{2R}{3} \right)^2 = \frac{5}{9} \pi R^2$$

The area of the inner ring is:

$$\pi \left(\frac{2R}{3} \right)^2 - \pi \left(\frac{R}{3} \right)^2 = \frac{3}{9} \pi R^2$$

So, the ratio of the inner ring’s area to the outer ring’s area is $\frac{3}{5}$.

In our bullseye diagram, if we pair up any red ring with the blue ring just inside it, the blue ring has $\frac{3}{5}$ the area of the red ring. So, the blue portion of the bullseye has $\frac{3}{5}$ the area of the red portion.

Since the circle has area 1, if we let the red portion have area A , then

$$1 = A + \frac{3}{5}A = \frac{8}{5}A$$

So, the red portion has area $\frac{5}{8}$.

Solutions to Exercises 3.4 — Jump to [TABLE OF CONTENTS](#)

S-1: False. For example if $b_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges by the alternating series test, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p -test.

Remark: if we had added that $\{b_n\}$ is a sequence of *alternating terms*, then by Theorem 3.4.2 in the text, the statement would have been true. This is because $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ would either be equal to $\sum_{n=1}^{\infty} |b_n|$ or $-\sum_{n=1}^{\infty} |b_n|$.

S-2: Absolute convergence describes the situation where $\sum |a_n|$ converges (see Definition 3.4.1 in the text). By Theorem 3.4.2 in the text, this guarantees that also $\sum a_n$ converges.

Conditional convergence describes the situation where $\sum |a_n|$ diverges but $\sum a_n$ converges (see again Definition 3.4.1 in the text).

If $\sum a_n$ diverges, we just say it diverges. The reason is that if $\sum a_n$ diverges, we automatically know $\sum |a_n|$ diverges as well, so there’s no need for a special name.

	$\sum a_n$ converges	$\sum a_n$ diverges
$\sum a_n $ converges	converges absolutely	not possible
$\sum a_n $ diverges	converges conditionally	diverges

S-3: The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{9n+5}$ converges by the alternating series test. On the other hand the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{9n+5} \right| = \sum_{n=1}^{\infty} \frac{1}{9n+5}$ diverges by the limit comparison test with $b_n = \frac{1}{n}$. So the given series is conditionally convergent.

S-4: Note that $(-1)^{2n+1} = (-1) \cdot (-1)^{2n} = -1$. So we can simplify

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{1+n} = - \sum_{n=1}^{\infty} \frac{1}{1+n}$$

Since $\frac{1}{1+n} \geq \frac{1}{n+n} = \frac{1}{2n}$, $\sum_{n=1}^{\infty} \frac{1}{1+n}$ diverges by the comparison test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. The extra overall factor of -1 in the original series does not change the conclusion of divergence.

S-5: Since

$$\lim_{n \rightarrow \infty} \frac{1+4^n}{3+2^{2n}} = \lim_{n \rightarrow \infty} \frac{1+4^n}{3+4^n} = 1$$

the alternating series test cannot be used. Indeed, $\lim_{n \rightarrow \infty} (-1)^{n-1} \frac{1+4^n}{3+2^{2n}}$ does not exist (for very large n , $(-1)^{n-1} \frac{1+4^n}{3+2^{2n}}$ alternates between a number close to $+1$ and a number close to -1) so the divergence test says that the series diverges. (Note that “none of the above” cannot possibly be the correct answer — *every* series either converges absolutely, converges conditionally, or diverges.)

S-6: First, we’ll develop some intuition. For very large n

$$\left| \frac{\sqrt{n} \cos(n)}{n^2 - 1} \right| \approx \left| \frac{\sqrt{n} \cos(n)}{n^2} \right| = \left| \frac{\cos(n)}{n^{3/2}} \right| \leq \frac{1}{n^{3/2}}$$

since $|\cos(n)| \leq 1$ for all n . By the p -test, which is in Example 3.3.7 in the text, the series $\sum_{n=5}^{\infty} \frac{1}{n^p}$ converges for all $p > 1$. So we would expect the given series to converge absolutely.

Now, to confirm that our intuition is correct, we'll first try the limit comparison theorem, which is Theorem 3.3.14 in the text, with $a_n = \left| \frac{\sqrt{n} \cos(n)}{n^2 - 1} \right|$ and $b_n = \frac{1}{n^{3/2}}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\left| \frac{\sqrt{n} \cos(n)}{n^2 - 1} \right|}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n} \cdot \sqrt{n^3} |\cos n|}{n^2 - 1} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 |\cos n|}{n^2 - 1} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 - 1/n^2} \right) |\cos n| \\ &= \lim_{n \rightarrow \infty} 1 \cdot |\cos n| \end{aligned}$$

Unfortunately, this limit doesn't exist, so this attempt to use the limit comparison theorem has failed. Fortunately, having seen that the $\cos n$ caused the failure, it is not hard to adjust our strategy to get a successful proof of absolute convergence. First, in step 1 below, we use the comparison test to eliminate the $\cos n$ and then, in step 2 below, we apply the limit comparison test.

Step 1: Since $|\cos n| \leq 1$, we have

$$\left| \frac{\sqrt{n} \cos(n)}{n^2 - 1} \right| \leq \frac{\sqrt{n}}{n^2 - 1}$$

for all $n > 1$. So, by part (a) of the comparison test, which is Theorem 3.3.11 in the text, if the series $\sum_{n=5}^{\infty} \frac{\sqrt{n}}{n^2 - 1}$ converges, then we will have that the series

$\sum_{n=5}^{\infty} \left| \frac{\sqrt{n} \cos(n)}{n^2 - 1} \right|$ also converges, and hence that the series $\sum_{n=5}^{\infty} \frac{\sqrt{n} \cos(n)}{n^2 - 1}$ converges absolutely.

Step 2: Now, to prove that the series $\sum_{n=5}^{\infty} \frac{\sqrt{n}}{n^2 - 1}$ converges, we apply the limit

comparison test with $a_n = \frac{\sqrt{n}}{n^2 - 1}$ and $b_n = \frac{1}{n^{3/2}}$ (for $n \geq 5$). Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n}}{n^2 - 1}}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n} \cdot \sqrt{n^3}}{n^2 - 1} \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{n^2 - 1} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/n^2} \\ &= 1 \end{aligned}$$

and since $\sum_{n=5}^{\infty} \frac{1}{n^{3/2}}$ converges by the p -test, the limit comparison test tells us that the series $\sum_{n=5}^{\infty} \frac{\sqrt{n}}{n^2 - 1}$ converges. So, by step 1, $\sum_{n=5}^{\infty} \frac{\sqrt{n} \cos(n)}{n^2 - 1}$ converges absolutely.

S-7: We first develop some intuition about $\sum_{n=1}^{\infty} \left| \frac{n^2 - \sin n}{n^6 + n^2} \right|$, where we take the absolute value of the summands to consider whether the series converges *absolutely*. For very large n , n^2 dominates $\sin n$ and n^6 dominates n^2 so that

$$\left| \frac{n^2 - \sin n}{n^6 + n^2} \right| \approx \frac{n^2}{n^6} = \frac{1}{n^4}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by the p -test with $p = 4 > 1$. We expect the given series to converge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_n = \frac{n^2 - \sin n}{n^6 + n^2} \quad \text{and} \quad b_n = \frac{1}{n^4}$$

which is valid since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left| \frac{(n^2 - \sin n)}{n^6 + n^2} \right| \cdot \frac{n^4}{1} = \lim_{n \rightarrow \infty} \frac{|n^6 - n^4 \sin n|}{n^6 + n^2} = \lim_{n \rightarrow \infty} \frac{1 - n^{-2} \sin n}{1 + n^{-4}} = 1$$

exists and is nonzero. Since the series $\sum_{n=1}^{\infty} b_n$ converges, the series $\sum_{n=1}^{\infty} \frac{|n^2 - \sin n|}{n^6 + n^2}$ converges too. Therefore, the series $\sum_{n=1}^{\infty} \frac{n^2 - \sin n}{n^6 + n^2}$ converges absolutely.

S-8: You might think that this series converges by the alternating series test. But you would be wrong. The problem is that $\{a_n\}$ does not converge to zero as $n \rightarrow \infty$, so that the series actually diverges by the divergence test. To verify that the n^{th} term does not converge to zero as $n \rightarrow \infty$ let's write $a_n = \frac{(2n)!}{(n^2+1)(n!)^2}$ (i.e. a_n is the n^{th} term without the sign) and check to see whether a_{n+1} is bigger than or smaller than a_n .

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{((n+1)^2+1)((n+1)!)^2} \cdot \frac{(n^2+1)(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot \frac{n^2+1}{(n+1)^2+1} \\ &= \frac{2(2n+1)}{(n+1)} \cdot \frac{1+1/n^2}{(1+1/n)^2+1/n^2} = 4 \frac{1+1/2n}{1+1/n} \cdot \frac{1+1/n^2}{(1+1/n)^2+1/n^2} \end{aligned}$$

So

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 4$$

and, in particular, for large n , $a_{n+1} > a_n$. Thus, for large n , a_n increases with n and so cannot converge to 0. So the series diverges by the divergence test.

S-9: This series converges by the alternating series test. We want to know whether it converges absolutely, so we consider the series $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\log n)^{101}} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{101}}$.

We've seen similar function before (e.g. Example 3.3.8 in the text, with $p = 101 > 1$) and it yields nicely to the integral test. Let $f(x) = \frac{1}{x(\log x)^{101}}$. Note $f(x)$ is positive and decreasing for $x \geq 3$. Then by the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{101}}$ converges if and only if the integral $\int_2^{\infty} \frac{1}{x(\log x)^{101}} dx$ does. We evaluate the integral using the substitution $u = \log x$, $du = \frac{1}{x} dx$.

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\log x)^{101}} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\log x)^{101}} dx \\ &= \lim_{b \rightarrow \infty} \int_{\log 2}^{\log b} \frac{1}{u^{101}} du \\ &= \lim_{b \rightarrow \infty} \left[\frac{-1}{100u^{100}} \right]_{\log 2}^{\log b} \\ &= \frac{1}{100(\log 2)^{100}} \end{aligned}$$

Since the integral converges, the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{101}}$ converges, and therefore the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^{101}}$ converges absolutely.

S-10: The sequence has some positive terms and some negative terms, which limits the tests we can use. However, if we consider the series $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$, we can use the direct comparison test.

For every n , $|\sin n| < 1$, so $0 \leq \left| \frac{\sin n}{n^2} \right| < \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by the direct comparison test, $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ converges as well. Then $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely—in particular, it converges.

S-11: The terms of this series are sometimes negative (for odd values of n where $\sin n < \frac{1}{2}$) and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.

$$\begin{aligned}
& -\frac{1}{4} \leq \frac{\sin n}{4} \leq \frac{1}{4} \\
\Rightarrow & \left(-\frac{1}{4} - \frac{1}{8}\right) \leq \left(\frac{\sin x}{4} - \frac{1}{8}\right) < \left(\frac{1}{4} - \frac{1}{8}\right) \\
\Rightarrow & -\frac{3}{8} \leq \left(\frac{\sin x}{4} - \frac{1}{8}\right) < \frac{1}{8} \\
\Rightarrow & 0 \leq \left|\frac{\sin x}{4} - \frac{1}{8}\right| < \frac{3}{8} \\
\Rightarrow & 0 \leq \left|\left(\frac{\sin x}{4} - \frac{1}{8}\right)^n\right| < \left(\frac{3}{8}\right)^n
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \left(\frac{3}{8}\right)^n$ converges (it's a geometric sum with $|r| < 1$), by the direct comparison test, $\sum_{n=1}^{\infty} \left|\left(\frac{\sin x}{4} - \frac{1}{8}\right)^n\right|$ converges as well.

Then $\sum_{n=1}^{\infty} \left(\frac{\sin x}{4} - \frac{1}{8}\right)^n$ converges absolutely—and so it converges.

S-12: The terms of this series are sometimes negative and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.

$$\left| \frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n} \right| \leq \frac{1 + 1 + \frac{1}{2}}{2^n} = \frac{5}{2^{n+1}}$$

The series $\sum_{n=1}^{\infty} \frac{5}{2^{n+1}}$ converges, because it's a geometric series with $r = \frac{1}{2}$. By the direct comparison test, $\sum_{n=1}^{\infty} \left| \frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n} \right|$ converges as well. Then $\sum_{n=1}^{\infty} \frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n}$ converges absolutely, so it converges.

S-13: (a)

Solution 1: We need to show that $\sum_{n=1}^{\infty} 24n^2 e^{-n^3}$ converges. If we replace n by x in the summand, we get $f(x) = 24x^2 e^{-x^3}$, which we can integrate. (Just substitute $u = x^3$.) So let's try the integral test. First, we have to check that $f(x)$ is positive and decreasing. It is certainly positive. To determine if it is decreasing, we compute

$$\frac{df}{dx} = 48x e^{-x^3} - 24 \times 3x^4 e^{-x^3} = 24x(2 - 3x^3)e^{-x^3}$$

which is negative for $x \geq 1$. Therefore $f(x)$ is decreasing for $x \geq 1$, and the integral test applies. The substitution $u = x^3$, $du = 3x^2 dx$, yields

$$\int f(x) dx = \int 24x^2 e^{-x^3} dx = \int 8e^{-u} du = -8e^{-u} + C = -8e^{-x^3} + C.$$

Therefore

$$\begin{aligned} \int_1^{\infty} f(x) dx &= \lim_{R \rightarrow \infty} \int_1^R f(x) dx = \lim_{R \rightarrow \infty} \left[-8e^{-x^3} \right]_1^R \\ &= \lim_{R \rightarrow \infty} (-8e^{-R^3} + 8e^{-1}) = 8e^{-1} \end{aligned}$$

Since the integral is convergent, the series $\sum_{n=1}^{\infty} 24n^2 e^{-n^3}$ converges and the series $\sum_{n=1}^{\infty} (-1)^{n-1} 24n^2 e^{-n^3}$ converges absolutely.

Solution 2: Alternatively, we can use the ratio test with $a_n = 24n^2 e^{-n^3}$. We calculate

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{24(n+1)^2 e^{-(n+1)^3}}{24n^2 e^{-n^3}} \right| \\ &= \lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{n^2} \frac{e^{n^3}}{e^{(n+1)^3}} \right) \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 e^{-(3n^2 + 3n + 1)} = 1 \cdot 0 = 0 < 1, \end{aligned}$$

and therefore the series converges absolutely.

Solution 3: Alternatively, alternatively, we can use the limiting comparison test. First a little intuition building. Recall that we need to show that $\sum_{n=1}^{\infty} 24n^2 e^{-n^3}$ converges.

