UBC Math 101: Integral Calculus
Textbook for AY 2025/26

This textbook is intended for use in the course UBC-V Math 101, Integral Calculus, in AY 2025/26. It is primarily a remix of other open-source materials.

- The vast majority of this text is from *CLP-2 Integral Calculus* by Joel Feldman, Andrew Rechnitzer, and Elyse Yeager
- Sections on probability are adapted from the probability chapter of *Optimal, Integral, Likely* compiled by Bruno Belevan, Parham Hamidi, Nisha Malhotra, and Elyse Yeager, which in turn contains content from *Introductory Statistics* by Barbara Ilowski and Susan Dean.

This document was typeset on Tuesday 31st March, 2026.

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- Links to the source files will be released when it is in a more final form

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This document contains original content as well as content remixed from the sources listed above. More details about the relationship between this text and its primary sources are below.

- Chapter 1 is adapted from Chapter 1, Integration.
  - Section 1.1 is adapted from 1.1, Definition of the Integral. Theorem 1.1.4 is taken from CLP–1 Theorem 1.4.17.
  - Subsection 1.1.5 is original content, 2025.
  - Section 1.2 is adapted from 1.2, Basic Properties of the Definite Integral.
  - Section 1.3 is adapted from 1.3, The Fundamental Theorem of Calculus.
  - Section 1.4 is original content, 2025.
  - Section 1.5 is adapted from 1.4, Substitution.
  - Section 1.6 is adapted from 1.5, Area Between Curves.
  - Section 1.7 is adapted from 1.8, Trigonometric Integrals. The tangent-secant cases have been truncated to match learning objectives. Subsection 1.7.3 is original content, 2025.

  - Section 1.8 is adapted from 1.9, Trigonometric Substitution.
  - Section 1.9 is adapted from 1.10, Partial Fractions, with some cases removed.
  - Section 1.10 is adapted from 1.6, Volumes.
  - Section 1.11 is adapted from 1.7, Integration by Parts.
  - Section 1.12 is adapted from 1.11, Numerical Integration. Subsection 1.12.4 is original content, 2025. Midpoint rule has been replaced with right Riemann sums.
  - Section 1.13 is adapted from 1.12, Improper Integrals.

- Chapter 2 is adapted from Optimal, Integral, Likely (OIL) Chapter 4, Probability. That chapter uses some material from Introductory Statistics by Ilowsky and Dean. Mentions of discrete random variables, and the definition of the CDF, have been removed; the definition of the PDF has been changed.
  - Section 2.1 is adapted from OIL 4.1, Introduction.
  - Section 2.2 is adapted from OIL 4.4, Probability Density.
  - Section 2.3 is adapted from OIL 4.5, Expected Value.
  - Section 2.4 is adapted from OIL 4.6, Variance and Standard Deviation.

- Chapter 3 is adapted from CLP–2 Chapter 3, Sequences and Series. Interval of convergence and telescoping series have been removed.
  - Section 3.1 is adapted from 3.1, Sequences.
  - Section 3.2 is adapted from 3.2, Series, with content about telescoping series removed, and geometric series moved to 3.3. Example 3.2.4 is original content, 2025.
  - Section 3.3 is adapted from 3.3, Convergence Tests. Some optional content removed. Example 3.3.1 is original content, 2025.
  - Section 3.4 is adapted from 3.4, Absolute and Conditional Convergence.
  - Section 3.5 is adapted from 3.5, Power Series, with content related to the interval of convergence removed.
  - Section 3.6 is adapted from 3.6, Taylor Series.

▸▸ Errors

If you find an error in this textbook or the accompanying practice book, please first check the list of known errors, `https://personal.math.ubc.ca/~elyse/Math101Text/errata.html`. If it has not yet been reported, please email Elyse Yeager at elyse math.ubc.ca with the location of the error (page number, and possibly location on the page) and a description of the error.

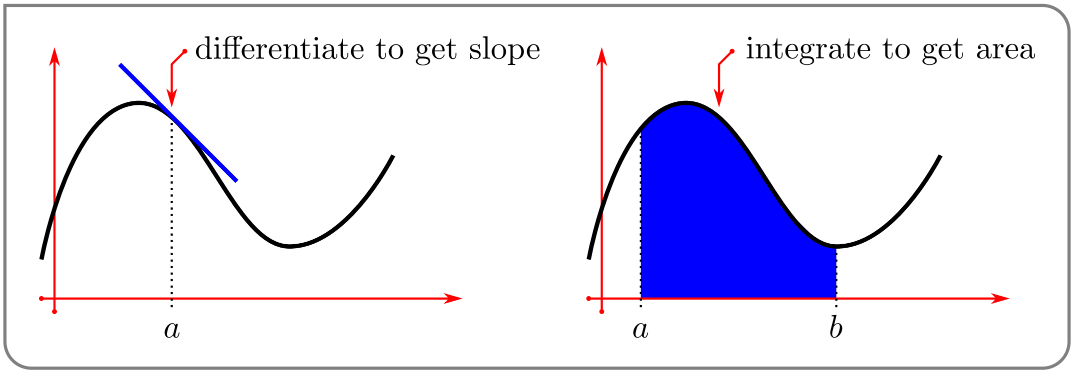
CONTENTS

Chapter 1

INTEGRATION

Calculus is built on two operations — differentiation and integration.

- Differentiation — as we saw last term, differentiation allows us to compute and study the instantaneous rate of change of quantities. At its most basic it allows us to compute tangent lines and velocities, but it also led us to quite sophisticated applications including approximation of functions through Taylor polynomials and optimisation of quantities by studying critical and singular points.
- Integration — at its most basic, allows us to analyse the area under a curve. Of course, its application and importance extend far beyond areas and it plays a central role in solving differential equations.



It is not immediately obvious that these two topics are related to each other. However, as we shall see, they are indeed intimately linked.

1.1▲ Definition of the integral

Learning Objectives

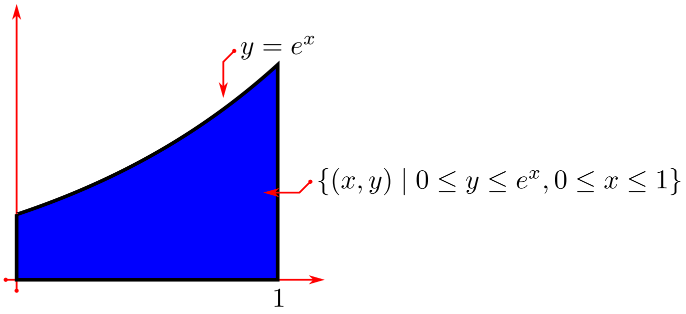
- Interpret the definite integral $\int_a^b f(x)\,\mathrm{d}x$ as signed area when $a < b$.

- Understand why the definite integral sometimes gives negative numbers, even though areas cannot be negative.
- Evaluate certain definite integrals using geometry and the interpretation of definite integral as "area under the curve."
- Understand that the areas of curved shapes can be approximated by cutting up those shapes into many small rectangles and/or triangles.
- Understand what an area function of the form $\int_a^x f(t)\,\mathrm{d}t$ is, and compute them for simple functions using geometry
- Explain using a picture how to approximate area using left- and right Riemann sums both theoretically (for unknown "n") and concretely for a small number of rectangles.
- Understand sigma notation (review from high school).
  - Express sums using sigma notation.
  - Manipulate sums using arithmetic properties: constant sums, factoring and addition.
- Write right Riemann sums in sigma notation.
- Understand the definition of a definite integral as the limit of a Riemann sum.
- Compute Riemann sums with a spreadsheet.

Arguably the easiest way to introduce integration is by considering the area between the graph of a given function and the x-axis, between two specific vertical lines — such as is shown in the figure above. We'll follow this route by starting with a motivating example.

1.1.1 ▸▸ A motivating example

Let us find the area under the curve $y = e^x$ (and above the x–axis) for $0 \leqslant x \leqslant 1$. That is, the area of $\{\ (x,y)\ |\ 0 \leqslant y \leqslant e^x, 0 \leqslant x \leqslant 1\ \}$.



This area is equal to the “definite integral”

$$\text{Area} = \int_0^1 e^x \,\mathrm{d}x$$

Do not worry about this notation or terminology just yet. We discuss it at length below. In different applications this quantity will have different interpretations — not just area. For example, if x is time and e^x is your velocity at time x, then we’ll see later (in Example 1.1.21) that the specified area is the net distance travelled between time 0 and time 1. After we finish with the example, we’ll mimic it to give a general definition of the integral $\int_a^b f(x)\,\mathrm{d}x$.

Example 1.1.1

We wish to compute the area of $\{\ (x,y)\ \big|\ 0 \leqslant y \leqslant e^x, 0 \leqslant x \leqslant 1\ \}$. We know, from our experience with e^x in differential calculus, that the curve $y = e^x$ is not easily written in terms of other simpler functions, so it is very unlikely that we would be able to model the area exactly as a combination of simpler geometric objects such as triangles, rectangles or circles.

So, rather than trying to write down the area exactly, our strategy is to approximate the area and then make our approximation more and more precise[1]. We choose[2] to approximate the area as a union of a large number of tall thin (vertical) rectangles. As we take more and more rectangles, we get better and better approximations. Taking the limit as the number of rectangles goes to infinity gives the exact area[3].

As a warm up exercise, we’ll start with four rectangles. (In Example 1.1.2, below, we’ll consider an arbitrary number of rectangles and then take the limit as the number of rectangles goes to infinity.) Here is the plan for now:

- Subdivide the interval $0 \leqslant x \leqslant 1$ into 4 equal subintervals each of width $1/4$, and

- subdivide the area of interest into four corresponding vertical strips, as in the figure below.

The area we want (the area under the curve) is exactly the sum of the areas of all four strips.

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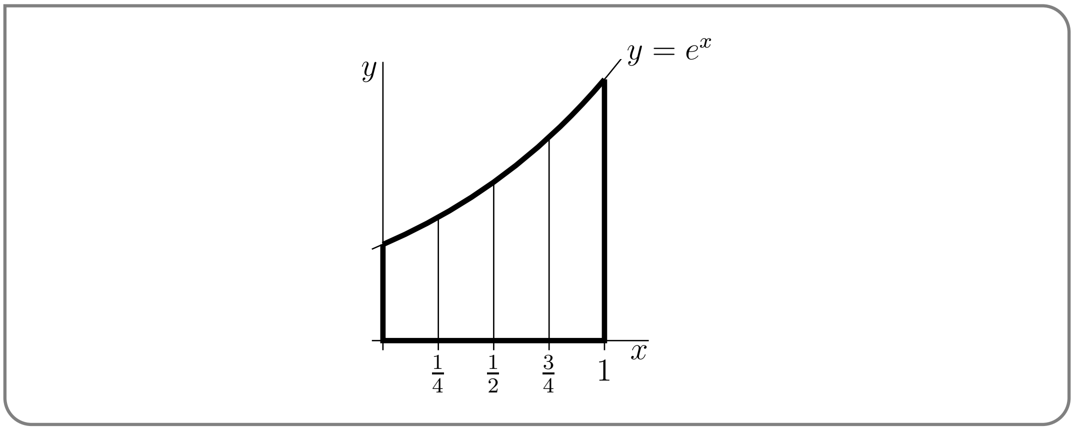
[1] This should remind the reader of the approach taken to compute the slope of a tangent line — way, way back at the start of differential calculus.

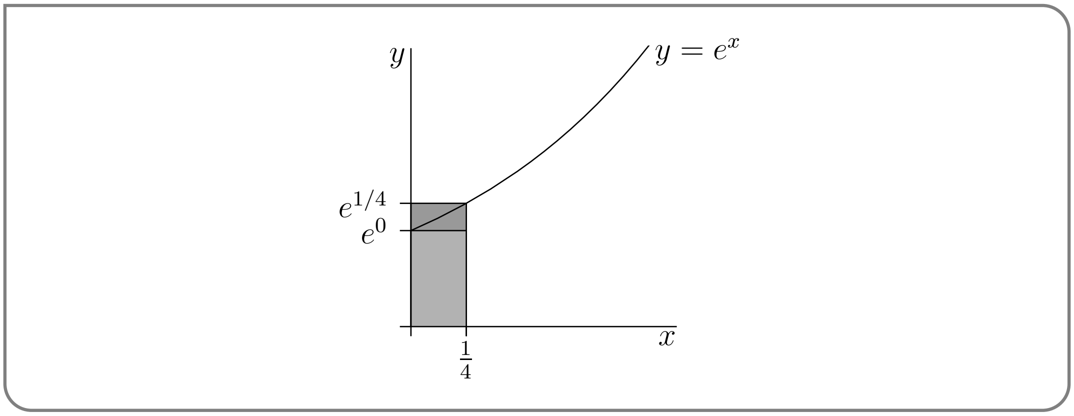
[2] Approximating the area in this way leads to a definition of integration that is called Riemann integration. This is the most commonly used approach to integration. However, we could also approximate the area by using long thin *horizontal* strips. This leads to a definition of integration that is called Lebesgue integration. We will not be covering Lebesgue integration in these notes.

[3] If we want to be more careful here, we should construct two approximations, one that is always a little smaller than the desired area and one that is a little larger. We can then take a limit using the Squeeze Theorem and arrive at the exact area. More on this later.



Each of these strips can be approximated by a rectangle. While the bottom and sides are fine (the sides are at right-angles to the base), the top of the strip is not horizontal. This is where we must start to approximate. We can replace each strip by a rectangle by just levelling off the top. But now we have to make a choice — at what height do we level off the top?

Consider, for example, the leftmost strip. On this strip, x runs from 0 to $1/4$. As x runs from 0 to $1/4$, the height y runs from e^0 to $e^{1/4}$. It would be reasonable to choose the height of the approximating rectangle to be somewhere between e^0 and $e^{1/4}$. Which height should we choose? Well, actually, it doesn't matter. When we eventually take the limit of infinitely many approximating rectangles, all of those different choices give exactly the same final answer. We'll say more about this later.



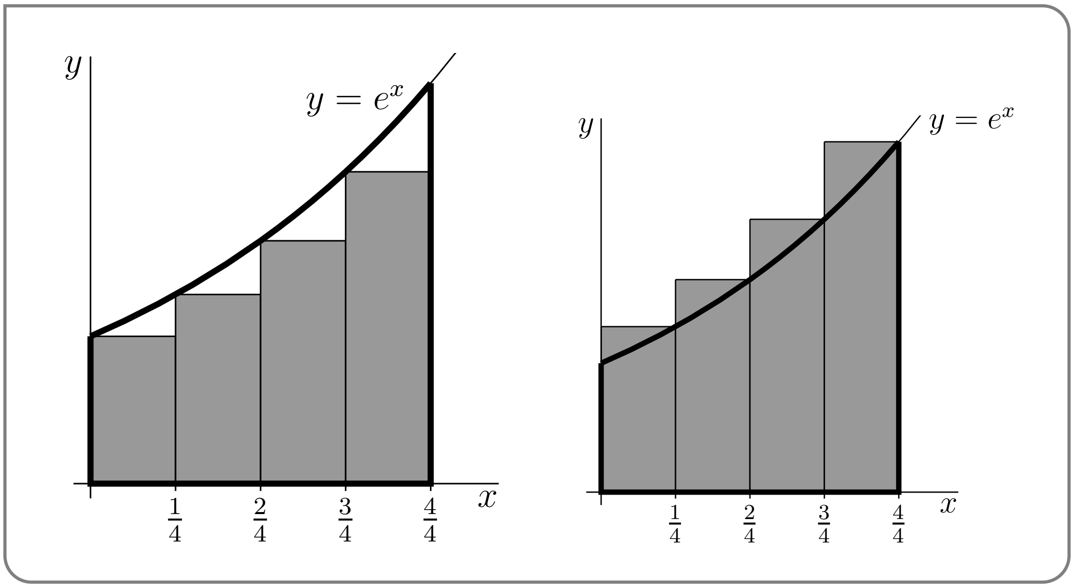
In this example, we'll do two sample computations.

- For the first computation, we approximate each slice by a rectangle whose height is the height of the *left* side of the slice.
  - On the first slice, x runs from 0 to $1/4$, and the height y runs from e^0 (on the left side) to $e^{1/4}$ (on the right side).
  - So, we approximate the first slice by the rectangle of height e^0 and width $1/4$, and hence of area $\frac{1}{4}e^0 = \frac{1}{4}$.
  - On the second slice, x runs from $1/4$ to $1/2$, and the height y runs from $e^{1/4}$ and $e^{1/2}$.

  - So, we approximate the second slice by the rectangle of height $e^{1/4}$ and width $1/4$, and hence of area $\frac{1}{4}\,e^{1/4}$.
  - And so on.
  - All together, we approximate the area of interest by the sum of the areas of the four approximating rectangles, which is

$$\left[1 + e^{1/4} + e^{1/2} + e^{3/4}\right]\frac{1}{4} \approx 1.5124$$

  - This particular approximation is called the "left Riemann sum approximation to $\int_0^1 e^x\,\mathrm{d}x$ with 4 subintervals." We'll explain this terminology later.
  - This particular approximation represents the shaded area in the figure on the left below. Note that, because e^x *increases* as x increases, this approximation is definitely smaller than the true area.



- For the second computation we approximate each slice by a rectangle whose height is the height of the *right* side of the slice.
  - On the first slice, x runs from 0 to $1/4$, and the height y runs from e^0 (on the left side) to $e^{1/4}$ (on the right side).
  - So, we approximate the first slice by the rectangle of height $e^{1/4}$ and width $1/4$, and hence of area $\frac{1}{4}\,e^{1/4}$.
  - On the second slice, x runs from $1/4$ to $1/2$, and the height y runs from $e^{1/4}$ and $e^{1/2}$.
  - So, we approximate the second slice by the rectangle of height $e^{1/2}$ and width $1/4$, and hence of area $\frac{1}{4}\,e^{1/2}$.
  - And so on.
  - All together, we approximate the area of interest by the sum of the areas of the four approximating rectangles, which is

$$\left[e^{1/4} + e^{1/2} + e^{3/4} + e^{1}\right]\frac{1}{4} = 1.9420$$

  - This particular approximation is called the "right Riemann sum approximation to $\int_0^1 e^x \, \mathrm{d}x$ with 4 subintervals".
  - This particular approximation represents the shaded area in the figure on the right above. Note that, because e^x *increases* as x increases, this approximation is definitely larger than the true area.

We conclude that the *actual* area of the region is between 1.5124 and 1.9420.

Example 1.1.1

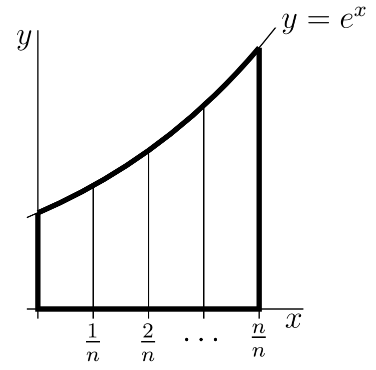
Now for the full computation that gives the exact area.

Example 1.1.2

Recall that we wish to compute the area of $\{ (x, y) \mid 0 \leqslant y \leqslant e^x, 0 \leqslant x \leqslant 1 \}$ and that our strategy is to approximate this area by the area of a union of a large number of very thin rectangles, and then take the limit as the number of rectangles goes to infinity. In Example 1.1.1, we used just four rectangles. Now, we'll consider a general number of rectangles, which we'll call n. Then, we'll take the limit $n \to \infty$. All together, our plan is to:

- pick a natural number n,
- subdivide the interval $0 \leqslant x \leqslant 1$ into n equal subintervals each of width $1/n$, and
- subdivide the area of interest into corresponding thin strips, as in the figure below.

The area we want is exactly the sum of the areas of all of the thin strips.

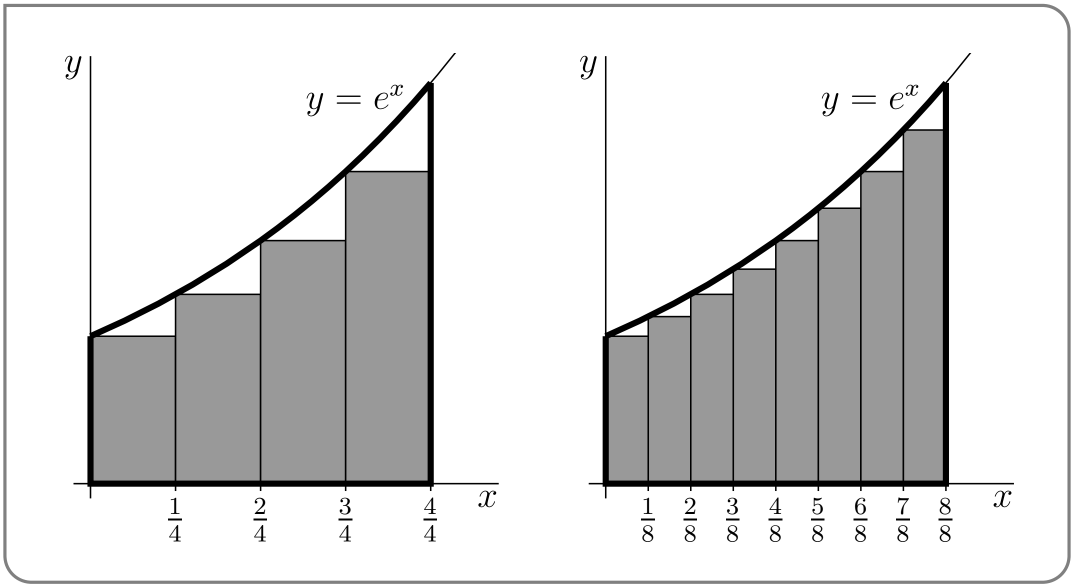


Each of these strips is almost, but not quite, a rectangle. As in Example 1.1.1, the only problem is that the top is not horizontal. So we approximate each strip by a rectangle, just by levelling off the top. Again, we have to make a choice — at what height do we level off the top?

Consider, for example, the leftmost strip. On this strip, x runs from 0 to $1/n$. As x runs from 0 to $1/n$, the height y runs from e^0 to $e^{1/n}$. It would be reasonable to choose the height of the approximating rectangle to be somewhere between e^0 and $e^{1/n}$. Which height should we choose?

Well, as we said in Example 1.1.1, it doesn't matter. We shall shortly take the limit $n \to \infty$ and, in that limit, all of those different choices give exactly the same final answer.

(The justificaion of such a claim is beyond the learning objectives for this course.) For this example we arbitrarily choose the height of each rectangle to be the height of the graph $y = e^x$ at the smallest value of x in the corresponding strip[4]. The figure on the left below shows the approximating rectangles when $n = 4$, and the figure on the right shows the approximating rectangles when $n = 8$.



Now we compute the approximating area when there are n strips.

- We approximate the leftmost strip by a rectangle of height e^0. All of the rectangles have width $1/n$. So the leftmost rectangle has area $\frac{1}{n}e^0$.
- On strip number 2, x runs from $\frac{1}{n}$ to $\frac{2}{n}$. So the smallest value of x on strip number 2 is $\frac{1}{n}$, and we approximate strip number 2 by a rectangle of height $e^{1/n}$ and hence of area $\frac{1}{n}e^{1/n}$.
- And so on.
- On the last strip, x runs from $\frac{n-1}{n}$ to $\frac{n}{n} = 1$. So the smallest value of x on the last strip is $\frac{n-1}{n}$, and we approximate the last strip by a rectangle of height $e^{(n-1)/n}$ and hence of area $\frac{1}{n}e^{(n-1)/n}$.

The total area of all of the approximating rectangles is:

$$\begin{aligned}\text{Total approximating area} &= \frac{1}{n}e^0 + \frac{1}{n}e^{1/n} + \frac{1}{n}e^{2/n} + \frac{1}{n}e^{3/n} + \cdots + \frac{1}{n}e^{(n-1)/n}\\ &= \frac{1}{n}\left(1 + e^{1/n} + e^{2/n} + e^{3/n} + \cdots + e^{(n-1)/n}\right)\end{aligned}$$

Now the sum in the brackets might look a little intimidating because of all the exponentials, but it actually has a pretty simple structure that can be easily seen if we rename $e^{1/n} = r$, then:

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[4] Notice that since e^x is an increasing function, this choice of heights means that each of our rectangles is smaller than the strip it came from.

- the first term is $1 = r^0$,
- the second term is $e^{1/n} = r^1$,
- the third term is $e^{2/n} = r^2$,
- the fourth term is $e^{3/n} = r^3$,
- and so on, and
- the last term is $e^{(n-1)/n} = r^{n-1}$.

This leads to the cleaner expression:

$$\text{Total approximating area} = \frac{1}{n}\left(1 + r + r^2 + \cdots + r^{n-1}\right) .$$

The sum in brackets is known as a geometric sum and satisfies a nice simple formula:

Equation 1.1.3 (Geometric sum)**.**

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1} \qquad \text{provided } r \neq 1$$

The derivation of the above formula is not too difficult. So, let's derive it in a little aside. (You'll see this formula again in 3.3.3.)

▸▸▸ Geometric Sum

Denote the sum as

$$S = 1 + r + r^2 + \cdots + r^{n-1}$$

Notice that if we multiply the whole sum by r, we get back almost the same thing:

$$\begin{aligned} rS &= r\left(1 + r + r^2 + \cdots + r^{n-1}\right) \\ &= r + r^2 + r^3 + \cdots + r^n \end{aligned}$$

This right-hand side differs from the original sum S in two ways:

- The right-hand side, which starts with "$r+$", is missing the "$1+$" that S starts with, and
- the right-hand side has an extra "$+r^n$" at the end that does not appear in S.

That is,

$$rS = S - 1 + r^n .$$

Moving this around a little gives

$$\begin{aligned} (r-1)S &= (r^n - 1) \\ S &= \frac{r^n - 1}{r - 1} , \end{aligned}$$

as shown in Equation 1.1.3. Notice that the last step in the manipulations only works for $r \neq 1$.

▸▸▸ Back to Approximating Areas

Now we can go back to our area approximation armed with the above result about geometric sums.

$$\begin{aligned}\text{Total approximating area} &= \frac{1}{n}\left(1+r+r^2+\cdots+r^{n-1}\right)\\ &= \frac{1}{n}\frac{r^n-1}{r-1} && \text{remember that } r=e^{1/n}\\ &= \frac{1}{n}\frac{e^{n/n}-1}{e^{1/n}-1}\\ &= \frac{1}{n}\frac{e-1}{e^{1/n}-1}\end{aligned}$$

To get the exact area[5] all we need to do is make the approximation better and better by taking the limit $n \to \infty$. The limit will look more familiar if we rename $1/n$ to X. As n tends to infinity, X tends to 0, so

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\frac{1}{n}\,\frac{e-1}{e^{1/n}-1}\\ &= (e-1)\lim_{n\to\infty}\frac{1/n}{e^{1/n}-1}\\ &= (e-1)\lim_{X\to 0}\frac{X}{e^X-1} && \text{(with } X=1/n\text{)}\end{aligned}$$

Examining this limit we see that both numerator and denominator tend to zero as $X \to 0$, and so we cannot evaluate this limit by computing the limits of the numerator and denominator separately and then dividing the results. Despite this, the limit is not too hard to evaluate. Two ways are shown below.

- Perhaps the easiest way to compute the limit is by using l'Hôpital's rule[6]. Since both numerator and denominator go to zero, this is a $0/0$ indeterminate form. Thus

  $$\lim_{X\to 0}\frac{X}{e^X-1} = \lim_{X\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}X}X}{\frac{\mathrm{d}}{\mathrm{d}X}(e^X-1)} = \lim_{X\to 0}\frac{1}{e^X} = 1$$

- Another way[7] to evaluate the same limit is to observe that it can be massaged into the form of the limit definition of the derivative. First notice that

  $$\lim_{X\to 0}\frac{X}{e^X-1} = \left[\lim_{X\to 0}\frac{e^X-1}{X}\right]^{-1}$$

---

[5] We haven't proved that this will give us the exact area, but it should be clear that taking this limit will give us a lower bound on the area. To complete things rigorously we also need an upper bound and the squeeze theorem.

[6] If you do not recall L'Hôpital's rule and indeterminate forms then we recommend you skim over your differential calculus notes on the topic.

[7] Say if you don't recall l'Hôpital's rule and have not had time to revise it.

provided this second limit exists and is nonzero. This second limit should look a little familiar:

$$\lim_{X\to 0}\frac{e^X-1}{X}=\lim_{X\to 0}\frac{e^X-e^0}{X-0}$$

which is just the definition of the derivative of e^x at $x = 0$. Hence we have

$$\begin{aligned}\lim_{X\to 0}\frac{X}{e^X-1}&=\left[\lim_{X\to 0}\frac{e^X-e^0}{X-0}\right]^{-1}\\&=\left[\frac{\mathrm{d}}{\mathrm{d}X}e^X\Big|_{X=0}\right]^{-1}\\&=\left[e^X\big|_{X=0}\right]^{-1}\\&=1\end{aligned}$$

So, after this short aside into limits, we may now conclude that

$$\begin{aligned}\text{Area}&=(e-1)\lim_{X\to 0}\frac{X}{e^X-1}\\&=e-1\end{aligned}$$

Example 1.1.2

1.1.2 ▸▸ Optional — a more rigorous area computation

In Example 1.1.1 above we considered the area of the region $\{\ (x,y)\ \big|\ 0\leqslant y\leqslant e^x,\ 0\leqslant x\leqslant 1\ \}$. We approximated that area by the area of a union of n thin rectangles. We then claimed that upon taking the number of rectangles to infinity, the approximation of the area became the exact area. However we did not justify the claim. The purpose of this optional section is to make that calculation rigorous.

The broad set-up is the same. We divide the region up into n vertical strips, each of width $1/n$ and we then approximate those strips by rectangles. However rather than an uncontrolled approximation, we construct two sets of rectangles — one set always smaller than the original area and one always larger. This then gives us lower and upper bounds on the area of the region. Finally we make use of the squeeze theorem to establish the result.

Theorem 1.1.4 (Squeeze theorem (or sandwich theorem or pinch theorem)).

Let $a \in \mathbb{R}$ and let f, g, h be three functions so that

$$f(x) \leqslant g(x) \leqslant h(x)$$

for all x in an interval around a, except possibly exactly at $x = a$. Then if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then it is also the case that

$$\lim_{x \to a} g(x) = L$$

- To find our upper and lower bounds we make use of the fact that e^x is an increasing function. We know this because the derivative $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$ is always positive. Consequently, the smallest and largest values of e^x on the interval $a \leqslant x \leqslant b$ are e^a and e^b, respectively.

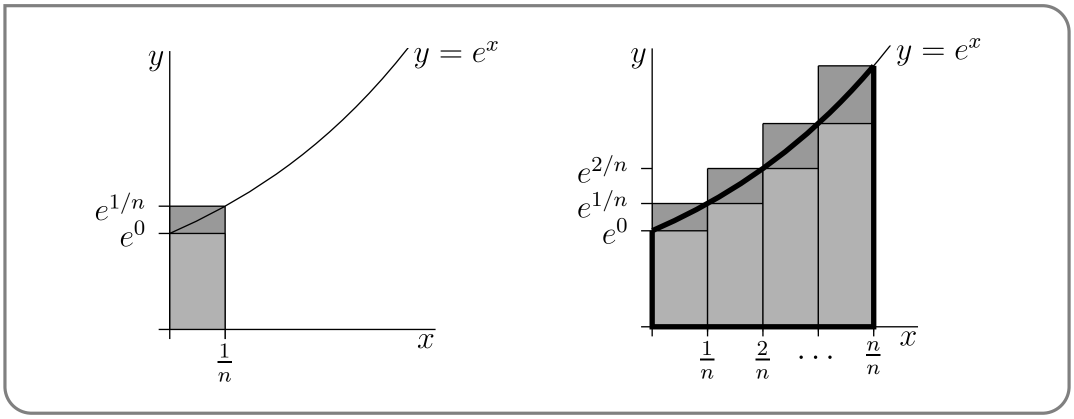
- In particular, for $0 \leqslant x \leqslant {}^1\!/_n$, e^x takes values only between e^0 and $e^{1/n}$. As a result, the first strip

$$\left\{ (x, y) \,\middle|\, 0 \leqslant x \leqslant {}^1\!/_n,\ 0 \leqslant y \leqslant e^x \right\}$$

  - contains the rectangle of $0 \leqslant x \leqslant {}^1\!/_n$, $0 \leqslant y \leqslant e^0$ (the lighter rectangle in the figure on the left below) and
  - is contained in the rectangle $0 \leqslant x \leqslant {}^1\!/_n$, $0 \leqslant y \leqslant e^{1/n}$ (the largest rectangle in the figure on the left below).