The n^{th} term in this series is

$$a_n = 24n^2 e^{-n^3} = \frac{24n^2}{e^{n^3}}$$

It is a ratio with both the numerator and denominator growing with n . A good rule of thumb is that exponentials grow a lot faster than powers. For example, if $n = 10$ the numerator is $2400 = 2.4 \times 10^3$ and the denominator is about 2×10^{434} . So we would guess that a_n tends to zero as $n \rightarrow \infty$. The question is “does a_n tend to zero fast enough with n that our series converges?”. For example, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by the p -test with $p = 2$). So if a_n tends to zero faster than $\frac{1}{n^2}$ does, our series will converge. So let’s try the limiting convergence test with $a_n = 24n^2 e^{-n^3} = \frac{24n^2}{e^{n^3}}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{24n^2 e^{-n^3}}{1/n^2} = \lim_{n \rightarrow \infty} \frac{24n^4}{e^{n^3}}$$

By l'Hôpital's rule, twice,

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{24x^4}{e^{x^3}} &= \lim_{x \rightarrow \infty} \frac{4 \times 24x^3}{3x^2 e^{x^3}} && \text{by l'Hôpital} \\
 &= \lim_{x \rightarrow \infty} \frac{32x}{e^{x^3}} && \text{just cleaning up} \\
 &= \lim_{x \rightarrow \infty} \frac{32}{3x^2 e^{x^3}} && \text{by l'Hôpital, again} \\
 &= 0
 \end{aligned}$$

That's it. The limit comparison test now tells us that $\sum_{n=1}^{\infty} a_n$ converges.

(b) In part (a) we saw that $24n^2e^{-n^3}$ is positive and decreasing. The limit of this sequence equals 0 (as can be shown with l'Hôpital's Rule, just as we did at the end of the third solution of part (a)). Therefore, we can use the alternating series test, so that the error made in approximating the infinite sum $S = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} 24n^2e^{-n^3}$ by the sum of its first N terms, $S_N = \sum_{n=1}^N a_n$, lies between 0 and the first omitted term, a_{N+1} . If we use 5 terms, the error satisfies

$$|S - S_5| \leq |a_6| = 24 \times 36e^{-6^3} \approx 1.3 \times 10^{-91}$$

S-14: The error in our approximation using through term N is at most $\frac{1}{(2(N+1))!}$. We want $\frac{1}{(2(N+1))!} < \frac{1}{1000}$. By checking small values of N , we see that $8! = 40320 > 1000$, so if $N = 3$, then $\frac{1}{2(N+1)!} = \frac{1}{40320} < \frac{1}{1000}$. So, for our approximation, it suffices to consider the first four terms of our series.

$$\begin{aligned}
 \cos(1) &\approx \sum_{N=0}^3 \frac{(-1)^n}{(2n)!} = \frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} \\
 &= 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \\
 &= \frac{720 - 360 + 30 - 1}{720} = \frac{389}{720}
 \end{aligned}$$

When we use a calculator, we see

$$\begin{aligned}
 \frac{389}{720} &= 0.5402\overline{77} \\
 \cos(1) &\approx 0.540302 \\
 \cos(1) - \frac{389}{720} &\approx 0.000024528 \approx \frac{1}{40770}
 \end{aligned}$$

So, our error is reasonably close to our bound of $\frac{1}{40320}$, and far smaller than $\frac{1}{1000}$.

S-15: The terms of this series are sometimes negative and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.

If n is prime, then

$$\left| \frac{a_n}{e^n} \right| = \left| -\frac{e^{n/2}}{e^n} \right| = \frac{1}{e^{n/2}} = \left(\frac{1}{\sqrt{e}} \right)^n$$

If n is not prime, then

$$\left| \frac{a_n}{e^n} \right| = \left| -\frac{n^2}{e^n} \right| = \frac{n^2}{e^n}$$

For n sufficiently large, $n^2 < e^{n/2}$, so for n sufficiently large,

$$\frac{n^2}{e^n} \leq \left(\frac{1}{\sqrt{e}} \right)^n.$$

Since $e > 1$, then $\sqrt{e} > 1$, so the geometric series $\sum \left(\frac{1}{\sqrt{e}} \right)^n$ has $|r| = r = \frac{1}{\sqrt{e}} < 1$, so it converges. By the direct comparison test, $\sum_{n=1}^{\infty} \left| \frac{a_n}{e^n} \right|$ converges as well. Then $\sum_{n=1}^{\infty} \frac{a_n}{e^n}$ converges absolutely, so it converges.

Solutions to Exercises 3.5 — Jump to [TABLE OF CONTENTS](#)

S-1:

$$\begin{aligned} f(1) &= \sum_{n=0}^{\infty} \left(\frac{3-1}{4} \right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \end{aligned}$$

This is a geometric series with $r = \frac{1}{2}$, so we know that it converges and

$$= \frac{1}{1 - \frac{1}{2}} = 2$$

S-2: By Theorem 3.5.11 in the text, we may differentiate our function term-by-term for all x obeying $|x - 5| < R$, where R is the radius of convergence of the power series. The series defining $f(x)$ is reminiscent of the exponential series $\sum_{n=0}^{\infty} \frac{X^n}{n!}$ of Example 3.5.5 in the text. In that example, we showed that $\sum_{n=0}^{\infty} \frac{X^n}{n!}$ has radius of convergence ∞ . Since

$$\left| \frac{(x-5)^n}{n! + 2} \right| \leq \frac{X^n}{n!} \quad \text{with } X = |x - 5|$$

the comparison test, Theorem 3.3.11 in the text, tells us that $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n!+2}$ converges for all x . So we may differentiate our function term-by-term.

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{(x-5)^n}{n!+2} \\ f'(x) &= \sum_{n=1}^{\infty} \frac{d}{dx} \left\{ \frac{(x-5)^n}{n!+2} \right\} \\ &= \sum_{n=1}^{\infty} \frac{n(x-5)^{n-1}}{n!+2} \end{aligned}$$

Keep in mind that x is our variable, and *for each term*, n is constant.

S-3: If $x = c$, then

$$\begin{aligned} f(x) &= A_a(c-c)^a + A_{a+1}(c-c)^{a+1} + A_{a+2}(c-c)^{a+2} + \cdots \\ &= A_a \cdot 0 + A_{a+1} \cdot 0 + A_{a+2} \cdot 0 + \cdots \\ &= 0 \end{aligned}$$

So, $f(x)$ converges (to the constant 0) when $x = c$. (Had we allowed $a = 0$, it would be possible for $f(x)$ to converge to a nonzero number A_0 , because we use the convention $0^0 = 1$.)

Depending on the sequence $\{A_n\}$, it's possible that $f(x)$ diverges for all $x \neq c$. For example, suppose $A_n = n!$, so $f(x) = \sum_{n=0}^{\infty} n!(x-c)^n$. If $x \neq c$, then the limit

$\lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-c)^{n+1}}{n!(x-c)^n} \right| = \lim_{n \rightarrow \infty} (n+1)|x-c|$ is infinity, since $x-c \neq 0$. So, the series diverges.

We've now shown that the series definitely converges at $x = c$, but at any other point, it may fail to converge.

S-4: According to Theorem 3.5.9 in the text, because $f(x)$ diverges somewhere, and because it converges at a point other than its centre, $f(x)$ has a positive radius of convergence R . That is, $f(x)$ converges whenever $|x-5| < R$, and it diverges whenever $|x-5| > R$.

Since we are told that the series *diverges* at $x = 11$, the statement $|11-5| < R$ must be false. That is, we must have $R \leq |11-5| = 6$.

Since we are told that the series *converges* at $x = -1$, the statement $|-1-5| > R$ must be false. That is, we must have $R \geq |-1-5| = 6$.

Therefore, $R = 6$.

S-5: (a) We give two solutions. For the first solution, we apply the ratio test for the series whose k^{th} term is $a_k = (-1)^k 2^{k+1} x^k$. Then

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} 2^{k+2} x^{k+1}}{(-1)^k 2^{k+1} x^k} \right| \\ &= \lim_{k \rightarrow \infty} |2x| = |2x| \end{aligned}$$

Therefore, by the ratio test, the series converges for all x obeying $|2x| < 1$, i.e. $|x| < \frac{1}{2}$, and diverges for all x obeying $|2x| > 1$, i.e. $|x| > \frac{1}{2}$. So the radius of convergence is $R = \frac{1}{2}$.

For the second solution we apply (3.5.2) in the text. To do so, we set $A_k = (-1)^k 2^{k+1}$ and compute

$$A = \lim_{k \rightarrow \infty} \left| \frac{A_{k+1}}{A_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} 2^{k+2}}{(-1)^k 2^{k+1}} \right| = \lim_{k \rightarrow \infty} 2 = 2$$

so that $R = \frac{1}{A} = \frac{1}{2}$, again.

(b) The series is

$$\sum_{k=0}^{\infty} (-1)^k 2^{k+1} x^k = 2 \sum_{k=0}^{\infty} (-2x)^k = 2 \sum_{k=0}^{\infty} r^k \Big|_{r=-2x} = 2 \times \frac{1}{1-r} = \frac{2}{1+2x}$$

for all $|r| = |2x| < 1$, i.e. all $|x| < \frac{1}{2}$.

S-6: We give two solutions. For the first solution, we apply the ratio test for the series whose k^{th} term is $a_k = \frac{x^k}{10^{k+1}(k+1)!}$. Then

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{x^{k+1}}{10^{k+2}(k+2)!} \cdot \frac{10^{k+1}(k+1)!}{x^k} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{10^{k+1}}{10^{k+2}} \right| \cdot \left| \frac{(k+1)!}{(k+2)!} \right| \cdot \left| \frac{x^{k+1}}{x^k} \right| \\ &= \lim_{k \rightarrow \infty} \frac{1}{10(k+2)} |x| = 0 < 1 \end{aligned}$$

for all x . Therefore, by the ratio test, the series converges for all x and the radius of convergence is $R = \infty$.

For the second solution we apply (3.5.2) in the text. To do so, we set $A_k = \frac{1}{10^{k+1}(k+1)!}$ and compute

$$A = \lim_{k \rightarrow \infty} \left| \frac{A_{k+1}}{A_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{10^{k+1}}{10^{k+2}} \right| \cdot \left| \frac{(k+1)!}{(k+2)!} \right| = 0$$

So, by the bullet just after (3.5.2) in the text, R is again $+\infty$.

S-7: We give two solutions. For the first solution, we apply the ratio test with

$$a_n = \frac{(x-2)^n}{n^2+1}.$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n^2+1}{(n+1)^2+1} |x-2| \\ &= \lim_{n \rightarrow \infty} \frac{1+1/n^2}{(1+1/n)^2+1/n^2} |x-2| \\ &= |x-2|\end{aligned}$$

So, the series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$. That is, the radius of convergence is 1.

For the second solution we apply (3.5.2) in the text. To do so, we set $A_n = \frac{1}{n^2+1}$ and compute

$$A = \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n \rightarrow \infty} \frac{n^2+1}{(n+1)^2+1} = \lim_{n \rightarrow \infty} \frac{1+1/n^2}{(1+1/n)^2+1/n^2} = 1$$

so that $R = \frac{1}{A} = 1$, again.

S-8:

To determine the radius of convergence, we apply (3.5.2) in the text. To do so, we set $A_n = \frac{(-1)^n}{\sqrt{n}}$ and compute

$$\begin{aligned}A &= \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+1/n}} \\ &= 1\end{aligned}$$

so that $R = \frac{1}{A} = 1$.

So the series must converge when $|x+2| < 1$ and must diverge when $|x+2| > 1$.

S-9:

We could determine the radius of convergence by applying (3.5.2), from the text, with $A_n = \frac{(-1)^n}{(n+1)3^n}$. As an alternate method, we apply the ratio test for the series whose n^{th}

term is $a_n = \frac{(-1)^n}{n+1} \left(\frac{x+1}{3}\right)^n$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1}}{n+2} \left(\frac{x+1}{3}\right)^{n+1}}{\frac{(-1)^n}{n+1} \left(\frac{x+1}{3}\right)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{n+1}{n+2} \cdot \frac{(x+1)^{n+1}}{(x+1)^n} \cdot \frac{3^n}{3^{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right) \cdot \left| \frac{x+1}{3} \right| = \frac{|x+1|}{3} \end{aligned}$$

Therefore, by the ratio test, the series converges when $\frac{|x+1|}{3} < 1$ and diverges when $\frac{|x+1|}{3} > 1$. In particular, it converges when

$$|x+1| < 3 \iff -3 < x+1 < 3 \iff -4 < x < 2$$

and the radius of convergence is $R = 3$.

S-10:

To determine the radius of convergence, we apply (3.5.2) in the text. To do so, we set $A_n = \frac{1}{n^{4/5}(5^n-4)}$ and compute

$$\begin{aligned} A \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| &= \lim_{n \rightarrow \infty} \frac{n^{4/5}(5^n-4)}{(n+1)^{4/5}(5^{n+1}-4)} \\ &= \lim_{n \rightarrow \infty} \frac{(1-4/5^n)}{(1+1/n)^{4/5}(5-4/5^n)} \\ &= \frac{1}{5} \end{aligned}$$

so that $R = \frac{1}{A} = 5$.

Therefore the series converges if $|x-2| < 5$ and diverges if $|x-2| > 5$.

S-11: We determine the radius of convergence by applying (3.5.2) in the text. To do so, we set $A_n = \frac{1}{n^2}$ and compute

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^2} \\ &= 1 \end{aligned}$$

So we have convergence for

$$|x + 2| < 1 \iff -1 < x + 2 < 1 \iff -3 < x < -1$$

and divergence for $|x + 2| > 1$. For $|x + 2| = 1$, i.e. for $x + 2 = \pm 1$, i.e. for $x = -3, -1$, the series reduces to $\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^2}$, which converges absolutely, because $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p = 2 > 1$. So the given series converges if and only if $-3 \leq x \leq -1$.

S-12: (a) We could determine the radius of convergence by applying (3.5.2), from the text, with $A_n = \frac{4^n}{n}$. As an alternate method, we apply the ratio test for the series whose n^{th} term is $a_n = \frac{4^n}{n}(x - 1)^n$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}(x - 1)^{n+1}/(n + 1)}{4^n(x - 1)^n/n} \right| \\ &= \lim_{n \rightarrow \infty} 4|x - 1| \frac{n}{n + 1} \\ &= 4|x - 1| \lim_{n \rightarrow \infty} \frac{n}{n + 1} = 4|x - 1| \cdot 1. \end{aligned}$$

the series converges if

$$4|x - 1| < 1 \iff -1 < 4(x - 1) < 1 \iff -\frac{1}{4} < x - 1 < \frac{1}{4} \iff \frac{3}{4} < x < \frac{5}{4}$$

and diverges if $4|x - 1| > 1$, i.e. when $|x - 1| > \frac{1}{4}$. So, $R = \frac{1}{4}$.

(b) We could determine the radius of convergence by applying (3.5.2), from the text. But to do so, we would first have to rewrite the series as $\sum_{n=1}^{\infty} \frac{8^n}{n} \left(x + \frac{1}{2}\right)^n$ to get the n^{th} term into the form $A_n(x - c)^n$ required by (3.5.2). Instead, we use the alternate method that consists of applying the ratio test with $a_n = \frac{4^n}{n}(2x + 1)^n$. Repeating the computation of part (a), just with $x - 1$ replaced by $2x + 1$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}(2x + 1)^{n+1}/(n + 1)}{4^n(2x + 1)^n/n} \right| \\ &= 4|2x + 1| \lim_{n \rightarrow \infty} \frac{n}{n + 1} = 4|2x + 1| \cdot 1. \end{aligned}$$

So the series converges if

$$\begin{aligned} 4|2x + 1| < 1 &\iff -1 < 4(2x + 1) < 1 \\ &\iff -\frac{1}{4} < 2x + 1 < \frac{1}{4} \\ &\iff -\frac{5}{4} < 2x < -\frac{3}{4} \\ &\iff -\frac{5}{8} < x < -\frac{3}{8} \end{aligned}$$

and diverges if $4|2x + 1| > 1$. That is, when $|x + \frac{1}{2}| > \frac{1}{8}$. So, $R = \frac{1}{8}$.

(c) This time we cannot determine the radius of convergence by applying (3.5.2), from the text, because the n^{th} term is not in the form $A_n(x - c)^n$ required by (3.5.2). But we may still apply the ratio test with $a_n = \frac{4^n}{n}(2x + 1)^{2n}$. Repeating the computation of part (a), just with $x - 1$ replaced by $(2x + 1)^2$,

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}(2x + 1)^{2(n+1)} / (n + 1)}{4^n(2x + 1)^{2n} / n} \right| \\ &= 4|2x + 1|^2 \lim_{n \rightarrow \infty} \frac{n}{n + 1} = 4|2x + 1|^2 \cdot 1.\end{aligned}$$

So the series converges if

$$\begin{aligned}4|2x + 1|^2 < 1 &\iff -1 < 2(2x + 1) < 1 \\ &\iff -\frac{1}{2} < 2x + 1 < \frac{1}{2} \\ &\iff -\frac{3}{2} < 2x < -\frac{1}{2} \\ &\iff -\frac{3}{4} < x < -\frac{1}{4}\end{aligned}$$

and diverges if $4|2x + 1|^2 > 1$. That is, when $|x + \frac{1}{2}| > \frac{1}{4}$. So, $R = \frac{1}{4}$.

S-13: We determine the radius of convergence by applying (3.5.2) in the text. To do so, we set $A_n = \frac{(-1)^n}{2^n(n+2)}$ and compute

$$\begin{aligned}A &= \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2^n(n+2)}{2^{n+1}(n+3)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \frac{n+2}{n+3} \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1 + 2/n}{1 + 3/n} = \frac{1}{2}\end{aligned}$$

So the radius of convergence is $R = \frac{1}{A} = 2$ and the series converges if

$$|x - 1| < 2 \iff -2 < (x - 1) < 2 \iff -1 < x < 3$$

and diverges if $|x - 1| > 2$.

S-14: We cannot determine the radius of convergence by applying (3.5.2), from the text, because the n^{th} term is not in the form $A_n(x - c)^n$ required by (3.5.2). But we may still apply the ratio test with $a_n = (-1)^n n^2 (x - a)^{2n}$. Since

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}(n+1)^2(x-a)^{2(n+1)}}{(-1)^n n^2 (x-a)^{2n}} \right| \\ &= \lim_{n \rightarrow \infty} |x-a|^2 \frac{(n+1)^2}{n^2} \\ &= |x-a|^2 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = |x-a|^2 \cdot 1.\end{aligned}$$

the series converges if

$$|x - a|^2 < 1 \iff |x - a| < 1 \iff -1 < x - a < 1 \iff a - 1 < x < a + 1$$

and diverges if $|x - a| > 1$. So, the radius of convergence is 1.

S-15: (a) We could determine the radius of convergence by applying (3.5.2), from the text, with $A_n = \frac{1}{n^2 9^n}$. As an alternate method, we apply the ratio test for the series whose k^{th} term is $a_k = \frac{(x+1)^k}{k^2 9^k}$. Then

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{(x+1)^{k+1}}{(k+1)^2 9^{k+1}} \frac{k^2 9^k}{(x+1)^k} \right| \\ &= \lim_{k \rightarrow \infty} |x+1| \frac{1}{9} \frac{k^2}{(k+1)^2} \\ &= \lim_{k \rightarrow \infty} |x+1| \frac{1}{9} \frac{1}{(1+1/k)^2} \\ &= \frac{|x+1|}{9} \end{aligned}$$

So the series must converge when $|x+1| < 9$ and must diverge when $|x+1| > 9$. So, $R = 9$

(b) The partial sum

$$\sum_{k=1}^N \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right) = \left(\frac{a_1}{a_2} - \frac{a_2}{a_3} \right) + \left(\frac{a_2}{a_3} - \frac{a_3}{a_4} \right) + \cdots + \left(\frac{a_N}{a_{N+1}} - \frac{a_{N+1}}{a_{N+2}} \right) = \frac{a_1}{a_2} - \frac{a_{N+1}}{a_{N+2}}$$

We are told that $\sum_{k=1}^{\infty} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right) = \frac{a_1}{a_2}$. This means that the above partial sum converges to $\frac{a_1}{a_2}$ as $N \rightarrow \infty$, or equivalently, that

$$\lim_{N \rightarrow \infty} \frac{a_{N+1}}{a_{N+2}} = 0$$

and hence that

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}(x-1)^{k+1}|}{|a_k(x-1)^k|} = |x-1| \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|}$$

is infinite for any $x \neq 1$. So, by the ratio test, this series converges only for $x = 1$. Therefore, its radius of convergence is 0.

S-16: Using the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$,

$$\frac{x^3}{1-x} = x^3 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+3} = \sum_{n=3}^{\infty} x^n$$

S-17: We can find $f(x)$ by differentiating its integral, or antidifferentiating its derivative. In the latter case, we'll have to solve for the arbitrary constant of integration; in the former case, we do not. (Remember that many different functions have the same derivative, but a single function has only one derivative.) To avoid the necessity of finding the arbitrary constant, we can ignore the given equation for $f'(x)$, which makes the problem much simpler. This is the method used in Solution 1.

Solution 1 Using the Fundamental Theorem of Calculus Part 1:

$$\begin{aligned}\frac{d}{dx} \left\{ \int_5^x f(t) dt \right\} &= f(x) \\ \text{So, } f(x) &= \frac{d}{dx} \left\{ 3x + \sum_{n=0}^{\infty} \frac{(x-1)^{n+1}}{n(n+1)^2} \right\} \\ &= 3 + \sum_{n=1}^{\infty} \frac{(n+1)(x-1)^n}{n(n+1)^2} \\ &= 3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)}\end{aligned}$$

Solution 2 Suppose we had used $f'(x)$ instead. We would antidifferentiate to find:

$$\begin{aligned}f(x) &= \int \left(\sum_{n=0}^{\infty} \frac{(x-1)^n}{n+2} \right) dx \\ &= \left(\sum_{n=0}^{\infty} \frac{(x-1)^{n+1}}{(n+1)(n+2)} \right) + C \\ &= \left(\sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)} \right) + C\end{aligned}$$

Notice $f(1) = 0 + C$. So, to find C , we must find $f(1)$. We can't get that information from $f'(x)$, so our only option is to consider the given formula for $\int_5^x f(t) dt$. Using the Fundamental Theorem of Calculus Part 1:

$$\begin{aligned}f(1) &= \frac{d}{dx} \left\{ \int_5^x f(t) dt \right\} \Big|_{x=1} \\ &= \frac{d}{dx} \left\{ 3x + \sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n(n+1)^2} \right\} \Big|_{x=1} \\ &= \left[3 + \sum_{n=1}^{\infty} \frac{(n+1)(x-1)^n}{n(n+1)^2} \right]_{x=1} \\ &= \left[3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)} \right]_{x=1} \\ &= 3 + \sum_{n=1}^{\infty} \frac{0^n}{n(n+1)} \\ &= 3\end{aligned}$$

$$\text{So, } f(x) = 3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)}.$$

Note that in Solution 2, we did the same calculation as Solution 1, and more.

S-18: We'll start by applying the ratio test twice,

- once for the series whose n^{th} term is $a_n = \frac{x^n}{3^{2n} \log n}$, and
- once for the series whose n^{th} term is $a_n = \left| \frac{x^n}{3^{2n} \log n} \right|$.

For both series, i.e. for both a_n 's,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{3^{2(n+1)} \log(n+1)} \cdot \frac{3^{2n} \log n}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x \log n}{3^2 \log(n+1)} \right| = \lim_{n \rightarrow \infty} \left| \frac{x \log n}{3^2 [\log(n) + \log(1 + 1/n)]} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{3^2 [1 + \log(1 + 1/n) / \log(n)]} \right| \\ &= \frac{|x|}{9} \end{aligned}$$

Therefore, by the ratio test, when $|x| < 9$, the series $\sum_{n=2}^{\infty} \left| \frac{x^n}{3^{2n} \log n} \right|$ converges and the series $\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \log n}$ converges absolutely, and when $|x| > 9$ the series $\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \log n}$ diverges. That just leaves $x = \pm 9$.

For $x = -9$, $\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \log n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$ which converges by the alternating series test.

For $x = +9$, $\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \log n} = \sum_{n=2}^{\infty} \frac{1}{\log n}$ which is the same series as $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\log n} \right|$. We shall shortly show that $n \geq \log n$, and hence $\frac{1}{\log n} \geq \frac{1}{n}$ for all $n \geq 1$. This implies that the series $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges by comparison with the divergent series $\sum_{n=2}^{\infty} \frac{1}{n^p} \Big|_{p=1}$. This yields both divergence for $x = 9$ and also the failure of absolute convergence for $x = -9$.

Finally, we show that $n - \log n > 0$, for all $n \geq 1$. Set $f(x) = x - \log x$. Then $f(1) = 1 > 0$ and

$$f'(x) = 1 - \frac{1}{x} \geq 0 \quad \text{for all } x \geq 1$$

So $f(x)$ is (strictly) positive when $x = 1$ and is increasing for all $x \geq 1$. So $f(x)$ is (strictly) positive for all $x \geq 1$.

S-19: (a) Applying $\frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n$ with $r = x^3$ gives

$$\int \frac{1}{1+x^3} dx = \sum_{n=0}^{\infty} (-1)^n \int x^{3n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} + C$$

(b) By part (a),

$$\int_0^{1/4} \frac{1}{1+x^3} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} \Big|_0^{1/4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(3n+1)4^{3n+1}}$$

This is an alternating series with successively smaller terms that converge to zero as $n \rightarrow \infty$. So truncating it introduces an error no larger than the magnitude of the first dropped term. We want that first dropped term to obey

$$\frac{1}{(3n+1)4^{3n+1}} < 10^{-5} = \frac{1}{10^5}$$

So let's check the first few terms.

$$\begin{aligned} \left. \frac{1}{(3n+1)4^{3n+1}} \right|_{n=0} &= \frac{1}{4} > \frac{1}{10^5} \\ \left. \frac{1}{(3n+1)4^{3n+1}} \right|_{n=1} &= \frac{1}{4^5} > \frac{1}{10^5} \\ \left. \frac{1}{(3n+1)4^{3n+1}} \right|_{n=2} &= \frac{1}{7 \times 4^7} = \frac{1}{7 \times 2^{14}} = \frac{1}{7 \times 16 \times 1024} = \frac{1}{112 \times 1024} < \frac{1}{10^5} \end{aligned}$$

So we need to keep two terms (the $n = 0$ and $n = 1$ terms).

S-20: (a) Differentiating both sides of

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

gives

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

Now multiplying both sides by x gives

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

as desired.

(b) Differentiating both sides of the conclusion of part (a) gives

$$\sum_{n=0}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2 - 2x(x-1)}{(1-x)^4} = \frac{(1-x)(1-x+2x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

Now multiplying both sides by x gives

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$

We know that differentiation preserves the radius of convergence of power series. So this series has radius of convergence 1 (the radius of convergence of the original geometric series). At $x = \pm 1$ the series diverges by the divergence test. So the series converges for $-1 < x < 1$.

S-21: By the divergence test, the fact that $\sum_{n=0}^{\infty} (1 - b_n)$ converges guarantees that $\lim_{n \rightarrow \infty} (1 - b_n) = 0$, or equivalently, that $\lim_{n \rightarrow \infty} b_n = 1$. So, by equation (3.5.2) in the text, the radius of convergence is

$$R = \left[\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| \right]^{-1} = \left[\frac{1}{1} \right]^{-1} = 1 \quad (3.6.1)$$

S-22: We know that the radius of convergence R obeys

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} \frac{(n+1)a_{n+1}}{na_n} = 1 \frac{C}{C} = 1$$

because we are told that $\lim_{n \rightarrow \infty} na_n = C$. So $R = 1$.

S-23: First, we differentiate.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n (x-c)^n \\ f'(x) &= \sum_{n=0}^{\infty} n A_n (x-c)^{n-1} \\ &= \sum_{n=1}^{\infty} n A_n (x-c)^{n-1} \\ f'(c) &= \sum_{n=1}^{\infty} n A_n \cdot 0^{n-1} \\ &= A_1 \cdot 1 + 2A_2 \cdot 0 + 3A_3 \cdot 0 + \cdots \\ &= A_1 \end{aligned}$$

So, if $A_1 = 0$, then $f'(c) = 0$. That is, $f(x)$ has a critical point at $x = c$.

To determine the behaviour of this critical point, we use the second derivative test.

$$\begin{aligned}
 f'(x) &= \sum_{n=1}^{\infty} nA_n(x-c)^{n-1} \\
 f''(x) &= \sum_{n=1}^{\infty} n(n-1)A_n(x-c)^{n-2} \\
 &= \sum_{n=2}^{\infty} n(n-1)A_n(x-c)^{n-2} \\
 f''(c) &= \sum_{n=2}^{\infty} n(n-1)A_n \cdot 0^{n-2} \\
 &= 2(1)A_2 \cdot 0^0 + 3(2)A_3 \cdot 0^1 + 4(3)A_4 \cdot 0^2 + \dots \\
 &= 2A_2
 \end{aligned}$$

Following the second derivative test, $x = c$ is the location of a local maximum if $A_2 < 0$, and it is the location of a local minimum if $A_2 > 0$. (If $A_2 = 0$, the critical point may or may not be a local extremum.)