  Hence

$$\frac{1}{n}e^0 \leqslant \text{Area}\left\{ (x, y) \,\middle|\, 0 \leqslant x \leqslant {}^1\!/_n,\ 0 \leqslant y \leqslant e^x \right\} \leqslant \frac{1}{n}e^{1/n}$$



- Similarly, for the second, third, …, last strips, as in the figure on the right above,

$$\begin{array}{rcl}
\frac{1}{n}e^{1/n} \leqslant \text{Area}\left\{ (x,y) \mid {}^{1}\!/\!{}_{n} \leqslant x \leqslant {}^{2}\!/\!{}_{n},\ 0 \leqslant y \leqslant e^x \right\} & \leqslant & \frac{1}{n}e^{2/n} \\
\frac{1}{n}e^{2/n} \leqslant \text{Area}\left\{ (x,y) \mid {}^{2}\!/\!{}_{n} \leqslant x \leqslant {}^{3}\!/\!{}_{n},\ 0 \leqslant y \leqslant e^x \right\} & \leqslant & \frac{1}{n}e^{3/n} \\
\vdots \qquad\qquad \vdots & & \vdots \\
\frac{1}{n}e^{(n-1)/n} \leqslant \text{Area}\left\{ (x,y) \mid {}^{(n-1)}\!/\!{}_{n} \leqslant x \leqslant {}^{n}\!/\!{}_{n},\ 0 \leqslant y \leqslant e^x \right\} & \leqslant & \frac{1}{n}e^{n/n}
\end{array}$$

- Adding these n inequalities together gives

$$\begin{aligned}
&\frac{1}{n}\left(1 + e^{1/n} + \cdots + e^{(n-1)/n}\right) \\
&\qquad\qquad \leqslant \text{Area}\left\{ (x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant e^x \right\} \\
&\qquad\qquad\qquad\qquad \leqslant \frac{1}{n}\left(e^{1/n} + e^{2/n} + \cdots + e^{n/n}\right)
\end{aligned}$$

- We can then recycle equation (1.1.3) with $r = e^{1/n}$, so that $r^n = \left(e^{1/n}\right)^n = e$. Thus we have

$$\frac{1}{n}\frac{e-1}{e^{1/n}-1} \leqslant \text{Area}\left\{ (x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant e^x \right\} \leqslant \frac{1}{n}e^{1/n}\frac{e-1}{e^{1/n}-1}$$

  where we have used the fact that the upper bound is a simple multiple of the lower bound:

$$\left(e^{1/n} + e^{2/n} + \cdots + e^{n/n}\right) = e^{1/n}\left(1 + e^{1/n} + \cdots + e^{(n-1)/n}\right).$$

- We now apply the squeeze theorem to the above inequalities. In particular, the limits of the lower and upper bounds are $\lim_{n\to\infty} \frac{1}{n}\frac{e-1}{e^{1/n}-1}$ and $\lim_{n\to\infty} \frac{1}{n}e^{1/n}\frac{e-1}{e^{1/n}-1}$, respectively. As we did near the end of Example 1.1.2, we make these limits look more familiar by renaming ${}^{1}\!/\!{}_{n}$ to X. As n tends to infinity, X tends to 0, so the limits of the lower and upper bounds are

$$\lim_{n\to\infty} \frac{1}{n}\frac{e-1}{e^{1/n}-1} = (e-1) \lim_{X={}^{1}\!/\!{}_{n}\to 0} \frac{X}{e^X - 1} = e - 1$$

  (by l'Hôpital's rule) and

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}e^{1/n}\frac{e-1}{e^{1/n}-1} &= (e-1) \lim_{X={}^{1}\!/\!{}_{n}\to 0} \cdot \frac{Xe^X}{e^X - 1} \\
&= (e-1) \lim_{X\to 0} e^X \cdot \lim_{X=\to 0} \frac{X}{e^X - 1} \\
&= (e-1) \cdot 1 \cdot 1
\end{aligned}$$

  Thus, since the exact area is trapped between the lower and upper bounds, the squeeze theorem then implies that

$$\text{Exact area} = e - 1.$$

1.1.3 ▸▸ Summation notation

As you can see from the above example (and the more careful rigorous computation), our discussion of integration will involve a fair bit of work with sums of quantities. To this end, we make a quick aside into summation notation. While one can work through the material below without this notation, proper summation notation is well worth learning, so we advise the reader to persevere.

Writing out the summands explicitly can become quite impractical — for example, say we need the sum of the first 11 squares:

$$1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2$$

This becomes tedious. Where the pattern is clear, we will often skip the middle few terms and instead write

$$1 + 2^2 + \cdots + 11^2.$$

A far more precise way to write this is using Σ (capital-sigma) notation. For example, we can write the above sum as

$$\sum_{k=1}^{11} k^2$$

This is read as

The sum from k equals 1 to 11 of k^2.

More generally:

Notation 1.1.5.

Let $m \leqslant n$ be integers and let $f(x)$ be a function defined on the integers. Then we write

$$\sum_{k=m}^{n} f(k)$$

to mean the sum of $f(k)$ for k from m to n:

$$f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n).$$

Similarly, we write

$$\sum_{i=m}^{n} a_i$$

to mean

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

for some set of coefficients $\{a_m, \ldots, a_n\}$.

Consider the example

$$\sum_{k=3}^{7} \frac{1}{k^2} = \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}$$

It is important to note that the right-hand side of this expression evaluates to a number[8]; it does not contain "k". The summation index k is just a "dummy" variable and it does not have to be called k. For example:

$$\sum_{k=3}^{7} \frac{1}{k^2} = \sum_{i=3}^{7} \frac{1}{i^2} = \sum_{j=3}^{7} \frac{1}{j^2} = \sum_{\ell=3}^{7} \frac{1}{\ell^2}$$

The summation index has no meaning outside the sum.

A sum can be represented using summation notation in many different ways. If you are unsure as to whether or not two summation notations represent the same sum, just write out the first few terms and the last couple of terms. For example,

$$\sum_{m=3}^{15} \frac{1}{m^2} = \overbrace{\frac{1}{3^2}}^{m=3} + \overbrace{\frac{1}{4^2}}^{m=4} + \overbrace{\frac{1}{5^2}}^{m=5} + \cdots + \overbrace{\frac{1}{14^2}}^{m=14} + \overbrace{\frac{1}{15^2}}^{m=15}$$

$$\sum_{m=4}^{16} \frac{1}{(m-1)^2} = \overbrace{\frac{1}{3^2}}^{m=4} + \overbrace{\frac{1}{4^2}}^{m=5} + \overbrace{\frac{1}{5^2}}^{m=6} + \cdots + \overbrace{\frac{1}{14^2}}^{m=15} + \overbrace{\frac{1}{15^2}}^{m=16}$$

are equal.

Here is a theorem that gives a few rules for manipulating summation notation.

Theorem 1.1.6 (Arithmetic of Summation Notation)**.**

Let $n \geqslant m$ be integers. Then for all real numbers c and a_i, b_i, $m \leqslant i \leqslant n$.

(a) $\sum_{i=m}^{n} ca_i = c\left(\sum_{i=m}^{n} a_i\right)$

(b) $\sum_{i=m}^{n} (a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right)$

(c) $\sum_{i=m}^{n} (a_i - b_i) = \left(\sum_{i=m}^{n} a_i\right) - \left(\sum_{i=m}^{n} b_i\right)$

Proof. We can prove this theorem by just writing out both sides of each equation, and observing that they are equal, by the usual laws of arithmetic[9]. For the example, for the first

---

[8] Some careful addition shows it is $\frac{46181}{176400}$.

[9] Since all the sums are finite, this isn't too hard. More care must be taken when the sums involve an infinite number of terms. We will examine this in Chapter 3.

equation, the left and right-hand sides are

$$\sum_{i=m}^{n} ca_i = ca_m + ca_{m+1} + \cdots + ca_n \quad \text{and} \quad c\left(\sum_{i=m}^{n} a_i\right) = c(a_m + a_{m+1} + \cdots + a_n)$$

They are equal by the usual distributive law. The "distributive law" is the fancy name for $c(a+b) = ca + cb$. ☐

Not many sums can be computed exactly[10]. Here are some that can. The first few are used a lot.

Theorem 1.1.7.

(a) $\sum\limits_{i=0}^{n} ar^i = a\frac{1-r^{n+1}}{1-r}$, for all real numbers a and $r \neq 1$ and all integers $n \geqslant 0$.

(b) $\sum\limits_{i=1}^{n} 1 = n$, for all integers $n \geqslant 1$.

(c) $\sum\limits_{i=1}^{n} i = \frac{1}{2}n(n+1)$, for all integers $n \geqslant 1$.

(d) $\sum\limits_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$, for all integers $n \geqslant 1$.

(e) $\sum\limits_{i=1}^{n} i^3 = \left[\frac{1}{2}n(n+1)\right]^2$, for all integers $n \geqslant 1$.

▸▸▸ Proof of Theorem 1.1.7 (Optional)

Proof. (a) The first sum is

$$\sum_{i=0}^{n} ar^i = ar^0 + ar^1 + ar^2 + \cdots + ar^n$$

---

[10] Of course, any finite sum can be computed exactly — just sum together the terms. What we mean by "computed exactly" in this context is that we can rewrite the sum as a simple, and easily evaluated, formula involving the terminals of the sum. For example,

$$\sum_{k=m}^{n} r^k = \frac{r^{n+1} - r^m}{r-1} \qquad \text{provided } r \neq 1\,.$$

No matter what finite integers we choose for m and n, we can quickly compute the sum in just a few arithmetic operations. On the other hand, the sums

$$\sum_{k=m}^{n} \frac{1}{k} \qquad \text{and} \qquad \sum_{k=m}^{n} \frac{1}{k^2}$$

cannot be expressed in such clean formulas (though you can rewrite them quite cleanly using integrals). To explain more exactly, we would need to go into a more detailed and careful discussion that is beyond the scope of this course.

which is just the left-hand side of equation (1.1.3), with n replaced by $n+1$ and then multiplied by a.

(b) The second sum is just n copies of 1 added together, so of course the sum is n.

(c) We'll derive the third sum using a trick that generalises to the fourth and fifth sums (and also to higher powers). The trick uses the generating function[11] $S(x)$:

Equation 1.1.8.

$$S(x) = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$$

Notice that this is just the geometric sum given by equation 1.1.3 with n replaced by $n+1$.

Now, consider the limit

$$\begin{aligned} \lim_{x\to 1} S(x) &= \lim_{x\to 1}\left(1 + x + x^2 + \cdots + x^n\right) = n+1 && \text{but also} \\ &= \lim_{x\to 1}\frac{x^{n+1}-1}{x-1} && \text{now use l'Hôpital's rule} \\ &= \lim_{x\to 1}\frac{(n+1)x^n}{1} = n+1. \end{aligned}$$

This is not so hard (or useful). But now consider the derivative of $S(x)$:

$$\begin{aligned} S'(x) &= 1 + 2x + 3x^2 + \cdots + nx^{n-1} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^{n+1}-1}{x-1}\right] && \text{use the quotient rule} \\ &= \frac{(x-1)\cdot(n+1)x^n - (x^{n+1}-1)\cdot 1}{(x-1)^2} && \text{now clean it up} \\ &= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}. \end{aligned}$$

---

[11] Generating functions are frequently used in mathematics to analyse sequences and series, but are beyond the scope of the course. The interested reader should take a look at "Generatingfunctionology" by Herb Wilf. It is an excellent book and is also free to download.

Hence if we take the limit of the above expression as $x \to 1$ we recover

$$\begin{aligned}
\lim_{x\to 1} S'(x) &= 1+2+3+\cdots+n \\
&= \lim_{x\to 1} \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2} && \text{now use l'Hôpital's rule} \\
&= \lim_{x\to 1} \frac{n(n+1)x^n - n(n+1)x^{n-1}}{2(x-1)} && \text{l'Hôpital's rule again} \\
&= \lim_{x\to 1} \frac{n^2(n+1)x^{n-1} - n(n+1)(n-1)x^{n-2}}{2} \\
&= \frac{n^2(n+1)-n(n-1)(n+1)}{2} = \frac{n(n+1)}{2}
\end{aligned}$$

as required. This computation can be done without l'Hôpital's rule, but the manipulations required are a fair bit messier.

(c) The derivation of the fourth and fifth sums is similar to, but even more tedious than, that of the third sum. One takes two or three derivatives of the generating function.

□

1.1.4 ▸▸ The definition of the definite integral

In this section we give a definition of the definite integral $\int_a^b f(x)\,\mathrm{d}x$, generalising the machinery we used in Example 1.1.1. But first, we introduce some terminology, and make a few remarks to better motivate the definition.

Notation 1.1.9.

The symbols $\int_a^b f(x)\,\mathrm{d}x$ are read "the definite integral of the function $f(x)$ from a to b". The function $f(x)$ is called the integrand of $\int_a^b f(x)\,\mathrm{d}x$ and a and b are called[12] the limits of integration. The interval $a \leqslant x \leqslant b$ is called the interval of integration and is also called the domain of integration.

Before we explain more precisely what the definite integral actually is, a few remarks (actually — a few interpretations) are in order.

- If $f(x) \geqslant 0$ and $a \leqslant b$, one interpretation of the symbol $\int_a^b f(x)\,\mathrm{d}x$ is "the area of the region $\{\ (x,y) \mid a \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x)\ \}$".

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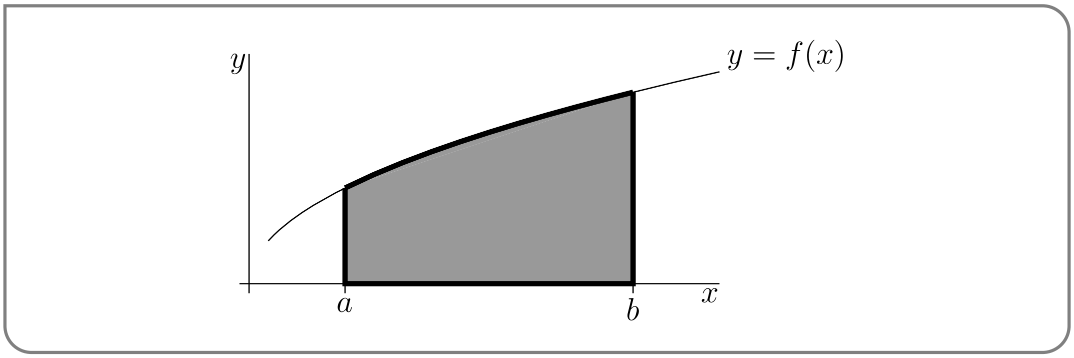
[12] a and b are also called the bounds of integration.



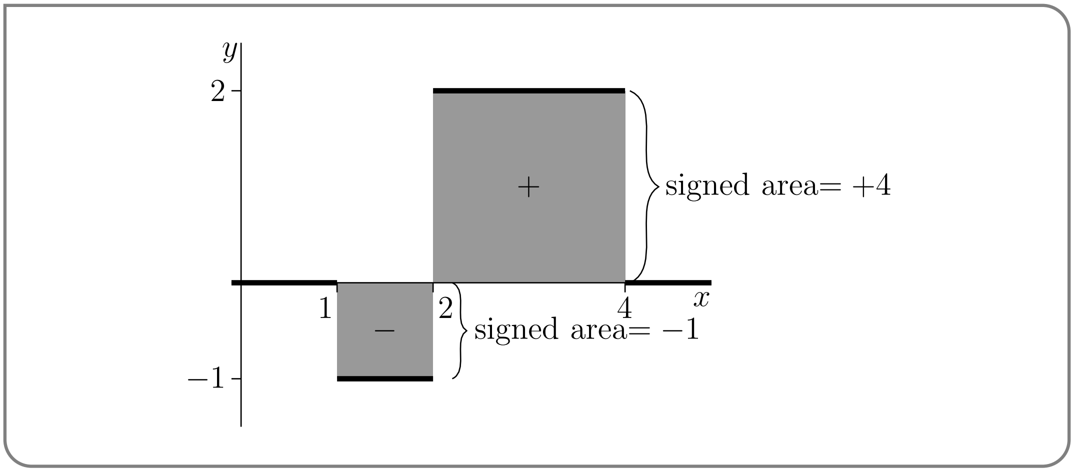
In this way we can rewrite the area in Example 1.1.1 as the definite integral $\int_0^1 e^x \,\mathrm{d}x$.

- This interpretation breaks down when either $a > b$ or $f(x)$ is not always positive, but it can be repaired by considering "signed areas".
- If $a \leqslant b$, one interpretation of $\int_a^b f(x)\,\mathrm{d}x$ is "the signed area between $y = f(x)$ and the x–axis for $a \leqslant x \leqslant b$". For "signed area" (which is also called the "net area"), areas above the x–axis count as positive while areas below the x–axis count as negative. In the example below, we have the graph of the function

$$f(x) = \begin{cases} -1 & \text{if } 1 \leqslant x \leqslant 2 \\ 2 & \text{if } 2 < x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

The 2×2 shaded square above the x–axis has signed area $+2 \times 2 = +4$. The 1×1 shaded square below the x–axis has signed area $-1 \times 1 = -1$. So, for this $f(x)$,

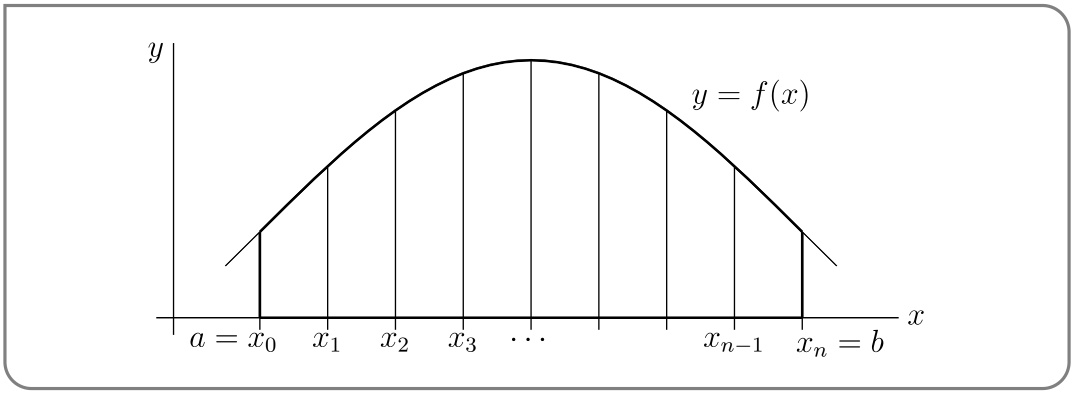
$$\int_0^5 f(x)\,\mathrm{d}x = +4 - 1 = 3$$



- We'll come back to the case $b < a$ later.

We're now ready to define $\int_a^b f(x)\,\mathrm{d}x$. The definition is a little involved, but essentially mimics what we did in Example 1.1.1 (which is why we did the example before the definition). The main differences are that we replace the function e^x by a generic function $f(x)$ and we replace the interval from 0 to 1 by the generic interval[13] from a to b.

- We start by selecting any natural number n and subdividing the interval from a to b into n equal subintervals. Each subinterval has width $\frac{b-a}{n}$.
- Just as was the case in Example 1.1.1, we will eventually take the limit as $n \to \infty$, which squeezes the width of each subinterval down to zero.
- For each integer $0 \leqslant i \leqslant n$, define $x_i = a + i \cdot \frac{b-a}{n}$. Note that this means that $x_0 = a$ and $x_n = b$. It is worth keeping in mind that these numbers x_i do depend on n, even though our choice of notation hides this dependence.
- Subinterval number i is $x_{i-1} \leqslant x \leqslant x_i$. In particular, on the first subinterval, x runs from $x_0 = a$ to $x_1 = a + \frac{b-a}{n}$. On the second subinterval, x runs from x_1 to $x_2 = a + 2\frac{b-a}{n}$.



- On each subinterval, we now pick $x_{i,n}^*$ between x_{i-1} and x_i. We then approximate $f(x)$ on the i^{th} subinterval by the constant function $y = f(x_{i,n}^*)$. We include n in the subscript to remind ourselves that these numbers depend on n.

  Geometrically, we're approximating the region

  $$\{\ (x,y)\ \big|\ x \text{ is between } x_{i-1} \text{ and } x_i, \text{ and } y \text{ is between } 0 \text{ and } f(x)\ \}$$

  by the rectangle

  $$\{\ (x,y)\ \big|\ x \text{ is between } x_{i-1} \text{ and } x_i, \text{ and } y \text{ is between } 0 \text{ and } f(x_{i,n}^*)\ \}$$

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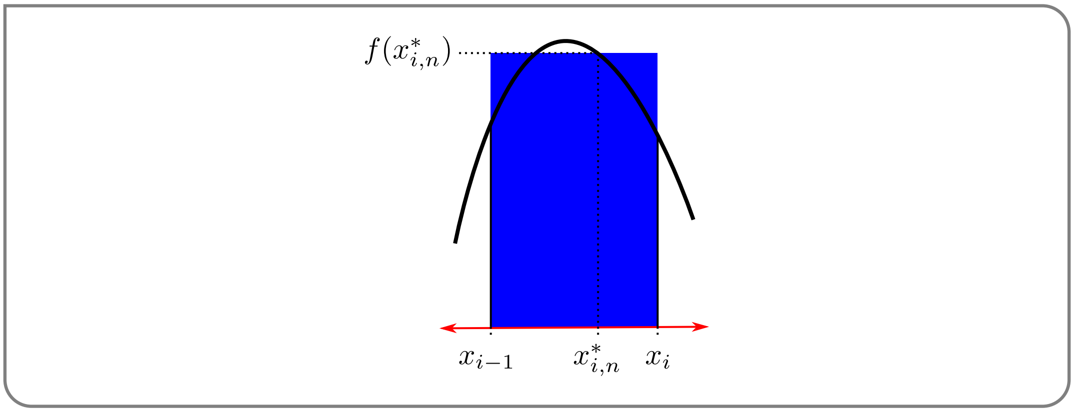
[13] We'll eventually allow a and b to be any two real numbers, not even requiring $a < b$. But it is easier to start off assuming $a < b$, and that's what we'll do.



In Example 1.1.1 we chose $x^*_{i,n} = x_{i-1}$ and so we approximated the function e^x on each subinterval by the value it took at the leftmost point in that subinterval.

- So, when there are n subintervals our approximation to the signed area between the curve $y = f(x)$ and the x–axis, with x running from a to b, is

$$\sum_{i=1}^{n} f(x^*_{i,n}) \cdot \frac{b-a}{n}$$

  We interpret this as the signed area since the summands $f(x^*_{i,n}) \cdot \frac{b-a}{n}$ need not be positive.

- Finally we define the definite integral by taking the limit of this sum as $n \to \infty$.

Oof! This is quite an involved process, but we can now write down the definition we need.

Definition 1.1.10.

Let a and b be two real numbers and let $f(x)$ be a function that is defined for all x between a and b. Then we define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x^*_{i,n}) \cdot \frac{b-a}{n}$$

when the limit exists and takes the same value for all choices of the $x^*_{i,n}$'s. In this case, we say that f is integrable on the interval from a to b.

Of course, it is not immediately obvious when this limit should exist. Thankfully it is easier for a function to be “integrable” than it is for it to be “differentiable”.

Theorem 1.1.11.

Let $f(x)$ be a function on the interval $[a,b]$. If

- $f(x)$ is continuous on $[a,b]$, or
- $f(x)$ has a finite number of jump discontinuities on $[a,b]$ (and is otherwise continuous)

then $f(x)$ is integrable on $[a,b]$.

We will not justify this theorem. But a slightly weaker statement is proved in (the optional) Section 1.1.8. Of course this does not tell us how to actually evaluate any definite integrals — but we will get to that in time.

Some comments:

- Note that, in Definition 1.1.10, we allow a and b to be any two real numbers. We do not require that $a < b$. That is, even when $a > b$, the symbol $\int_a^b f(x)\,\mathrm{d}x$ is still defined by the formula of Definition 1.1.10. We'll get an interpretation for $\int_a^b f(x)\,\mathrm{d}x$, when $a > b$, later.

- It is important to note that the definite integral $\int_a^b f(x)\,\mathrm{d}x$ represents a number, not a function of x. The integration variable x is another "dummy" variable, just like the summation index i in $\sum_{i=m}^n a_i$ (see Section 1.1.3). The integration variable does not have to be called x. For example:

  $$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(u)\,\mathrm{d}u\,.$$

  Just as with summation variables, the integration variable x has no meaning outside of $f(x)\,\mathrm{d}x$.

The sum inside definition 1.1.10 is named after Bernhard Riemann[14] who made the first rigorous definition of the definite integral and so placed integral calculus on rigorous footings.

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[14] Bernhard Riemann was a 19th century German mathematician who made extremely important contributions to many different areas of mathematics — far too many to list here. Arguably two of the most important (after Riemann sums) are now called Riemann surfaces and the Riemann hypothesis. (He didn't name them after himself.)

Definition 1.1.12.

The sum inside definition 1.1.10

$$\sum_{i=1}^{n} f(x_{i,n}^*)\, \frac{b-a}{n}$$

is called a Riemann sum. It is also often written as

$$\sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

where $\Delta x = \frac{b-a}{n}$.

- If we choose each $x_{i,n}^* = x_{i-1} = a + (i-1)\frac{b-a}{n}$ to be the left-hand end point of the i^{th} interval, $[x_{i-1}, x_i]$, we get the approximation

  $$\sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right) \frac{b-a}{n}$$

  which is called the "left Riemann sum approximation to $\int_a^b f(x)\,\mathrm{d}x$ with n subintervals". This is the approximation used in Example 1.1.1.

- In the same way, if we choose $x_{i,n}^* = x_i = a + i\frac{b-a}{n}$ we obtain the approximation

  $$\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \frac{b-a}{n}$$

  which is called the "right Riemann sum approximation to $\int_a^b f(x)\,\mathrm{d}x$ with n subintervals". The word "right" signifies that, on each subinterval $[x_{i-1}, x_i]$ we approximate f by its value at the right–hand end–point, $x_i = a + i\frac{b-a}{n}$, of the subinterval.

- A third commonly used approximation is

  $$\sum_{i=1}^{n} f\left(a + (i-1/2)\frac{b-a}{n}\right) \frac{b-a}{n}$$

  which is called the "midpoint Riemann sum approximation to $\int_a^b f(x)\,\mathrm{d}x$ with n subintervals". The word "midpoint" signifies that, on each subinterval $[x_{i-1}, x_i]$ we approximate f by its value at the midpoint, $\frac{x_{i-1}+x_i}{2} = a + (i-1/2)\frac{b-a}{n}$, of the subinterval.

In order to compute a definite integral using Riemann sums we need to be able to compute the limit of the sum as the number of summands goes to infinity. This approach is not always feasible and we will soon arrive at other means of computing definite integrals. However, Riemann sums also provide us with a good means of approximating definite integrals — if we take n to be a large, but finite, integer, then the corresponding Riemann sum can be a good approximation of the definite integral. Under certain circumstances this can be strengthened to give rigorous bounds on the integral.

Let us revisit Example 1.1.1.

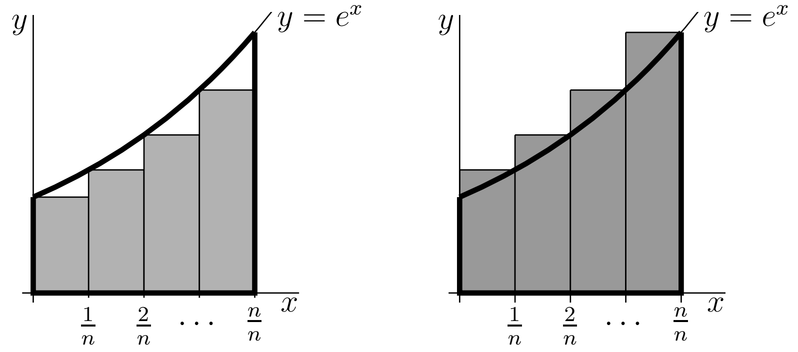
Example 1.1.13

Let's say we are again interested in the integral $\int_0^1 e^x \,\mathrm{d}x$. We can follow the same procedure as we used previously to construct Riemann sum approximations. However, since the integrand $f(x) = e^x$ is an increasing function, we can make our approximations into upper and lower bounds without much extra work.

More precisely, we approximate $f(x)$ on each subinterval $x_{i-1} \leqslant x \leqslant x_i$:

- by its smallest value on the subinterval, namely $f(x_{i-1})$, when we compute the left Riemann sum approximation, and
- by its largest value on the subinterval, namely $f(x_i)$, when we compute the right Riemann sum approximation.

This is illustrated in the two figures below. The shaded region in the left-hand figure is the left Riemann sum approximation and the shaded region in the right-hand figure is the right Riemann sum approximation.



We can see that exactly because $f(x)$ is increasing, the left Riemann sum describes an area smaller than the definite integral, while the right Riemann sum gives an area larger[15] than the integral.

When we approximate the integral $\int_0^1 e^x \,\mathrm{d}x$ using n subintervals, then, on interval number i,

- x runs from $\frac{i-1}{n}$ to $\frac{i}{n}$ and

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[15] When a function is decreasing the situation is reversed — the left Riemann sum is always larger than the integral while the right Riemann sum is smaller than the integral. For more general functions that both increase and decrease it is perhaps easiest to study each increasing (or decreasing) interval separately.

- $y = e^x$ runs from $e^{(i-1)/n}$, when x is at the left-hand end point of the interval, to $e^{i/n}$, when x is at the right-hand end point of the interval.

Consequently, the left Riemann sum approximation to $\int_0^1 e^x\,\mathrm{d}x$ is $\sum_{i=1}^n e^{(i-1)/n}\,\frac{1}{n}$ and the right Riemann sum approximation is $\sum_{i=1}^n e^{i/n}\cdot\frac{1}{n}$. So,

$$\sum_{i=1}^n e^{(i-1)/n}\,\frac{1}{n} \leqslant \int_0^1 e^x\,\mathrm{d}x \leqslant \sum_{i=1}^n e^{i/n}\cdot\frac{1}{n}\,.$$

Thus $L_n = \sum_{i=1}^n e^{(i-1)/n}\,\frac{1}{n}$, which for any n can be evaluated by computer, is a lower bound on the exact value of $\int_0^1 e^x\,\mathrm{d}x$ and $R_n = \sum_{i=1}^n e^{i/n}\,\frac{1}{n}$, which for any n can also be evaluated by computer, is an upper bound on the exact value of $\int_0^1 e^x\,\mathrm{d}x$. For example, when $n = 1000$, $L_n = 1.7174$ and $R_n = 1.7191$ (both to four decimal places) so that, again to four decimal places,

$$1.7174 \leqslant \int_0^1 e^x\,\mathrm{d}x \leqslant 1.7191\,.$$

Recall that the exact value is $e - 1 = 1.718281828\ldots$.

Example 1.1.13

1.1.5 ▸▸ Using spreadsheets to evaluate Riemann sums

In Math 100, you learned some basics of spreadsheet use. Spreadsheets are natural tools for evaluating Riemann sums.

Example 1.1.14

Find a decimal approximation for $\int_0^1 e^x\,\mathrm{d}x$ using a right Riemann sum with 20 rectangles.

Solution. We can write the answer in sigma notation easily enough, but that doesn't automatically translate to what's asked for – a decimal.

The number we want, in sigma notation, is:

$$\sum_{i=1}^{20} \frac{1}{20}\cdot e^{\frac{i}{20}}$$

(as we saw in Example 1.1.13).

Let's set up a spreadsheet to evaluate the sum. Row 1 contains labels. Column A contains the x-values where the function is being approximated. Column B contains the function's values, e^x.