S-24: We recognize $\sum_{n=3}^{\infty} \frac{n}{5^{n-1}}$ as $f(x) = \sum_{n=3}^{\infty} n \cdot x^{n-1}$, evaluated at $x = \frac{1}{5}$. We should figure out what $f(x)$ is in equation form (as opposed to power series form). Notice that this looks similar to the derivative of the geometric series $\sum x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{when } |x| < 1$$

$$\frac{d}{dx} \left\{ \frac{1}{1-x} \right\} = \frac{d}{dx} \left\{ \sum_{n=0}^{\infty} x^n \right\}$$

$$\begin{aligned} \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1} \\ &= 1x^0 + 2x^1 + \sum_{n=3}^{\infty} nx^{n-1} \\ &= 1 + 2x + \sum_{n=3}^{\infty} nx^{n-1} \end{aligned}$$

$$\text{So,} \quad \frac{1}{(1-x)^2} - 1 - 2x = \sum_{n=3}^{\infty} nx^{n-1}$$

$$\begin{aligned} \text{Setting } x = \frac{1}{5}: \quad \frac{1}{(1-1/5)^2} - 1 - \frac{2}{5} &= \sum_{n=3}^{\infty} n \left(\frac{1}{5} \right)^{n-1} \\ \left(\frac{5}{4} \right)^2 - 1 - \frac{2}{5} &= \sum_{n=3}^{\infty} \frac{n}{5^{n-1}} \end{aligned}$$

So, our series evaluates to $\frac{25}{16} - 1 - \frac{2}{5} = \frac{13}{80}$.

S-25: As we saw in in Example 3.5.18 of the text,

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

which is an alternating series when x is positive. If we use its partial sum S_N to approximate $\log(1+x)$, the absolute error involved is no more than

$$\frac{x^{(N+1)+1}}{(N+1)+1} = \frac{x^{N+2}}{N+2}$$

We want this error to be at most 10^{-5} whenever $0 < x < \frac{1}{10}$. For this range of x values, $\frac{x^{N+2}}{N+2} < \frac{1}{(N+2)10^{N+2}}$, so we want N that satisfies the inequality:

$$\begin{aligned} \frac{1}{(N+2)10^{N+2}} &\leq \frac{1}{10^5} \\ \Rightarrow (N+2)10^{N+2} &\geq 10^5 \end{aligned}$$

We see $N = 3$ suffices.

So, the partial sum

$$\sum_{n=0}^3 (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

approximates $\log(1+x)$ to within an error of $\frac{x^5}{5}$.

When x is between 0 and $\frac{1}{10}$, that error is at most $\frac{1}{5 \cdot 10^5} < 10^{-5}$, as desired.

Now we can approximate $\log(1.05)$.

$$\begin{aligned} \log(1.05) &= \log\left(1 + \frac{1}{20}\right) \\ &\approx \left(\frac{1}{20}\right) - \frac{\left(\frac{1}{20}\right)^2}{2} + \frac{\left(\frac{1}{20}\right)^3}{3} - \frac{\left(\frac{1}{20}\right)^4}{4} \\ &= \frac{12 \times 20^3 - 6 \times 20^2 + 4 \times 20 - 3}{12 \times 20^4} = \frac{93677}{1920000} \end{aligned}$$

We note that a computer approximates $\frac{93677}{1920000} \approx 0.04879010$ and $\log(1.05) \approx 0.04879016$. So, our actual error is around 6×10^{-8} .

S-26: As we saw in in Example 3.5.19 of the text,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

which is an alternating series when x is nonzero. If we use its partial sum S_N to approximate $\arctan x$, the absolute error involved is no more than

$$\frac{|x|^{2(N+1)+1}}{2(N+1)+1} = \frac{|x|^{2N+3}}{2N+3}$$

We want this error to be at most 10^{-5} whenever $-\frac{1}{4} < x < \frac{1}{4}$. For this range of x values, $\frac{|x|^{2N+3}}{2N+3} < \frac{1}{(2N+3)4^{2N+3}}$, so we want N that satisfies the inequality:

$$\frac{1}{(2N+3)4^{2N+3}} \leq \frac{1}{10^5}$$

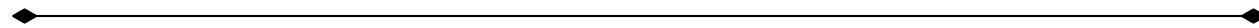
A quick check with a calculator shows that $N = 2$ suffices.

So, the partial sum

$$\sum_{n=0}^2 (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5}$$

approximates $\arctan x$ to within an error of $\frac{x^7}{7}$.

When x is between $-\frac{1}{4}$ and $\frac{1}{4}$, that error is at most $\frac{1}{7 \cdot 4^7} = \frac{1}{114688} < \frac{1}{100000} = 10^{-5}$, as desired. (When $x = 0$, our approximation is 0, the exact value of $\arctan 0$.)



Solutions to Exercises 3.6 — Jump to [TABLE OF CONTENTS](#)

S-1: All functions A , B , and C intersect the function $y = f(x)$ when $x = 2$. B is a constant function, so this is the constant approximation. A is the tangent line, so A is the linear approximation. C is a tangent parabola, so C is the quadratic approximation.

S-2: Following how a Taylor series is constructed, the Taylor series and the function agree at the point chosen as the centre. So, $T(5) = \arctan^3(e^5 + 7)$.

If we were evaluating a Taylor series at a point *other than its centre*, we would generally need to check that (a) the series converges, and (b) it converges to the same value as the function we used to create it.

S-3: These are listed in Theorem 3.6.5 in the text. However, it's possible to figure out many of them without a lot of memorization. For example, $e^0 = \cos(0) = \frac{1}{1-0} = 1$, while $\sin(0) = \log(1+0) = \arctan(0) = 0$. So by plugging in $x = 0$ to the series listed, we can divide them into these two categories.

The derivative of sine is cosine, so we can also look for one series that is the derivative of another. The derivative of e^x is e^x , so we can look for a series that is its own derivative.

Furthermore, sine and arctangent are odd functions and only II and IV are odd. Cosine is an even function and only III is even.

Alternately, we can find the first few terms of each series using the definition of a Taylor series, and match them up.

In any event, here is what Theorem 3.6.5 in the text gives.

- (A) The Taylor series representation of $\frac{1}{1-x}$ is given in V. The series converges for $-1 < x < 1$. So by Definition 3.5.3 in the text, the series has radius of convergence 1.
- (B) The Taylor series representation of $\log(1+x)$ is given in I. The series converges for $-1 < x \leq 1$. In particular, it converges for all $|x| < 1$ and diverges for all $|x| > 1$. So by Definition 3.5.3 in the text, the series has radius of convergence 1.
- (C) The Taylor series representation of $\arctan x$ is given in IV. The series converges for $-1 \leq x \leq 1$. In particular, it converges for all $|x| < 1$ and diverges for all $|x| > 1$. So it has radius of convergence 1.
- (D) The Taylor series representation of e^x is given in VI. The series converges for all x . So it has infinite radius of convergence.

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- (E) The Taylor series representation of $\sin x$ is given in II. The series converges for all x . So it has infinite radius of convergence.
- (F) The Taylor series representation of $\cos x$ is given in III. The series converges for all x . So it has infinite radius of convergence.

S-4:

- (a) Using the definition of a Taylor series, we know

$$\sum_{n=0}^{\infty} \frac{n^2}{(n! + 1)} (x - 3)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} (x - 3)^n$$

So, the coefficient of $(x - 3)^{20}$ is $\frac{f^{(20)}(3)}{20!}$ (using the definition). Using the given series, the coefficient of $(x - 3)^{20}$ is $\frac{20^2}{20! + 1}$. So,

$$\begin{aligned} \frac{f^{(20)}(3)}{20!} &= \frac{20^2}{20! + 1} \\ \Rightarrow f^{(20)}(3) &= 20^2 \left(\frac{20!}{20! + 1} \right) \end{aligned}$$

(which is extremely close to 20^2).

- (b) Using the definition of a Taylor series, we know

$$\sum_{n=0}^{\infty} \frac{n^2}{(n! + 1)} (x - 3)^{2n} = \sum_{k=0}^{\infty} \frac{g^{(k)}(3)}{k!} (x - 3)^k$$

So, the coefficient of $(x - 3)^{20}$ is $\frac{g^{(20)}(3)}{20!}$ (using the definition). Looking at the given series, the coefficient of $(x - 3)^{20}$ occurs when $n = 10$, so it is $\frac{10^2}{10! + 1}$. So,

$$\begin{aligned} \frac{g^{(20)}(3)}{20!} &= \frac{10^2}{10! + 1} \\ \Rightarrow g^{(20)}(3) &= 10^2 \left(\frac{20!}{10! + 1} \right) \end{aligned}$$

- (c) With the previous two examples in mind, we find the Maclaurin series for $h(x)$. (Using the series representation will be much easier than differentiating $h(x)$ directly

twenty times.) Recall from the text that we know the Maclaurin series for $\arctan x$.

$$\begin{aligned}\arctan(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\ \arctan(5x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(5x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{4n+2} \\ \frac{\arctan(5x^2)}{x^2} &= \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{4n} \\ \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k &= \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{4n}\end{aligned}$$

Using the definition of a Maclaurin series, the coefficient of x^{20} is $\frac{h^{(20)}(0)}{20!}$. This occurs in the given series when $n = 5$, so

$$\begin{aligned}\frac{h^{(20)}(0)}{20!} &= (-1)^5 \frac{5^{2 \times 5 + 1}}{2 \times 5 + 1} = -\frac{5^{11}}{11} \\ \Rightarrow h^{(20)}(0) &= -\frac{20! \cdot 5^{11}}{11}\end{aligned}$$

Similarly, the coefficient of x^{22} in the Maclaurin series is $\frac{h^{(22)}(0)}{22!}$. Since no term x^{22} occurs in our series, that coefficient is 0, so $h^{(22)}(0) = 0$.

S-5: The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n$$

That is, we need a general description of $f^{(n)}(1)$. To find this, we take a few derivatives, and look for a pattern.

$f(x) = \log(x)$	$f(1) = 0$
$f'(x) = x^{-1}$	$f'(1) = 1$
$f''(x) = (-1)x^{-2}$	$f''(1) = -1$
$f^{(3)}(x) = (-2)(-1)x^{-3}$	$f^{(3)}(1) = 2!$
$f^{(4)}(x) = (-3)(-2)(-1)x^{-4}$	$f^{(4)}(1) = -3!$
$f^{(5)}(x) = (-4)(-3)(-2)(-1)x^{-5}$	$f^{(5)}(1) = 4!$
$f^{(6)}(x) = (-5)(-4)(-3)(-2)(-1)x^{-6}$	$f^{(6)}(1) = -5!$
$\vdots$	$\vdots$
$f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$	$f^{(n)}(1) = (-1)^{n-1}(n-1)!$

Using the convention $0! = 1$, our pattern for $f^{(n)}(1)$ begins when $n = 1$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$$

S-6: To find the Taylor series for sine, centred at $a = \pi$, we'll need to know the various derivatives of sine at π .

$f(x) = \sin x$	$f(\pi) = 0$
$f'(x) = \cos x$	$f'(\pi) = -1$
$f''(x) = -\sin x$	$f''(\pi) = 0$
$f'''(x) = -\cos x$	$f'''(\pi) = 1$
$f^{(4)}(x) = \sin x = f(x)$	$f^{(4)}(\pi) = 0$

Even derivatives are 0; odd derivatives alternate between -1 and $+1$. (If you're following along with the derivation of the Maclaurin series for sine in the text, note $f^{(n)}(\pi) = -f^{(n)}(0)$.)

In our Taylor series, every even-indexed term will be zero, and we will be left with only odd-indexed terms. If we let n be our index, then the term $2n + 1$ will capture all the odd numbers. Since the signs alternate, $f^{(2n+1)}(\pi) = (-1)^{n+1}$. So, our Taylor series is:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi)}{k!} (x-\pi)^k &= \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(\pi)}{(2n+1)!} (x-\pi)^{2n+1} && \text{(since the even terms are all zero)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1} \end{aligned}$$

S-7: The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(10)}{n!} (x-10)^n$$

That is, we need a general description of $g^{(n)}(10)$. To find this, we take a few derivatives, and look for a pattern.

$g(x) = x^{-1}$	$g(10) = \frac{1}{10}$
$g'(x) = (-1)x^{-2}$	$g'(10) = \frac{-1}{10^2}$
$g''(x) = (-2)(-1)x^{-3}$	$g''(10) = \frac{(-1)^2 2!}{10^3}$
$g^{(3)}(x) = (-3)(-2)(-1)x^{-4}$	$g^{(3)}(10) = \frac{(-1)^3 3!}{10^4}$
$g^{(4)}(x) = (-4)(-3)(-2)(-1)x^{-5}$	$g^{(4)}(10) = \frac{(-1)^4 4!}{10^5}$
$g^{(5)}(x) = (-5)(-4)(-3)(-2)(-1)x^{-6}$	$g^{(5)}(10) = \frac{(-1)^5 5!}{10^6}$
$\vdots$	$\vdots$
$g^{(n)}(x) = (-1)^n n! x^{-(n+1)}$	$g^{(n)}(10) = \frac{(-1)^n n!}{10^{n+1}}$

Using the convention $0! = 1$, our pattern for $g^{(n)}(10)$ begins when $n = 0$.

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{g^{(n)}(10)}{n!} (x-10)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n! 10^{n+1}} (x-10)^n \\
 &= - \sum_{n=0}^{\infty} \frac{(x-10)^n}{(-10)^{n+1}} \\
 &= \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{10-x}{10} \right)^n
 \end{aligned}$$

For fixed x , we recognize this as a geometric series with $r = \frac{10-x}{10}$. So it converges precisely when $|r| < 1$, i.e.

$$\begin{aligned}
 \left| \frac{10-x}{10} \right| &< 1 \\
 |10-x| &< 10
 \end{aligned}$$

So, its radius of convergence is 10.

S-8: The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!} (x-a)^n$$

That is, we need a general description of $h^{(n)}(a)$. To find this, we take a few derivatives, and look for a pattern.

$$\begin{array}{ll}
h(x) = e^{3x} & h(a) = e^{3a} \\
h'(x) = 3e^{3x} & h'(a) = 3e^{3a} \\
h''(x) = 3^2 e^{3x} & h''(a) = 3^2 e^{3a} \\
h'''(x) = 3^3 e^{3x} & h'''(a) = 3^3 e^{3a} \\
\vdots & \vdots \\
h^{(n)}(x) = 3^n e^{3x} & h^{(n)}(a) = 3^n e^{3a}
\end{array}$$

The pattern for $h^{(n)}(a)$ holds for all (whole numbers) $n \geq 0$. So, our Taylor series for $h(x)$ is

$$\sum_{n=0}^{\infty} \frac{3^n e^{3a}}{n!} (x-a)^n$$

To find its radius of convergence, we use the ratio test.

$$\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{3^{n+1} e^{3a} (x-a)^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n e^{3a} (x-a)^n} \right| \\
&= \left| \frac{3^{n+1}}{3^n} \cdot \frac{e^{3a}}{e^{3a}} \cdot \frac{n!}{(n+1)!} \cdot \frac{(x-a)^{n+1}}{(x-a)^n} \right| \\
&= 3 \cdot \frac{1}{n+1} \cdot |x-a| \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[\frac{3}{n+1} \cdot |x-a| \right] = 0
\end{aligned}$$

Our series converges for every value of x , so its radius of convergence is ∞ .