	A	B
1	x	exp(x)
2	`=1/20`	`=exp(A2)`
3	`=A2+1/20`	↓

We copy the values down through Row 21. Now, we evaluate

$$\frac{1}{20}\sum_{i=1}^{20} e^{i/20}$$

by writing

```
=SUM(B2:B21)/20
```

in an empty cell. The result is the decimal 1.761596835.

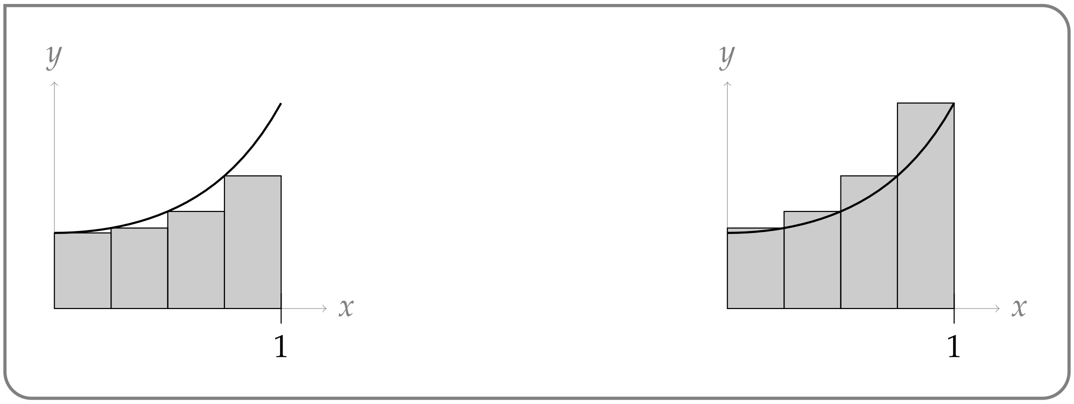
▦ Google sheets link

Example 1.1.14

Example 1.1.15

Let's now approximate another important value: $\int_0^1 e^{x^2}\,\mathrm{d}x$. (You'll see this integral a lot in this course. It's an example of a quantity that's surprisingly difficult to find.)

As in Example 1.1.13, we can bound the quantity between its left and right Riemann sums.



Let's say we want the difference between the left and right Riemann sums to be no more than 0.01. The difference between them is easy to compute without a calculator. For a Riemann sum approximating $\int_a^b f(x)\,\mathrm{d}x$ with n rectangles, the difference (right RS)-(left RS) is just the rightmost rectangle of the right Riemann sum ($\frac{b-a}{n}\cdot f(b)$) minus the leftmost rectangle of the left Riemann sum ($\frac{b-a}{n}\cdot f(a)$). You can see this with a picture, like that above, or with sigma notation:

$$\text{(RRS)} - \text{(LRS)} = \sum_{i=1}^{n}\frac{b-a}{n}\cdot f\left(a+\frac{b-a}{n}i\right) - \sum_{i=1}^{n}\frac{b-a}{n}\cdot f\left(a+\frac{b-a}{n}(i-1)\right)$$

Set the index $j = i - 1$ for the left RS.

$$\begin{aligned}
&= \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + \frac{b-a}{n} i\right) - \sum_{j=0}^{n-1} \frac{b-a}{n} \cdot f\left(a + \frac{b-a}{n}(j)\right) \\
&= \sum_{i=1}^{n-1} \frac{b-a}{n} \cdot f\left(a + \frac{b-a}{n} i\right) + \frac{b-a}{n} \cdot f\left(a + \frac{b-a}{n} n\right) \\
&\qquad - \left[\frac{b-a}{n} \cdot f\left(a + \frac{b-a}{n}(0)\right) + \sum_{j=1}^{n-1} \frac{b-a}{n} \cdot f\left(a + \frac{b-a}{n}(j)\right)\right] \\
&= \frac{b-a}{n} \cdot f\,(b) - \frac{b-a}{n} \cdot f(a)
\end{aligned}$$

So, if we want the difference between the two approximations to be at most 0.01, we should have:

$$\begin{aligned}
& & \frac{1}{n}e - \frac{1}{n} &\leqslant 0.01 \\
&\iff & \frac{1}{n}(e-1) &\leqslant \frac{1}{100} \\
&\iff & \frac{1}{n} &\leqslant \frac{1}{100(e-1)} \\
&\iff & n &\geqslant 100(e-1) \approx 171.8
\end{aligned}$$

So, we can let $n = 172$ (or any larger number).

We set up a spreadsheet to evaluate both Riemann sums.

	A	B
1	x	exp(x∧2)
2	`=0`	`=EXP(A2*A2)`
3	`=A2+1/172`	↓

We copy the above down through row 174. For the left Riemann sum, we write `=SUM(B2:B173)/172` into a blank cell, and get the decimal 1.457672054. For the right Riemann sum, we write `=SUM(B3:B174)/172` into another blank cell, and get the decimal 1.467662065.

We conclude:

$$1.457672054 \leqslant \int_0^1 e^{x^2}\, dx \leqslant 1.467662065$$

Google sheets link

Example 1.1.15

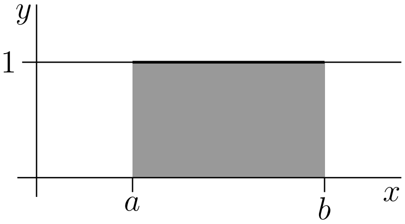
1.1.6 ▸▸ Using known areas to evaluate integrals

One of the main aims of this course is to build up general machinery for computing definite integrals (as well as interpreting and applying them). We shall start on this soon, but not quite yet. We have already seen one concrete, if laborious, method for computing definite integrals — taking limits of Riemann sums as we did in Example 1.1.1. A second method, which will work for some special integrands, works by interpreting the definite integral as “signed area”. This approach will work nicely when the area under the curve decomposes into simple geometric shapes like triangles, rectangles and circles. Here are some examples of this second method.

Example 1.1.16

The integral $\int_a^b 1\mathrm{d}x$ (which is also written as just $\int_a^b \mathrm{d}x$) is the area of the shaded rectangle (of width $b-a$ and height 1) in the figure on the right below. So,

$$\int_a^b \mathrm{d}x = (b-a)\times(1) = b-a$$

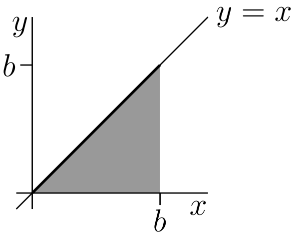


Example 1.1.16

Example 1.1.17

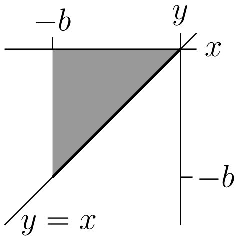
Let $b>0$. The integral $\int_0^b x\,\mathrm{d}x$ is the area of the shaded triangle (of base b and of height b) in the figure on the right below. So,

$$\int_0^b x\,\mathrm{d}x = \frac{1}{2}b\times b = \frac{b^2}{2}$$



The integral $\int_{-b}^0 x\,\mathrm{d}x$ is the signed area of the shaded triangle (again of base b and of height b) in the figure on the right below. So,

$$\int_{-b}^0 x\,\mathrm{d}x = -\frac{b^2}{2}$$



Example 1.1.17

Notice that it is very easy to extend this example to the integral $\int_0^b cx\,\mathrm{d}x$ for any real numbers $b, c > 0$ and find

$$\int_0^b cx\,\mathrm{d}x = \frac{c}{2}b^2.$$

Example 1.1.18

In this example, we shall evaluate $\int_{-1}^{1}(1-|x|)\,\mathrm{d}x$. Recall that

$$|x| = \begin{cases} -x & \text{if } x \leqslant 0 \\ x & \text{if } x \geqslant 0 \end{cases}$$

so that

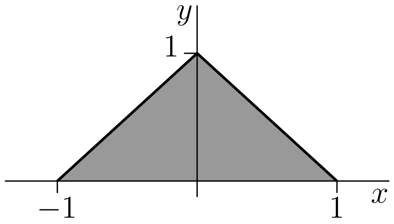
$$1-|x| = \begin{cases} 1+x & \text{if } x \leqslant 0 \\ 1-x & \text{if } x \geqslant 0 \end{cases}.$$

To picture the geometric figure whose area the integral represents, observe that:

- at the left end of the domain of integration $x = -1$ and the integrand $1-|x| = 1-|-1| = 1-1 = 0$;
- as x increases from -1 towards 0, the integrand $1-|x| = 1+x$ increases linearly, until $x = 0$;
- when x hits 0, the integrand hits $1-|x| = 1-|0| = 1$; and then
- as x increases from 0, the integrand $1-|x| = 1-x$ decreases linearly, until
- when x hits $+1$, the right-hand end of the domain of integration, the integrand hits $1-|x| = 1-|1| = 0$.

So, the integral $\int_{-1}^{1}(1-|x|)\,\mathrm{d}x$ is the area of the shaded triangle (of base 2 and of height 1) in the figure on the right below and

$$\int_{-1}^{1}(1-|x|)\,\mathrm{d}x = \frac{1}{2}\times 2\times 1 = 1\,.$$



Example 1.1.18

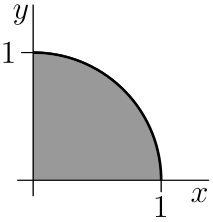
Example 1.1.19

The integral $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$ has integrand $f(x) = \sqrt{1-x^2}$. So, it represents the area under $y = \sqrt{1-x^2}$ with x running from 0 to 1. Note that we may rewrite

$$y = \sqrt{1-x^2} \qquad \text{as} \qquad x^2+y^2 = 1, y \geqslant 0\,.$$

This is the (implicit) equation for a circle — the extra condition that $y \geqslant 0$ makes it the equation for the semi-circle centred at the origin with radius 1 lying on and above the x-axis. Thus the integral represents the area of the quarter circle of radius 1, as shown in the figure on the right below. So,

$$\int_0^1 \sqrt{1-x^2}\mathrm{d}x = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4} .$$



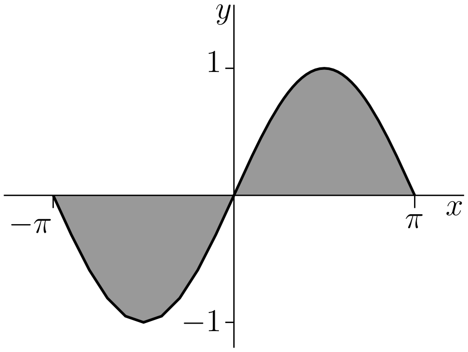
Example 1.1.19

This next one relies on us knowing the symmetries of the sine function.

Example 1.1.20

The integral $\int_{-\pi}^{\pi} \sin x \, \mathrm{d}x$ is the signed area of the shaded region in the figure on the right below. It naturally splits into two regions, one on either side of the y-axis. We don't know the formula for the area of either of these regions (yet), but the two regions are very nearly the same. In fact, the part of the shaded region below the x–axis is exactly the reflection, in the x–axis, of the part of the shaded region above the x–axis. So the signed area of part of the shaded region below the x–axis is the negative of the signed area of part of the shaded region above the x–axis and

$$\int_{-\pi}^{\pi} \sin x \, \mathrm{d}x = 0 .$$



Example 1.1.20

1.1.7 ▸▸ Another interpretation of definite integrals

So far, we have only a single interpretation for definite integrals — namely areas under graphs. If this were the *only* interpretation, then integrals would be a nice mathematical curiosity and unlikely to be the core topic of a large first year mathematics course. In the following example, we develop a second interpretation. More interpretations will follow in Section 1.4.

Example 1.1.21

Suppose that a particle is moving along the x–axis, and suppose that at time t its velocity is $v(t)$ (with $v(t) > 0$ indicating rightward motion and $v(t) < 0$ indicating leftward motion). What is the change in its x–coordinate between time a and time $b > a$?

Solution. We'll work this out using a procedure similar to our definition of the integral. First pick a natural number n and divide the time interval from a to b into n equal subintervals, each of width $\frac{b-a}{n}$. We are working our way towards a Riemann sum (as we have done several times above) and so we will eventually take the limit $n \to \infty$.

- The first time interval runs from a to $a + \frac{b-a}{n}$. If we think of n as some large number, the width of this interval, $\frac{b-a}{n}$ is very small and over this time interval, the velocity does not change very much. Hence we can approximate the velocity over the first subinterval as being essentially constant at its value at the start of the time interval — $v(a)$. Over the subinterval the x-coordinate changes by velocity times time, namely $v(a) \cdot \frac{b-a}{n}$.
- Similarly, the second interval runs from time $a + \frac{b-a}{n}$ to time $a + 2\frac{b-a}{n}$. Again, we can assume that the velocity does not change very much and so we can approximate the velocity as being essentially constant at its value at the start of the subinterval — namely $v\left(a + \frac{b-a}{n}\right)$. So during the second subinterval the particle's x–coordinate changes by approximately $v\left(a + \frac{b-a}{n}\right)\frac{b-a}{n}$.
- In general, time subinterval number i runs from $a + (i-1)\frac{b-a}{n}$ to $a + i\frac{b-a}{n}$ and during this subinterval the particle's x–coordinate changes, essentially, by
$$v\left(a + (i-1)\frac{b-a}{n}\right)\frac{b-a}{n}.$$

So the net change in x–coordinate from time a to time b is approximately

$$\begin{aligned} &v(a)\,\frac{b-a}{n} + v\Big(a + \frac{b-a}{n}\Big)\,\frac{b-a}{n} + \cdots + v\Big(a + (i-1)\frac{b-a}{n}\Big)\,\frac{b-a}{n} + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad + v\Big(a + (n-1)\frac{b-a}{n}\Big)\,\frac{b-a}{n} \\ &= \sum_{i=1}^{n} v\Big(a + (i-1)\frac{b-a}{n}\Big)\,\frac{b-a}{n} \end{aligned}$$

This is exactly the left Riemann sum approximation to the integral of v from a to b with n subintervals. The limit as $n \to \infty$ is exactly the definite integral $\int_a^b v(t)\mathrm{d}t$. Following tradition, we have called the (dummy) integration variable t rather than x to remind us that it is time that is running from a to b.

The conclusion of the above discussion is that if a particle is moving along the x–axis and its x–coordinate and velocity at time t are $x(t)$ and $v(t)$, respectively, then, for all $b > a$,

$$x(b) - x(a) = \int_a^b v(t)\mathrm{d}t.$$

Example 1.1.21

1.1.8 ▸▸ Optional — careful definition of the integral

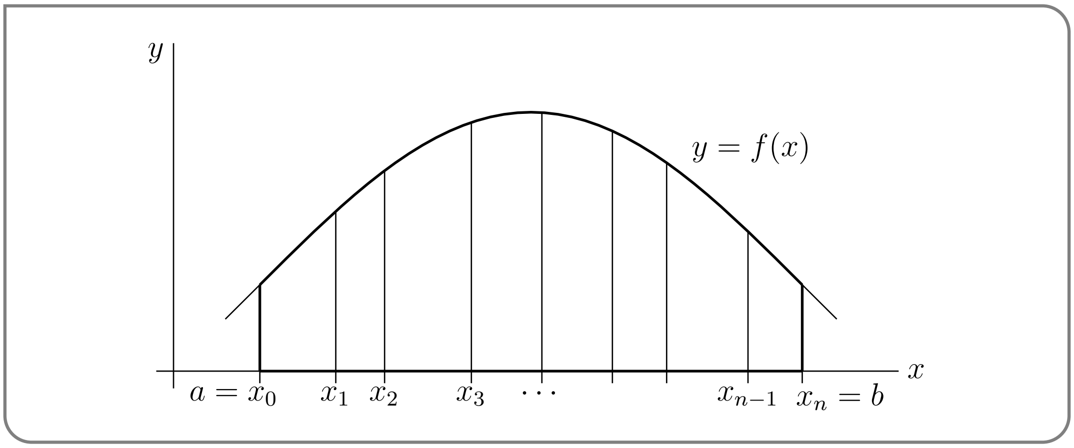
In this optional section we give a more mathematically rigorous definition of the definite integral $\int_a^b f(x)\,\mathrm{d}x$. Some textbooks use a sneakier, but equivalent, definition. The integral will be defined as the limit of a family of approximations to the area between the graph of $y = f(x)$ and the x–axis, with x running from a to b. We will then show conditions under which this limit is guaranteed to exist. We should state up front that these conditions are more restrictive than is strictly necessary — this is done so as to keep the proof accessible.

The family of approximations needed is slightly more general than that used to define Riemann sums in the previous sections, though it is quite similar. The main difference is that we do not require that all the subintervals have the same size.

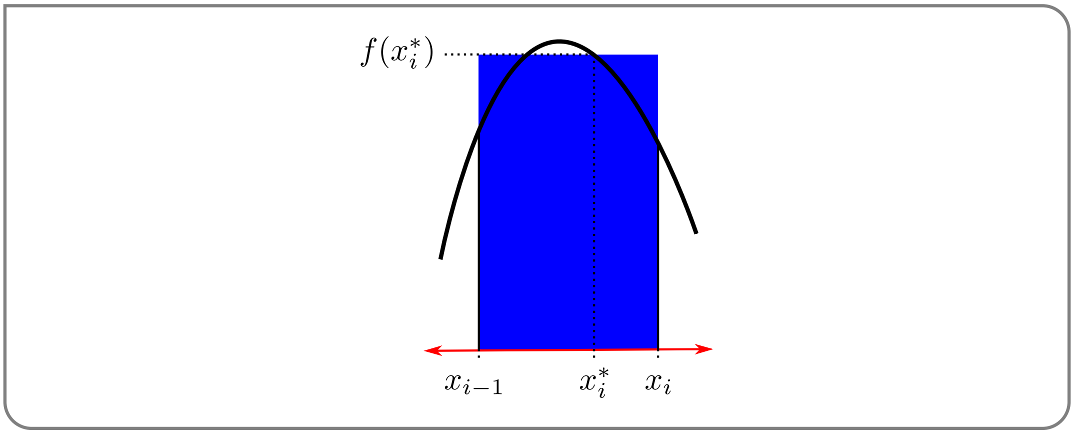
- We start by selecting a positive integer n. As was the case previously, this will be the number of subintervals used in the approximation and eventually we will take the limit as $n \to \infty$.
- Now subdivide the interval from a to b into n subintervals by selecting $n+1$ values of x that obey

  $$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

  The subinterval number i runs from x_{i-1} to x_i. This formulation does not require the subintervals to have the same size. However we will eventually require that the widths of the subintervals shrink towards zero as $n \to \infty$.
- Then for each subinterval we select a value of x in that interval. That is, for $i = 1, 2, \ldots, n$, choose x_i^* satisfying $x_{i-1} \leqslant x_i^* \leqslant x_i$. We will use these values of x to help approximate $f(x)$ on each subinterval.
- The area between the graph of $y = f(x)$ and the x–axis, with x running from x_{i-1}



to x_i, i.e. the contribution, $\int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x$, from interval number i to the integral, is approximated by the area of a rectangle. The rectangle has width $x_i - x_{i-1}$ and height $f(x_i^*)$.



- Thus the approximation to the integral, using all n subintervals, is

$$\int_a^b f(x)\,\mathrm{d}x \approx f(x_1^*)[x_1 - x_0] + f(x_2^*)[x_2 - x_1] + \cdots + f(x_n^*)[x_n - x_{n-1}]$$

- Of course every different choice of n and $x_1, x_2, \ldots, x_{n-1}$ and $x_1^*, x_2^*, \ldots, x_n^*$ gives a different approximation. So to simplify the discussion that follows, let us denote a particular choice of all these numbers by $\mathbb{P}$:

$$\mathbb{P} = (n, x_1, x_2, \cdots, x_{n-1}, x_1^*, x_2^*, \cdots, x_n^*)\,.$$

  Similarly let us denote the resulting approximation by $\mathcal{I}(\mathbb{P})$:

$$\mathcal{I}(\mathbb{P}) = f(x_1^*)[x_1 - x_0] + f(x_2^*)[x_2 - x_1] + \cdots + f(x_n^*)[x_n - x_{n-1}]$$

- We claim that, for any reasonable[16] function $f(x)$, if you take any reasonable[17] sequence of these approximations you always get the exactly the same limiting value. We define $\int_a^b f(x)\,\mathrm{d}x$ to be this limiting value.

- Let's be more precise. We can take the limit of these approximations in two equivalent ways. Above we did this by taking the number of subintervals n to infinity. When we did this, the width of all the subintervals went to zero. With the formulation we are now using, simply taking the number of subintervals to be very large does not imply that they will all shrink in size. We could have one very large subinterval and a large number of tiny ones. Thus we take the limit we need by taking the width of the subintervals to zero. So for any choice $\mathbb{P}$, we define

$$M(\mathbb{P}) = \max\left\{x_1 - x_0\ ,\ x_2 - x_1\ ,\ \cdots\ ,\ x_n - x_{n-1}\right\}$$

  that is the maximum width of the subintervals used in the approximation determined by $\mathbb{P}$. By forcing the maximum width to go to zero, the widths of all the subintervals go to zero.

---

[16] We'll be more precise about what "reasonable" means shortly.

[17] Again, we'll explain this "reasonable" shortly

- We then define the definite integral as the limit

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P}).$$

Of course, one is now left with the question of determining when the above limit exists. A proof of the very general conditions which guarantee existence of this limit is beyond the scope of this course, so we instead give a weaker result (with stronger conditions) which is far easier to prove.

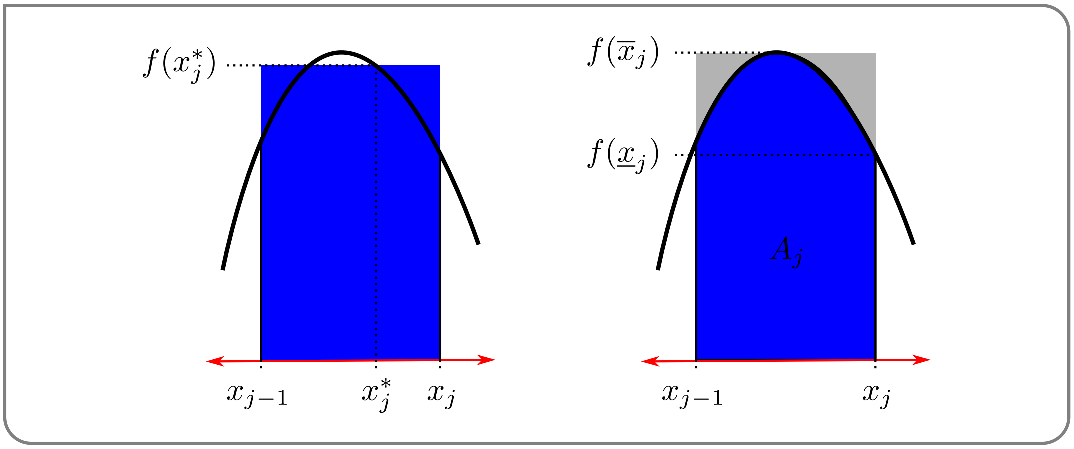
For the rest of this section, assume

- that $f(x)$ is continuous for $a \leqslant x \leqslant b$,
- that $f(x)$ is differentiable for $a < x < b$, and
- that $f'(x)$ is bounded — ie $|f'(x)| \leqslant F$ for some constant F.

We will now show that, under these hypotheses, as $M(\mathbb{P})$ approaches zero, $\mathcal{I}(\mathbb{P})$ always approaches the area, A, between the graph of $y = f(x)$ and the x–axis, with x running from a to b.

These assumptions are chosen to make the argument particularly transparent. With a little more work one can weaken the hypotheses considerably. We are cheating a little by implicitly assuming that the area A exists. In fact, one can adjust the argument below to remove this implicit assumption.

- Consider A_j, the part of the area coming from $x_{j-1} \leqslant x \leqslant x_j$.



We have approximated this area by $f(x_j^*)[x_j - x_{j-1}]$ (see figure left).

- Let $f(\overline{x}_j)$ and $f(\underline{x}_j)$ be the largest and smallest values[18] of $f(x)$ for $x_{j-1} \leqslant x \leqslant x_j$. Then the true area is bounded by

$$f(\underline{x}_j)[x_j - x_{j-1}] \leqslant A_j \leqslant f(\overline{x}_j)[x_j - x_{j-1}].$$

(see figure right).

---

[18] Here we are using the extreme value theorem — its proof is beyond the scope of this course. The theorem says that any continuous function on a closed interval must attain a minimum and maximum at least once. In this situation this implies that for any continuous function $f(x)$, there are $x_{j-1} \leqslant \overline{x}_j, \underline{x}_j \leqslant x_j$ such that $f(\underline{x}_j) \leqslant f(x) \leqslant f(\overline{x}_j)$ for all $x_{j-1} \leqslant x \leqslant x_j$.

- Now since $f(\underline{x}_j) \leqslant f(x_j^*) \leqslant f(\overline{x}_j)$, we also know that

$$f(\underline{x}_j)[x_j - x_{j-1}] \leqslant f(x_j^*)[x_j - x_{j-1}] \leqslant f(\overline{x}_j)[x_j - x_{j-1}].$$

- So both the true area, A_j, and our approximation of that area $f(x_j^*)[x_j - x_{j-1}]$ have to lie between $f(\overline{x}_j)[x_j - x_{j-1}]$ and $f(\underline{x}_j)[x_j - x_{j-1}]$. Combining these bounds we have that the difference between the true area and our approximation of that area is bounded by

$$\left|A_j - f(x_j^*)[x_j - x_{j-1}]\right| \leqslant [f(\overline{x}_j) - f(\underline{x}_j)] \cdot [x_j - x_{j-1}].$$

  (To see this think about the smallest the true area can be and the largest our approximation can be and vice versa.)

- Now since our function $f(x)$ is differentiable we can apply the Mean Value Theorem, which states that for a function continuous on $[a, b]$ and differentiable on (a, b), there exists a number c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  The MVT implies that there exists a c between $\underline{x}_j$ and $\overline{x}_j$ such that

$$f(\overline{x}_j) - f(\underline{x}_j) = f'(c) \cdot [\overline{x}_j - \underline{x}_j]$$

- By the assumption that $|f'(x)| \leqslant F$ for all x and the fact that $\underline{x}_j$ and $\overline{x}_j$ must both be between x_{j-1} and x_j

$$\left|f(\overline{x}_j) - f(\underline{x}_j)\right| \leqslant F \cdot \left|\overline{x}_j - \underline{x}_j\right| \leqslant F \cdot [x_j - x_{j-1}]$$

  Hence the error in this part of our approximation obeys

$$\left|A_j - f(x_j^*)[x_j - x_{j-1}]\right| \leqslant F \cdot [x_j - x_{j-1}]^2.$$

- That was just the error in approximating A_j. Now we bound the total error by combining the errors from approximating on all the subintervals. This gives

$$\begin{aligned}
|A - \mathcal{I}(\mathbb{P})| &= \left|\sum_{j=1}^{n} A_j - \sum_{j=1}^{n} f(x_j^*)[x_j - x_{j-1}]\right| \\
&= \left|\sum_{j=1}^{n} \Big(A_j - f(x_j^*)[x_j - x_{j-1}]\Big)\right| && \text{triangle inequality} \\
&\leqslant \sum_{j=1}^{n} \left|A_j - f(x_j^*)[x_j - x_{j-1}]\right| \\
&\leqslant \sum_{j=1}^{n} F \cdot [x_j - x_{j-1}]^2 && \text{from above}
\end{aligned}$$

Now do something a little sneaky. Replace one of these factors of $[x_j - x_{j-1}]$ (which is just the width of the j^{th} subinterval) by the maximum width of the subintervals:

$$
\begin{aligned}
&\leqslant \sum_{j=1}^{n} F \cdot M(\mathbb{P}) \cdot [x_j - x_{j-1}] && F \text{ and } M(\mathbb{P}) \text{ are constant} \\
&\leqslant F \cdot M(\mathbb{P}) \cdot \sum_{j=1}^{n} [x_j - x_{j-1}] && \text{sum is total width} \\
&= F \cdot M(\mathbb{P}) \cdot (b-a).
\end{aligned}
$$

- Since a, b and F are fixed, this tends to zero as the maximum rectangle width $M(\mathbb{P})$ tends to zero.

Thus, we have proven:

Theorem 1.1.22.

Assume that $f(x)$ is continuous for $a \leqslant x \leqslant b$, and is differentiable for all $a < x < b$ with $|f'(x)| \leqslant F$, for some constant F. Then, as the maximum rectangle width $M(\mathbb{P})$ tends to zero, $\mathcal{I}(\mathbb{P})$ always converges to A, the area between the graph of $y = f(x)$ and the x–axis, with x running from a to b.

1.2▲ Basic properties of the definite integral

Learning Objectives

- Explain (using pictures, words, equations, and inequalities) the arithmetic of integrals, as well as properties of definite integrals.
- Understand that antidifferentiation is linear, in the same way that differentiation is linear.

When we studied limits and derivatives, we developed methods for taking limits or derivatives of “complicated functions” like $f(x) = x^2 + \sin(x)$ by understanding how limits and derivatives interact with basic arithmetic operations like addition and subtraction. This allowed us to reduce the problem into one of of computing derivatives of simpler functions like x^2 and $\sin(x)$. Along the way, we established simple rules such as

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}\,.$$

Some of these rules have very natural analogues for integrals and we discuss them below. Unfortunately, the analogous rules for integrals of products of functions or integrals of compositions of functions are more complicated than those for limits or derivatives. We discuss those rules at length in subsequent sections. For now, let us consider some of the simpler rules of the arithmetic of integrals.

Theorem 1.2.1 (Arithmetic of Integration)**.**

Let a, b and A, B, C be real numbers. Let the functions $f(x)$ and $g(x)$ be integrable on an interval that contains a and b. Then

(a) $$\int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

(b) $$\int_a^b (f(x)-g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$$

(c) $$\int_a^b Cf(x)\,\mathrm{d}x = C\cdot\int_a^b f(x)\,\mathrm{d}x$$

Combining these three rules we have

(d) $$\int_a^b (Af(x)+Bg(x))\,\mathrm{d}x = A\int_a^b f(x)\,\mathrm{d}x + B\int_a^b g(x)\,\mathrm{d}x$$

That is, integrals depend linearly on the integrand.

(e) $$\int_a^b \mathrm{d}x = \int_a^b 1\cdot\mathrm{d}x = b-a$$

It is not too hard to prove this theorem from the definition of the definite integral. Additionally, we only really need to prove (d) and (e) since

- (a) follows from (d) by setting $A = B = 1$,
- (b) follows from (d) by setting $A = 1, B = -1$, and
- (c) follows from (d) by setting $A = C, B = 0$.

Proof. As noted above, it suffices for us to prove (d) and (e). Since (e) is easier, we will start with that. It is also a good warm-up for (d).

- The definite integral in (e), $\int_a^b 1\mathrm{d}x$, can be interpreted geometrically as the area of the rectangle with height 1 running from $x = a$ to $x = b$; this area is clearly $b - a$. We can also prove this formula from the definition of the integral (Definition 1.1.10):

$$\begin{aligned}
\int_a^b \mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i,n}^*)\,\frac{b-a}{n} && \text{by definition}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} 1\,\frac{b-a}{n} && \text{since } f(x)=1\\
&= \lim_{n\to\infty}(b-a)\sum_{i=1}^{n}\frac{1}{n} && \text{since } a,b \text{ are constants}\\
&= \lim_{n\to\infty}(b-a)\\
&= b-a
\end{aligned}$$

as required.

- To prove (d), let us start by defining $h(x) = Af(x) + Bg(x)$. Then, we need to express the integral of $h(x)$ in terms of those of $f(x)$ and $g(x)$. We use Definition 1.1.10 and some algebraic manipulations[19] to arrive at the result.

$$\begin{aligned}
\int_a^b h(x)\mathrm{d}x &= \sum_{i=1}^{n} h(x_{i,n}^*)\cdot\frac{b-a}{n} && \text{by Definition 1.1.10}\\
&= \sum_{i=1}^{n}\big(Af(x_{i,n}^*)+Bg(x_{i,n}^*)\big)\cdot\frac{b-a}{n}\\
&= \sum_{i=1}^{n}\left(Af(x_{i,n}^*)\cdot\frac{b-a}{n}+Bg(x_{i,n}^*)\cdot\frac{b-a}{n}\right)\\
&= \left(\sum_{i=1}^{n}Af(x_{i,n}^*)\cdot\frac{b-a}{n}\right)+\left(\sum_{i=1}^{n}Bg(x_{i,n}^*)\cdot\frac{b-a}{n}\right) && \text{by Theorem 1.1.6(b)}\\
&= A\left(\sum_{i=1}^{n}f(x_{i,n}^*)\cdot\frac{b-a}{n}\right)+B\left(\sum_{i=1}^{n}g(x_{i,n}^*)\cdot\frac{b-a}{n}\right) && \text{by Theorem 1.1.6(a)}\\
&= A\int_a^b f(x)\,\mathrm{d}x + B\int_a^b g(x)\mathrm{d}x && \text{by Definition 1.1.10}
\end{aligned}$$

as required.

□

This theorem allows us to integrate sums, differences and constant multiples of functions that we know how to integrate.

Example 1.2.2

---

[19] Now is a good time to look back at Theorem 1.1.6.

In Example 1.1.1 we saw that $\int_0^1 e^x\,\mathrm{d}x = e-1$. So:

$$\begin{aligned}\int_0^1 (e^x+7)\,\mathrm{d}x &= \int_0^1 e^x\,\mathrm{d}x + 7\int_0^1 1\mathrm{d}x \\ &\qquad\text{by Theorem 1.2.1(d) with } A=1, f(x)=e^x, B=7, g(x)=1 \\ &= (e-1)+7\times(1-0) \\ &\qquad\text{by Example 1.1.1 and Theorem 1.2.1(e)} \\ &= e+6\end{aligned}$$

Example 1.2.2

When we gave the formal definition of $\int_a^b f(x)\,\mathrm{d}x$ in Definition 1.1.10 we explained that the integral could be interpreted as the signed area between the curve $y=f(x)$ and the x-axis on the interval $[a,b]$. In order for this interpretation to make sense we required that $a<b$. Though we remarked that the integral makes sense when $a>b$, we did not explain any further. Thankfully, there is an easy way to express the integral $\int_a^b f(x)\,\mathrm{d}x$ in terms of $\int_b^a f(x)\,\mathrm{d}x$ — making it always possible to write an integral so the lower limit of integration is less than the upper limit of integration. Theorem 1.2.3 below tell us that, for example, $\int_7^3 e^x\,\mathrm{d}x = -\int_3^7 e^x\,\mathrm{d}x$. The same theorem also provides us with two other simple manipulations of the limits of integration.

Theorem 1.2.3 (Arithmetic for the Domain of Integration).

Let a,b,c be real numbers. Let the function $f(x)$ be integrable on an interval that contains a, b and c. Then

(a) $$\int_a^a f(x)\,\mathrm{d}x = 0$$

(b) $$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

(c) $$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

The proof of this statement is not too difficult.

Proof. Let us prove the statements in order.

- Consider the definition of the definite integral

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^n f(x_{i,n}^*)\cdot\frac{b-a}{n}$$

If we now substitute $b = a$ in this expression we have

$$\begin{aligned}\int_a^a f(x)\,\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \underbrace{\frac{a-a}{n}}_{=0} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_{i,n}^*) \cdot 0}_{=0} \\ &= \lim_{n\to\infty} 0 \\ &= 0\end{aligned}$$

as required.

- Consider now the definite integral $\int_a^b f(x)\,\mathrm{d}x$. We will sneak up on the proof by first examining Riemann sum approximations to both this and $\int_b^a f(x)\,\mathrm{d}x$. The midpoint Riemann sum approximation to $\int_a^b f(x)\,\mathrm{d}x$ with 4 subintervals (so that each subinterval has width $\frac{b-a}{4}$) is

$$\begin{aligned}&\left\{f\Big(a+\frac{1}{2}\frac{b-a}{4}\Big) + f\Big(a+\frac{3}{2}\frac{b-a}{4}\Big) + f\Big(a+\frac{5}{2}\frac{b-a}{4}\Big) + f\Big(a+\frac{7}{2}\frac{b-a}{4}\Big)\right\} \cdot \frac{b-a}{4} \\ &= \left\{f\Big(\frac{7}{8}a+\frac{1}{8}b\Big) + f\Big(\frac{5}{8}a+\frac{3}{8}b\Big) + f\Big(\frac{3}{8}a+\frac{5}{8}b\Big) + f\Big(\frac{1}{8}a+\frac{7}{8}b\Big)\right\} \cdot \frac{b-a}{4}\end{aligned}$$

Now we do the same for $\int_b^a f(x)\,\mathrm{d}x$ with 4 subintervals. Note that b is now the lower limit on the integral and a is now the upper limit on the integral. This is likely to cause confusion when we write out the Riemann sum, so we'll temporarily rename b to A and a to B. The midpoint Riemann sum approximation to $\int_A^B f(x)\,\mathrm{d}x$ with 4 subintervals is

$$\begin{aligned}&\left\{f\Big(A+\frac{1}{2}\frac{B-A}{4}\Big) + f\Big(A+\frac{3}{2}\frac{B-A}{4}\Big) + f\Big(A+\frac{5}{2}\frac{B-A}{4}\Big) + f\Big(A+\frac{7}{2}\frac{B-A}{4}\Big)\right\} \cdot \frac{B-A}{4} \\ &= \left\{f\Big(\frac{7}{8}A+\frac{1}{8}B\Big) + f\Big(\frac{5}{8}A+\frac{3}{8}B\Big) + f\Big(\frac{3}{8}A+\frac{5}{8}B\Big) + f\Big(\frac{1}{8}A+\frac{7}{8}B\Big)\right\} \cdot \frac{B-A}{4}\end{aligned}$$

Now recalling that $A = b$ and $B = a$, we have that the midpoint Riemann sum approximation to $\int_b^a f(x)\,\mathrm{d}x$ with 4 subintervals is

$$\left\{f\Big(\frac{7}{8}b+\frac{1}{8}a\Big) + f\Big(\frac{5}{8}b+\frac{3}{8}a\Big) + f\Big(\frac{3}{8}b+\frac{5}{8}a\Big) + f\Big(\frac{1}{8}b+\frac{7}{8}a\Big)\right\} \cdot \frac{a-b}{4}$$

Thus we see that the Riemann sums for the two integrals are nearly identical — the only difference being the factor of $\frac{b-a}{4}$ versus $\frac{a-b}{4}$. Hence the two Riemann sums are negatives of each other.

The same computation with n subintervals shows that the midpoint Riemann sum approximations to $\int_b^a f(x)\,\mathrm{d}x$ and $\int_a^b f(x)\,\mathrm{d}x$ with n subintervals are negatives of each other. Taking the limit $n \to \infty$ gives $\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$.

- Finally consider (c) — we will not give a formal proof of this, but instead will interpret it geometrically. Indeed one can also interpret (a) geometrically. In both cases these become statements about areas:

  $$\int_a^a f(x)\,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

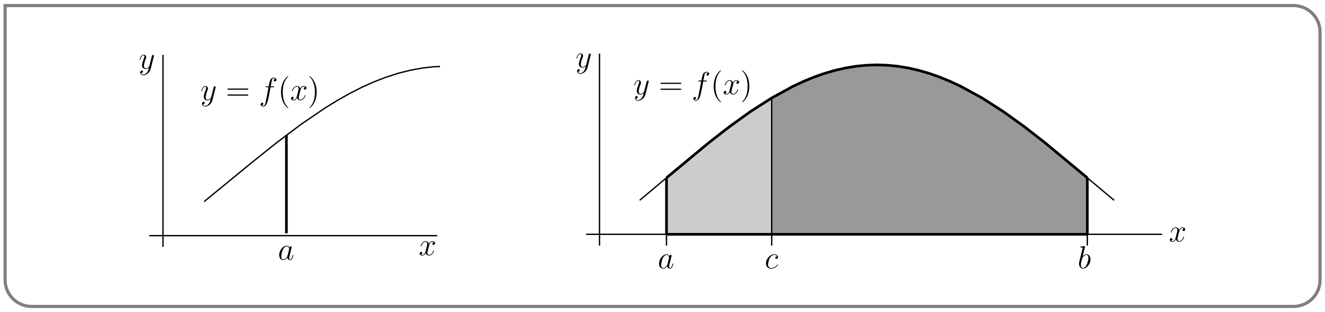
  are

  $$\text{Area}\{\ (x,y)\ \big|\ a \leqslant x \leqslant a,\ 0 \leqslant y \leqslant f(x)\ \} = 0$$

  and

  $$\begin{aligned}\text{Area}\{\ (x,y)\ \big|\ a \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x)\ \} &= \text{Area}\{\ (x,y)\ \big|\ a \leqslant x \leqslant c,\ 0 \leqslant y \leqslant f(x)\ \}\\ &\quad + \text{Area}\{\ (x,y)\ \big|\ c \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x)\ \}\end{aligned}$$

  respectively. Both of these geometric statements are intuitively obvious. See the figures below.

  Note that we have assumed that $a \leqslant c \leqslant b$ and that $f(x) \geqslant 0$. One can remove these restrictions and also make the proof more formal, but it becomes quite tedious and less intuitive.

$\square$

Remark 1.2.4. For notational simplicity, let's assume that $a \leqslant c \leqslant b$ and $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$. The geometric interpretations of the identities

$$\int_a^a f(x)\,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

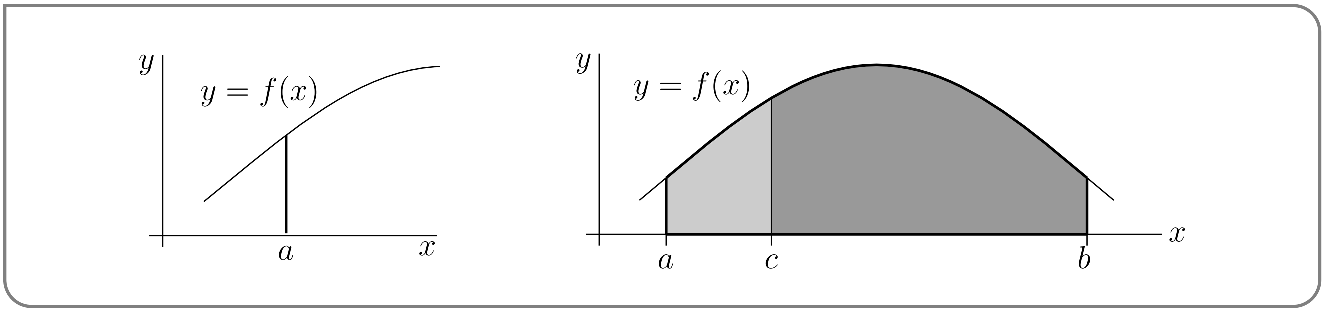
are

$$\text{Area}\{\ (x,y)\ \big|\ a \leqslant x \leqslant a,\ 0 \leqslant y \leqslant f(x)\ \} = 0$$

and

$$\begin{aligned}\text{Area}\{\ (x,y)\ \big|\ a \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x)\ \} &= \text{Area}\{\ (x,y)\ \big|\ a \leqslant x \leqslant c,\ 0 \leqslant y \leqslant f(x)\ \}\\ &\quad + \text{Area}\{\ (x,y)\ \big|\ c \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x)\ \}\end{aligned}$$

respectively. Both of these geometric statements are intuitively obvious. See the figures below.



Example 1.2.5

Back in Example 1.1.17 we saw that when $b > 0$, $\int_0^b x\,\mathrm{d}x = \frac{b^2}{2}$. We'll now verify that $\int_0^b x\,\mathrm{d}x = \frac{b^2}{2}$ is still true when $b = 0$, and also when $b < 0$.

- First consider $b = 0$. Then the statement $\int_0^b x\,\mathrm{d}x = \frac{b^2}{2}$ becomes

  $$\int_0^0 x\,\mathrm{d}x = 0$$

  This is an immediate consequence of Theorem 1.2.3(a).

- Now consider $b < 0$. Let us write $B = -b$, so that $B > 0$. In Example 1.1.17 we saw that

  $$\int_{-B}^0 x\,\mathrm{d}x = -\frac{B^2}{2}.$$

  So we have

  $$\begin{aligned} \int_0^b x\,\mathrm{d}x &= \int_0^{-B} x\,\mathrm{d}x = -\int_{-B}^0 x\,\mathrm{d}x && \text{by Theorem 1.2.3(b)} \\ &= -\left(-\frac{B^2}{2}\right) && \text{by Example 1.1.17} \\ &= \frac{B^2}{2} = \frac{b^2}{2} \end{aligned}$$

We have now shown that

$$\int_0^b x\,\mathrm{d}x = \frac{b^2}{2} \qquad \text{for all real numbers } b.$$

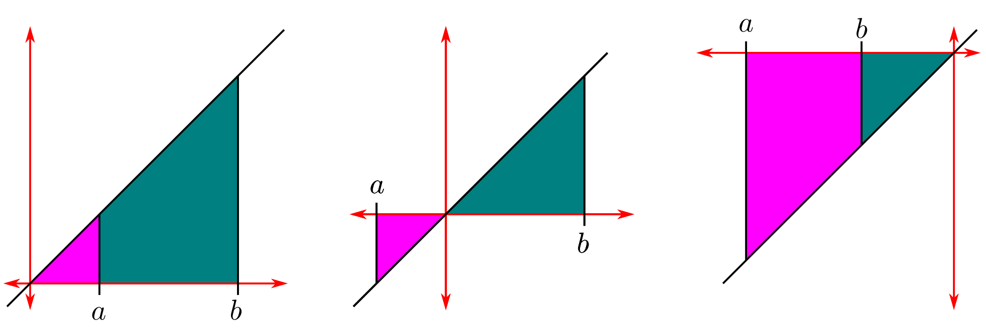
Example 1.2.5

Example 1.2.6

Applying Theorem 1.2.3 yet again, we have, for all real numbers a and b,

$$\begin{aligned}\int_a^b x\,\mathrm{d}x &= \int_a^0 x\,\mathrm{d}x + \int_0^b x\,\mathrm{d}x && \text{by Theorem 1.2.3(c) with } c=0\\ &= \int_0^b x\,\mathrm{d}x - \int_0^a x\,\mathrm{d}x && \text{by Theorem 1.2.3(b)}\\ &= \frac{b^2-a^2}{2} && \text{by Example 1.2.5, twice}\end{aligned}$$

We can also understand this result geometrically.



- (left) When $0 < a < b$, the integral represents the area in green which is the difference of two right–angle triangles — the larger with area $b^2/2$ and the smaller with area $a^2/2$.
- (centre) When $a < 0 < b$, the integral represents the signed area of the two displayed triangles. The one above the axis has area $b^2/2$ while the one below has area $-a^2/2$ (since it is below the axis).
- (right) When $a < b < 0$, the integral represents the signed area in purple of the difference between the two triangles — the larger with area $-a^2/2$ and the smaller with area $-b^2/2$.

Example 1.2.6

Theorem 1.2.3(c) shows us how we can split an integral over a larger interval into one over two (or more) smaller intervals. This is particularly useful for dealing with piecewise functions, like $|x|$.

Example 1.2.7

Using Theorem 1.2.3, we can readily evaluate integrals involving $|x|$. First, recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0\\ -x & \text{if } x < 0\end{cases}$$

Now consider (for example) $\int_{-2}^{3} |x| \mathrm{d}x$. Since the integrand changes at $x = 0$, it makes sense to split the interval of integration at that point:

$$\begin{aligned} \int_{-2}^{3} |x| \mathrm{d}x &= \int_{-2}^{0} |x| \mathrm{d}x + \int_{0}^{3} |x| \mathrm{d}x && \text{by Theorem 1.2.3} \\ &= \int_{-2}^{0} (-x) \mathrm{d}x + \int_{0}^{3} x \, \mathrm{d}x && \text{by definition of } |x| \\ &= -\int_{-2}^{0} x \, \mathrm{d}x + \int_{0}^{3} x \, \mathrm{d}x && \text{by Theorem 1.2.1(c)} \\ &= -(-2^2/2) + (3^2/2) = (4+9)/2 \\ &= 13/2 \end{aligned}$$

We can go further still — given a function $f(x)$ we can rewrite the integral of $f(|x|)$ in terms of the integral of $f(x)$ and $f(-x)$.

$$\begin{aligned} \int_{-1}^{1} f\big(|x|\big) \mathrm{d}x &= \int_{-1}^{0} f\big(|x|\big) \mathrm{d}x + \int_{0}^{1} f\big(|x|\big) \mathrm{d}x \\ &= \int_{-1}^{0} f(-x) \mathrm{d}x + \int_{0}^{1} f(x) \, \mathrm{d}x \end{aligned}$$

Example 1.2.7

Here is a more concrete example.

Example 1.2.8

Let us compute $\int_{-1}^{1} \big(1 - |x|\big) \mathrm{d}x$ again. In Example 1.1.18 we evaluated this integral by interpreting it as the area of a triangle. This time we are going to use *only* the properties given in Theorems 1.2.1 and 1.2.3 and the facts that

$$\int_{a}^{b} \mathrm{d}x = b - a \qquad \text{and} \qquad \int_{a}^{b} x \, \mathrm{d}x = \frac{b^2 - a^2}{2}$$

That $\int_a^b \mathrm{d}x = b - a$ is part (e) of Theorem 1.2.1. We saw that $\int_a^b x \, \mathrm{d}x = \frac{b^2-a^2}{2}$ in Example 1.2.6.

First we are going to get rid of the absolute value signs by splitting the interval over which we integrate. Recalling that $|x| = x$ whenever $x \geqslant 0$ and $|x| = -x$ whenever $x \leqslant 0$, we split the interval by Theorem 1.2.3(c),

$$\begin{aligned} \int_{-1}^{1} \big(1 - |x|\big) \mathrm{d}x &= \int_{-1}^{0} \big(1 - |x|\big) \mathrm{d}x + \int_{0}^{1} \big(1 - |x|\big) \mathrm{d}x \\ &= \int_{-1}^{0} \big(1 - (-x)\big) \mathrm{d}x + \int_{0}^{1} (1 - x) \mathrm{d}x \\ &= \int_{-1}^{0} (1 + x) \mathrm{d}x + \int_{0}^{1} (1 - x) \mathrm{d}x \end{aligned}$$

Now we apply parts (a) and (b) of Theorem 1.2.1, and then

$$\begin{aligned}\int_{-1}^{1}\left[1-|x|\right]\mathrm{d}x &= \int_{-1}^{0} 1\mathrm{d}x + \int_{-1}^{0} x\,\mathrm{d}x + \int_{0}^{1} 1\mathrm{d}x - \int_{0}^{1} x\,\mathrm{d}x \\ &= [0-(-1)] + \frac{0^2-(-1)^2}{2} + [1-0] - \frac{1^2-0^2}{2} \\ &= 1\end{aligned}$$

Example 1.2.8

1.2.1 ▸▸ More properties of integration: even and odd functions

Recall[20] the following definition

Definition 1.2.9.

Let $f(x)$ be a function. Then,

- we say that $f(x)$ is even when $f(x) = f(-x)$ for all x, and
- we say that $f(x)$ is odd when $f(x) = -f(-x)$ for all x.

Of course most functions are neither even nor odd, but many of the standard functions you know are.

Example 1.2.10 (Even functions)

- Three examples of even functions are $f(x) = |x|$, $f(x) = \cos x$ and $f(x) = x^2$. In fact, if $f(x)$ is any even power of x, then $f(x)$ is an even function.
- The part of the graph $y = f(x)$ with $x \leqslant 0$ may be constructed by drawing the part of the graph with $x \geqslant 0$ (as in the figure on the left below) and then reflecting it in the y–axis (as in the figure on the right below).

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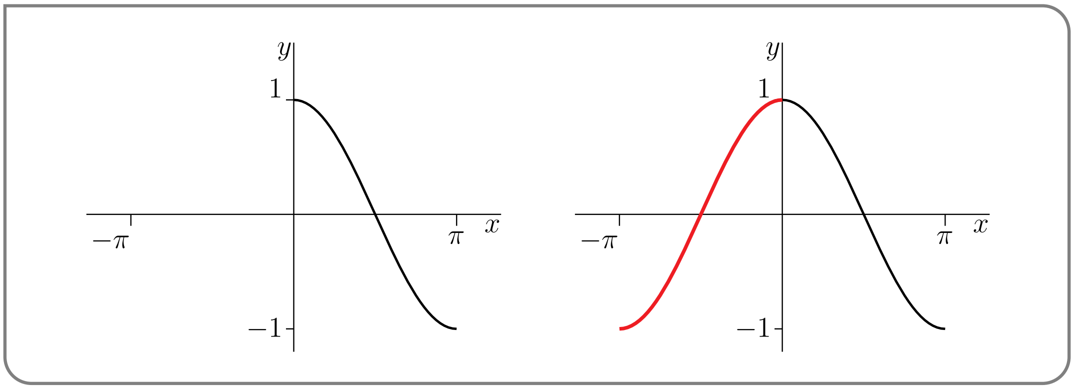
[20] You should have seen this definition in your differential calculus course, or perhaps even earlier.



- In particular, if $f(x)$ is an even function and $a > 0$, then the two sets

$$\left\{ (x,y) \bigm| 0 \leqslant x \leqslant a \text{ and } y \text{ is between } 0 \text{ and } f(x) \right\}$$
$$\left\{ (x,y) \bigm| -a \leqslant x \leqslant 0 \text{ and } y \text{ is between } 0 \text{ and } f(x) \right\}$$

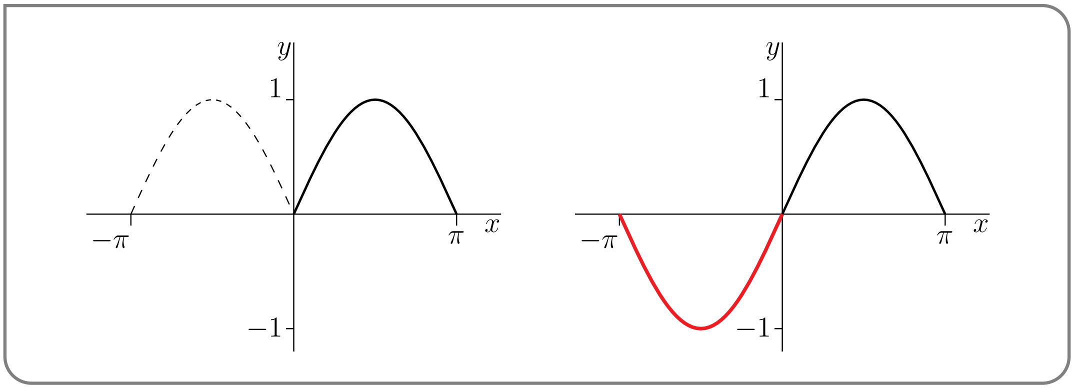
  are reflections of each other in the y–axis and so have the same signed area. That is

$$\int_0^a f(x)\,\mathrm{d}x = \int_{-a}^0 f(x)\,\mathrm{d}x$$

Example 1.2.10

Example 1.2.11 (Odd functions)

- Three examples of odd functions are $f(x) = \sin x$, $f(x) = \tan x$ and $f(x) = x^3$. In fact, if $f(x)$ is any odd power of x, then $f(x)$ is an odd function.
- The part of the graph $y = f(x)$ with $x \leqslant 0$, may be constructed by drawing the part of the graph with $x \geqslant 0$ (like the solid line in the figure on the left below) and then reflecting it in the y–axis (like the dashed line in the figure on the left below) and then reflecting the result in the x–axis (i.e. flipping it upside down, like in the figure on the right, below).



- In particular, if $f(x)$ is an odd function and $a > 0$, then the signed areas of the two sets

$$\left\{ (x,y) \,\middle|\, 0 \leqslant x \leqslant a \text{ and } y \text{ is between } 0 \text{ and } f(x) \right\}$$
$$\left\{ (x,y) \,\middle|\, -a \leqslant x \leqslant 0 \text{ and } y \text{ is between } 0 \text{ and } f(x) \right\}$$

are negatives of each other — to get from the first set to the second set, you flip it upside down, in addition to reflecting it in the y–axis. That is

$$\int_0^a f(x)\,\mathrm{d}x = -\int_{-a}^0 f(x)\,\mathrm{d}x$$

Example 1.2.11

We can exploit the symmetries noted in the examples above. Namely:

$$\int_0^a f(x)\,\mathrm{d}x = \int_{-a}^0 f(x)\,\mathrm{d}x \qquad \text{for } f \text{ even}$$
$$\int_0^a f(x)\,\mathrm{d}x = -\int_{-a}^0 f(x)\,\mathrm{d}x \qquad \text{for } f \text{ odd}$$

together with Theorem 1.2.3

$$\int_{-a}^a f(x)\,\mathrm{d}x = \int_{-a}^0 f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x$$

can simplify the integration of even and odd functions over intervals of the form $[-a, a]$.

Theorem 1.2.12 (Even and Odd).

Let $a > 0$.

(a) If $f(x)$ is an even function, then

$$\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$$

(b) If $f(x)$ is an odd function, then

$$\int_{-a}^a f(x)\,\mathrm{d}x = 0$$

Proof. For any function

$$\int_{-a}^a f(x)\,\mathrm{d}x = \int_0^a f(x)\,\mathrm{d}x + \int_{-a}^0 f(x)\,\mathrm{d}x$$

When f is even, the two terms on the right-hand side are equal. When f is odd, the two terms on the right-hand side are negatives of each other. ☐

1.2.2 ▸▸ More properties of integration: inequalities for integrals

We are still unable to integrate many functions, but with a little work we can infer bounds on integrals from bounds on their integrands.

Theorem 1.2.13 (Inequalities for Integrals)**.**

Let $a \leqslant b$ be real numbers and let the functions $f(x)$ and $g(x)$ be integrable on the interval $a \leqslant x \leqslant b$.

(a) If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,\mathrm{d}x \geqslant 0$$

(b) If $f(x) \leqslant g(x)$ for all $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$$

(c) If there are constants m and M such that $m \leqslant f(x) \leqslant M$ for all $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,\mathrm{d}x \leqslant M(b-a)$$

(d) We have

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\,\mathrm{d}x$$

Proof. (a) By interpreting the integral as the signed area, this statement simply says that if the curve $y = f(x)$ lies above the x–axis and $a \leqslant b$, then the signed area of the set $\{\, (x,y) \mid a \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x) \,\}$ is at least zero. This is quite clear. Alternatively, we could argue more algebraically from Definition 1.1.10. We observe that when we define $\int_a^b f(x)\,\mathrm{d}x$ via Riemann sums, every summand, $f(x_{i,n}^*)\,\frac{b-a}{n} \geqslant 0$. Thus the whole sum is nonnegative and consequently, so is the limit, and thus so is the integral.

(b) We are assuming that $g(x) - f(x) \geqslant 0$, so part (a) gives

$$\begin{aligned}\int_a^b \big[g(x) - f(x)\big]\mathrm{d}x \geqslant 0 &\implies \int_a^b g(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \geqslant 0 \\ &\implies \int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x\end{aligned}$$

(c) Applying part (b) with $g(x) = M$ for all $a \leqslant x \leqslant b$ gives

$$\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b M\,\mathrm{d}x = M(b-a)$$

Similarly, viewing m as a (constant) function, and applying part (b) gives

$$m \leqslant f(x) \implies \overbrace{\int_a^b m\,\mathrm{d}x}^{=m(b-a)} \leqslant \int_a^b f(x)\,\mathrm{d}x$$

(d) For any x, $|f(x)|$ is either $f(x)$ or $-f(x)$ (depending on whether $f(x)$ is positive or negative), so we certainly have

$$f(x) \leqslant |f(x)| \qquad \text{and} \qquad -f(x) \leqslant |f(x)|$$

Applying part (c) to each of those inequalities gives

$$\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b |f(x)|\,\mathrm{d}x \qquad \text{and} \qquad -\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b |f(x)|\,\mathrm{d}x$$

Now $\left|\int_a^b f(x)\,\mathrm{d}x\right|$ is either equal to $\int_a^b f(x)\,\mathrm{d}x$ or $-\int_a^b f(x)\,\mathrm{d}x$ (depending on whether the integral is positive or negative). In either case we can apply the above two inequalities to get the same result, namely

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\,\mathrm{d}x.$$

□

Example 1.2.14 $\left(\int_0^{\pi/3} \sqrt{\cos x}\mathrm{d}x\right)$

Consider the integral

$$\int_0^{\pi/3} \sqrt{\cos x}\mathrm{d}x$$

This is not so easy to compute exactly[21], but we can bound it quite quickly.

For x between 0 and $\frac{\pi}{3}$, the function $\cos x$ takes values[22] between 1 and $\frac{1}{2}$. Thus the function $\sqrt{\cos x}$ takes values between 1 and $\frac{1}{\sqrt{2}}$. That is:

$$\frac{1}{\sqrt{2}} \leqslant \sqrt{\cos x} \leqslant 1 \qquad \text{for } 0 \leqslant x \leqslant \frac{\pi}{3}.$$

---

[21] It is not too hard to use Riemann sums and a computer to evaluate it numerically: 0.948025319….

[22] You know the graphs of sine and cosine, so you should be able to work this out without too much difficulty.

Consequently, by Theorem 1.2.13(c) with $a = 0$, $b = \frac{\pi}{3}$, $m = \frac{1}{\sqrt{2}}$ and $M = 1$,

$$\frac{\pi}{3\sqrt{2}} \leqslant \int_0^{\pi/3} \sqrt{\cos x}\mathrm{d}x \leqslant \frac{\pi}{3}$$

Plugging these expressions into a calculator gives us

$$0.7404804898 \leqslant \int_0^{\pi/3} \sqrt{\cos x}\mathrm{d}x \leqslant 1.047197551 \ .$$

Example 1.2.14

1.3▲ The fundamental theorem of calculus

Learning Objectives

- Understand what an area function of the form $\int_a^x f(t)\,\mathrm{d}t$ is, and compute them for simple functions using geometry.
- Given a function, sketch the area function $A(x)$. Pay special attention to where the slope of $A(x)$ is relatively large and relatively small.
- Produce a compelling argument that $A(x) = \int_a^x f(t)\,\mathrm{d}t$ should satisfy $A'(x) = f(x)$ if f is continuous at x; illustrate what can go wrong if f has a simple jump discontinuity at x.
- State the fundamental theorem of calculus, part 1 (FTC1).
- Use FTC1 to differentiate a function defined as a definite integral (area function).
- Use FTC1 to prove the fundamental theorem of calculus, part 2 (FTC2).
- Use FTC2 to compute definite integrals.
- Define the indefinite integral and explain how it differs from the definite integral.
- Explain why antiderivatives are not unique.
- Understand what an antiderivative (also called indefinite integral) is.
- Find antiderivatives of basic functions by inspection. (In particular, those important integrals listed in Theorem 1.3.16.)
- Find antiderivatives of polynomials using the power rule.

We have spent quite a few pages (and lectures) talking about definite integrals, what they are (Definition 1.1.10), when they exist (Theorem 1.1.11), how to compute some special cases (Section 1.1.6), some ways to manipulate them (Theorem 1.2.1 and 1.2.3) and how to bound them (Theorem 1.2.13). Conspicuously missing from all of this has been a discussion of how to compute them in general. It is high time we rectified that.

The single most important tool used to evaluate integrals is called "the fundamental theorem of calculus". Its grand name is justified — it links the two branches of calculus by connecting derivatives to integrals. In so doing it also tells us how to compute integrals. Very roughly speaking, the derivative of an integral is the original function. This fact allows us to compute integrals using antiderivatives[23]. Of course "very rough" is not enough — let's be precise.

Theorem 1.3.1 (Fundamental theorem of calculus)**.**

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$.