S-9: Substituting $y = 2x$ into $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is valid for all $-1 < y < 1$) gives

$$f(x) = \frac{1}{2x-1} = -\frac{1}{1-2x} = -\sum_{n=0}^{\infty} (2x)^n = -\sum_{n=0}^{\infty} 2^n x^n \quad \text{for all } -\frac{1}{2} < x < \frac{1}{2}$$

S-10: Substituting first $y = -x$ and then $y = 2x$ into $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is valid for all $-1 < y < 1$) gives

$$\begin{aligned}
\frac{1}{1-(-x)} &= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{for all } -1 < x < 1 \\
\frac{1}{1-(2x)} &= \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n \quad \text{for all } -\frac{1}{2} < x < \frac{1}{2}
\end{aligned}$$

Hence, for all $-\frac{1}{2} < x < \frac{1}{2}$,

$$\begin{aligned} f(x) &= \frac{3}{x+1} - \frac{1}{2x-1} = \frac{3}{1-(-x)} + \frac{1}{1-2x} = 3 \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} 2^n x^n \\ &= \sum_{n=0}^{\infty} (3(-1)^n + 2^n) x^n \end{aligned}$$

So $b_n = 3(-1)^n + 2^n$.

S-11: We found the Taylor series for e^{3x} from scratch in Question 8. If we hadn't just done that, we could easily find it by modifying the series for e^x .

Substituting $y = 3x$ into the exponential series

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

gives

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

so that c_5 , the coefficient of x^5 , which appears only in the $n = 5$ term, is $c_5 = \frac{3^5}{5!}$

S-12: Since

$$f'(t) = \frac{d}{dt} \log(1+2t) = \frac{2}{1+2t} = 2 \sum_{n=0}^{\infty} (-2t)^n \quad \text{if } |2t| < 1 \text{ i.e. } |t| < \frac{1}{2}$$

and $f(0) = 0$, we have

$$f(x) = \int_0^x f'(t) dt = 2 \sum_{n=0}^{\infty} \int_0^x (-1)^n 2^n t^n dt = \sum_{n=0}^{\infty} (-1)^n 2^{n+1} \frac{x^{n+1}}{n+1} \quad \text{for all } |x| < \frac{1}{2}$$

S-13: We just need to substitute $y = x^3$ into the known Maclaurin series for $\sin y$, to get the Maclaurin series for $\sin(x^3)$, and then multiply the result by x^2 .

$$\begin{aligned} \sin y &= y - \frac{y^3}{3!} + \cdots \\ \sin(x^3) &= x^3 - \frac{x^9}{3!} + \cdots \\ x^2 \sin(x^3) &= x^5 - \frac{x^{11}}{3!} + \cdots \end{aligned}$$

so $a = 1$ and $b = -\frac{1}{3!} = -\frac{1}{6}$.

S-14: Recall that

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \cdots$$

Setting $y = -x^2$, we have

$$\begin{aligned} e^{-x^2} &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots \\ e^{-x^2} - 1 &= -x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots \\ \frac{e^{-x^2} - 1}{x} &= -x + \frac{x^3}{2} - \frac{x^5}{6} + \cdots \\ \int \frac{e^{-x^2} - 1}{x} dx &= C - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{36} + \cdots \end{aligned}$$

S-15: Recall that

$$\arctan(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{2n+1}$$

Setting $y = 2x$, we have

$$\begin{aligned} \int x^4 \arctan(2x) dx &= \int \left(x^4 \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} \right) dx \\ &= \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+5}}{2n+1} \right) dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+6}}{(2n+1)(2n+6)} + C \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+6}}{(2n+1)(n+3)} + C \end{aligned}$$

S-16: Substituting $y = -3x^3$ into $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ gives

$$\frac{df}{dx} = x \cdot \frac{1}{1+3x^3} = x \sum_{n=0}^{\infty} (-3x^3)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+1}$$

Now integrating,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{3n+2}}{3n+2} + C$$

To have $f(0) = 1$, we need $C = 1$. So, finally

$$f(x) = 1 + \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{3n+2} x^{3n+2}$$

S-17: We're given a big hint: that our series resembles the Taylor series for arctangent.

The terms of arctangent are $(-1)^n \frac{x^{2n+1}}{2n+1}$. Our terms resemble those terms, with x^{2n+1} replaced by $\frac{1}{3^n}$.

Since $3^n = (\sqrt{3})^{2n} = \frac{1}{\sqrt{3}} (\sqrt{3})^{2n+1}$:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} &= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \sqrt{3} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \Big|_{x=\frac{1}{\sqrt{3}}} = \sqrt{3} \arctan \frac{1}{\sqrt{3}} \\ &= \sqrt{3} \frac{\pi}{6} = \frac{\pi}{2\sqrt{3}} \end{aligned}$$

S-18: Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{x^n}{n!} \right]_{x=-1} = [e^x]_{x=-1} = e^{-1}$$

S-19: Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. So

$$\sum_{k=0}^{\infty} \frac{1}{e^k k!} = \left[\sum_{k=0}^{\infty} \frac{x^k}{k!} \right]_{x=1/e} = [e^x]_{x=1/e} = e^{1/e}$$

S-20: Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. So

$$\sum_{k=0}^{\infty} \frac{1}{\pi^k k!} = \left[\sum_{k=0}^{\infty} \frac{x^k}{k!} \right]_{x=1/\pi} = [e^x]_{x=1/\pi} = e^{1/\pi}$$

This series differs from the given one only in that it starts with $k = 0$ while the given series starts with $k = 1$. So

$$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = \sum_{k=0}^{\infty} \frac{1}{\pi^k k!} - \underbrace{1}_{k=0} = e^{1/\pi} - 1$$

S-21: Recall, from Theorem 3.6.5 in the text, that, for all $-1 < x \leq 1$,

$$\log(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

(To get from the first sum to the second sum we substituted $n = k + 1$. If you don't see why the two sums are equal, write out the first few terms of each.) So

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n} = \left[\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \right]_{x=1/2} = \left[\log(1+x) \right]_{x=1/2} = \log(3/2)$$

S-22: Write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n+2}{n!} e^n &= \sum_{n=1}^{\infty} \frac{n}{n!} e^n + \sum_{n=1}^{\infty} \frac{2}{n!} e^n \\ &= \sum_{n=1}^{\infty} \frac{e^n}{(n-1)!} + 2 \sum_{n=1}^{\infty} \frac{e^n}{n!} \\ &= e \sum_{n=1}^{\infty} \frac{e^{n-1}}{(n-1)!} + 2 \sum_{n=1}^{\infty} \frac{e^n}{n!} \\ &= e \sum_{n=0}^{\infty} \frac{e^n}{n!} + 2 \sum_{n=1}^{\infty} \frac{e^n}{n!} \end{aligned}$$

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n+2}{n!} e^n &= e \left[\sum_{n=0}^{\infty} \frac{x^n}{n!} \right]_{x=e} + 2 \left[\sum_{n=1}^{\infty} \frac{x^n}{n!} \right]_{x=e} = e \left[e^x \right]_{x=e} + 2 \left[e^x - 1 \right]_{x=e} = e^{e+1} + 2(e^e - 1) \\ &= (e+2)e^e - 2 \end{aligned}$$

S-23: Let's use the ratio test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} \right| \\ &= \lim_{n \rightarrow \infty} 2 \frac{n}{n+1} = 2 > 1 \end{aligned}$$

So, the series diverges.

Remark: it's tempting to note that $\log(1+y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{n+1}}{n+1} = - \sum_{n=1}^{\infty} \frac{(-y)^n}{n}$, and try to substitute in $y = -2$. But, the Maclaurin series for $\log(1+y)$ has radius of convergence $R = 1$, so it doesn't converge at $y = -2$. Furthermore, $\log(1+(-2)) = \log(-1)$, but this is undefined.

S-24: Our series looks something like the Taylor series for sine,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} (1 + 2^{2n+1}) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\left(\frac{\pi}{4}\right)^{2n+1} + \left(\frac{\pi}{2}\right)^{2n+1} \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n+1} \\ &= \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) \\ &= \frac{1}{\sqrt{2}} + 1 = \frac{1 + \sqrt{2}}{\sqrt{2}} \end{aligned}$$

S-25: (a)

Solution 1: The naive strategy is to set $a_n = \frac{x^{2n}}{(2n)!}$ and apply the ratio test.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}} \right| = \left| \frac{x^{2n+2}}{x^{2n}} \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{x^2}{(2n+2)(2n+1)} \\ &= 0 \end{aligned}$$

This is smaller than 1 no matter what x is. So the series converges for all x .

Solution 2: Alternatively, the sneaky way is to observe that both $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and

$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$ are known to converge for all x . So

$$\frac{1}{2}(e^x + e^{-x}) = \sum_{n \text{ even}} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

also converges for all x .

(b) Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then:

$$\begin{aligned} e &= \sum_{n=0}^{\infty} \frac{1}{n!} \\ e^{-1} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \\ e + e^{-1} &= \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n!} = 2 \sum_{n \text{ even}} \frac{1}{n!} = 2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \end{aligned}$$

Hence $\sum_{n=0}^{\infty} \frac{1}{(2n)!} = \frac{1}{2} \left(e + \frac{1}{e} \right)$.

S-26: All three series we're adding up are alternating, so we can bound the absolute error in the approximation S_N (the N -th partial sum) by $|a_{N+1}|$.

The Taylor series for arctangent is

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for every real x .

(a) Using the Taylor series for arctangent when $x = 1$, we see

$$\begin{aligned} \frac{\pi}{4} &= \arctan(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \\ \pi &= \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1} \end{aligned}$$

The error involved in approximating π with the partial sum S_N is at most $|a_{N+1}| = \frac{4}{2N+3}$. In order for this to be at most 4×10^{-5} , we need:

$$\begin{aligned} \frac{4}{2N+3} &\leq 4 \times 10^{-5} \\ 2N+3 &\geq 10^5 \\ N &\geq \frac{10^5 - 3}{2} = 5 \times 10^4 - \frac{3}{2} = 50,000 - 1.5 \end{aligned}$$

Since n must be an integer, we need to add up the terms from $n = 0$ to $n = 49,999$. That is, we add up the first 50,000 terms.

(b) Using the Taylor series for arctangent:

$$\begin{aligned} \pi &= 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239} = 16 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)5^{2n+1}} - 4 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1) \cdot 239^{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{16}{5^{2n+1}} - \frac{4}{239^{2n+1}} \right) \end{aligned}$$

This is an alternating sum, so the absolute error in using the partial sum S_N is at most:

$$|a_{N+1}| = \frac{1}{2N+3} \left(\frac{16}{5^{2N+3}} - \frac{4}{239^{2N+3}} \right)$$

So, we want to find a value of N that makes this at most 4×10^{-5} . Several values of N are given below.

N	$ a_{N+1} $
1	$\frac{1}{5} \left(\frac{16}{5^5} - \frac{4}{239^5} \right) \approx 0.001$
2	$\frac{1}{7} \left(\frac{16}{5^7} - \frac{4}{239^7} \right) \approx 0.000029 < 4 \times 10^{-5}$

So, it suffices to add up the first three terms ($n = 0$, $n = 1$, and $n = 2$) of the series.

(c) Again, we use the Taylor series for arctangent.

$$\begin{aligned} \arctan \frac{1}{2} + \arctan \frac{1}{3} &= \arctan \left(\frac{3+2}{2 \cdot 3 - 1} \right) = \arctan(1) = \frac{\pi}{4} \\ \pi &= 4 \left(\arctan \frac{1}{2} + \arctan \frac{1}{3} \right) \\ &= 4 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)2^{2n+1}} + 4 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)3^{2n+1}} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1} \left(\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} \right) \end{aligned}$$

If we use the partial sum S_N , our absolute error is at most

$$|a_{N+1}| = \frac{4}{2N+3} \left(\frac{1}{2^{2N+3}} + \frac{1}{3^{2N+3}} \right).$$

Several of these values are given below.

N	$ a_{N+1} $
1	$\frac{4}{5} \left(\frac{1}{2^5} + \frac{1}{3^5} \right) \approx 0.028$
2	$\frac{4}{7} \left(\frac{1}{2^7} + \frac{1}{3^7} \right) \approx 0.0047$
3	$\frac{4}{9} \left(\frac{1}{2^9} + \frac{1}{3^9} \right) \approx 0.00089$
4	$\frac{4}{11} \left(\frac{1}{2^{11}} + \frac{1}{3^{11}} \right) \approx 0.00018$
5	$\frac{4}{13} \left(\frac{1}{2^{13}} + \frac{1}{3^{13}} \right) \approx 0.000038 < 4 \times 10^{-5}$

So, it suffices to add the first six terms ($n = 0$ to $n = 5$) of the series.

Remark: if we actually wanted to approximate π this way, the series from part (a) is probably not ideal—adding 50,000 terms sounds rough. The series from (b) and (c) seem much more practical.

S-27: Using the Taylor series for $\log(1+x)$:

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$
$$\log(1.5) = \log\left(1 + \frac{1}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n2^n}$$

Since this is an alternating series, the error involved in using the partial sum S_N is at most

$$|a_{N+1}| = \frac{1}{(N+1)2^{N+1}}.$$

We want this to be at most 5×10^{-11} .

N	$ a_{N+1} $
10	$\frac{1}{11 \cdot 2^{11}} \approx 4 \times 10^{-5}$
15	$\frac{1}{16 \cdot 2^{16}} \approx 9.5 \times 10^{-7}$
20	$\frac{1}{21 \cdot 2^{21}} \approx 2 \times 10^{-8}$
25	$\frac{1}{26 \cdot 2^{26}} \approx 6 \times 10^{-10}$
26	$\frac{1}{27 \cdot 2^{27}} \approx 3 \times 10^{-10}$
27	$\frac{1}{28 \cdot 2^{28}} \approx 1 \times 10^{-10}$
28	$\frac{1}{29 \cdot 2^{29}} \approx 6 \times 10^{-11}$
29	$\frac{1}{30 \cdot 2^{30}} \approx 3 \times 10^{-11}$

So, it suffices to add up the first 29 terms.

S-28: The Taylor Series for e^x is not alternating, so we'll use Theorem 3.6.1-b in the text to bound the error in a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!} (x-a)^{N+1}$$

for some c strictly between a and x . In our case, $a = 0$ and $x = 1$. So, we want to find a value of N such that

$$\left| \frac{f^{(N+1)}(c)}{(N+1)!} (1-0)^{N+1} \right| = \frac{e^c}{(N+1)!} < 5 \times 10^{-11}$$

for all c in $(0, 1)$.