Part 1: Let $F(x) = \displaystyle\int_a^x f(t)\mathrm{d}t$ for any $x \in [a, b]$. Then the function $F(x)$ is differentiable and further

$$F'(x) = f(x)$$

Part 2: Let $G(x)$ be any function which is defined and continuous on $[a, b]$. Further let $G(x)$ be differentiable with $G'(x) = f(x)$ for all $a < x < b$. Then

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a) \quad \text{or equivalently} \quad \int_a^b G'(x)\,\mathrm{d}x = G(b) - G(a)$$

Before we prove this theorem and look at a bunch of examples of its application, it is important that we recall one definition from differential calculus — antiderivatives. If $F'(x) = f(x)$ on some interval, then $F(x)$ is called an antiderivative of $f(x)$ on that interval. So Part 2 of the fundamental theorem of calculus tells us how to evaluate the definite integral of $f(x)$ in terms of any of its antiderivatives — if $G(x)$ is any antiderivative of $f(x)$ then

$$\int_a^b f(x)\mathrm{d}x = G(b) - G(a)$$

The form $\int_a^b G'(x)\,\mathrm{d}x = G(b) - G(a)$ of the fundamental theorem relates the rate of change of $G(x)$ over the interval $a \leqslant x \leqslant b$ to the net change of G between $x = a$ and $x = b$. For that reason, it is sometimes called the "net change theorem".

We'll start with a simple example. Then we'll see why the fundamental theorem is true and then we'll do many more, and more involved, examples.

---

[23] You learned these near the end of your differential calculus course. Now is a good time to revise — but we'll go over them here since they are so important in what follows.

Example 1.3.2 (A first example)

Consider the integral $\int_a^b x\,\mathrm{d}x$ which we have explored previously in Example 1.2.6.

- The integrand is $f(x) = x$.
- We can readily verify that $G(x) = \frac{x^2}{2}$ satisfies $G'(x) = f(x)$ and so is an antiderivative of the integrand.
- Part 2 of Theorem 1.3.1 then tells us that

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= G(b) - G(a)\\ \int_a^b x\,\mathrm{d}x &= \frac{b^2}{2} - \frac{a^2}{2}\end{aligned}$$

which is precisely the result we obtained (with more work) in Example 1.2.6.

Example 1.3.2

We do not give completely rigorous proofs of the two parts of the theorem — that is not really needed for this course. We just give the main ideas of the proofs so that you can understand why the theorem is true.

Part 1. We wish to show that if

$$F(x) = \int_a^x f(t)\mathrm{d}t \qquad \text{then} \qquad F'(x) = f(x)$$

- Assume that F is the above integral and then consider $F'(x)$. By definition

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

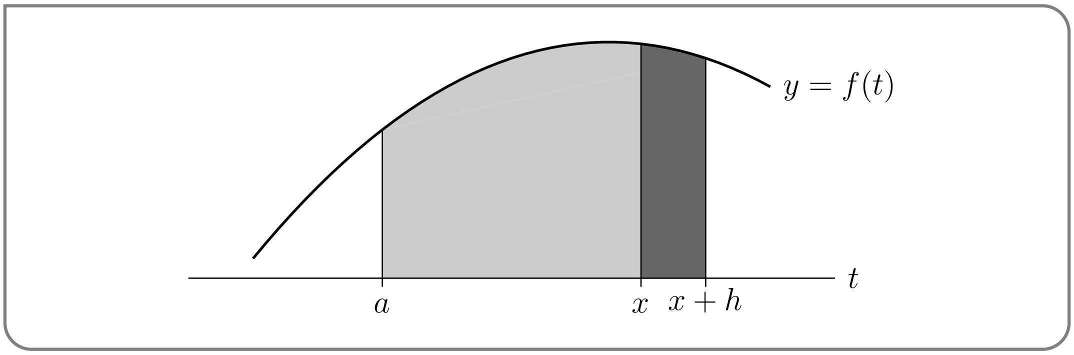
- To understand this limit, we interpret the terms $F(x), F(x+h)$ as signed areas. To simplify this further, let's only consider the case that f is always nonnegative and that $h > 0$. These restrictions are not hard to remove, but the proof ideas are a bit cleaner if we keep them in place. Then we have

$$\begin{aligned}F(x+h) &= \text{the area of the region } \left\{ (t,y) \;\middle|\; a \leqslant t \leqslant x+h,\ 0 \leqslant y \leqslant f(t) \right\}\\ F(x) &= \text{the area of the region } \left\{ (t,y) \;\middle|\; a \leqslant t \leqslant x, \quad 0 \leqslant y \leqslant f(t) \right\}\end{aligned}$$

- Then the numerator

$$F(x+h) - F(x) = \text{the area of the region } \left\{ (t,y) \;\middle|\; x \leqslant t \leqslant x+h,\ 0 \leqslant y \leqslant f(t) \right\}$$

This is just the more darkly shaded region in the figure



- We will be taking the limit $h \to 0$. So suppose that h is very small. Then, as t runs from x to $x+h$, $f(t)$ runs only over a very narrow range of values[24], all close to $f(x)$.
- So the darkly shaded region is almost a rectangle of width h and height $f(x)$ and so has an area which is very close to $f(x)h$. Thus $\frac{F(x+h)-F(x)}{h}$ is very close to $f(x)$.
- In the limit $h \to 0$, $\frac{F(x+h)-F(x)}{h}$ becomes exactly $f(x)$, which is precisely what we want.

$\square$

Part 2. We want to show that $\int_a^b f(t)\mathrm{d}t = G(b) - G(a)$. To do this we exploit the fact that the derivative of a constant is zero.

- Let

  $$H(x) = \int_a^x f(t)\mathrm{d}t - G(x) + G(a)$$

  Then the result we wish to prove is that $H(b) = 0$. We will do this by showing that $H(x) = 0$ for all x between a and b.

- We first show that $H(x)$ is constant by computing its derivative:

  $$H'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}(G(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(G(a))$$

  Since $G(a)$ is a constant, its derivative is 0 and by assumption the derivative of $G(x)$ is just $f(x)$, so

  $$= \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t - f(x)$$

  Now Part 1 of the theorem tells us that this derivative is just $f(x)$, so

  $$= f(x) - f(x) = 0$$

  Hence H is constant.

---

[24] Notice that if f were discontinuous, then this might be false.

- To determine which constant we just compute $H(a)$:

$$\begin{aligned} H(a) &= \int_a^a f(t)\mathrm{d}t - G(a) + G(a) \\ &= \int_a^a f(t)\mathrm{d}t && \text{by Theorem 1.2.3(a)} \\ &= 0 \end{aligned}$$

as required.

□

The simple example we did above (Example 1.3.2) demonstrates the application of part 2 of the fundamental theorem of calculus. Before we do more examples (and there will be many more over the coming sections), we should do some examples illustrating the use of part 1 of the fundamental theorem of calculus.

Example 1.3.3 $\left(\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t\mathrm{d}t\right)$

Consider the integral $\int_0^x t\,\mathrm{d}t$. We know how to evaluate this — it is just Example 1.3.2 with $a = 0$, $b = x$. So we have two ways to compute the derivative. We can evaluate the integral and then take the derivative, or we can apply Part 1 of the fundamental theorem. We'll do both, and check that the two answers are the same.

First, Example 1.3.2 gives

$$F(x) = \int_0^x t\,\mathrm{d}t = \frac{x^2}{2}$$

So, of course, $F'(x) = x$. Second, Part 1 of the fundamental theorem of calculus tells us that the derivative of $F(x)$ is just the integrand. That is, Part 1 of the fundamental theorem of calculus also gives $F'(x) = x$.

Example 1.3.3

In the previous example we were able to evaluate the integral explicitly, so we did not need the fundamental theorem to determine its derivative. Here is an example that really does require the use of the fundamental theorem.

Example 1.3.4 $\left(\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{-t^2}\mathrm{d}t\right)$

We would like to find $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{-t^2}\mathrm{d}t$. In the previous example, we were able to compute the corresponding derivative in two ways — we could explicitly compute the integral and then differentiate the result, or we could apply part 1 of the fundamental theorem of calculus. In this example we do not know the integral explicitly. Indeed, it is not possible to express[25] the integral $\int_0^x e^{-t^2}\mathrm{d}t$ as a finite combination of standard functions such as

---

[25] The integral $\int_0^x e^{-t^2}\mathrm{d}t$ is closely related to the "error function" which is an extremely important function in mathematics. While we cannot express this integral (or the error function) as a *finite* combination of polynomials, exponentials etc, we can express it as an infinite series:

$$\int_0^x e^{-t^2}\mathrm{d}t = x - \frac{x^3}{3\cdot 1} + \frac{x^5}{5\cdot 2} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \cdots + (-1)^k\frac{x^{2k+1}}{(2k+1)\cdot k!} + \cdots$$

polynomials, exponentials, trigonometric functions and so on.

Despite this, we can find its derivative by just applying the first part of the fundamental theorem of calculus with $f(t) = e^{-t^2}$ and $a = 0$. That gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{-t^2}\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\mathrm{d}t \\ &= f(x) = e^{-x^2}\end{aligned}$$

Example 1.3.4

Let us ratchet up the complexity of the previous example — we can make the limits of the integral more complicated functions. So consider the previous example with the upper limit x replaced by x^2:

Example 1.3.5 $\left(\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\mathrm{d}t\right)$

Consider the integral $\int_0^{x^2} e^{-t^2}\mathrm{d}t$. We would like to compute its derivative with respect to x using part 1 of the fundamental theorem of calculus.

The fundamental theorem tells us how to compute the derivative of functions of the form $\int_a^x f(t)\mathrm{d}t$ but the integral at hand is *not* of the specified form because the upper limit we have is x^2, rather than x, — so more care is required. Thankfully we can deal with this obstacle with only a little extra work. The trick is to define an auxiliary function by simply changing the upper limit to x. That is, define

$$E(x) = \int_0^x e^{-t^2}\mathrm{d}t$$

Then the integral we want to work with is

$$E(x^2) = \int_0^{x^2} e^{-t^2}\mathrm{d}t$$

The derivative $E'(x)$ can be found via part 1 of the fundamental theorem of calculus (as we did in Example 1.3.4) and is $E'(x) = e^{-x^2}$. We can then use this fact with the chain rule to compute the derivative we need:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}E(x^2) && \text{use the chain rule} \\ &= 2xE'(x^2) \\ &= 2xe^{-x^4}\end{aligned}$$

Example 1.3.5

But more on this in Chapter 3.

What if both limits of integration are functions of x? We can still make this work, but we have to split the integral using Theorem 1.2.3.

Example 1.3.6 $\left(\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{-t^2}\mathrm{d}t\right)$

Consider the integral

$$\int_x^{x^2} e^{-t^2}\mathrm{d}t$$

As was the case in the previous example, we have to do a little pre-processing before we can apply the fundamental theorem.

This time (by design), not only is the upper limit of integration x^2 rather than x, but the lower limit of integration also depends on x — this is different from the integral $\int_a^x f(t)\mathrm{d}t$ in the fundamental theorem where the *lower* limit of integration is a constant.

Fortunately we can use the basic properties of integrals (Theorem 1.2.3(b) and (c)) to split $\int_x^{x^2} e^{-t^2}\mathrm{d}t$ into pieces whose derivatives we already know.

$$\begin{aligned}\int_x^{x^2} e^{-t^2}\mathrm{d}t &= \int_x^{0} e^{-t^2}\mathrm{d}t + \int_0^{x^2} e^{-t^2}\mathrm{d}t && \text{by Theorem 1.2.3(c)}\\ &= -\int_0^{x} e^{-t^2}\mathrm{d}t + \int_0^{x^2} e^{-t^2}\mathrm{d}t && \text{by Theorem 1.2.3(b)}\end{aligned}$$

With this pre-processing, both integrals are of the right form. Using what we have learned in the previous two examples,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{-t^2}\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_0^{x} e^{-t^2}\mathrm{d}t + \int_0^{x^2} e^{-t^2}\mathrm{d}t\right)\\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x} e^{-t^2}\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\mathrm{d}t\\ &= -e^{-x^2} + 2xe^{-x^4}\end{aligned}$$

Example 1.3.6

Before we start to work with part 2 of the fundamental theorem, we need a little terminology and notation. First, some terminology.

Definition 1.3.7 (Antiderivatives).

Let $f(x)$ and $F(x)$ be functions. If $F'(x) = f(x)$ on an interval, then we say that $F(x)$ is an antiderivative of $f(x)$ on that interval.

As we saw above, an antiderivative of $f(x) = x$ is $F(x) = x^2/2$ — we can easily verify this by differentiation. Notice that $x^2/2 + 3$ is also an antiderivative of x, as is $x^2/2 + C$ for any constant C. This observation gives us the following simple lemma.

Lemma 1.3.8.

Let $f(x)$ be a function and let $F(x)$ be an antiderivative of $f(x)$. Then $F(x) + C$ is also an antiderivative for any constant C. Further, every antiderivative of $f(x)$ must be of this form.

Proof. There are two parts to the lemma and we prove each in turn.

- Let $F(x)$ be an antiderivative of $f(x)$ and let C be some constant. Then

  $$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(F(x)+C) &= \frac{\mathrm{d}}{\mathrm{d}x}(F(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(C) \\ &= f(x) + 0 \end{aligned}$$

  since the derivative of a constant is zero, and by definition the derivative of $F(x)$ is just $f(x)$. Thus $F(x) + C$ is also an antiderivative of $f(x)$.

- Now let $F(x)$ and $G(x)$ both be antiderivatives of $f(x)$ — we will show that $G(x) = F(x) + C$ for some constant C. To do this let $H(x) = G(x) - F(x)$. Then

  $$\frac{\mathrm{d}}{\mathrm{d}x}H(x) = \frac{\mathrm{d}}{\mathrm{d}x}(G(x) - F(x)) = \frac{\mathrm{d}}{\mathrm{d}x}G(x) - \frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x) - f(x) = 0$$

  Since the derivative of $H(x)$ is zero, $H(x)$ must be a constant function[26]. Thus $H(x) = G(x) - F(x) = C$ for some constant C and the result follows.

□

Based on the above lemma we have the following definition.

Definition 1.3.9.

The "indefinite integral of $f(x)$" is denoted by $\int f(x)\,\mathrm{d}x$ and should be regarded as the general antiderivative of $f(x)$. In particular, if $F(x)$ is an antiderivative of $f(x)$ then

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

where the C is an arbitrary constant. In this context, the constant C is also often called a "constant of integration".

---

[26] This follows from the Mean Value Theorem. Indeed, fix any number x_0. Then, for each $x \neq x_0$, the MVT gives us a number c between x_0 and x with

$$H(x) - H(x_0) = H'(c)\,(x - x_0) = 0$$

since the derivative of H is zero everywhere. Thus $H(x) = H(x_0)$ for all x and $H(x)$ is a constant function.

Now we just need a tiny bit more notation.

Notation 1.3.10.

The symbol

$$\int f(x)\,\mathrm{d}x\Big|_a^b$$

denotes the change in an antiderivative of $f(x)$ from $x = a$ to $x = b$. More precisely, let $F(x)$ be any antiderivative of $f(x)$. Then

$$\int f(x)\,\mathrm{d}x\Big|_a^b = F(x)\big|_a^b = F(b) - F(a)$$

Notice that this notation allows us to write part 2 of the fundamental theorem as

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \int f(x)\,\mathrm{d}x\Big|_a^b \\ &= F(x)\big|_a^b = F(b) - F(a)\end{aligned}$$

An equivalent notation uses square brackets:

$$\int_a^b f(x)\,\mathrm{d}x = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

You should be familiar with both notations.

We'll soon develop some strategies for computing more complicated integrals. But for now, we'll try a few integrals that are simple enough that we can just guess the answer. Of course, any antiderivative that we can guess we can also check — simply differentiate the guess and verify you get back to the original function:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\,\mathrm{d}x = f(x).$$

We do these examples in some detail to help us become comfortable finding indefinite integrals.

Example 1.3.11

Compute the definite integral $\int_1^2 x\,\mathrm{d}x$.

Solution. We have already seen, in Example 1.2.6, that $\int_1^2 x\,\mathrm{d}x = \frac{2^2-1^2}{2} = \frac{3}{2}$. We shall now rederive that result using the fundamental theorem of calculus.

- The main difficulty in this approach is finding the indefinite integral (an antiderivative) of x. That is, we need to find a function $F(x)$ whose derivative is x. So think back to all the derivatives you computed last term[27] and try to remember a function

---

[27] Of course, this assumes that you did your differential calculus course last term. If you did that course at a different time then please think back to that point in time. If it is long enough ago that you don't quite remember when it was, then you should probably do some revision of derivatives of simple functions before proceeding further.

whose derivative was something like x.

- This shouldn't be too hard — we recall that the derivatives of polynomials are polynomials. More precisely, we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

So if we want to end up with just $x = x^1$, we need to take $n = 2$. However this gives us

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$$

- This is pretty close to what we want except for the factor of 2. Since this is a constant we can just divide both sides by 2 to obtain:

$$\begin{aligned} \frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}x^2 &= \frac{1}{2}\cdot 2x && \text{which becomes} \\ \cdot\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{2} &= x \end{aligned}$$

which is exactly what we need. It tells us that $x^2/2$ is an antiderivative of x.

- Once one has an antiderivative, it is easy to compute the indefinite integral

$$\int x\,\mathrm{d}x = \frac{1}{2}x^2 + C$$

as well as the definite integral:

$$\begin{aligned} \int_1^2 x\,\mathrm{d}x &= \frac{1}{2}x^2\Big|_1^2 && \text{since } x^2/2 \text{ is the antiderivative of } x \\ &= \frac{1}{2}2^2 - \frac{1}{2}1^2 = \frac{3}{2} \end{aligned}$$

Example 1.3.11

While the previous example could be computed using signed areas, the following example would be very difficult to compute without using the fundamental theorem of calculus.

Example 1.3.12

Compute $\int_0^{\pi/2} \sin x\,\mathrm{d}x$.

Solution.

- Once again, the crux of the solution is guessing the antiderivative of $\sin x$ — that is finding a function whose derivative is $\sin x$.

- The standard derivative that comes closest to $\sin x$ is

  $$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

  which is the derivative we want, multiplied by a factor of -1.

- Just as we did in the previous example, we multiply this equation by a constant to remove this unwanted factor:

  $$\begin{aligned} (-1)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\cos x &= (-1)\cdot(-\sin x) && \text{giving us} \\ \frac{\mathrm{d}}{\mathrm{d}x}\big(-\cos x\big) &= \sin x \end{aligned}$$

  This tells us that $-\cos x$ is an antiderivative of $\sin x$.

- Now it is straightforward to compute the integral:

  $$\begin{aligned} \int_0^{\pi/2}\sin x\,\mathrm{d}x &= -\cos x\big|_0^{\pi/2} && \text{since } -\cos x \text{ is the antiderivative of } \sin x \\ &= -\cos\frac{\pi}{2}+\cos 0 \\ &= 0+1=1 \end{aligned}$$

Example 1.3.12

Example 1.3.13

Find $\int_1^2 \frac{1}{x}\mathrm{d}x$.

Solution.

- Once again, the crux of the solution is guessing a function whose derivative is $\frac{1}{x}$. Our standard way to differentiate powers of x, namely

  $$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1},$$

  doesn't work in this case — since it would require us to pick $n=0$ and this would give

  $$\frac{\mathrm{d}}{\mathrm{d}x}x^0 = \frac{\mathrm{d}}{\mathrm{d}x}1 = 0.$$

- Fortunately, we also know[28] that

$$\frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}$$

  which is exactly the derivative we want.

- We're now ready to compute the prescribed integral.

$$\begin{aligned} \int_1^2 \frac{1}{x}\mathrm{d}x &= \log x\big|_1^2 && \text{since } \log x \text{ is an antiderivative of } 1/x \\ &= \log 2 - \log 1 && \text{since } \log 1 = 0 \\ &= \log 2 \end{aligned}$$

Example 1.3.13

Example 1.3.14

Find $\int_{-2}^{-1} \frac{1}{x}\mathrm{d}x$.

Solution.

- As we saw in the last example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}$$

  and if we naively use this here, then we will obtain

$$\int_{-2}^{-1} \frac{1}{x}\mathrm{d}x = \log(-1) - \log(-2)$$

  which makes no sense since the logarithm is only defined for positive numbers[29].

- We can work around this problem using a slight variation of the logarithm — $\log|x|$.
  - When $x > 0$, we know that $|x| = x$ and so we have

$$\begin{aligned} \log|x| &= \log x && \text{differentiating gives us} \\ \frac{\mathrm{d}}{\mathrm{d}x}\log|x| &= \frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}. \end{aligned}$$

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[28] Recall that in most mathematics courses (especially this one) we use $\log x$ without any indicated base to denote the natural logarithm — the logarithm base e. Many widely used computer languages, like Java, C, Python, MATLAB, $\cdots$, use $\log(x)$ to denote the logarithm base e too. But many texts also use $\ln x$ to denote the natural logarithm

$$\log x = \log_e x = \ln x.$$

The reader should be comfortable with all three notations for this function. They should also be aware that in different contexts — such as in chemistry or physics — it is common to use $\log x$ to denote the logarithm base 10, while in computer science often $\log x$ denotes the logarithm base 2. Context is key.

[29] This is not entirely true — one can extend the definition of the logarithm to negative numbers, but to do so one needs to understand complex numbers which is a topic beyond the scope of this course.

  - When $x < 0$ we have that $|x| = -x$ and so

$$\begin{aligned} \log|x| &= \log(-x) && \text{differentiating with the chain rule gives} \\ \frac{\mathrm{d}}{\mathrm{d}x}\log|x| &= \frac{\mathrm{d}}{\mathrm{d}x}\log(-x) \\ &= \frac{1}{(-x)}\cdot(-1) = \frac{1}{x} \end{aligned}$$

  - Indeed, more generally we should write the indefinite integral of $1/x$ as

$$\int \frac{1}{x}\mathrm{d}x = \log|x| + C$$

    which is valid for all positive and negative x. It is, however, undefined at $x = 0$.

- We're now ready to compute the prescribed integral.

$$\begin{aligned} \int_{-2}^{-1}\frac{1}{x}\mathrm{d}x &= \log|x|\bigg|_{-2}^{-1} && \text{since } \log|x| \text{ is an antiderivative of } 1/x \\ &= \log|-1| - \log|-2| = \log 1 - \log 2 \\ &= -\log 2 = \log 1/2. \end{aligned}$$

Example 1.3.14

This next example raises a nasty issue that requires a little care. We know that the function $1/x$ is not defined at $x = 0$ — so can we integrate over an interval that contains $x = 0$ and still obtain an answer that makes sense? More generally can we integrate a function over an interval on which that function has discontinuities?

Example 1.3.15

Find $\int_{-1}^{1}\frac{1}{x^2}\mathrm{d}x$.

Solution. Beware that this is a particularly nasty example, which illustrates a booby trap hidden in the fundamental theorem of calculus. The booby trap explodes when the theorem is applied sloppily.

- The sloppy solution starts, as our previous examples have, by finding an antiderivative of the integrand. In this case we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = -\frac{1}{x^2}$$

  which means that $-x^{-1}$ is an antiderivative of x^{-2}.

- This suggests (if we proceed naively) that

$$\begin{aligned} \int_{-1}^{1} x^{-2}\mathrm{d}x &= -\frac{1}{x}\bigg|_{-1}^{1} && \text{since } -1/x \text{ is an antiderivative of } 1/x^2 \\ &= -\frac{1}{1} - \Big(-\frac{1}{-1}\Big) \\ &= -2 \end{aligned}$$

  Unfortunately,

- At this point we should really start to be concerned. This answer cannot be correct. Our integrand, being a square, is positive everywhere. So our integral represents the area of a region above the x–axis and must be positive.

- So what has gone wrong? The flaw in the computation is that the fundamental theorem of calculus, which says that

$$\text{if } F'(x) = f(x) \text{ then } \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

  is *only* applicable when $F'(x)$ exists and equals $f(x)$ for *all* x between a and b.

- In this case $F'(x) = \frac{1}{x^2}$ does not exist for $x = 0$. So we cannot apply the fundamental theorem of calculus as we tried to above.

An integral, like $\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x$, whose integrand is undefined somewhere in the domain of integration is called improper. We'll give a more thorough treatment of improper integrals later in the text (1.13). For now, we'll just say that the correct way to define (and evaluate) improper integrals is as a limit of well–defined approximating integrals. We shall later see that, not only is $\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x$ not negative, it is infinite.

Example 1.3.15

The above examples have illustrated how we can use the fundamental theorem of calculus to convert knowledge of derivatives into knowledge of integrals. We are now in a position to easily build a table of integrals. Here is a short table of important derivatives that we know.

$F(x)$	1	x^n	$\sin x$	$\cos x$	$\tan x$	e^x	$\log_e \lvert x\rvert$	$\arcsin x$	$\arctan x$
$f(x) = F'(x)$	0	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$	e^x	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

Of course we know other derivatives, such as those of $\sec x$ and $\cot x$, but the ones listed above are arguably the most important. From this table (with a very little massaging) we can write down a short table of indefinite integrals.

Theorem 1.3.16 (Important indefinite integrals).

$f(x)$	$F(x) = \int f(x)\,\mathrm{d}x$
1	$x + C$
x^n	$\frac{1}{n+1}x^{n+1} + C$ provided that $n \neq -1$
$\frac{1}{x}$	$\log_e \lvert x\rvert + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{1}{1+x^2}$	$\arctan x + C$

Example 1.3.17

Find the following integrals

(i) $\int_2^7 e^x\,\mathrm{d}x$

(ii) $\int_{-2}^2 \frac{1}{1+x^2}\mathrm{d}x$

(iii) $\int_0^3 (2x^3 + 7x - 2)\mathrm{d}x$

Solution. We can proceed with each of these as before — find the antiderivative and then apply the fundamental theorem. The third integral is a little more complicated, but we can split it up into monomials using Theorem 1.2.1 and do each separately.

(i) An antiderivative of e^x is just e^x, so

$$\begin{aligned}\int_2^7 e^x \, \mathrm{d}x &= e^x \Big|_2^7 \\ &= e^7 - e^2 = e^2(e^5 - 1).\end{aligned}$$

(ii) An antiderivative of $\frac{1}{1+x^2}$ is $\arctan(x)$, so

$$\begin{aligned}\int_{-2}^2 \frac{1}{1+x^2} \mathrm{d}x &= \arctan(x) \Big|_{-2}^2 \\ &= \arctan(2) - \arctan(-2)\end{aligned}$$

We can simplify this a little further by noting that $\arctan(x)$ is an odd function, so $\arctan(-2) = -\arctan(2)$ and thus our integral is

$$= 2\arctan(2)$$

(iii) We can proceed by splitting the integral using Theorem 1.2.1(d)

$$\begin{aligned}\int_0^3 (2x^3 + 7x - 2)\mathrm{d}x &= \int_0^3 2x^3 \mathrm{d}x + \int_0^3 7x \, \mathrm{d}x - \int_0^3 2\mathrm{d}x \\ &= 2\int_0^3 x^3 \mathrm{d}x + 7\int_0^3 x \, \mathrm{d}x - 2\int_0^3 \mathrm{d}x\end{aligned}$$

and because we know that $x^4/4, x^2/2, x$ are antiderivatives of $x^3, x, 1$ respectively, this becomes

$$\begin{aligned}&= \left[\frac{x^4}{2}\right]_0^3 + \left[\frac{7x^2}{2}\right]_0^3 - [2x]_0^3 \\ &= \frac{81}{2} + \frac{7 \cdot 9}{2} - 6 \\ &= \frac{81 + 63 - 12}{2} = \frac{132}{2} = 66.\end{aligned}$$

We can also just find the antiderivative of the whole polynomial by finding the antiderivatives of each term of the polynomial and then recombining them. This is equivalent to what we have done above, but perhaps a little neater:

$$\begin{aligned}\int_0^3 (2x^3 + 7x - 2)\mathrm{d}x &= \left[\frac{x^4}{2} + \frac{7x^2}{2} - 2x\right]_0^3 \\ &= \frac{81}{2} + \frac{7 \cdot 9}{2} - 6 = 66.\end{aligned}$$

Example 1.3.17

1.4▲ Interpretations of the definite integral

Learning Objectives

- Apply knowledge of integration (approximation via rectangles, antidifferentiation, FTC) in context (i.e. word problems).

The fundamental theorem of calculus part 2 tells us, subject to some find print, that

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)\,.$$

So, for some function $f(x)$, the definite integral of its *rate of change*, $f'(x)$, gives us *net change* or *amount of change*, $f(b) - f(a)$. This is your first hint that integration can do more than just find areas under curves.

Example 1.4.1 (Velocity and position)

Suppose a particle moves along the x-axis. Its velocity at time t is given by $v(t) = \sin t$. Let its position at time t be $s(t)$.

(a) Give an interpretation, in words, of the quantity $s(2\pi) - s(0)$.

(b) Find the quantity $s(2\pi) - s(0)$.

(c) If we ignore the direction the particle was travelling, how far did the particle travel from $t = 0$ to $t = 2\pi$? We are interested in the *total* distance travelled, not the *net* distance travelled.

Solution.

(a) The quantity $s(2\pi) - s(0)$ is the distance along the x-axis from the particle's position at time $t = 0$ to its position at time $t = 2\pi$. That is, it is the net distance travelled by the particle from $t = 0$ to $t = 2\pi$.

(b) Note $v(t) = s'(t)$, so we use the fundamental theorem of calculus, part 2.

$$\begin{aligned} s(2\pi) - s(0) &= \int_0^{2\pi} s'(t)\,\mathrm{d}t = \int_0^{2\pi} \sin t\,\mathrm{d}t \\ &= [-\cos t]_0^{2\pi} \\ &= -1 - (-1) = 0 \end{aligned}$$

(That is, it is in the same position at time $t = 0$ and time $t = 2\pi$.)

(c) We note that $\sin t \geqslant 0$ for $0 \leqslant t \leqslant \pi$, and $\sin t \leqslant 0$ for $\pi \leqslant t \leqslant 2\pi$. So, the particle travels to the right from $t = 0$ to $t = \pi$, then it turns around and travels to the left from $t = \pi$ to $t = 2\pi$.

So, to find the total distance travelled, we can compute $|s(\pi)-s(0)|+|s(2\pi)-s(\pi)| = 2[s(\pi)-s(0)]$.

$$\begin{aligned} 2[s(\pi)-s(0)] &= 2\int_0^{\pi} s'(t)\,\mathrm{d}t \\ &= 2\int_0^{\pi} \sin t\,\mathrm{d}t \\ &= 2\left[-\cos t\right]_0^{\pi} \\ &= 2[-\cos\pi+\cos 0] = 2[1+1] = 4 \end{aligned}$$

The total distance travelled by the particle is 4 units.

Example 1.4.1

Example 1.4.2 (Population change)

A population has $N(t)$ individuals at time t. The population changes only through births and deaths (and not, say, migration). Its birth rate at time t is $b(t)=t$, and its death rate at time t is $d(t)=t(t-24)$.

Between $t=0$ and $t=25$, has the population increased or decreased? By how much?

Solution.

The rate of change of $N(t)$ is precisely the rate of births minus the rate of deaths, $N'(t)=b(t)-d(t)$. We have rate of change, and we're interested in net change, so the tool to use is a definite integral.

$$\begin{aligned} N(25)-N(0) &= \int_0^{25} N'(t)\,\mathrm{d}t = \int_0^{25} (b(t)-d(t))\,\mathrm{d}t \\ &= \int_0^{25} [t-t(t-24)]\,\mathrm{d}t \\ &= \int_0^{25} [-t^2+25t]\,\mathrm{d}t \\ &= \left[-\frac{t^3}{3}+\frac{25t^2}{2}\right]_0^{25} \\ &= -\frac{25^3}{3}+\frac{25^4}{2} = 25^3\left(\frac{25}{2}-\frac{1}{3}\right) \end{aligned}$$

This is a positive number, so the population grew by $\frac{25^4}{2}-\frac{25^3}{3}$ individuals.

Example 1.4.2

Example 1.4.3 (Asset valuations)

A certain speculative asset has been changing value at a rate of

$$f(t) = t^3 - 11t^2 + 24t$$

dollars per day. How do you expect its value at the end of this week ($t = 7$) to compare to its value right now ($t = 0$)?

Solution.

As before, we have rate of change, and we want to know net change, so the tool to use is the definite integral. Let $F(t)$ be the value of the asset at time t. Then $F'(t) = f(t)$.

$$\begin{aligned} F(7) - F(0) &= \int_0^7 (t^3 - 11t^2 + 24t)\,\mathrm{d}t \\ &= \left[\frac{1}{4}t^4 - \frac{11}{3}t^3 + 12t^2\right]_0^7 \\ &= \frac{7^4}{4} - \frac{11 \cdot 7^3}{3} + 12(7^2) \\ &= 49\left[\frac{49}{4} - \frac{77}{3} + 12\right] = -\frac{49 \cdot 17}{12} \end{aligned}$$

So, the asset will *decrease* in value, and the amount of the decrease will be $\frac{49 \cdot 17}{3}$ dollars.

Example 1.4.3

In the next example, we interpret an integral as a sum. You'll see more of this trick in Section 1.10.

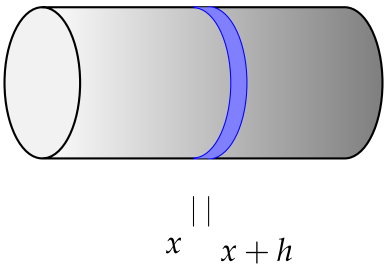
Example 1.4.4 (Density)

A long, thin, straight rod has varying density. Its length is one metre. At a distance of x metres from the left end of the rod, the density of the rod is given by $\rho(x) = 1 + x$ kg per metre.

What is the mass of the rod?

Solution.

We approximated areas by slicing them into rectangles. Let's approximate mass by slicing our rod into segments.



From position x to $x+h$, when $h \approx 0$, the density of the rod is about $1+x$ kg per metre, and the slice has a length of h metres, so the mass of the slice is approximately $(1+x)h$ kg. Adding up N of these slices (where each has length $h = \frac{1}{N}$), from $x=0$ to $x=1$, gives us a total mass of

$$\sum_{i=0}^{N} x_i \cdot h = \sum_{i=0}^{N} \underbrace{\left(1+\frac{1}{N}\right)}_{\rho(x_i)} \cdot \underbrace{\frac{1}{N}}_{h}$$

We note that this is also the Riemann sum for the integral $\int_0^1 (1+x)\,\mathrm{d}x$, so the total mass is

$$\int_0^1 (1+x)\,\mathrm{d}x = \left[x + \frac{x^2}{2}\right]_0^1 = \frac{3}{2}$$

kg.

Remark: To avoid the slicing argument, you could also think of $\rho(x)$ as a rate of change. Imagine moving from $x=0$ to $x=1$ along the rod, and adding up the amount of mass you've passed. The rate at which the mass is changing would be $\rho(x)$ kg/m, so the net amount of change (from none of the rod's mass to all of it) would be $\int_0^1 \rho(x)\,\mathrm{d}x$.

Example 1.4.4

1.5▲ Substitution

Learning Objectives

- Explain how the chain rule for derivatives corresponds to the substitution method for antiderivatives.
- Use a given substitution to evaluate an indefinite integral.
- Show how a given substitution affects the bounds of integration when used with a definite integral.
- Recognize when a substitution will simplify a given integral (definite or indefinite), and determine the form of an effective substitution.
- Compute integrals where the integrand requires manipulation to reveal an effective substitution.

- Compute integrals using a sequence of substitutions. For example, the integral $\int \sin^2(x^2)\cos(x^2)[2x]\,\mathrm{d}x$.

In the Section 1.3, we explored the fundamental theorem of calculus and the link it provides between definite integrals and antiderivatives. Indeed, integrals with simple integrands are usually evaluated via this link. In this section we start to explore methods for integrating more complicated integrals. We have already seen — via Theorem 1.2.1 — that integrals interact very nicely with addition, subtraction and multiplication by constants:

$$\int_a^b (Af(x)+Bg(x))\,\mathrm{d}x = A\int_a^b f(x)\,\mathrm{d}x + B\int_a^b g(x)\,\mathrm{d}x$$

for A, B constants. By combining this with the list of indefinite integrals in Theorem 1.3.16, we can compute integrals of linear combinations of simple functions. For example,

$$\begin{aligned}\int_1^4 \left(e^x - 2\sin x + 3x^2\right)\mathrm{d}x &= \int_1^4 e^x\,\mathrm{d}x - 2\int_1^4 \sin x\,\mathrm{d}x + 3\int_1^4 x^2\mathrm{d}x\\ &= \left(e^x + (-2)\cdot(-\cos x) + 3\frac{x^3}{3}\right)\Big|_1^4 \qquad \text{and so on.}\end{aligned}$$

Of course there are a great many functions that can be approached in this way, but there are some very simple examples that cannot.

$$\int \sin(\pi x)\mathrm{d}x \qquad \int xe^x\,\mathrm{d}x \qquad \int \frac{x}{x^2-5x+6}\mathrm{d}x$$

In each case the integrands are not linear combinations of simpler functions; in order to compute them we need to understand how integrals (and antiderivatives) interact with compositions, products and quotients. We reached a very similar point in our differential calculus course where we understood the linearity of the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Af(x)+Bg(x)\right) = A\frac{\mathrm{d}f}{\mathrm{d}x} + B\frac{\mathrm{d}g}{\mathrm{d}x},$$

but had not yet seen the chain, product and quotient rules[30]. While we will develop tools to find the second and third integrals in later sections, we should really start with how to integrate compositions of functions.

It is important to state up front that in general one cannot write down the integral of the composition of two functions — even if those functions are simple. This is not because the integral does not exist. Rather it is because the integral cannot be written down as a finite combination of the standard functions we know. A very good example of this, which we encountered in Example 1.3.4, is the composition of e^x and $-x^2$. Even though we know

$$\int e^x\,\mathrm{d}x = e^x + C \qquad \text{and} \qquad \int -x^2\mathrm{d}x = -\frac{1}{3}x^3 + C$$

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[30] If your memory of these rules is a little hazy then you really should go back and revise them before proceeding. You will definitely need a good grasp of the chain rule for what follows in this section.

there is no simple function that is equal to the indefinite integral

$$\int e^{-x^2} \mathrm{d}x$$

even though the indefinite integral exists. In this way integration is very different from differentiation.

With that caveat out of the way, we can introduce the substitution rule. The substitution rule is obtained by antidifferentiating the chain rule. In some sense it is the chain rule in reverse. For completeness, let us restate the chain rule:

Theorem 1.5.1 (The chain rule)**.**

Let $F(u)$ and $u(x)$ be differentiable functions and form their composition $F(u(x))$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}F\big(u(x)\big) = F'\big(u(x)\big) \cdot u'(x)$$

Equivalently, if $y(x) = F(u(x))$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}F}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Consider a function $f(u)$, which has antiderivative $F(u)$. We know that

$$\int f(u)\,\mathrm{d}u = \int F'(u)\,\mathrm{d}u = F(u) + C\,.$$

Now take the above equation and substitute into it $u = u(x)$ — i.e. replace the variable u with any (differentiable) function of x to get

$$\int f(u)\,\mathrm{d}u\bigg|_{u=u(x)} = F(u(x)) + C\,.$$

But now the right-hand side is a function of x, so we can differentiate it with respect to x to get

$$\frac{\mathrm{d}}{\mathrm{d}x}F(u(x)) = F'(u(x)) \cdot u'(x)\,.$$

This tells us that $F(u(x))$ is an antiderivative of the function $F'(u(x)) \cdot u'(x) = f(u(x))u'(x)$. Thus we know

$$\int f\big(u(x)\big) \cdot u'(x)\,\mathrm{d}x = F\big(u(x)\big) + C = \int f(u)\,\mathrm{d}u\bigg|_{u=u(x)}.$$

This is the substitution rule for indefinite integrals.

Theorem 1.5.2 (The substitution rule — indefinite integral version).

For any differentiable function $u(x)$:

$$\int f(u(x))u'(x)\mathrm{d}x = \int f(u)\,\mathrm{d}u\Big|_{u=u(x)}$$

In order to apply the substitution rule successfully we will have to write the integrand in the form $f(u(x)) \cdot u'(x)$. To do this we need to make a good choice of the function $u(x)$; after that it is not hard to then find $f(u)$ and $u'(x)$. Unfortunately there is no one strategy for choosing $u(x)$. This can make applying the substitution rule more art than science[31]. Here we suggest two possible strategies for picking $u(x)$:

(1) Factor the integrand and choose one of the factors to be $u'(x)$. For this to work, you must be able to easily find the antiderivative of the chosen factor. The antiderivative will be $u(x)$.

(2) Look for a factor in the integrand that is a function with an argument that is more complicated than just "x". That factor will play the role of $f\big(u(x)\big)$ Choose $u(x)$ to be the complicated argument.

Here are two examples which illustrate each of those strategies in turn.

Example 1.5.3

Consider the integral

$$\int 9\sin^8(x)\cos(x)\mathrm{d}x$$

We want to massage this into the form of the integrand in the substitution rule — namely $f(u(x)) \cdot u'(x)$. Our integrand can be written as the product of the two factors

$$\underbrace{9\sin^8(x)}_{\text{first factor}} \cdot \underbrace{\cos(x)}_{\text{second factor}}$$

and we start by determining (or guessing) which factor plays the role of $u'(x)$. We can choose $u'(x) = 9\sin^8(x)$ or $u'(x) = \cos(x)$.

- If we choose $u'(x) = 9\sin^8(x)$, then antidifferentiating this to find $u(x)$ is really not very easy. So it is perhaps better to investigate the other choice before proceeding further with this one.
- If we choose $u'(x) = \cos(x)$, then we know (Theorem 1.3.16) that $u(x) = \sin(x)$. This also works nicely because it makes the other factor simplify quite a bit $9\sin^8(x) = 9u^8$. This looks like the right way to go.

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[31] Thankfully this does become easier with experience and we recommend that the reader read some examples and then practice a LOT.

So we go with the second choice. Set $u'(x) = \cos(x), u(x) = \sin(x)$, then

$$\begin{aligned}\int 9\sin^8(x)\cos(x)\mathrm{d}x &= \int 9u(x)^8 \cdot u'(x)\mathrm{d}x \\ &= \int 9u^8\mathrm{d}u\Big|_{u=\sin(x)} && \text{by the substitution rule}\end{aligned}$$

We are now left with the problem of antidifferentiating a monomial; this we can do with Theorem 1.3.16.

$$\begin{aligned}&= \left(u^9 + C\right)\Big|_{u=\sin(x)} \\ &= \sin^9(x) + C\end{aligned}$$

Note that $9\sin^8(x)\cos(x)$ is a function of x. So our answer, which is the indefinite integral of $9\sin^8(x)\cos(x)$, must also be a function of x. This is why we have substituted $u = \sin(x)$ in the last step of our solution — it makes our solution a function of x.

Example 1.5.3

Example 1.5.4

Evaluate the integral

$$\int 3x^2\cos(x^3)\mathrm{d}x$$

Solution. Again we are going to use the substitution rule. Helpfully, our integrand is a product of two factors:

$$\underbrace{3x^2}_{\text{first factor}} \cdot \underbrace{\cos(x^3)}_{\text{second factor}}$$

The second factor, $\cos\left(x^3\right)$, is a function (namely cos) with a complicated argument (namely x^3). So we try $u(x) = x^3$. Then $u'(x) = 3x^2$, which is the other factor in the integrand. So the integral becomes

$$\begin{aligned}\int 3x^2\cos(x^3)\mathrm{d}x &= \int u'(x)\cos\left(u(x)\right)\mathrm{d}x && \text{just swap order of factors} \\ &= \int \cos\left(u(x)\right)u'(x)\mathrm{d}x && \text{by the substitution rule} \\ &= \int \cos(u)\,\mathrm{d}u\Big|_{u=x^3} \\ &= (\sin(u) + C)\Big|_{u=x^3} && \text{using Theorem 1.3.16)} \\ &= \sin(x^3) + C\end{aligned}$$

Example 1.5.4

One more — we'll use this to show how to use the substitution rule with definite integrals.

Example 1.5.5 $\left(\int_0^1 e^x \sin(e^x)\mathrm{d}x\right)$

Compute

$$\int_0^1 e^x \sin\left(e^x\right)\mathrm{d}x.$$

Solution. Again we use the substitution rule.

- The integrand is again the product of two factors and we can choose $u'(x) = e^x$ or $u'(x) = \sin(e^x)$.
- If we choose $u'(x) = e^x$ then $u(x) = e^x$ and the other factor becomes $\sin(u)$ — this looks promising. Notice that if we applied the other strategy of looking for a complicated argument then we would arrive at the same choice.
- So we try $u'(x) = e^x$ and $u(x) = e^x$. This gives (if we ignore the limits of integration for a moment)

$$\begin{aligned}\int e^x \sin\left(e^x\right)\mathrm{d}x &= \int \sin\big(u(x)\big)u'(x)\mathrm{d}x && \text{apply the substitution rule}\\ &= \int \sin(u)\,\mathrm{d}u\bigg|_{u=e^x}\\ &= \left(-\cos(u)+C\right)\bigg|_{u=e^x}\\ &= -\cos\left(e^x\right)+C\end{aligned}$$

- But what happened to the limits of integration? We can incorporate them now. We have just shown that the indefinite integral is $-\cos(e^x)$, so by the fundamental theorem of calculus

$$\begin{aligned}\int_0^1 e^x \sin\left(e^x\right)\mathrm{d}x &= \left[-\cos\left(e^x\right)\right]_0^1\\ &= -\cos(e^1) - (-\cos(e^0))\\ &= -\cos(e) + \cos(1)\end{aligned}$$

Example 1.5.5

Theorem 1.5.2, the substitution rule for indefinite integrals, tells us that if $F(u)$ is any antiderivative for $f(u)$, then $F\big(u(x)\big)$ is an antiderivative for $f\big(u(x)\big)u'(x)$. So the fundamental theorem of calculus gives us

$$\begin{aligned} \int_a^b f\big(u(x)\big)u'(x)\,\mathrm{d}x &= F\big(u(x)\big)\Big|_{x=a}^{x=b} \\ &= F\big(u(b)\big) - F\big(u(a)\big) \\ &= \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u && \text{since } F(u) \text{ is an antiderivative for } f(u) \end{aligned}$$

and we have just found

Theorem 1.5.6 (The substitution rule — definite integral version)**.**

For any differentiable function $u(x)$:

$$\int_a^b f(u(x))u'(x)\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u$$

Notice that to get from the integral on the left-hand side to the integral on the right hand side, you:

- substitute[32] $u(x) \to u$ and $u'(x)\,\mathrm{d}x \to \mathrm{d}u$,
- set the lower limit for the u integral to the value of u (namely $u(a)$) that corresponds to the lower limit of the x integral (namely $x = a$), and
- set the upper limit for the u integral to the value of u (namely $u(b)$) that corresponds to the upper limit of the x integral (namely $x = b$).

Also note that we now have two ways to evaluate definite integrals of the form $\int_a^b f\big(u(x)\big)u'(x)\,\mathrm{d}x$.

- We can find the indefinite integral $\int f\big(u(x)\big)u'(x)\,\mathrm{d}x$, using Theorem 1.5.2, and then evaluate the result between $x = a$ and $x = b$. This is what was done in Example 1.5.5.
- Or we can apply Theorem 1.5.2. This entails finding the indefinite integral $\int f(u)\,\mathrm{d}u$ and evaluating the result between $u = u(a)$ and $u = u(b)$. This is what we will do in the following example.

Example 1.5.7 $\left(\int_0^1 x^2 \sin(x^3+1)\mathrm{d}x\right)$

Compute

$$\int_0^1 x^2 \sin\left(x^3+1\right)\mathrm{d}x$$

Solution.

---

[32] A good way to remember this last step is that we replace $\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ by just $\mathrm{d}u$ — which looks like we cancelled out the $\mathrm{d}x$ terms: $\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \mathrm{d}u$. While using "cancel the $\mathrm{d}x$" is a good mnemonic (memory aid), you should not think of the derivative $\frac{\mathrm{d}u}{\mathrm{d}x}$ as a fraction — you are not dividing $\mathrm{d}u$ by $\mathrm{d}x$.

- In this example the integrand is already neatly factored into two pieces. While we could deploy either of our two strategies, it is perhaps easier in this case to choose $u(x)$ by looking for a complicated argument.
- The second factor of the integrand is $\sin\left(x^3+1\right)$, which is the function sin evaluated at x^3+1. So set $u(x)=x^3+1$, giving $u'(x)=3x^2$ and $f(u)=\sin(u)$
- The first factor of the integrand is x^2, which is not quite $u'(x)$. However, we can easily massage the integrand into the required form by multiplying and dividing by 3:

$$x^2\sin\left(x^3+1\right)=\frac{1}{3}\cdot 3x^2\cdot\sin\left(x^3+1\right).$$

- We want this in the form of the substitution rule, so we do a little massaging:

$$\begin{aligned}\int_0^1 x^2\sin\left(x^3+1\right)\mathrm{d}x &= \int_0^1 \frac{1}{3}\cdot 3x^2\cdot\sin\left(x^3+1\right)\mathrm{d}x\\ &= \frac{1}{3}\int_0^1 \sin\left(x^3+1\right)\cdot 3x^2\mathrm{d}x && \text{by Theorem 1.2.1(c)}\end{aligned}$$

- Now we are ready for the substitution rule:

$$\begin{aligned}\frac{1}{3}\int_0^1 \sin\left(x^3+1\right)\cdot 3x^2\mathrm{d}x &= \frac{1}{3}\int_0^1 \underbrace{\sin\left(x^3+1\right)}_{=f(u(x))}\cdot\underbrace{3x^2}_{=u'(x)}\mathrm{d}x\\ &= \frac{1}{3}\int_0^1 f(u(x))u'(x)\mathrm{d}x && \text{with } u(x)=x^3+1 \text{ and } f(u)=\sin(u)\\ &= \frac{1}{3}\int_{u(0)}^{u(1)} f(u)\,\mathrm{d}u && \text{by the substitution rule}\\ &= \frac{1}{3}\int_1^2 \sin(u)\,\mathrm{d}u && \text{since } u(0)=1 \text{ and } u(1)=2\\ &= \frac{1}{3}\big[-\cos(u)\big]_1^2\\ &= \frac{1}{3}\big(-\cos(2)-(-\cos(1))\big)\\ &= \frac{\cos(1)-\cos(2)}{3}.\end{aligned}$$

Example 1.5.7

There is another, and perhaps easier, way to view the manipulations in the previous example. Once you have chosen $u(x)$ you

- make the substitution $u(x)\rightarrow u$,

- replace $\mathrm{d}x \to \dfrac{1}{u'(x)}\mathrm{d}u$.

In so doing, we take the integral

$$\begin{aligned}\int_a^b f(u(x))\cdot u'(x)\mathrm{d}x &= \int_{u(a)}^{u(b)} f(u)\cdot u'(x)\cdot\frac{1}{u'(x)}\mathrm{d}u\\ &= \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u \qquad \text{exactly the substitution rule}\end{aligned}$$

but we do not have to manipulate the integrand so as to make $u'(x)$ explicit. Let us redo the previous example by this approach.

Example 1.5.8 (*Example 1.5.7 revisited*)

Compute the integral

$$\int_0^1 x^2\sin\left(x^3+1\right)\mathrm{d}x$$

Solution.

- We have already observed that one factor of the integrand is $\sin\left(x^3+1\right)$, which is sin evaluated at x^3+1. Thus we try setting $u(x)=x^3+1$.
- This makes $u'(x)=3x^2$, and we replace $u(x)=x^3+1\to u$ and $\mathrm{d}x\to\frac{1}{u'(x)}\mathrm{d}u=\frac{1}{3x^2}\mathrm{d}u$:

$$\begin{aligned}\int_0^1 x^2\sin\left(x^3+1\right)\mathrm{d}x &= \int_{u(0)}^{u(1)} x^2\underbrace{\sin\left(x^3+1\right)}_{=\sin(u)}\frac{1}{3x^2}\mathrm{d}u\\ &= \int_1^2 \sin(u)\frac{x^2}{3x^2}\mathrm{d}u\\ &= \int_1^2 \frac{1}{3}\sin(u)\,\mathrm{d}u\\ &= \frac{1}{3}\int_1^2 \sin(u)\,\mathrm{d}u\end{aligned}$$

which is precisely the integral we found in Example 1.5.7.

Example 1.5.8

Example 1.5.9

Compute the indefinite integrals

$$\int\sqrt{2x+1}\mathrm{d}x \qquad \text{and} \qquad \int e^{3x-2}\mathrm{d}x$$

Solution.

- Starting with the first integral, we see that it is not too hard to spot the complicated argument. If we set $u(x) = 2x+1$ then the integrand is just $\sqrt{u}$.

- Hence we substitute $2x+1 \to u$ and $\mathrm{d}x \to \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{2}\mathrm{d}u$:

$$\begin{aligned}\int \sqrt{2x+1}\,\mathrm{d}x &= \int \sqrt{u}\frac{1}{2}\mathrm{d}u \\ &= \int u^{1/2}\frac{1}{2}\mathrm{d}u \\ &= \left(\frac{2}{3}u^{3/2}\cdot\frac{1}{2} + C\right)\Big|_{u=2x+1} \\ &= \frac{1}{3}(2x+1)^{3/2} + C\end{aligned}$$

- We can evaluate the second integral in much the same way. Set $u(x) = 3x-2$ and replace $\mathrm{d}x$ by $\frac{1}{u'(x)}\mathrm{d}u = \frac{1}{3}\mathrm{d}u$:

$$\begin{aligned}\int e^{3x-2}\mathrm{d}x &= \int e^{u}\frac{1}{3}\mathrm{d}u \\ &= \left(\frac{1}{3}e^{u} + C\right)\Big|_{u=3x-2} \\ &= \frac{1}{3}e^{3x-2} + C\end{aligned}$$

Example 1.5.9

This last example illustrates that substitution can be used to easily deal with arguments of the form $ax+b$ (with a, b constants and $a \neq 0$), i.e. that are linear functions of x, and suggests the following theorem.

Theorem 1.5.10.

Let $F(u)$ be an antiderivative of $f(u)$ and let a, b be constants with $a \neq 0$. Then

$$\int f(ax+b)\mathrm{d}x = \frac{1}{a}F(ax+b) + C$$

Proof. We can show this using the substitution rule. Let $u(x) = ax + b$ so $u'(x) = a$, then

$$\begin{aligned}
\int f(ax+b)\mathrm{d}x &= \int f(u) \cdot \frac{1}{u'(x)}\mathrm{d}u \\
&= \int \frac{1}{a} f(u)\, \mathrm{d}u \\
&= \frac{1}{a}\int f(u)\, \mathrm{d}u && \text{since } a \text{ is a constant} \\
&= \frac{1}{a}F(u)\Big|_{u=ax+b} + C && \text{since } F(u) \text{ is an antiderivative of } f(u) \\
&= \frac{1}{a}F(ax+b) + C.
\end{aligned}$$

☐

We can do the next example using the substitution rule or the above theorem.

Example 1.5.11 $\left(\int_0^{\pi/2}\cos(3x)\mathrm{d}x\right)$

Compute $\int_0^{\pi/2}\cos(3x)\mathrm{d}x$.

- In this example we should set $u = 3x$, and substitute $\mathrm{d}x \to \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{3}\mathrm{d}u$. When we do this we also have to convert the limits of the integral: $u(0) = 0$ and $u(\pi/2) = 3\pi/2$. This gives

$$\begin{aligned}
\int_0^{\pi/2}\cos(3x)\mathrm{d}x &= \int_0^{3\pi/2}\cos(u)\frac{1}{3}\mathrm{d}u \\
&= \left[\frac{1}{3}\sin(u)\right]_0^{3\pi/2} \\
&= \frac{\sin(3\pi/2) - \sin(0)}{3} \\
&= \frac{-1-0}{3} = -\frac{1}{3}.
\end{aligned}$$

- We can also do this example more directly using the above theorem. Since $\sin(x)$ is an antiderivative of $\cos(x)$, Theorem 1.5.10 tells us that $\frac{\sin(3x)}{3}$ is an antiderivative of $\cos(3x)$. Hence

$$\begin{aligned}
\int_0^{\pi/2}\cos(3x)\mathrm{d}x &= \left[\frac{\sin(3x)}{3}\right]_0^{\pi/2} \\
&= \frac{\sin(3\pi/2) - \sin(0)}{3} \\
&= -\frac{1}{3}.
\end{aligned}$$

Example 1.5.11

The rest of this section is just more examples of the substitution rule. We recommend that you after reading these that you practice many examples by yourself under exam conditions.

Example 1.5.12 $\left(\int_0^1 x^2\sin(1-x^3)\mathrm{d}x\right)$

This integral looks a lot like that of Example 1.5.7. It makes sense to try $u(x) = 1-x^3$ since it is the argument of $\sin(1-x^3)$. We

- substitute $u = 1-x^3$ and
- replace $\mathrm{d}x$ with $\frac{1}{u'(x)}\mathrm{d}u = \frac{1}{-3x^2}\mathrm{d}u$,
- when $x=0$, we have $u = 1-0^3 = 1$ and
- when $x=1$, we have $u=1-1^3=0$.

So

$$\begin{aligned}\int_0^1 x^2\sin\left(1-x^3\right)\cdot\mathrm{d}x &= \int_1^0 x^2\sin(u)\cdot\frac{1}{-3x^2}\mathrm{d}u\\ &= \int_1^0 -\frac{1}{3}\sin(u)\,\mathrm{d}u.\end{aligned}$$

Note that the lower limit of the u–integral, namely 1, is larger than the upper limit, which is 0. There is absolutely nothing wrong with that. We can simply evaluate the u–integral in the normal way. Since $-\cos(u)$ is an antiderivative of $\sin(u)$:

$$\begin{aligned}&= \left[\frac{\cos(u)}{3}\right]_1^0\\ &= \frac{\cos(0)-\cos(1)}{3}\\ &= \frac{1-\cos(1)}{3}.\end{aligned}$$

Example 1.5.12

Example 1.5.13 $\left(\int_0^1 \frac{1}{(2x+1)^3}\mathrm{d}x\right)$

Compute $\int_0^1 \frac{1}{(2x+1)^3}\mathrm{d}x$.

We could do this one using Theorem 1.5.10, but its not too hard to do without. We can think of the integrand as the function "one over a cube" with the argument $2x+1$. So it makes sense to substitute $u=2x+1$. That is

- set $u = 2x+1$ and
- replace $\mathrm{d}x \rightarrow \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{2}\mathrm{d}u$.

- When $x = 0$, we have $u = 2 \times 0 + 1 = 1$ and
- when $x = 1$, we have $u = 2 \times 1 + 1 = 3$.

So

$$\begin{aligned}\int_0^1 \frac{1}{(2x+1)^3}\mathrm{d}x &= \int_1^3 \frac{1}{u^3} \cdot \frac{1}{2}\mathrm{d}u \\ &= \frac{1}{2}\int_1^3 u^{-3}\mathrm{d}u \\ &= \frac{1}{2}\left[\frac{u^{-2}}{-2}\right]_1^3 \\ &= \frac{1}{2}\left(\frac{1}{-2}\cdot\frac{1}{9} - \frac{1}{-2}\cdot\frac{1}{1}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{18}\right) = \frac{1}{2}\cdot\frac{8}{18} \\ &= \frac{2}{9}\end{aligned}$$

Example 1.5.13

Example 1.5.14 $\left(\int_0^1 \frac{x}{1+x^2}\mathrm{d}x\right)$

Evaluate $\int_0^1 \frac{x}{1+x^2}\mathrm{d}x$.

Solution.

- The integrand can be rewritten as $x \cdot \frac{1}{1+x^2}$. This second factor suggests that we should try setting $u = 1 + x^2$ — and so we interpret the second factor as the function "one over" evaluated at argument $1 + x^2$.
- With this choice we
  - set $u = 1 + x^2$,
  - substitute $\mathrm{d}x \to \frac{1}{2x}\mathrm{d}u$, and
  - translate the limits of integration: when $x = 0$, we have $u = 1 + 0^2 = 1$ and when $x = 1$, we have $u = 1 + 1^2 = 2$.
- The integral then becomes

$$\begin{aligned}\int_0^1 \frac{x}{1+x^2}\mathrm{d}x &= \int_1^2 \frac{x}{u}\frac{1}{2x}\mathrm{d}u \\ &= \int_1^2 \frac{1}{2u}\mathrm{d}u \\ &= \frac{1}{2}\big[\log|u|\big]_1^2 \\ &= \frac{\log 2 - \log 1}{2} = \frac{\log 2}{2}.\end{aligned}$$

Remember that we are using the notation "log" for the natural logarithm, i.e. the logarithm with base e. You might also see it written as "$\ln x$", or with the base made explicit as "$\log_e x$".

Example 1.5.14

Example 1.5.15 $\left(\int x^3 \cos\left(x^4+2\right)\mathrm{d}x\right)$

Compute the integral $\int x^3 \cos\left(x^4+2\right)\mathrm{d}x$.

Solution.

- The integrand is the product of cos evaluated at the argument x^4+2 times x^3, which aside from a factor of 4, is the derivative of the argument x^4+2.
- Hence we set $u = x^4+2$ and then substitute $\mathrm{d}x \to \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{4x^3}\mathrm{d}u$.
- Before proceeding further, we should note that this is an indefinite integral so we don't have to worry about the limits of integration. However we do need to make sure our answer is a function of x — we cannot leave it as a function of u.
- With this choice of u, the integral then becomes

$$\begin{aligned}
\int x^3 \cos\left(x^4+2\right)\mathrm{d}x &= \int x^3 \cos(u)\frac{1}{4x^3}\mathrm{d}u\bigg|_{u=x^4+2}\\
&= \int \frac{1}{4}\cos(u)\,\mathrm{d}u\bigg|_{u=x^4+2}\\
&= \left(\frac{1}{4}\sin(u)+C\right)\bigg|_{u=x^4+2}\\
&= \frac{1}{4}\sin(x^4+2)+C.
\end{aligned}$$

Example 1.5.15

The next two examples are more involved and require more careful thinking.

Example 1.5.16 $\left(\int \sqrt{1+x^2}\,x^3\mathrm{d}x\right)$

Compute $\int \sqrt{1+x^2}\,x^3\mathrm{d}x$.

- An obvious choice of u is the argument inside the square root. So substitute $u = 1+x^2$ and $\mathrm{d}x \to \frac{1}{2x}\mathrm{d}u$.
- When we do this we obtain

$$\begin{aligned}
\int \sqrt{1+x^2}\cdot x^3\mathrm{d}x &= \int \sqrt{u}\cdot x^3\cdot\frac{1}{2x}\mathrm{d}u\\
&= \int \frac{1}{2}\sqrt{u}\cdot x^2\mathrm{d}u
\end{aligned}$$

Unlike all our previous examples, we have not cancelled out all of the x's from the integrand. However before we do the integral with respect to u, the integrand must be expressed solely in terms of u — no x's are allowed. (Look that integrand on the right-hand side of Theorem 1.5.2.)

- But all is not lost. We can rewrite the factor x^2 in terms of the variable u. We know that $u = 1 + x^2$, so this means $x^2 = u - 1$. Substituting this into our integral gives

$$\begin{aligned}\int \sqrt{1+x^2}\cdot x^3 \mathrm{d}x &= \int \frac{1}{2}\sqrt{u}\cdot x^2 \mathrm{d}u \\ &= \int \frac{1}{2}\sqrt{u}\cdot (u-1)\mathrm{d}u \\ &= \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right) \mathrm{d}u \\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\bigg|_{u=x^2+1} + C \\ &= \left(\frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2}\right)\bigg|_{u=x^2+1} + C \\ &= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C.\end{aligned}$$

Oof!

- Don't forget that you can always check the answer by differentiating:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{5}(x^2+1)^{5/2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}(x^2+1)^{3/2}\right) \\ &= \frac{1}{5}\cdot 2x\cdot\frac{5}{2}\cdot(x^2+1)^{3/2} - \frac{1}{3}\cdot 2x\cdot\frac{3}{2}\cdot(x^2+1)^{1/2} \\ &= x(x^2+1)^{3/2} - x(x^2+1)^{1/2} \\ &= x\left[(x^2+1)-1\right]\cdot\sqrt{x^2+1} \\ &= x^3\sqrt{x^2+1}.\end{aligned}$$

which is the original integrand ✓.

Example 1.5.16

Example 1.5.17 ($\int \tan x \, \mathrm{d}x$)

Evaluate the indefinite integral $\int \tan(x)\mathrm{d}x$.