If c is between 0 and 1, then e^c is between 1 and e . However, since the purpose of this problem is to approximate e precisely, it doesn't make much sense to use e in our bound. Since e is less than 3, then $e^c < 3$ for all c in $(0, 1)$. Now we can search for an appropriate value of N .

N	$\frac{3}{(N+1)!}$
10	$\frac{3}{11!} = \frac{1}{9^{10}} \approx 8 \times 10^{-8}$
11	$\frac{3}{12!} \approx 6 \times 10^{-9}$
12	$\frac{3}{13!} \approx 5 \times 10^{-10}$
13	$\frac{3}{14!} \approx 3 \times 10^{-11}$

So, it suffices to use the partial sum S_{13} .

S-29: The Taylor Series for $\log(1-x)$ is not alternating, so we'll use Theorem 3.6.1-b in the text to bound the error in a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!} (x-a)^{N+1}$$

for some c strictly between a and x . In our case, $a = 0$ and $x = \frac{1}{10}$. So, we want to find a value of N such that

$$\left| \frac{f^{(N+1)}(c)}{(N+1)!} \left(\frac{1}{10} \right)^{N+1} \right| < 5 \times 10^{-11}$$

for *all* c in $(0, \frac{1}{10})$.

To find this N , we to know $f^{(N+1)}(x)$. Just like when we create a Taylor polynomial from scratch, we'll differentiate $f(x)$ several times, and look for a pattern.

$$\begin{array}{ll}
 f(x) = \log(1-x) & f^{(6)}(x) = \frac{-2(3)(4)(5)}{(1-x)^6} \\
 f'(x) = \frac{-1}{1-x} & f^{(7)}(x) = \frac{-2(3)(4)(5)(6)}{(1-x)^7} \\
 f''(x) = \frac{-1}{(1-x)^2} & \vdots \\
 f'''(x) = \frac{-2}{(1-x)^3} & f^{(N+1)}(x) = \frac{-N!}{(1-x)^{N+1}} \\
 f^{(4)}(x) = \frac{-2(3)}{(1-x)^4} & \\
 f^{(5)}(x) = \frac{-2(3)(4)}{(1-x)^5} &
 \end{array}$$

Now we want a reasonable bound on $f^{(N+1)}(c)$, when c is in $(0, \frac{1}{10})$. Note that in this

range, $1 - c > 0$.

$$\begin{aligned}
& 0 < c < \frac{1}{10} \\
\Rightarrow & \frac{9}{10} < 1 - c < 1 \\
\Rightarrow & \left(\frac{9}{10}\right)^{N+1} < (1 - c)^{N+1} < 1 \\
\Rightarrow & 1 < \frac{1}{(1 - c)^{N+1}} < \left(\frac{10}{9}\right)^{N+1} \\
\Rightarrow & N! < \frac{N!}{(1 - c)^{N+1}} < N! \left(\frac{10}{9}\right)^{N+1}
\end{aligned}$$

This bound provides us with a “worst-case scenario” error. We don’t know exactly what c is, but we don’t need to—the bound above holds for *all* c between 0 and $\frac{1}{10}$.

Now we’re ready to choose an N that results in a sufficiently small error bound.

$$\left| \frac{f^{(N+1)}(c)}{(N+1)!} \left(\frac{1}{10}\right)^{N+1} \right| < \frac{N! \left(\frac{10}{9}\right)^{N+1}}{(N+1)!} \left(\frac{1}{10}\right)^{N+1} = \frac{1}{9^{N+1} \cdot (N+1)}$$

So, we want: $\frac{1}{9^{N+1} \cdot (N+1)} < 5 \times 10^{-11}$

To find an appropriate N , we test several values.

N	$\frac{1}{9^{N+1} \cdot (N+1)}$
8	$\frac{1}{9 \cdot 9^9} = \frac{1}{9^{10}} \approx 3 \times 10^{-10}$
9	$\frac{1}{10 \cdot 9^{10}} \approx 3 \times 10^{-11}$

So, it suffices to use the partial sum S_9 .

S-30: We’ll use Theorem 3.6.1-b in the text to bound the error of a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!} (x - a)^{N+1}$$

for some c strictly between a and x . In our case, $a = 0$ and x is in $(-2, 1)$. So, we want to find a value of N such that

$$\left| \frac{f^{(N+1)}(c)}{(N+1)!} (x)^{N+1} \right| < 5 \times 10^{-11}$$

for *all* x in $(-2, 1)$, and *all* c in $(-2, 1)$.

To find this N , we to know $f^{(N+1)}(x)$. Just like when we create a Taylor polynomial from scratch, we'll differentiate $f(x)$ several times, and look for a pattern.

$$\begin{aligned}f(x) &= \sinh(x) = \frac{e^x - e^{-x}}{2} \\f'(x) &= \frac{e^x + e^{-x}}{2} \\f''(x) &= \frac{e^x - e^{-x}}{2} \\f'''(x) &= \frac{e^x + e^{-x}}{2}\end{aligned}$$

That is, even derivatives of $f(x)$ are $f(x)$, and odd derivatives of $f(x)$ are $\frac{e^x + e^{-x}}{2}$ (which, incidentally, is the function called $\cosh x$).

Now we want a reasonable bound on $f^{(N+1)}(c)$, when c is in $(-2, 1)$. Since powers of e are always positive, we begin by noting that $0 < \frac{e^x - e^{-x}}{2} < \frac{e^x + e^{-x}}{2}$. So, all derivatives of $f(x)$ are bounded above by $\frac{e^x + e^{-x}}{2}$.

$$\begin{aligned}& -2 < c < 1 \\ \Rightarrow & e^{-2} < e^c < e \quad \text{and} \quad e^{-1} < e^{-c} < e^2 \\ \Rightarrow & f^{(N+1)}(c) < \frac{e^c + e^{-c}}{2} < \frac{e^2 + e^2}{2} = e^2 < 9\end{aligned}$$

This bound provides us with a “worst-case scenario” error. We don't know exactly what c is, but we don't need to—the bound above holds for *all* c between -2 and 1 .

We also don't know exactly what x will be, only that it's between -2 and 1 . So, we note $|x|^{N+1} < 2^{N+1}$.

Now we're ready to choose an N that results in a sufficiently small error bound.

$$\begin{aligned}& \left| \frac{f^{(N+1)}(c)}{(N+1)!} (x)^{N+1} \right| < \frac{9 \cdot 2^{N+1}}{(N+1)!} \\ \text{So, we want:} & \frac{9 \cdot 2^{N+1}}{(N+1)!} < 5 \times 10^{-11}\end{aligned}$$

To find an appropriate N , we test several values.

N	$\frac{9 \cdot 2^{N+1}}{(N+1)!}$
10	$\frac{9 \cdot 2^{11}}{(11)!} \approx 5 \times 10^{-4}$
15	$\frac{9 \cdot 2^{16}}{(16)!} \approx 3 \times 10^{-8}$
17	$\frac{9 \cdot 2^{18}}{(18)!} \approx 4 \times 10^{-10}$
18	$\frac{9 \cdot 2^{19}}{(19)!} \approx 4 \times 10^{-11} < 5 \times 10^{-11}$

So, it suffices to use the partial sum S_{18} .

S-31: We'll use Theorem 3.6.1-b in the text to bound the error in a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!} (x-a)^{N+1}$$

for some c strictly between a and x . In our case, $a = \frac{1}{2}$, $x = -\frac{1}{3}$, and we are given the n th derivative of $f(x)$:

$$\begin{aligned}
 E_6 &= \frac{f^{(7)}(c)}{7!} \left(-\frac{1}{3} - \frac{1}{2}\right)^7 \\
 &= \frac{1}{7!} \cdot \frac{6!}{2} \left[(1-c)^{-7} + (-1)^6 (1+c)^{-7} \right] \left(-\frac{5}{6}\right)^7 \\
 &= \frac{-5^7}{14 \cdot 6^7} \cdot \left[(1-c)^{-7} + (1+c)^{-7} \right] =
 \end{aligned}$$

for some c in $(-\frac{1}{3}, \frac{1}{2})$.

We want to provide actual numeric bounds for this expression. That is, we want to find the absolute max and min of

$$E(c) = \frac{-5^7}{14 \cdot 6^7} \cdot \left[(1-c)^{-7} + (1+c)^{-7} \right]$$

over the interval $(-\frac{1}{3}, \frac{1}{2})$. Absolute extrema occur at endpoints and critical points. So, we'll start by differentiating $E(c)$, and finding its critical points (if any) in the interval $(-\frac{1}{3}, \frac{1}{2})$.

$$\begin{aligned}
E(c) &= \frac{-5^7}{14 \cdot 6^7} \cdot \left[(1-c)^{-7} + (1+c)^{-7} \right] \\
E'(c) &= \frac{-5^7}{14 \cdot 6^7} \cdot \left[7(1-c)^{-8} - 7(1+c)^{-8} \right] = 0 \\
(1-c)^{-8} &= (1+c)^{-8} \\
1-c &= 1+c \\
c &= 0
\end{aligned}$$

Since $E'(c)$ is defined over our entire interval, its only critical point is $c = 0$.

- $E(0) = \frac{-5^7}{14 \cdot 6^7} [2]$
- $E\left(-\frac{1}{3}\right) = \frac{-5^7}{14 \cdot 6^7} \left[\left(\frac{4}{3}\right)^{-7} + \left(\frac{2}{3}\right)^{-7} \right] = \frac{-5^7}{14 \cdot 6^7} \left[\left(\frac{3}{4}\right)^7 + \left(\frac{3}{2}\right)^7 \right]$
- $E\left(\frac{1}{2}\right) = \frac{-5^7}{14 \cdot 6^7} \left[\left(\frac{1}{2}\right)^{-7} + \left(\frac{3}{2}\right)^{-7} \right] = \frac{-5^7}{14 \cdot 6^7} \left[2^7 + \left(\frac{2}{3}\right)^7 \right]$

We want to decide which of these numbers is biggest, and which smallest. Note that 2^7 is much, much bigger than $(3/2)^7$, and both $(3/4)^7$ and $(2/3)^7$ are less than one.

Furthermore, $(3/2)^7$ is much larger than 2. So: $[2^7 + (2/3)^7] > [(3/2)^7 + (3/4)^7] > 2$.

Therefore,

$$\frac{-5^7}{14 \cdot 6^7} \left[2^7 + \left(\frac{2}{3}\right)^7 \right] < \frac{-5^7}{14 \cdot 6^7} \left[\left(\frac{3}{4}\right)^7 + \left(\frac{3}{2}\right)^7 \right] < \frac{-5^7}{14 \cdot 6^7} [2]$$

We conclude that the error E_6 is in the interval

$$\left(\frac{-5^7}{14 \cdot 6^7} \left[2^7 + \left(\frac{2}{3}\right)^7 \right] , \frac{-5^7}{14 \cdot 6^7} [2] \right)$$

or, equivalently,

$$\left(\frac{-5^7}{14 \cdot 3^7} \left[1 + \frac{1}{3^7} \right] , \frac{-5^7}{7 \cdot 6^7} \right)$$

which is approximately $(-0.199, -0.040)$.

S-32: Using the Maclaurin series expansions of $\cos x$ and e^x ,

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\ 1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ 1 + x - e^x &= -\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \\ \frac{1 - \cos x}{1 + x - e^x} &= \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots}{-\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots} = \frac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots}\end{aligned}$$

we have

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots} = \frac{\frac{1}{2!}}{-\frac{1}{2!}} = -1$$

S-33: Using the Maclaurin series expansion of $\sin x$,

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \sin x - x + \frac{x^3}{6} &= \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \frac{\sin x - x + \frac{x^3}{6}}{x^5} &= \frac{1}{5!} - \frac{x^2}{7!} + \cdots\end{aligned}$$

we have

$$\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x \rightarrow 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \cdots \right) = \frac{1}{5!} = \frac{1}{120}$$

Remark: to solve this using l'Hôpital's rule we would differentiate five times, making series a practical alternative.

S-34: Our limit has the indeterminate form 1^∞ ; as with l'Hôpital's rule, we can change it to a friendlier form using the natural logarithm.

$$\begin{aligned}f(x) &= \left(1 + x + x^2\right)^{2/x} \\ \log(f(x)) &= \log \left[\left(1 + x + x^2\right)^{2/x} \right] = \frac{2}{x} \log \left(1 + x + x^2\right)\end{aligned}$$

Recall $\log(1 + y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} y^n}{n}$, and set $y = x + x^2$. The series converges when $|y| < 1$, and since we only consider values of x that are very close to 0, we can assume

$$|x + x^2| < 1.$$

$$\begin{aligned}
\log(f(x)) &= \frac{2}{x} \log\left(1 + (x + x^2)\right) = \frac{2}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x + x^2)^n}{n} \\
&= \frac{2}{x} \left[(x + x^2) - \frac{(x + x^2)^2}{2} + \frac{(x + x^2)^3}{3} - \dots \right] \\
&= 2 + 2x - \frac{(x + x^2)^2}{2x} + \frac{(x + x^2)^3}{3x} - \dots \\
&= 2 + 2x - \frac{(x^2 + x)(1 + x)}{2} + \frac{(x^2 + x)^2(1 + x)}{3} - \dots \\
\lim_{x \rightarrow 0} \log(f(x)) &= \lim_{x \rightarrow 0} \left[2 + 2x - \frac{(x^2 + x)(1 + x)}{2} + \frac{(x^2 + x)^2(1 + x)}{3} - \dots \right] \\
&= 2 + 0 + 0 \dots = 2 \\
\lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} e^{\log f(x)} = e^2
\end{aligned}$$

S-35: We have an indeterminate form 1^∞ . We can use a natural logarithm to change this to a friendlier form. Furthermore, to avoid negative powers, we substitute $y = \frac{1}{2x}$. As x grows larger and larger, y gets closer and closer to zero, while staying positive.

$$\begin{aligned}
\log \left[\left(1 + \frac{1}{2x} \right)^x \right] &= x \log \left(1 + \frac{1}{2x} \right) = \frac{1}{2y} \log(1 + y) \\
&= \frac{1}{2y} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} y^n \\
&= \frac{1}{2y} \left[y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \right] \\
&= \left[\frac{1}{2} - \frac{y}{4} + \frac{y^2}{6} - \frac{y^3}{8} + \dots \right] \\
\lim_{x \rightarrow \infty} \log \left[\left(1 + \frac{1}{2x} \right)^x \right] &= \lim_{y \rightarrow 0^+} \left[\frac{1}{2} - \frac{y}{4} + \frac{y^2}{6} - \frac{y^3}{8} + \dots \right] = \frac{1}{2} \\
\lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{2x} \right)^x \right] &= e^{1/2} = \sqrt{e}
\end{aligned}$$

S-36: The factor $(n + 1)(n + 2)$ reminds us of a derivative. Start with the geometric

series.