Solution.

- At first glance there is nothing to manipulate here and so very little to go on. However we can rewrite $\tan x$ as $\frac{\sin x}{\cos x}$, making the integral $\int \frac{\sin x}{\cos x}\mathrm{d}x$. This gives us more to work with.

- Now think of the integrand as being the product $\frac{1}{\cos x} \cdot \sin x$. This suggests that we set $u = \cos x$ and that we interpret the first factor as the function "one over" evaluated at $u = \cos x$.
- Substitute $u = \cos x$ and $\mathrm{d}x \to \frac{1}{-\sin x}\mathrm{d}u$ to give:

$$\begin{aligned}\int \frac{\sin x}{\cos x}\mathrm{d}x &= \int \frac{\sin x}{u}\frac{1}{-\sin x}\mathrm{d}u\Big|_{u=\cos x}\\ &= \int -\frac{1}{u}\mathrm{d}u\Big|_{u=\cos x}\\ &= -\log|\cos x| + C && \text{and if we want to go further}\\ &= \log\left|\frac{1}{\cos x}\right| + C\\ &= \log|\sec x| + C.\end{aligned}$$

Example 1.5.17

In all of the above substitution examples we expressed the new integration variable, u, as a function, $u(x)$, of the old integration variable x. It is also possible to express the old integration variable, x, as a function, $x(u)$, of the new integration variable u. We shall see examples of this in §1.8.

1.6▲ Area between curves

Learning Objectives

- Set up and compute the area between two curves (perhaps with other geometric boundaries). This includes the case where the two curves intersect or where we integrate along the y axis.
- Determine whether it is more advantageous to compute an area by integrating in x or by integrating in y.

Before we continue our exploration of different methods for integrating functions, we have now have sufficient tools to examine some simple applications of definite integrals. One of the motivations for our definition of "integral" was the problem of finding the area between some curve and the x–axis for x running between two specified values. More precisely

$$\int_a^b f(x)\,\mathrm{d}x$$

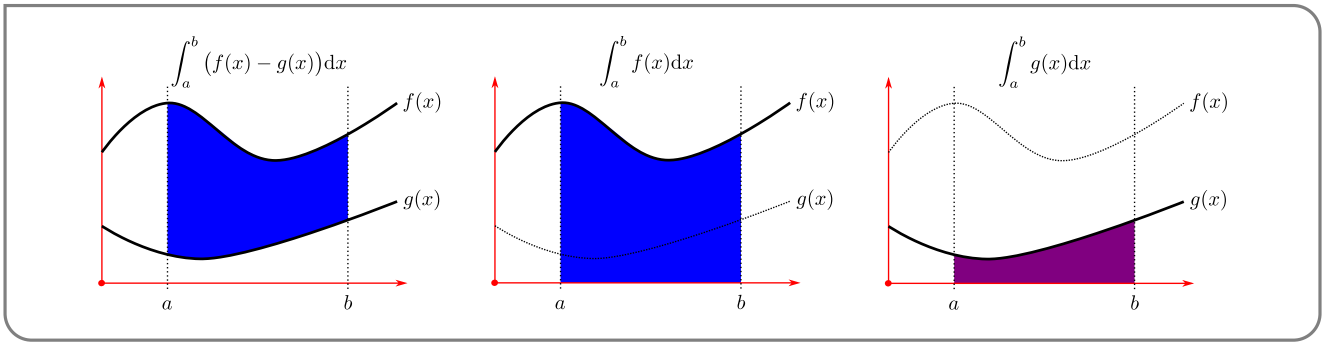
is equal to the signed area between the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

We found the area of this region by approximating it by the union of tall thin rectangles, and then found the exact area by taking the limit as the width of the approximating rectangles went to zero. We can use the same strategy to find areas of more complicated regions in the xy-plane.

As a preview of the material to come, let $f(x) > g(x) > 0$ and $a < b$ and suppose that we are interested in the area of the region

$$S_1 = \left\{ (x,y) \,\middle|\, a \leqslant x \leqslant b\,,\ g(x) \leqslant y \leqslant f(x) \right\}$$

that is sketched in the left-hand figure below.



We already know that $\int_a^b f(x)\,\mathrm{d}x$ is the area of the region

$$S_2 = \left\{ (x,y) \,\middle|\, a \leqslant x \leqslant b\,,\ 0 \leqslant y \leqslant f(x) \right\}$$

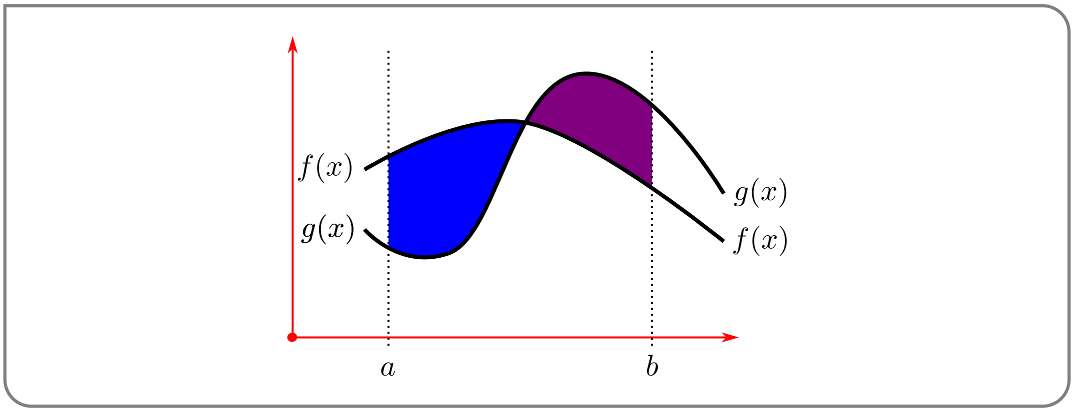
sketched in the middle figure above and that $\int_a^b g(x)\,\mathrm{d}x$ is the area of the region

$$S_3 = \left\{ (x,y) \,\middle|\, a \leqslant x \leqslant b\,,\ 0 \leqslant y \leqslant g(x) \right\}$$

sketched in the right-hand figure above. Now the region S_1 of the left-hand figure can be constructed by taking the region S_2 of center figure and removing from it the region S_3 of the right-hand figure. So the area of S_1 is exactly

$$\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x$$

This computation depended on the assumption that $f(x) > g(x)$ and, in particular, that the curves $y = g(x)$ and $y = f(x)$ did not cross. If they do cross, as in this figure



then we have to be a lot more careful. The idea is to separate the domain of integration depending on where $f(x) - g(x)$ changes sign — i.e. where the curves intersect. We will illustrate this in Example 1.6.5 below.

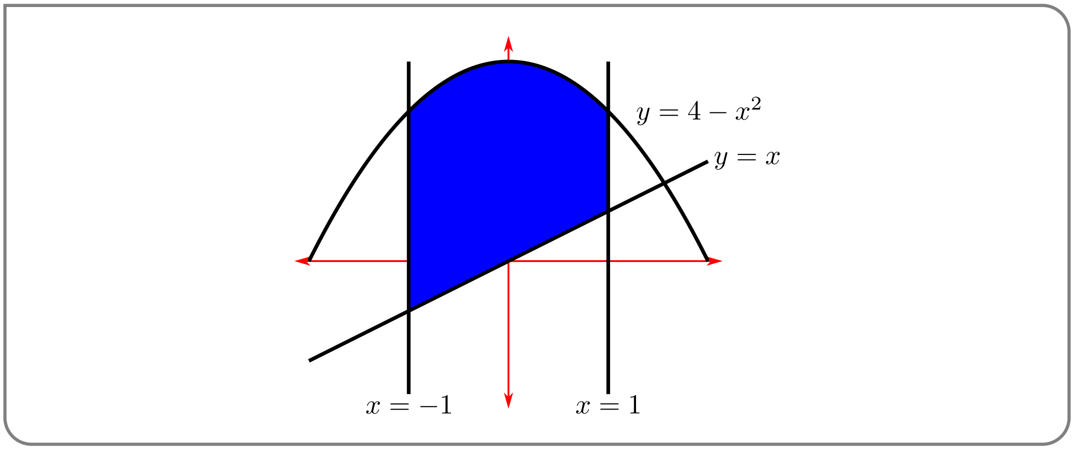
Let us start with an example that makes the link to Riemann sums and definite integrals quite explicit.

Example 1.6.1

Find the area bounded by the curves $y = 4 - x^2$, $y = x$, $x = -1$ and $x = 1$.

Solution.

- Before we do any calculus, it is a very good idea to make a sketch of the area in question. The curves $y = x$, $x = -1$ and $x = 1$ are all straight lines, while the curve $y = 4 - x^2$ is a parabola whose apex is at $(0, 4)$ and then curves down (because of the minus sign in $-x^2$) with x-intercepts at $(\pm 2, 0)$. Putting these together gives



  Notice that the curves $y = 4 - x^2$ and $y = x$ intersect when $4 - x^2 = x$, namely when $x = \frac{1}{2}\left(-1 \pm \sqrt{17}\right) \approx 1.56, -2.56$. Hence the curve $y = 4 - x^2$ lies above the line $y = x$ for all $-1 \leqslant x \leqslant 1$.

- We are to find the area of the shaded region. Each point (x, y) in this shaded region has $-1 \leqslant x \leqslant 1$ and $x \leqslant y \leqslant 4 - x^2$. When we were defining the integral (way back in Definition 1.1.10) we used a and b to denote the smallest and largest allowed values of x; let's do that here too. Let's also use $B(x)$ to denote the bottom curve (i.e. to denote the smallest allowed value of y for a given x) and use $T(x)$ to denote the top curve (i.e. to denote the largest allowed value of y for a given x). So in this example

$$a = -1 \qquad b = 1 \qquad B(x) = x \qquad T(x) = 4 - x^2$$

  and the shaded region is

$$\left\{ (x, y) \;\middle|\; a \leqslant x \leqslant b,\ B(x) \leqslant y \leqslant T(x) \right\}$$

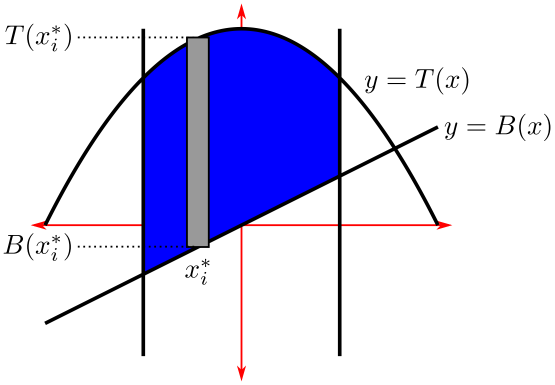
- We use the same strategy as we used when defining the integral in Section 1.1.4:

- Pick a natural number n (that we will later send to infinity), then

- subdivide the region into n narrow slices, each of width $\Delta x = \frac{b-a}{n}$.

- For each $i = 1, 2, \ldots, n$, slice number i runs from $x = x_{i-1}$ to $x = x_i$, and we approximate its area by the area of a rectangle. We pick a number x_i^* between x_{i-1} and x_i and approximate the slice by a rectangle whose top is at $y = T(x_i^*)$ and whose bottom is at $y = B(x_i^*)$.

- Thus the area of slice i is approximately $\left[T(x_i^*) - B(x_i^*)\right]\Delta x$ (as shown in the figure below).



- So the Riemann sum approximation of the area is

$$\text{Area} \approx \sum_{i=1}^{n} \left[T(x_i^*) - B(x_i^*)\right]\Delta x$$

- By taking the limit as $n \to \infty$ (i.e. taking the limit as the width of the rectangles goes to zero), we convert the Riemann sum into a definite integral (see Definition 1.1.10) and at the same time our approximation of the area becomes the exact area:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\left[T(x_i^*)-B(x_i^*)\right]\Delta x &= \int_a^b \left[T(x)-B(x)\right]\mathrm{d}x && \text{Riemann sum} \to \text{integral}\\
&= \int_{-1}^{1}\left[(4-x^2)-x\right]\mathrm{d}x\\
&= \int_{-1}^{1}\left[4-x-x^2\right]\mathrm{d}x\\
&= \left[4x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{1}\\
&= \left(4-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-\frac{1}{2}+\frac{1}{3}\right)\\
&= \frac{24-3-2}{6}-\frac{-24-3+2}{6}\\
&= \frac{19}{6}+\frac{25}{6}\\
&= \frac{44}{6}=\frac{22}{3}.
\end{aligned}$$

Example 1.6.1

Oof! Thankfully we generally do not need to go through the Riemann sum steps to get to the answer. Usually, provided we are careful to check where curves intersect and which curve lies above which, we can just jump straight to the integral

$$\text{Area} = \int_a^b \left[T(x)-B(x)\right]\mathrm{d}x. \tag{1.6.1}$$

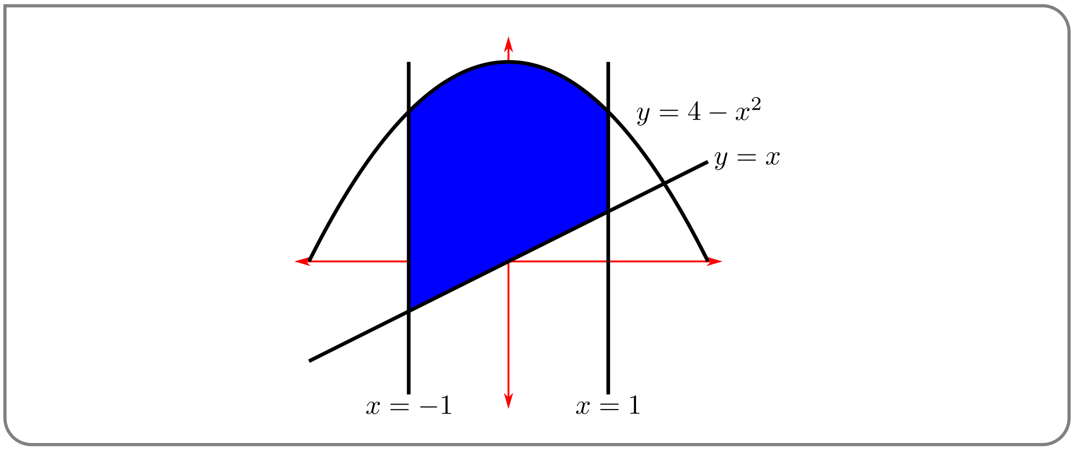
So let us redo the above example.

Example 1.6.2 (Example 1.6.1 revisited)

Find the area bounded by the curves $y=4-x^2$, $y=x$, $x=-1$ and $x=1$.

Solution.

- We first sketch the region



and verify[33] that $y = T(x) = 4 - x^2$ lies above the curve $y = B(x) = x$ on the region $-1 \leqslant x \leqslant 1$.

- The area between the curves is then

$$\begin{aligned} \text{Area} &= \int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x \\ &= \int_{-1}^{1} \big[4 - x - x^2\big]\,\mathrm{d}x \\ &= \left[4x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} \\ &= \frac{19}{6} + \frac{25}{6} = \frac{44}{6} = \frac{22}{3}. \end{aligned}$$

Example 1.6.2

Example 1.6.3

Find the area of the finite region bounded by $y = x^2$ and $y = 6x - 2x^2$.

Solution. This is a little different from the previous question, since we are not given bounding lines $x = a$ and $x = b$ — instead we have to determine the minimum and maximum allowed values of x by determining where the curves intersect. Hence our very first task is to get a good idea of what the region looks like by sketching it.

- Start by sketching the region:
  - The curve $y = x^2$ is a parabola. The point on this parabola with the smallest y–coordinate is $(0, 0)$. As $|x|$ increases, y increases so the parabola opens upward.
  - The curve $y = 6x - 2x^2 = -2(x^2 - 3x) = -2(x - \frac{3}{2})^2 + \frac{9}{2}$ is also a parabola. The point on this parabola with the largest value of y has $x = 3/2$ (so that the

---

[33] We should do this by checking where the curves intersect; that is by solving $T(x) = B(x)$ and seeing if any of the solutions lie in the range $-1 \leqslant x \leqslant 1$.

negative term in $-2(x-\frac{3}{2})^2+\frac{9}{2}$ is zero). So the point with the largest value of y is is $(3/2, 9/2)$. As x moves away from $3/2$, either to the right or to the left, y decreases. So the parabola opens downward. The parabola crosses the x–axis when $0 = 6x - 2x^2 = 2x(3-x)$. That is, when $x = 0$ and $x = 3$.

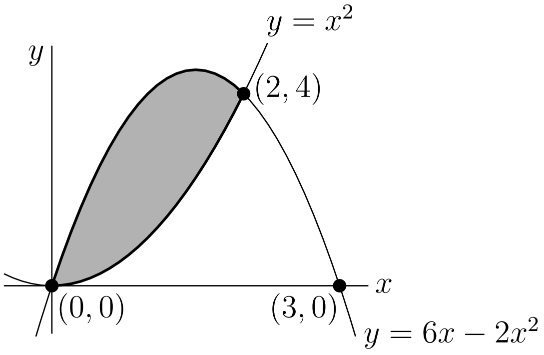
- The two parabolas intersect when $x^2 = 6x - 2x^2$, or

$$\begin{aligned} 3x^2 - 6x &= 0 \\ 3x(x-2) &= 0 \end{aligned}$$

  So there are two points of intersection, one being $x = 0$, $y = 0^2 = 0$ and the other being $x = 2$, $y = 2^2 = 4$.

- The finite region between the curves lies between these two points of intersection.

This leads us to the sketch



- So on this region we have $0 \leqslant x \leqslant 2$, the top curve is $T(x) = 6x - 2x^2$ and the bottom curve is $B(x) = x^2$. Hence the area is given by

$$\begin{aligned} \text{Area} &= \int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x \\ &= \int_0^2 \big[(6x - 2x^2) - (x^2)\big]\,\mathrm{d}x \\ &= \int_0^2 \big[6x - 3x^2\big]\,\mathrm{d}x \\ &= \left[6\frac{x^2}{2} - 3\frac{x^3}{3}\right]_0^2 \\ &= 3(2)^2 - 2^3 = 4 \end{aligned}$$

Example 1.6.3

Example 1.6.4

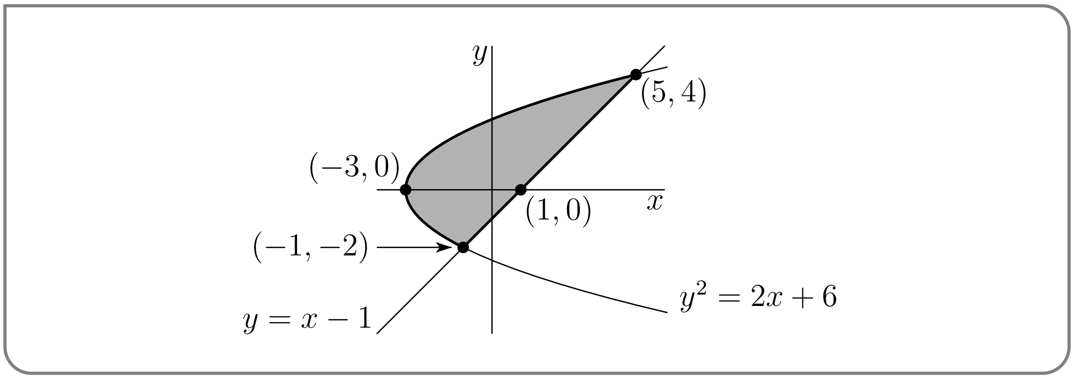
Find the area of the finite region bounded by $y^2 = 2x + 6$ and $y = x - 1$.

Solution. We show two different solutions to this problem. The first takes the approach we have in Example 1.6.3 but leads to messy algebra. The second requires a little bit of thinking at the beginning but then is quite straightforward. Before we get to that we should start by by sketching the region.

- The curve $y^2 = 2x + 6$, or equivalently $x = \frac{1}{2}y^2 - 3$ is a parabola. The point on this parabola with the smallest x–coordinate has $y = 0$ (so that the positive term in $\frac{1}{2}y^2 - 3$ is zero). So the point on this parabola with the smallest x–coordinate is $(-3, 0)$. As $|y|$ increases, x increases so the parabola opens to the right.
- The curve $y = x - 1$ is a straight line of slope 1 that passes through $x = 1$, $y = 0$.
- The two curves intersect when $\frac{y^2}{2} - 3 = y + 1$, or

  $$\begin{aligned} y^2 - 6 &= 2y + 2 \\ y^2 - 2y - 8 &= 0 \\ (y + 2)(y - 4) &= 0 \end{aligned}$$

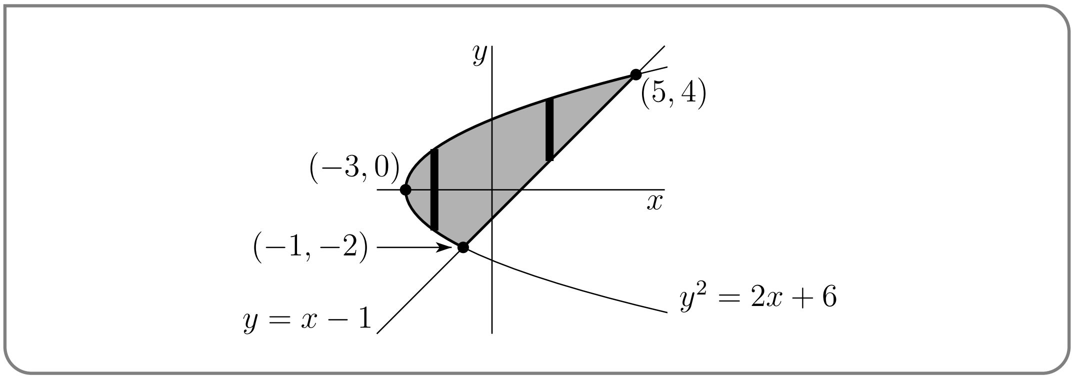
  So there are two points of intersection, one being $y = 4$, $x = 4 + 1 = 5$ and the other being $y = -2$, $x = -2 + 1 = -1$.
- Putting this all together gives us the sketch



As noted above, we can find the area of this region by approximating it by a union of narrow vertical rectangles, as we did in Example 1.6.3 — though it is a little harder. The easy way is to approximate it by a union of narrow horizontal rectangles. Just for practice, here is the hard solution. The easy solution is after it.

Harder solution:

- As we have done previously, we approximate the region by a union of narrow vertical rectangles, each of width Δx. Two of those rectangles are illustrated in the sketch



- In this region, x runs from $a = -3$ to $b = 5$. The curve at the top of the region is

  $$y = T(x) = \sqrt{2x+6}$$

  The curve at the bottom of the region is more complicated. To the left of $(-1,-2)$ the lower half of the parabola gives the bottom of the region while to the right of $(-1,-2)$ the straight line gives the bottom of the region. So

  $$B(x) = \begin{cases} -\sqrt{2x+6} & \text{if } -3 \leqslant x \leqslant -1 \\ x-1 & \text{if } -1 \leqslant x \leqslant 5 \end{cases}$$

- Just as before, the area is still given by the formula $\int_a^b \big[T(x) - B(x)\big]\mathrm{d}x$, but to accommodate our $B(x)$, we have to split up the domain of integration when we evaluate the integral.

$$\begin{aligned}\int_a^b \big[T(x) - B(x)\big]\mathrm{d}x &= \int_{-3}^{-1} \big[T(x) - B(x)\big]\mathrm{d}x + \int_{-1}^{5} \big[T(x) - B(x)\big]\mathrm{d}x \\ &= \int_{-3}^{-1} \big[\sqrt{2x+6} - (-\sqrt{2x+6})\big]\mathrm{d}x + \int_{-1}^{5} \big[\sqrt{2x+6} - (x-1)\big]\mathrm{d}x \\ &= 2\int_{-3}^{-1} \sqrt{2x+6}\mathrm{d}x + \int_{-1}^{5} \sqrt{2x+6} - \int_{-1}^{5} (x-1)\mathrm{d}x\end{aligned}$$

- The third integral is straightforward, while we evaluate the first two via the substitution rule. In particular, set $u = 2x+6$ and replace $\mathrm{d}x \to \frac{1}{2}\mathrm{d}u$. Also $u(-3) = 0, u(-1) = 4, u(5) = 16$. Hence

$$\begin{aligned}\text{Area} &= 2\int_0^4 \sqrt{u}\,\frac{\mathrm{d}u}{2} + \int_4^{16} \sqrt{u}\,\frac{\mathrm{d}u}{2} - \int_{-1}^{5} (x-1)\mathrm{d}x \\ &= 2\left[\frac{u^{3/2}}{3/2}\frac{1}{2}\right]_0^4 + \left[\frac{u^{3/2}}{3/2}\frac{1}{2}\right]_4^{16} - \left[\frac{x^2}{2} - x\right]_{-1}^{5} \\ &= \frac{2}{3}[8-0] + \frac{1}{3}[64-8] - \left[\left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} + 1\right)\right] \\ &= \frac{72}{3} - \frac{24}{2} + 6 \\ &= 18\end{aligned}$$

Oof!

Easier solution:
The easy way to determine the area of our region is to approximate by narrow horizontal rectangles, rather than narrow vertical rectangles. (Really we are just swapping the roles of x and y in this problem)

- Look at our sketch of the region again — each point (x, y) in our region has $-2 \leqslant y \leqslant 4$ and $\frac{1}{2}(y^2 - 6) \leqslant x \leqslant y + 1$.

- Let's use

  - c to denote the smallest allowed value of y,
  - d to denote the largest allowed value of y
  - $L(y)$ ("L" stands for "left") to denote the smallest allowed value of x, when the y–coordinate is y, and
  - $R(y)$ ("R" stands for "right") to denote the largest allowed value of x, when the y–coordinate is y.

  So, in this example,

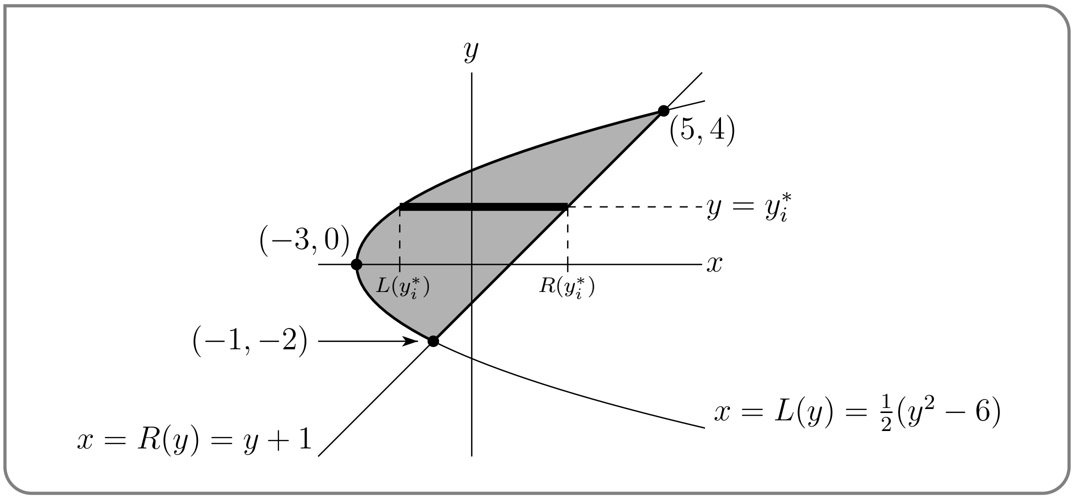
$$c = -2 \qquad d = 4 \qquad L(y) = \frac{1}{2}(y^2 - 6) \qquad R(y) = y + 1$$

  and the shaded region is

$$\left\{ (x, y) \mid c \leqslant y \leqslant d,\ L(y) \leqslant x \leqslant R(y) \right\}$$

- Our strategy is now nearly the same as that used in Example 1.6.1:

  - Pick a natural number n (that we will later send to infinity), then
  - subdivide the interval $c \leqslant y \leqslant d$ into n narrow subintervals, each of width $\Delta y = \frac{d-c}{n}$. Each subinterval cuts a thin horizontal slice from the region (see the figure below).
  - We approximate the area of slice number i by the area of a thin horizontal rectangle (indicated by the dark rectangle in the figure below). On this slice, the y–coordinate runs over a very narrow range. We pick a number y_i^*, somewhere in that range. We approximate slice i by a rectangle whose left side is at $x = L(y_i^*)$ and whose right side is at $x = R(y_i^*)$.
  - Thus the area of slice i is approximately $\left[R(x_i^*) - L(x_i^*)\right]\Delta y$.



- The desired area is

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\big[R(y_i^*)-L(y_i^*)\big]\Delta y &= \int_c^d \big[R(y)-L(y)\big]\,\mathrm{d}y && \text{Riemann sum} \to \text{integral}\\ &= \int_{-2}^{4}\big[(y+1)-\tfrac{1}{2}\big(y^2-6\big)\big]\,\mathrm{d}y\\ &= \int_{-2}^{4}\big[-\tfrac{1}{2}y^2+y+4\big]\,\mathrm{d}y\\ &= \Big[-\tfrac{1}{6}y^3+\tfrac{1}{2}y^2+4y\Big]_{-2}^{4}\\ &= -\tfrac{1}{6}\big(64-(-8)\big)+\tfrac{1}{2}(16-4)+4(4+2)\\ &= -12+6+24\\ &= 18\end{aligned}$$

Example 1.6.4

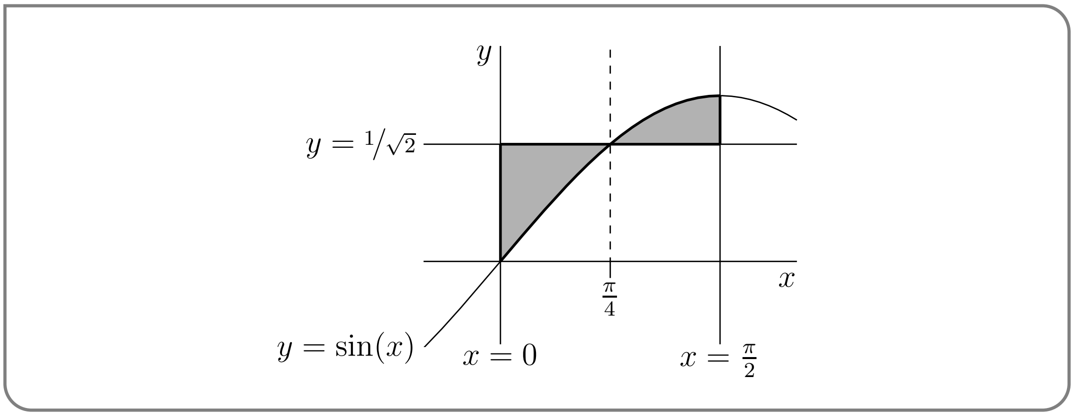
One last example.

Example 1.6.5

Find the area between the curves $y = \dfrac{1}{\sqrt{2}}$ and $y = \sin(x)$ with x running from 0 to $\pi/2$.

Solution. This one is a little trickier since (as we shall see) the region is split into two pieces and we need to treat them separately.

- Again we start by sketching the region.



We want the shaded area.