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ \frac{d}{dx} \left\{ \frac{1}{1-x} \right\} &= \frac{d}{dx} \left\{ \sum_{n=0}^{\infty} x^n \right\} \\ \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} \\ \frac{d}{dx} \left\{ \frac{1}{(1-x)^2} \right\} &= \frac{d}{dx} \left\{ \sum_{n=1}^{\infty} nx^{n-1} \right\} \\ \frac{2}{(1-x)^3} &= \sum_{n=1}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)x^n\end{aligned}$$

Let $x = \frac{1}{7}$. Then $|x| < 1$, so our series converges.

$$\begin{aligned}\frac{2}{(1-1/7)^3} &= \sum_{n=0}^{\infty} (n+2)(n+1) \left(\frac{1}{7}\right)^n \\ \frac{2}{(6/7)^3} &= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{7^n}\end{aligned}$$

S-37: Recall the Taylor series for arctangent is:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

There are similarities between this and our given series: skipping powers of x , and a denominator that's not factorial. We'll try to manipulate it to look like our series. First, we antidifferentiate, to get a factor of $(2n+2)$ on the bottom.

$$\int \arctan x \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} + C$$

We can find the antiderivative of arctangent using integration by parts. Let $u = \arctan x$ and $dv = dx$; then $du = \frac{1}{1+x^2} dx$ and $v = x$.

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} dx + C$$

Now, we use the substitution $w = 1 + x^2$, $dw = 2x dx$.

$$\begin{aligned} &= x \arctan x - \frac{1}{2} \log(1 + x^2) + C \\ \text{So, } \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} &= x \arctan x - \frac{1}{2} \log(1 + x^2) + C \end{aligned}$$

To find C , we evaluate both sides of the equation at $x = 0$.

$$\begin{aligned} 0 &= 0 \arctan 0 - \frac{1}{2} \log(1) + C = C \\ \text{Therefore, } \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} &= x \arctan x - \frac{1}{2} \log(1 + x^2) \end{aligned}$$

Multiplying both sides by x^2 ,

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+1)(2n+2)} = x^3 \arctan x - \frac{x^2}{2} \log(1 + x^2)$$

S-38:

(a) We'll start, as we usually do, by finding a pattern for $f^{(n)}(0)$.

$$\begin{aligned} f(x) &= (1 - x)^{-1/2} \\ f'(x) &= \frac{1}{2}(1 - x)^{-3/2} \\ f''(x) &= \frac{1 \cdot 3}{2^2}(1 - x)^{-5/2} \\ f'''(x) &= \frac{1 \cdot 3 \cdot 5}{2^3}(1 - x)^{-7/2} \\ f^{(4)}(x) &= \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4}(1 - x)^{-9/2} \\ &\vdots \\ f^{(n)}(x) &= \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n}(1 - x)^{-(2n+1)/2} \\ f^{(n)}(0) &= \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n} \end{aligned}$$

We could leave it like this, but we simplify, to make our work cleaner later on.

$$\begin{aligned} &= \frac{1}{2^n} \cdot \frac{(2n)!}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} \\ &= \frac{1}{2^n} \cdot \frac{(2n)!}{2^n \cdot n!} \\ &= \frac{(2n)!}{2^{2n} n!} \end{aligned}$$

This pattern holds for $n \geq 0$. Now, we can write our Maclaurin series for $f(x)$.

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} x^n$$

To find the radius of convergence, we use the ratio test.

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{(2n+2)!}{2^{2n+2} ((n+1)!)^2} \cdot \frac{2^{2n} (n!)^2}{(2n)!} \cdot |x| \\ &= \frac{(2n+2)!}{(2n)!} \left(\frac{n!}{(n+1)!} \right)^2 \cdot \frac{2^{2n}}{2^{2n+2}} |x| \\ &= (2n+2)(2n+1) \left(\frac{1}{n+1} \right)^2 \cdot \frac{1}{4} |x| \\ &= \frac{4n^2 + 4n + 2}{4n^2 + 8n + 4} |x| \\ \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[\frac{4n^2 + 4n + 2}{4n^2 + 8n + 4} |x| \right] = |x| \end{aligned}$$

So, the radius of convergence is $R = 1$.

- (b) We note the derivative of the arcsine function is $\frac{1}{\sqrt{1-x^2}} = f(x^2)$. With this insight, we can manipulate our Taylor series for $f(x)$ into a Taylor series for arcsine.

$$\begin{aligned} \frac{1}{\sqrt{1-x}} &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} x^n \\ \frac{1}{\sqrt{1-x^2}} &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} x^{2n} \\ \int \frac{1}{\sqrt{1-x^2}} dx &= \int \left(\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} x^{2n} \right) dx \\ \arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2 (2n+1)} x^{2n+1} + C \\ \arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2 (2n+1)} x^{2n+1} \end{aligned}$$

where we found the value of C by setting $x = 0$. Its radius of convergence is also 1, by Theorem 3.5.11.

S-39: We use that

$$\log(1+y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y^n}{n} \quad \text{for all } -1 < y \leq 1$$

with $y = \frac{x-2}{2}$ to give

$$\begin{aligned}\log(x) &= \log(2 + x - 2) = \log \left[2 \left(1 + \frac{x-2}{2} \right) \right] \\ &= \log 2 + \log \left(1 + \frac{x-2}{2} \right) = \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n} (x-2)^n\end{aligned}$$

It converges when $-1 < y \leq 1$, or equivalently, $0 < x \leq 4$. So, it converges when $|x-2| < 2$, and diverges when $|x-2| > 2$. So, its radius of convergence is 2.

S-40: (a) Using the geometric series expansion with $r = -t^4$,

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \implies \frac{1}{1+t^4} = \sum_{n=0}^{\infty} (-t^4)^n = \sum_{n=0}^{\infty} (-1)^n t^{4n}$$

Substituting this into our integral,

$$\begin{aligned}I(x) &= \int_0^x \frac{1}{1+t^4} dt \\ &= \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{4n} \right) dt \\ &= \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+1}}{4n+1} \right]_{t=0}^{t=x} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1}\end{aligned}$$

(b) Substituting in $x = \frac{1}{2}$,

$$\begin{aligned}I(1/2) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+1)2^{4n+1}} \\ &= \frac{1}{2} - \frac{1}{5 \times 2^5} + \frac{1}{9 \times 2^9} - \frac{1}{13 \times 2^{13}} + \cdots \\ &= 0.5 - 0.00625 + 0.000217 - 0.0000094 + \cdots = \boxed{0.493967} - 0.0000094 + \cdots\end{aligned}$$

See part (c) for the error analysis.

(c) The series for $I(x)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is between zero and the first omitted term. In this case, the first omitted term was negative (-0.0000094). So the exact value of $I(1/2)$ is the approximate value found in part (b) plus a negative number whose magnitude is smaller than $0.00001 = 10^{-5}$. So the approximate value of part (b) is larger than the true value of $I(1/2)$.

S-41: Expanding the exponential using its Maclaurin series,

$$\begin{aligned} I &= \int_0^1 x^4 e^{-x^2} dx = \sum_{n=0}^{\infty} \int_0^1 x^4 \frac{(-x^2)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n+4} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+5)} = \underbrace{\frac{1}{5}}_{n=0} - \underbrace{\frac{1}{7}}_{n=1} + \underbrace{\frac{1}{18}}_{n=2} - \underbrace{\frac{1}{3!(11)}}_{n=3} + \dots \end{aligned}$$

The signs of successive terms in this series alternate. Furthermore the magnitude of the n^{th} term decreases with n . Hence, by the alternating series test, I lies between $\frac{1}{5} - \frac{1}{7} + \frac{1}{18}$ and $\frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{3!(11)}$. So

$$|I - a| \leq \frac{1}{3!(11)} = \frac{1}{66}$$

S-42: Expanding the exponential using its Taylor series,

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} x^2 e^{-x^2} dx = \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} x^2 \frac{(-x^2)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\frac{1}{2}} x^{2n+2} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}} \end{aligned}$$

The signs of successive terms in this series alternate. Furthermore the magnitude of the n^{th} term decreases with n . Hence, by the alternating series test, I lies between

$\sum_{n=0}^N \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$ and $\sum_{n=0}^{N+1} \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$, for every N . The first few terms are, to five decimal places,

n	0	1	2	3
$\frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$	0.04167	-0.00625	0.00056	-0.00004

Allowing for a roundoff error of 0.000005 in each of these, I must be between

$$0.04167 - 0.00625 + 0.00056 + 0.000005 \times 3 = 0.035995$$

and

$$0.04167 - 0.00625 + 0.00056 - 0.00004 - 0.000005 \times 4 = 0.035920$$

where the multiples of 0.000005 are the maximum possible accumulated roundoff errors in the added terms.

S-43: (a) Using the Taylor series expansion of e^x with $x = -t$,

$$e^{-t} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \implies e^{-t} - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{t^n}{n!} \implies \frac{e^{-t} - 1}{t} = \sum_{n=1}^{\infty} (-1)^n \frac{t^{n-1}}{n!}$$

Substituting this into our integral,

$$I(x) = \int_0^x \frac{e^{-t} - 1}{t} dt = \sum_{n=1}^{\infty} (-1)^n \int_0^x \frac{t^{n-1}}{n!} dt = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \cdot n!}$$

(b) Substituting in $x = 1$,

$$\begin{aligned} I(1) &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n \cdot n!} \\ &= -1 + \frac{1}{2 \cdot 2!} - \frac{1}{3 \cdot 3!} + \frac{1}{4 \cdot 4!} - \frac{1}{5 \cdot 5!} + \cdots \\ &= -1 + 0.25 - 0.0556 + 0.0104 - 0.0017 + \cdots = -0.80 \end{aligned}$$

See part (c) for the error analysis.

(c) The series for $I(x)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is no larger than the first omitted term. So the magnitude of $-\frac{1}{5 \cdot 5!} + \cdots$ is no larger than 0.0017. Allowing for a roundoff error of at most 0.0001 in each of the two terms $-0.0556 + 0.0104$

$$I(1) = -1 + 0.25 - 0.0556 + 0.0104 \pm 0.0019 = -0.7952 \pm 0.0019$$

S-44: (a) Using the Taylor series expansion of $\sin x$ with $x = t$,

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \implies \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$$

So

$$\Sigma(x) = \int_0^x \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x \frac{t^{2n}}{(2n+1)!} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$$

(b) The critical points of $\Sigma(x)$ are the solutions of $\Sigma'(x) = 0$. By the fundamental theorem of calculus $\Sigma'(x) = \frac{\sin x}{x}$, so the critical points of $\Sigma(x)$ are $x = \pm\pi, \pm2\pi, \dots$. The absolute maximum occurs at $x = \pi$.

(c) Substituting in $x = \pi$,

$$\begin{aligned} \Sigma(\pi) &= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)(2n+1)!} \\ &= \pi - \frac{\pi^3}{3 \cdot 3!} + \frac{\pi^5}{5 \cdot 5!} - \frac{\pi^7}{7 \cdot 7!} + \cdots \\ &= 3.1416 - 1.7226 + 0.5100 - 0.0856 + 0.0091 - 0.0007 + \cdots \end{aligned}$$

The series for $\Sigma(\pi)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is no larger than the first omitted term. So

$$\Sigma(\pi) = 3.1416 - 1.7226 + 0.5100 - 0.0856 + 0.0091 = 1.8525$$

with an error of magnitude at most $0.0007 + 0.0005$ (the 0.0005 is the maximum possible accumulated roundoff error in all five retained terms).

S-45: (a) Using the Taylor series expansion of $\cos t$,

$$\begin{aligned}\cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} \\ \frac{\cos t - 1}{t^2} &= -\frac{1}{2!} + \frac{t^2}{4!} - \frac{t^4}{6!} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n-2}}{(2n)!} \\ I(x) &= \int_0^x \frac{\cos t - 1}{t^2} dt = -\frac{x}{2!} + \frac{x^3}{4!3} - \frac{x^5}{6!5} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!(2n-1)}\end{aligned}$$

(b), (c) Substituting in $x = 1$,

$$\begin{aligned}I(1) &= -\frac{1}{2} + \frac{1}{4!3} - \frac{1}{6!5} + \cdots \\ &= -0.5 + 0.0139 - 0.0003 - \cdots \\ &= -0.486 \pm 0.001\end{aligned}$$

The series for $I(1)$ is an alternating series with decreasing successive terms that converge to zero. So approximating $I(1)$ by $-\frac{1}{2} + \frac{1}{4!3}$ introduces an error between 0 and $-\frac{1}{6!5}$. Hence $I(1) < -\frac{1}{2} + \frac{1}{4!3}$.

S-46: (a) Using the Taylor series expansions of $\sin x$ and $\cos x$ with $x = t$,

$$\begin{aligned}\sin t &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} &&= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \\ t \sin t &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+2}}{(2n+1)!} &&= t^2 - \frac{t^4}{3!} + \frac{t^6}{5!} - \frac{t^8}{7!} + \cdots \\ &= -\sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n-1)!} \\ \cos t &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} &&= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots \\ \cos t - 1 &= \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} &&= -\frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots\end{aligned}$$

$$\begin{aligned}
\cos t + t \sin t - 1 &= \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} - \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n-1)!} \\
&= \sum_{n=1}^{\infty} (-1)^n t^{2n} \left(\frac{1}{(2n)!} - \frac{1}{(2n-1)!} \right) = \left(1 - \frac{1}{2!} \right) t^2 - \left(\frac{1}{3!} - \frac{1}{4!} \right) t^4 + \dots \\
&= \sum_{n=1}^{\infty} (-1)^n t^{2n} \left(\frac{1}{(2n)!} - \frac{2n}{(2n)!} \right) \\
&= \sum_{n=1}^{\infty} (-1)^n t^{2n} \left(\frac{1-2n}{(2n)!} \right) = \left(\frac{2}{2!} - \frac{1}{2!} \right) t^2 - \left(\frac{4}{4!} - \frac{1}{4!} \right) t^4 + \dots \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} t^{2n} \left(\frac{2n-1}{(2n)!} \right) = \frac{1}{2!} t^2 - \frac{3}{4!} t^4 + \frac{5}{6!} t^6 - \frac{7}{8!} t^8 + \dots \\
\frac{\cos t + t \sin t - 1}{t^2} &= \sum_{n=1}^{\infty} (-1)^{n+1} t^{2n-2} \left(\frac{2n-1}{(2n)!} \right) = \frac{1}{2!} t - \frac{3}{4!} t^2 + \frac{5}{6!} t^4 - \frac{7}{8!} t^6 + \dots
\end{aligned}$$

Now, we're ready to integrate.

$$\begin{aligned}
I(x) &= \int_0^x \left(\frac{\cos t + t \sin t - 1}{t^2} \right) dt = \int_0^x \left(\sum_{n=1}^{\infty} (-1)^{n+1} t^{2n-2} \left(\frac{2n-1}{(2n)!} \right) \right) dt \\
&= \left[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n-1}}{(2n)!} \right]_0^x \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!}
\end{aligned}$$

(b) $I(1) = \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \frac{1}{8!} + \dots = 0.5 - 0.041\dot{6} + 0.00139 - 0.000024 + \dots = \boxed{0.460}$. The error analysis is in part (c).

(c) The series for $I(1)$ is an alternating series with decreasing successive terms that converge to zero. So approximating $I(1)$ by $\frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!}$ introduces an error between 0 and $-\frac{1}{8!}$. So $I(1) < \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} < 0.460$.

S-47: (a) Substituting $x = -t$ into the known power series

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, we see that:

$$\begin{aligned}
e^{-t} &= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \dots \\
1 - e^{-t} &= t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \dots \\
\frac{1 - e^{-t}}{t} &= 1 - \frac{t}{2!} + \frac{t^2}{3!} - \frac{t^3}{4!} + \dots \\
\int \frac{1 - e^{-t}}{t} dt &= C + x - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} - \frac{x^4}{4 \cdot 4!} + \dots
\end{aligned}$$

Finally, $f(0) = 0$ (since $f(0)$ is an integral from 0 to 0) and so $C = 0$. Therefore

$$f(x) = \int_0^x \frac{1 - e^{-t}}{t} dt = x - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} - \frac{x^4}{4 \cdot 4!} + \cdots.$$

We can also do this calculation entirely in summation notation: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and so

$$\begin{aligned} e^{-t} &= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} \\ 1 - e^{-t} &= - \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n!} \\ \frac{1 - e^{-t}}{t} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{n-1}}{n!} \\ f(x) = \int_0^x \frac{1 - e^{-t}}{t} dt &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!} \end{aligned}$$

(b) We set $a_n = A_n x^n = \frac{(-1)^{n-1}}{n \cdot n!} x^n$ and apply the ratio test.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^n x^{n+1} / ((n+1) \cdot (n+1)!)}{(-1)^{n-1} x^n / (n \cdot n!)} \right| \\ &= \lim_{n \rightarrow \infty} \left(\frac{|x|^{n+1}}{|x|^n} \frac{n \cdot n!}{(n+1) \cdot (n+1)!} \right) \\ &= \lim_{n \rightarrow \infty} \left(|x| \frac{n}{(n+1)^2} \right) \quad \text{since } (n+1)! = (n+1) n! \\ &= 0 \end{aligned}$$

This is smaller than 1 no matter what x is. So the series converges for all x .

S-48:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \geq 1 + x \quad \text{for all } x \geq 0 \\ \implies e^x - 1 &\geq x \\ \implies \frac{x^3}{e^x - 1} &\leq \frac{x^3}{x} = x^2 \\ \implies \int_0^1 \frac{x^3}{e^x - 1} dx &\leq \int_0^1 x^2 dx = \frac{1}{3} \end{aligned}$$

S-49: (a) We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x . Replacing x by $-x$, we also have

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \text{ for all } x \text{ and hence}$$

$$\cosh(x) = \frac{1}{2}[e^x + e^{-x}] = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right] = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

for all x . In particular, it converges for all real numbers. So, its radius of convergence is ∞ .

(b) Using the power series expansion of part (a),

$$\cosh(2) = 1 + \frac{2^2}{2!} + \frac{2^4}{4!} + \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} = 3\frac{2}{3} + \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$$

So it suffices to show that $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} \leq 0.1$. Let's write $b_n = \frac{2^{2n}}{(2n)!}$. The first term in $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$ is

$$b_3 = \frac{2^6}{6!} = \frac{2^6}{6 \times 5 \times 4 \times 3 \times 2} = \frac{4}{45}$$

The ratio between successive terms in $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$ is

$$\frac{b_{n+1}}{b_n} = \frac{2^{2n+2}/2^{2n}}{(2n+2)!/(2n)!} = \frac{4}{(2n+2)(2n+1)} \leq \frac{4}{8 \times 7} = \frac{1}{14} \quad \text{for all } n \geq 3$$

Hence

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} &\leq \overbrace{\frac{4}{45}}^{b_3} + \overbrace{\frac{4}{45} \times \frac{1}{14}}^{b_4 \leq} + \overbrace{\frac{4}{45} \times \frac{1}{14^2}}^{b_5 \leq} + \overbrace{\frac{4}{45} \times \frac{1}{14^3}}^{b_6 \leq} + \dots \\ &= \frac{4}{45} \frac{1}{1 - \frac{1}{14}} = \frac{4}{45} \frac{14}{13} = \frac{56}{585} < \frac{1}{10} \end{aligned}$$

(c) Comparing

$$\cosh(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{(2n)!} \quad \text{and} \quad e^{\frac{1}{2}t^2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{2^n n!}$$

we see that it suffices to show that $(2n)! \geq 2^n n!$. Now, for all $n \geq 1$,

$$\begin{aligned} (2n)! &= \overbrace{1 \times 2 \times \dots \times n}^{n \text{ factors}} \overbrace{(n+1) \times (n+2) \times \dots \times 2n}^{n \text{ factors}} \\ &\geq \overbrace{1 \times 2 \times \dots \times n}^{n \text{ factors}} \overbrace{2 \times 2 \times \dots \times 2}^{n \text{ factors}} \\ &= 2^n n! \end{aligned}$$

S-50:

- (a) For Newton's method, recall we approximate a root of the function $g(x)$ in iterations: given an approximation x_n , our next approximation is $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$. In our case,

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2}{3} \left(x_n + \frac{1}{x_n^2} \right).$$

We want to start somewhere reasonably close to the actual root we want, so let's set $x_0 = 1$. (Your starting point may vary.)

$$\begin{array}{lll} x_0 = 1 & \implies x_1 = \frac{2}{3} \left(1 + \frac{1}{1} \right) = \frac{4}{3} & \approx 1.3333 \\ x_1 = \frac{4}{3} & \implies x_2 = \frac{2}{3} \left(\frac{4}{3} + \frac{9}{16} \right) = \frac{91}{72} & \approx 1.2639 \\ x_2 = \frac{91}{72} & \implies x_3 = \frac{2}{3} \left(\frac{91}{72} + \frac{72^2}{91^2} \right) = \frac{1126819}{894348} & \approx 1.2599 \\ x_3 = \frac{1126819}{894348} & \implies x_4 = \frac{2}{3} \left(\frac{1126819}{894348} + \frac{894348^2}{1126819^2} \right) & \approx 1.2599 \end{array}$$

So, $\sqrt[3]{2} \approx 1.26$.

- (b) We'll evaluate the given series at $x = 2$. This yields the series

$$\sqrt[3]{2} = 1 + \frac{1}{6} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2)(5)(8) \cdots (3n-4)}{3^n n!}.$$

This series is alternating, so if we use the partial sum S_N , our absolute error is at most

$$|a_{N+1}| = \frac{(2)(5)(8) \cdots (3N-1)}{3^{N+1} (N+1)!}$$

(if $N \geq 2$). We want to know which value of N makes this at most 0.01. We test several values.

N	$ a_{N+1} $
3	$\frac{(2)(5)(8)}{3^4 \cdot 4!} \approx 0.04$
4	$\frac{(2)(5)(8)(11)}{3^5 5!} \approx 0.03$
5	$\frac{(2)(5)(8)(11)(14)}{3^6 6!} \approx 0.023$
6	$\frac{(2)(5)(8)(11)(14)(17)}{3^7 7!} \approx 0.019$
7	$\frac{(2)(5)(8)(11)(14)(17)(20)}{3^8 8!} \approx 0.016$
8	$\frac{(2)(5)(8)(11)(14)(17)(20)(23)}{3^9 9!} \approx 0.013$
9	$\frac{(2)(5)(8)(11)(14)(17)(20)(23)(26)}{3^{10} 10!} \approx 0.012$
10	$\frac{(2)(5)(8)(11)(14)(17)(20)(23)(26)(29)}{3^{11} 11!} \approx 0.0103$
11	$\frac{(2)(5)(8)(11)(14)(17)(20)(23)(26)(29)(32)}{3^{12} 12!} \approx 0.009$

So, the approximation S_{11} has a sufficiently small error. That is, we would add up the first twelve terms.

S-51:

Our plan is as follows:

- Make a Taylor series for $f(x)$
- Calculate the tenth derivative of the Taylor series of $f(x)$.
- Decide how many terms we need to add to achieve the desired accuracy.
- Approximate $f^{(10)}\left(\frac{1}{5}\right)$ with a partial sum.

We know that the Taylor series for $\arctan x$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, which converges for $-1 \leq x \leq 1$. So, the Taylor series for $\arctan(x^3)$ is

$$f(x) = \arctan(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$$

It is much easier to differentiate this series many times than it is to differentiate

$\arctan(x^3)$ directly many times.

$$\begin{aligned}
 f'(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(6n+3)x^{6n+2}}{2n+1} \\
 f''(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(6n+3)(6n+2)x^{6n+1}}{2n+1} \\
 f'''(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(6n+3)(6n+2)(6n+1)x^{6n}}{2n+1} \\
 &\vdots \\
 f^{(10)}(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(6n+3)(6n+2)(6n+1)\cdots(6n-6)x^{6n-7}}{2n+1} \\
 &= \sum_{n=2}^{\infty} (-1)^n \frac{(6n+3)(6n+2)(6n+1)\cdots(6n-6)x^{6n-7}}{2n+1} \\
 &= \sum_{n=2}^{\infty} (-1)^n \frac{(6n+3)!}{(2n+1)(6n-7)!} x^{6n-7} \\
 f^{(10)}\left(\frac{1}{5}\right) &= \sum_{n=2}^{\infty} (-1)^n \frac{(6n+3)!}{(2n+1)(6n-7)! \cdot 5^{6n-7}}
 \end{aligned}$$

(Notice, after ten differentiations, the terms a_0 and a_1 are both zero.)

Since this is an alternating series, the absolute error involved in using the approximation S_N is at most

$$|a_{N+1}| = \frac{(6N+9)!}{(2N+3)(6N-1)! \cdot 5^{6N-1}}$$

By testing a few values of N , we find

$$|a_6| = |a_{5+1}| = \frac{39!}{(13)(29)! \cdot 5^{29}} \approx 0.00000095 < 10^{-6}$$

So, S_5 is a sufficient approximation. That is,

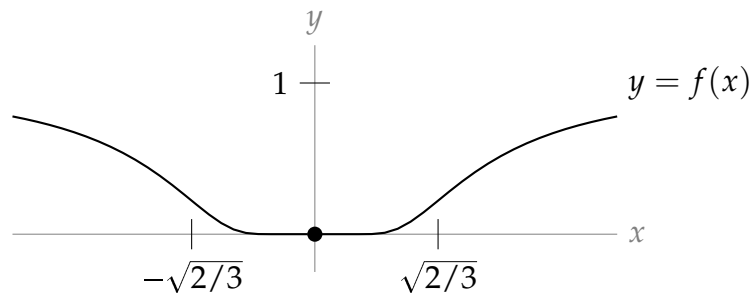
$$\begin{aligned}
 f^{(10)}\left(\frac{1}{5}\right) &\approx \sum_{n=2}^5 (-1)^n \frac{(6n+3)!}{(2n+1)(6n-7)! \cdot 5^{6n-7}} \\
 &= (-1)^2 \frac{15!}{5 \cdot 5! \cdot 5^5} + (-1)^3 \frac{21!}{7! \cdot 11! \cdot 5^{11}} + (-1)^4 \frac{27!}{9! \cdot 17! \cdot 5^{17}} + (-1)^5 \frac{33!}{11! \cdot 23! \cdot 5^{23}} \\
 &= \frac{15!}{5! \cdot 5^6} - \frac{21!}{7! \cdot 11! \cdot 5^{11}} + \frac{27!}{9! \cdot 17! \cdot 5^{17}} - \frac{33!}{11! \cdot 23! \cdot 5^{23}}
 \end{aligned}$$

Remark: if we had calculated $f^{(10)}(1/5)$ directly, using derivative rules instead of series, we would have found an exact value; however, our value here is easier to find, and is highly accurate (if not exact).

S-52:

(a) To sketch $y = f(x)$, we note the following:

- $f(x)$ is never negative.
- $\lim_{x \rightarrow \pm\infty} f(x) = e^0 = 1$, so the curve has horizontal asymptotes in both directions at $y = 1$.
- $\lim_{x \rightarrow \pm 0} f(x) = \lim_{x \rightarrow \pm 0} \frac{1}{e^{1/x^2}} = \lim_{u \rightarrow +\infty} \frac{1}{e^u} = 0 = f(0)$, so the curve is continuous at $x = 0$.
- For $x \neq 0$, $f'(x) = \frac{2}{x^3}e^{-1/x^2}$, so our curve is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.
- For $x \neq 0$, $f''(x) = 2x^{-6}(2 - 3x^2)e^{-1/x^2}$, so our curve is concave up on $(-\sqrt{2/3}, \sqrt{2/3})$, and concave down elsewhere.



(b) Since $f^{(n)}(0) = 0$ for all whole n (that is, the graph is really quite flat at the origin), and since $f(0) = 0$, the Maclaurin series for $f(x)$ is $\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$.

(c) The Maclaurin series converges for all real values of x (to the constant 0).

(d) Since $e^y > 0$ for any real y , we see $f(x) = 0$ only when $x = 0$. So, $f(x)$ is only equal to its Maclaurin series at the single point $x = 0$.

Remark: the function $f(x)$ is an example of a function whose Maclaurin series converges, but not to $f(x)$! To describe this behaviour, we say $f(x)$ is *non-analytic*.

S-53:

Solution 1: Since $f(x)$ is odd, $f(-x) = -f(x)$ for all x in its domain. We plug this into our power series, then consider the even-indexed terms and the odd-indexed terms

separately.

$$\begin{aligned} f(-x) &= -f(x) \\ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (-x)^n &= - \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} (-x)^{2n+1} + \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} (-x)^{2n} &= - \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} - \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} \\ - \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} + \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= - \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} - \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} \\ \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= - \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} \\ 2 \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= 0 \\ \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= 0 \end{aligned}$$

Solution 2: Alternately, we could note the following:

- Since all derivative of $f(x)$ exist, all its derivatives are continuous.
- The derivative of an odd function is even, and the derivative of an even function is odd.
- So, the even-indexed derivatives of $f(x)$ are continuous, odd functions.
- Every continuous, odd function passes through the origin. That is, $f^{(2n)}(0) = 0$.
- So, every term in the series is 0.