- Unlike our previous examples, the bounding curves $y = 1/\sqrt{2}$ and $y = \sin(x)$ cross in the middle of the region of interest. They cross when $y = 1/\sqrt{2}$ and $\sin(x) = y = 1/\sqrt{2}$, i.e. when $x = \pi/4$. So
  - to the left of $x = \pi/4$, the top boundary is part of the straight line $y = 1/\sqrt{2}$ and the bottom boundary is part of the curve $y = \sin(x)$
  - while to the right of $x = \pi/4$, the top boundary is part of the curve $y = \sin(x)$ and the bottom boundary is part of the straight line $y = 1/\sqrt{2}$.
- Thus the formulae for the top and bottom boundaries are

$$T(x) = \begin{Bmatrix} 1/\sqrt{2} & \text{if } 0 \leqslant x \leqslant \pi/4 \\ \sin(x) & \text{if } \pi/4 \leqslant x \leqslant \pi/2 \end{Bmatrix} \qquad B(x) = \begin{Bmatrix} \sin(x) & \text{if } 0 \leqslant x \leqslant \pi/4 \\ 1/\sqrt{2} & \text{if } \pi/4 \leqslant x \leqslant \pi/2 \end{Bmatrix}$$

  We may compute the area of interest using our canned formula

$$\text{Area} = \int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x$$

  but since the formulas for $T(x)$ and $B(x)$ change at the point $x = \pi/4$, we must split the domain of the integral in two at that point[34]

- Our integral over the domain $0 \leqslant x \leqslant \pi/2$ is split into an integral over $0 \leqslant x \leqslant \pi/4$

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[34] We are effectively computing the area of the region by computing the area of the two disjoint pieces separately. Alternatively, if we set $f(x) = \sin(x)$ and $g(x) = 1/\sqrt{2}$, we can rewrite the integral $\int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x$ as $\int_a^b \big|f(x) - g(x)\big|\,\mathrm{d}x$. To see that the two integrals are the same, split the domain of integration where $f(x) - g(x)$ changes sign.

and one over $\pi/4 \leqslant x \leqslant \pi/2$:

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/2} \big[T(x) - B(x)\big]\,\mathrm{d}x \\
&= \int_0^{\pi/4} \big[T(x) - B(x)\big]\,\mathrm{d}x + \int_{\pi/4}^{\pi/2} \big[T(x) - B(x)\big]\,\mathrm{d}x \\
&= \int_0^{\pi/4} \left[\frac{1}{\sqrt{2}} - \sin(x)\right]\mathrm{d}x + \int_{\pi/4}^{\pi/2} \left[\sin(x) - \frac{1}{\sqrt{2}}\right]\mathrm{d}x \\
&= \left[\frac{x}{\sqrt{2}} + \cos(x)\right]_0^{\pi/4} + \left[-\cos(x) - \frac{x}{\sqrt{2}}\right]_{\pi/4}^{\pi/2} \\
&= \left[\frac{1}{\sqrt{2}}\frac{\pi}{4} + \frac{1}{\sqrt{2}} - 1\right] + \left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{\pi}{4}\right] \\
&= \frac{2}{\sqrt{2}} - 1 \\
&= \sqrt{2} - 1
\end{aligned}$$

Example 1.6.5

1.7▲ Trigonometric integrals

Learning Objectives

- Memorize the derivative of tangent.
- Compute integrals involving powers of sine and cosine by utilizing an appropriate substitution.
- Use trigonometric identities to compute integrals involving even powers of sine and cosine. You must know: $\sin^2 x + \cos^2 x = 1$, $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$, $\sin(2x) = 2\sin x \cos x$, and $\tan^2 x + 1 = \sec^2 x$.
- Use the definitions of different trigonometric functions to convert integrals into an easier form, where appropriate. This includes antidifferentiating products of various trig functions by converting them into sines and cosines.

Integrals of polynomials of the trigonometric functions $\sin x$, $\cos x$, $\tan x$ and so on, are generally evaluated by using a combination of simple substitutions and trigonometric identities. There are of course a very large number[35] of trigonometric identities, but usually we use only a handful of them. The most important three are:

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[35] The more pedantic reader could construct an infinite list of them.

Equation 1.7.1.

$$\sin^2 x + \cos^2 x = 1$$

Equation 1.7.2.

$$\sin(2x) = 2\sin x \cos x$$

Equation 1.7.3.

$$\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$

Notice that the last two lines of Equation (1.7.3) follow from the first line by replacing either $\sin^2 x$ or $\cos^2 x$ using Equation (1.7.1). It is also useful to rewrite these last two lines:

Equation 1.7.4.

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

Equation 1.7.5.

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

These last two are particularly useful since they allow us to rewrite higher powers of sine and cosine in terms of lower powers. For example:

$$\begin{aligned}\sin^4(x) &= \left[\frac{1-\cos(2x)}{2}\right]^2 && \text{by Equation (1.7.4)}\\ &= \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\underbrace{\cos^2(2x)}_{\text{do it again}} && \text{use Equation (1.7.5)}\\ &= \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{8}\left(1+\cos(4x)\right)\\ &= \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\end{aligned}$$

So while it was hard to integrate $\sin^4(x)$ directly, the final expression is quite straightforward (with a little substitution rule).

There are many such tricks for integrating powers of trigonometric functions. Here we concentrate on two families

$$\int \sin^m x \cos^n x \, \mathrm{d}x \qquad \text{and} \qquad \int \tan^m x \sec^n x \, \mathrm{d}x$$

for nonnegative integer n, m. The details of the technique depend on the parity of n and m — that is, whether n and m are even or odd numbers.

1.7.1 ▸▸ Integrating $\int \sin^m x \cos^n x \, \mathrm{d}x$

▸▸▸ One of n and m is Odd

Consider the integral $\int \sin^2 x \cos x \, \mathrm{d}x$. We can integrate this by substituting $u = \sin x$ and $\mathrm{d}u = \cos x \, \mathrm{d}x$. This gives

$$\begin{aligned} \int \sin^2 x \cos x \, \mathrm{d}x &= \int u^2 \mathrm{d}u \\ &= \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C \end{aligned}$$

This method can be used whenever n is an odd integer.

- Substitute $u = \sin x$ and $\mathrm{d}u = \cos x \, \mathrm{d}x$.
- This leaves an even power of cosines — convert them using $\cos^2 x = 1 - \sin^2 x = 1 - u^2$.

Here is an example.

Example 1.7.6 ($\int \sin^2 x \cos^3 x \, \mathrm{d}x$)

Start by factoring off one power of $\cos x$ to combine with $\mathrm{d}x$ to get $\cos x \, \mathrm{d}x = \mathrm{d}u$.

$$\begin{aligned} \int \sin^2 x \cos^3 x \, \mathrm{d}x &= \int \underbrace{\sin^2 x}_{=u^2} \underbrace{\cos^2 x}_{=1-u^2} \underbrace{\cos x \, \mathrm{d}x}_{=\mathrm{d}u} && \text{set } u = \sin x \\ &= \int u^2 \, (1 - u^2) \mathrm{d}u \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \end{aligned}$$

Example 1.7.6

Of course if m is an odd integer we can use the same strategy with the roles of $\sin x$ and $\cos x$ exchanged. That is, we substitute $u = \cos x$, $\mathrm{d}u = -\sin x \, \mathrm{d}x$ and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

▸▸▸ Both n and m are Even

If m and n are both even whole numbers, the strategy is to use the trig identities (1.7.4) and (1.7.5) to get back to the m or n odd case. This is typically more laborious than the previous case we studied. Here are a couple of examples that arise quite commonly in applications.

Example 1.7.7 ($\int \cos^2 x\,\mathrm{d}x$)

By (1.7.5)

$$\int \cos^2 x\,\mathrm{d}x = \frac{1}{2}\int \big[1+\cos(2x)\big]\,\mathrm{d}x = \frac{1}{2}\Big[x+\frac{1}{2}\sin(2x)\Big]+C$$

Example 1.7.7

Example 1.7.8 ($\int \cos^4 x\,\mathrm{d}x$)

First we'll prepare the integrand $\cos^4 x$ for easy integration by applying (1.7.5) a couple times. We have already used (1.7.5) once to get

$$\cos^2 x = \frac{1}{2}\big[1+\cos(2x)\big]$$

Squaring it gives

$$\cos^4 x = \frac{1}{4}\big[1+\cos(2x)\big]^2 = \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\cos^2(2x)$$

Now by (1.7.5) a second time

$$\begin{aligned}\cos^4 x &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\,\frac{1+\cos(4x)}{2}\\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

Now it's easy to integrate

$$\begin{aligned}\int \cos^4 x\,\mathrm{d}x &= \frac{3}{8}\int \mathrm{d}x+\frac{1}{2}\int \cos(2x)\mathrm{d}x+\frac{1}{8}\int \cos(4x)\mathrm{d}x\\ &= \frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C\end{aligned}$$

Example 1.7.8

Example 1.7.9 ($\int \cos^2 x \sin^2 x \,\mathrm{d}x$)

Here we apply both (1.7.4) and (1.7.5).

$$\begin{aligned}\int \cos^2 x \sin^2 x \,\mathrm{d}x &= \frac{1}{4}\int \big[1+\cos(2x)\big]\big[1-\cos(2x)\big]\,\mathrm{d}x\\ &= \frac{1}{4}\int \big[1-\cos^2(2x)\big]\,\mathrm{d}x\end{aligned}$$

We can then apply (1.7.5) again

$$\begin{aligned}&= \frac{1}{4}\int \Big[1-\frac{1}{2}\left(1+\cos(4x)\right)\Big]\,\mathrm{d}x\\ &= \frac{1}{8}\int \big[1-\cos(4x)\big]\,\mathrm{d}x\\ &= \frac{1}{8}x-\frac{1}{32}\sin(4x)+C\end{aligned}$$

Oof! We could also have done this one using (1.7.2) to write the integrand as $\sin^2(2x)$ and then used (1.7.4) to write it in terms of $\cos(4x)$.

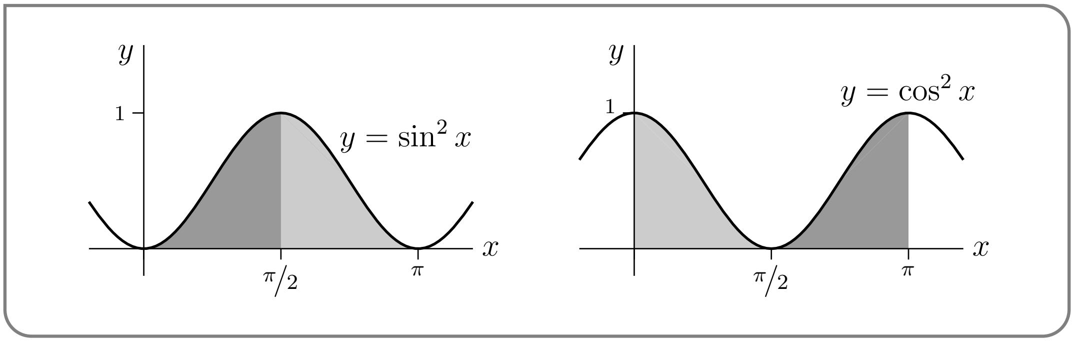
Example 1.7.9

Example 1.7.10 ($\int_0^\pi \cos^2 x \,\mathrm{d}x$ and $\int_0^\pi \sin^2 x \,\mathrm{d}x$)

Of course we can compute the definite integral $\int_0^\pi \cos^2 x \,\mathrm{d}x$ by using the antiderivative for $\cos^2 x$ that we found in Example 1.7.7. But here is a trickier way to evaluate that integral, and also the integral $\int_0^\pi \sin^2 x \,\mathrm{d}x$ at the same time, very quickly without needing the antiderivative of Example 1.7.7.

Solution.

- Observe that $\int_0^\pi \cos^2 x \,\mathrm{d}x$ and $\int_0^\pi \sin^2 x \,\mathrm{d}x$ are equal because they represent the same area — look at the graphs below — the darkly shaded regions in the two graphs have the same area and the lightly shaded regions in the two graphs have the same area.



- Consequently,

$$\begin{aligned}\int_0^\pi \cos^2 x\,\mathrm{d}x = \int_0^\pi \sin^2 x\,\mathrm{d}x &= \frac{1}{2}\left[\int_0^\pi \sin^2 x\,\mathrm{d}x + \int_0^\pi \cos^2 x\,\mathrm{d}x\right]\\ &= \frac{1}{2}\int_0^\pi \left[\sin^2 x + \cos^2 x\right]\mathrm{d}x\\ &= \frac{1}{2}\int_0^\pi \mathrm{d}x\\ &= \frac{\pi}{2}\end{aligned}$$

Example 1.7.10

1.7.2 ▸▸ Integrating $\int \tan^m x \sec^n x\,\mathrm{d}x$

The strategy for dealing with these integrals is similar to the strategy that we used to evaluate integrals of the form $\int \sin^m x \cos^n x\,\mathrm{d}x$ and again depends on the parity of the exponents n and m. However, there are more cases than in the previous section. We will only cover some of them.

First, a useful example.

Example 1.7.11 ($\int \tan^2 x\,\mathrm{d}x$)

The identity $\sin^2 x + \cos^2 x = 1$, divided through by $\cos x$ on both sides, yields the identity $\tan^2 x + 1 = \sec^2 x$.

$$\int \tan^2 x\,\mathrm{d}x = \int \left(\sec^2 x - 1\right)\mathrm{d}x = \tan x - x + C$$

Example 1.7.11

▸▸▸ m is Odd — Odd Power of Tangent

In this case we rewrite the integrand in terms of sine and cosine and then substitute $u = \cos x, \mathrm{d}u = -\sin x\,\mathrm{d}x$.

Example 1.7.12 ($\int \tan x\,\mathrm{d}x$)

Solution.

- Write the integrand $\tan x = \frac{1}{\cos x}\sin x$.

- Now substitute $u = \cos x$, $\mathrm{d}u = -\sin x \, \mathrm{d}x$ just as we did in treating integrands of the form $\sin^m x \cos^n x$ with m odd.

$$\begin{aligned}
\int \tan x \, \mathrm{d}x &= \int \frac{1}{\cos x} \sin x \, \mathrm{d}x && \text{substitute } u = \cos x \\
&= \int \frac{1}{u} \cdot (-1) \mathrm{d}u \\
&= -\log|u| + C \\
&= -\log|\cos x| + C && \text{can also write in terms of secant} \\
&= \log|\cos x|^{-1} + C = \log|\sec x| + C
\end{aligned}$$

Example 1.7.12

Example 1.7.13 ($\int \tan^3 x \, \mathrm{d}x$)

Solution.

- Write the integrand $\tan^3 x = \frac{\sin^2 x}{\cos^3 x} \sin x$.
- Again substitute $u = \cos x$, $\mathrm{d}u = -\sin x \, \mathrm{d}x$. We rewrite the remaining even powers of $\sin x$ using $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.
- Hence

$$\begin{aligned}
\int \tan^3 x \, \mathrm{d}x &= \int \frac{\sin^2 x}{\cos^3 x} \sin x \, \mathrm{d}x && \text{substitute } u = \cos x \\
&= \int \frac{1 - u^2}{u^3} (-1) \mathrm{d}u \\
&= \frac{u^{-2}}{2} + \log|u| + C \\
&= \frac{1}{2\cos^2 x} + \log|\cos x| + C && \text{can rewrite in terms of secant} \\
&= \frac{1}{2} \sec^2 x - \log|\sec x| + C
\end{aligned}$$

Example 1.7.13

▸▸▸ m is Odd and $n \geqslant 1$ — Odd Power of Tangent and at Least One Secant

When m is odd, the strategy discussed above will work, but it may not be the most efficient way. Here we collect a factor of $\tan x \sec x$ and then substitute $u = \sec x$ and $\mathrm{d}u = \sec x \tan x \, \mathrm{d}x$. We can then rewrite any remaining even powers of $\tan x$ in terms of $\sec x$ using $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.

Example 1.7.14 ($\int \tan^3 x \sec^4 x \, \mathrm{d}x$)

Solution.

- Start by factoring off one copy of $\sec x \tan x$ and combine it with $\mathrm{d}x$ to form $\sec x \tan x \, \mathrm{d}x$, which will be $\mathrm{d}u$.
- Now substitute $u = \sec x$, $\mathrm{d}u = \sec x \tan x \, \mathrm{d}x$ and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.
- This gives

$$\begin{aligned}\int \tan^3 x \sec^4 x \, \mathrm{d}x &= \int \underbrace{\tan^2 x}_{u^2-1} \; \underbrace{\sec^3 x}_{u^3} \; \underbrace{\sec x \tan x \, \mathrm{d}x}_{\mathrm{d}u} \\ &= \int [u^2 - 1] u^3 \mathrm{d}u \\ &= \frac{u^6}{6} - \frac{u^4}{4} + C \\ &= \frac{1}{6} \sec^6 x - \frac{1}{4} \sec^4 x + C\end{aligned}$$

Example 1.7.14

1.7.3 ▸▸ Integration by converting to sines and cosines

The trick of converting the product of trigonometric functions to sines and cosines is a nice one to try, if your integrand doesn't otherwise match a familiar form.

Example 1.7.15 ($\int \tan^3 x \csc^2 x \, \mathrm{d}x$)

We know that for an odd power of tangent and some number of *secant* functions, we can solve by converting to sines and cosines. So, let's try that trick here, where we have *cosecant* functions.

$$\begin{aligned}\int \tan^3 x \csc^2 x \, \mathrm{d}x &= \int \left(\frac{\sin x}{\cos x}\right)^3 \cdot \left(\frac{1}{\sin x}\right)^2 \mathrm{d}x \\ &= \int \frac{\sin x}{\cos^3 x} \mathrm{d}x\end{aligned}$$

Let $u = \cos x$, $-\mathrm{d}u = \sin x \, \mathrm{d}x$

$$\begin{aligned}&= \int -u^{-3} \, \mathrm{d}u \\ &= \frac{1}{2u^2} + C \\ &= \frac{1}{2\cos^2 x} + C \\ &= \frac{1}{2} \sec^2 x + C\end{aligned}$$

Example 1.7.15

Example 1.7.16 ($\int \tan x \cos^2 x \, \mathrm{d}x$)

We proceed as before.

$$\begin{aligned}\int \tan x \cos^2 x \, \mathrm{d}x &= \int \left(\frac{\sin x}{\cos x}\right) \cdot \cos^2 x \, \mathrm{d}x \\ &= \int \sin x \cdot \cos x \, \mathrm{d}x\end{aligned}$$

Let $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$

$$\begin{aligned}&= \int u \, \mathrm{d}x \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\sin^2 x + C\end{aligned}$$

Example 1.7.16

In the previous two examples, converting the integrand to sines and cosines left us with products of sines and cosines that we knew how to integrate. This works pretty well when the powers of sines and cosines are both nonnegative, or when there is one negative power and one odd positive power. But it isn't stong enough to solve all cases.

Example 1.7.17 (Rain on the parade)

Consider $\int \tan^2 x \sec^2 x \, \mathrm{d}x$. This is a classic case of the substitution rule: let $u = \tan x$ and $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$, and you're all but done. But, suppose you wanted to turn it into sines and cosines, like the other examples. If we convert it to sines and cosines, we get $\int \frac{\sin^2 x}{\cos^4 x} \, \mathrm{d}x$.

- Since neither sine nor cosine has an odd power, a substitution like $u = \sin x$ or $u = \cos x$ is hard to shoehorn in.

- Since both sine and cosine have even powers, we might want to try the half-angle formulas.

$$\begin{aligned}\frac{\sin^2 x}{\cos^4 x} &= \sin^2 x \cdot \left(\frac{1}{\cos^2 x}\right)^2\\ &= \frac{1-\cos(2x)}{2} \cdot \left(\frac{2}{1+\cos(2x)}\right)^2\\ &= 2 \cdot \frac{1-\cos(2x)}{1+2\cos(2x)+\cos^2(2x)}\\ &= 2 \cdot \frac{1-\cos(2x)}{1+2\cos(2x)+\left(\frac{1+\cos(4x)}{2}\right)^2}\end{aligned}$$

This doesn't seem to improve matters.

So, we see with this example that converting to sines and cosines is not a panacea.

Example 1.7.17

1.8▲ Trigonometric substitution

Learning Objectives

- Recognize when it's appropriate to use the method of trigonometric substitution when computing an integral.
- Identify which substitution and which trig identity is required during trig substitution. In particular, remember the Pythagorean identity three ways: $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $\sec^2\theta - 1 = \tan^2\theta$.
- Compute integrals using trig substitution.
- Reduce compositions like $\sin(\cos^{-1}(x))$ to radical functions of x, where no trigonometric functions appear. Extend this skill to simplify any composition of the form $f(g(x))$, where $f(\theta)$ is one of $\sin\theta$, $\cos\theta$, etc., and $g(x)$ is one of $\tan^{-1} x$, $\sec^{-1} x$, or $\sin^{-1} x$.
- Memorize (or be able to independently derive) exact values of $\sin\theta$ and $\cos\theta$ for all θ that are integer multiples of $\frac{\pi}{4}$ or $\frac{\pi}{6}$, and use them to simplify definite integrals.

In this section we discuss substitutions that simplify integrals containing square roots of the form

$$\sqrt{a^2-x^2} \qquad \sqrt{a^2+x^2} \qquad \sqrt{x^2-a^2}.$$

When the integrand contains one of these square roots, then we can use trigonometric substitutions to eliminate them. That is, we substitute

$$x = a\sin u \qquad \text{or} \qquad x = a\tan u \qquad \text{or} \qquad x = a\sec u$$

and then use trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{and} \quad 1 + \tan^2\theta = \sec^2\theta$$

to simplify the result. To be more precise, we can

- eliminate $\sqrt{a^2 - x^2}$ from an integrand by substituting $x = a\sin u$ to give
$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2 u} = \sqrt{a^2\cos^2 u} = |a\cos u|$$
- eliminate $\sqrt{a^2 + x^2}$ from an integrand by substituting $x = a\tan u$ to give
$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2 u} = \sqrt{a^2\sec^2 u} = |a\sec u|$$
- eliminate $\sqrt{x^2 - a^2}$ from an integrand by substituting $x = a\sec u$ to give
$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2 u - a^2} = \sqrt{a^2\tan^2 u} = |a\tan u|$$
  Be very careful with signs and absolute values when using this substitution. See Example 1.8.6.

When we have used substitutions before, we usually gave the new integration variable, u, as a function of the old integration variable x. Here we are doing the reverse — we are giving the old integration variable, x, in terms of the new integration variable u. We may do so, as long as we may invert to get u as a function of x. For example, with $x = a\sin u$, we may take $u = \arcsin\frac{x}{a}$. This is a good time for you to review the definitions of $\arcsin\theta$, $\arctan\theta$ and $\operatorname{arcsec}\theta$.

As a warm-up, consider the area of a quarter of the unit circle.

Example 1.8.1 (Quarter of the unit circle)

Compute the area of the unit circle lying in the first quadrant.

Solution. We know that the answer is $\pi/4$, but we can also compute this as an integral — we saw this way back in Example 1.1.19:

$$\text{area} = \int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x$$

- To simplify the integrand we substitute $x = \sin u$. With this choice $\frac{\mathrm{d}x}{\mathrm{d}u} = \cos u$ and so $\mathrm{d}x = \cos u\,\mathrm{d}u$.
- We also need to translate the limits of integration and it is perhaps easiest to do this by writing u as a function of x — namely $u(x) = \arcsin x$. Hence $u(0) = 0$ and $u(1) = \pi/2$.

- Hence the integral becomes

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\mathrm{d}x &= \int_0^{\pi/2} \sqrt{1-\sin^2 u}\cdot\cos u\mathrm{d}u\\ &= \int_0^{\pi/2}\sqrt{\cos^2 u}\cdot\cos u\mathrm{d}u\\ &= \int_0^{\pi/2}\cos^2 u\mathrm{d}u\end{aligned}$$

  Notice that here we have used that the *positive* square root $\sqrt{\cos^2 u} = |\cos u| = \cos u$ because $\cos(u) \geqslant 0$ for $0 \leqslant u \leqslant \pi/2$.

- To go further we use the techniques of Section 1.7.

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\mathrm{d}x &= \int_0^{\pi/2}\cos^2 u\mathrm{d}u && \text{and since } \cos^2 u = \frac{1+\cos 2u}{2}\\ &= \frac{1}{2}\int_0^{\pi/2}(1+\cos(2u))\mathrm{d}u\\ &= \frac{1}{2}\left[u+\frac{1}{2}\sin(2u)\right]_0^{\pi/2}\\ &= \frac{1}{2}\left(\frac{\pi}{2}-0+\frac{\sin\pi}{2}-\frac{\sin 0}{2}\right)\\ &= \frac{\pi}{4}\checkmark\end{aligned}$$

Example 1.8.1

Example 1.8.2 $\left(\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x\right)$

Solution. We proceed much as we did in the previous example.

- To simplify the integrand we substitute $x = \sin u$. With this choice $\frac{\mathrm{d}x}{\mathrm{d}u} = \cos u$ and so $\mathrm{d}x = \cos u\mathrm{d}u$. Also note that $u = \arcsin x$.

- The integral becomes

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x &= \int \frac{\sin^2 u}{\sqrt{1-\sin^2 u}}\cdot\cos u\mathrm{d}u\\ &= \int\frac{\sin^2 u}{\sqrt{\cos^2 u}}\cdot\cos u\mathrm{d}u\end{aligned}$$

- To proceed further we need to get rid of the square-root. Since $u = \arcsin x$ has domain $-1 \leqslant x \leqslant 1$ and range $-\pi/2 \leqslant u \leqslant \pi/2$, it follows that $\cos u \geqslant 0$ (since cosine is non-negative on these inputs). Hence

$$\sqrt{\cos^2 u} = \cos u \qquad \text{when } -\pi/2 \leqslant u \leqslant \pi/2$$

- So our integral now becomes

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x &= \int \frac{\sin^2 u}{\sqrt{\cos^2 u}} \cdot \cos u \mathrm{d}u \\
&= \int \frac{\sin^2 u}{\cos u} \cdot \cos u \mathrm{d}u \\
&= \int \sin^2 u \mathrm{d}u \\
&= \frac{1}{2}\int (1-\cos 2u)\mathrm{d}u && \text{by Equation (1.7.4)} \\
&= \frac{u}{2} - \frac{1}{4}\sin 2u + C \\
&= \frac{1}{2}\arcsin x - \frac{1}{4}\sin(2\arcsin x) + C
\end{aligned}$$

- We can simplify this further using a double-angle identity. Recall that $u = \arcsin x$ and that $x = \sin u$. Then

$$\sin 2u = 2\sin u \cos u$$

We can replace $\cos u$ using $\cos^2 u = 1 - \sin^2 u$. Taking a square-root of this formula gives $\cos u = \pm\sqrt{1-\sin^2 u}$. We need the positive branch here since $\cos u \geqslant 0$ when $-\pi/2 \leqslant u \leqslant \pi/2$ (which is exactly the range of $\arcsin x$). Continuing along:

$$\begin{aligned}
\sin 2u &= 2\sin u \cdot \sqrt{1-\sin^2 u} \\
&= 2x\sqrt{1-x^2}
\end{aligned}$$

Thus our solution is

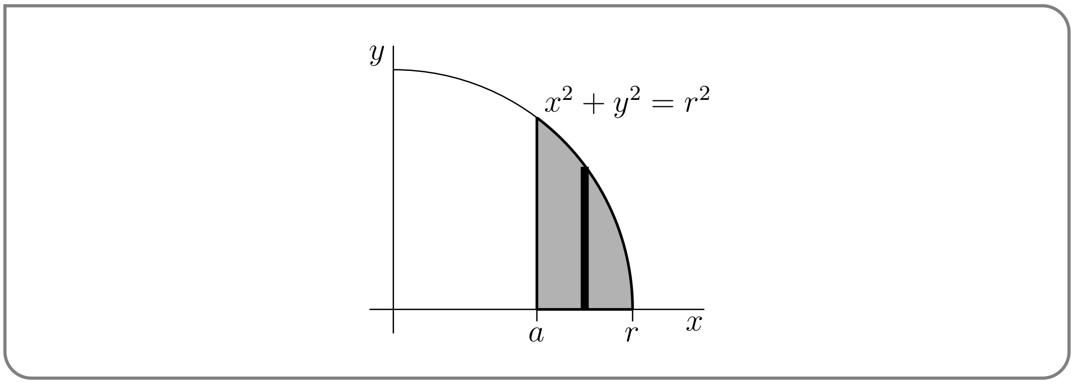
$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x &= \frac{1}{2}\arcsin x - \frac{1}{4}\sin(2\arcsin x) + C \\
&= \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C
\end{aligned}$$

Example 1.8.2

The above two example illustrate the main steps of the approach. The next example is similar, but with more complicated limits of integration.

Example 1.8.3 $\left(\int_a^r \sqrt{r^2-x^2}\,\mathrm{d}x\right)$

Let's find the area of the shaded region in the sketch below.



We'll set up the integral using vertical strips. The strip in the figure has width dx and height $\sqrt{r^2 - x^2}$. So the area is given by the integral

$$\text{area} = \int_a^r \sqrt{r^2 - x^2}\,\mathrm{d}x$$

Which is very similar to the previous example.

Solution.

- To evaluate the integral we substitute

  $$x = x(u) = r\sin u \qquad\qquad \mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u = r\cos u\,\mathrm{d}u$$

  It is also helpful to write u as a function of x — namely $u = \arcsin\frac{x}{r}$.

- The integral runs from $x = a$ to $x = r$. These correspond to

  $$\begin{aligned} u(r) &= \arcsin\frac{r}{r} = \arcsin 1 = \frac{\pi}{2} \\ u(a) &= \arcsin\frac{a}{r} \quad \text{which does not simplify further} \end{aligned}$$

- The integral then becomes

  $$\begin{aligned} \int_a^r \sqrt{r^2 - x^2}\mathrm{d}x &= \int_{\arcsin(a/r)}^{\pi/2} \sqrt{r^2 - r^2\sin^2 u} \cdot r\cos u\mathrm{d}u \\ &= \int_{\arcsin(a/r)}^{\pi/2} r^2\sqrt{1 - \sin^2 u} \cdot \cos u\mathrm{d}u \\ &= r^2\int_{\arcsin(a/r)}^{\pi/2} \sqrt{\cos^2 u} \cdot \cos u\mathrm{d}u \end{aligned}$$

  To proceed further (as we did in Examples 1.8.1 and 1.8.2) we need to think about whether $\cos u$ is positive or negative.

- Since a (as shown in the diagram) satisfies $0 \leqslant a \leqslant r$, we know that $u(a)$ lies between $\arcsin(0) = 0$ and $\arcsin(1) = \pi/2$. Hence the variable u lies between 0 and $\pi/2$, and on this range $\cos u \geqslant 0$. This allows us get rid of the square-root:

  $$\sqrt{\cos^2 u} = |\cos u| = \cos u$$

- Putting this fact into our integral we get

$$\begin{aligned}\int_a^r \sqrt{r^2-x^2}\mathrm{d}x &= r^2\int_{\arcsin(a/r)}^{\pi/2} \sqrt{\cos^2 u}\cdot \cos u\mathrm{d}u\\ &= r^2\int_{\arcsin(a/r)}^{\pi/2} \cos^2 u\mathrm{d}u\end{aligned}$$

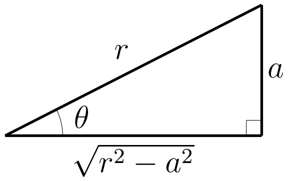
  Recall the identity $\cos^2 u = \frac{1+\cos 2u}{2}$ from Section 1.7

$$\begin{aligned}&= \frac{r^2}{2}\int_{\arcsin(a/r)}^{\pi/2}(1+\cos 2u)\mathrm{d}u\\ &= \frac{r^2}{2}\left[u+\frac{1}{2}\sin(2u)\right]_{\arcsin(a/r)}^{\pi/2}\\ &= \frac{r^2}{2}\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi - \arcsin(a/r)-\frac{1}{2}\sin(2\arcsin(a/r))\right)\\ &= \frac{r^2}{2}\left(\frac{\pi}{2}-\arcsin(a/r)-\frac{1}{2}\sin(2\arcsin(a/r))\right)\end{aligned}$$

  Oof! But there is a little further to go before we are done.

- We can again simplify the term $\sin(2\arcsin(a/r))$ using a double angle identity. Set $\theta = \arcsin(a/r)$. Then θ is the angle in the triangle on the right below. By the double angle formula for $\sin(2\theta)$ (Equation (1.7.2))

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\ \cos\theta\\ &= 2\,\frac{a}{r}\,\frac{\sqrt{r^2-a^2}}{r}.\end{aligned}$$



- So finally the area is

$$\begin{aligned}\text{area} &= \int_a^r \sqrt{r^2-x^2}\mathrm{d}x\\ &= \frac{r^2}{2}\left(\frac{\pi}{2}-\arcsin(a/r)-\frac{1}{2}\sin(2\arcsin(a/r))\right)\\ &= \frac{\pi r^2}{4}-\frac{r^2}{2}\arcsin(a/r)-\frac{a}{2}\sqrt{r^2-a^2}\end{aligned}$$

- This is a relatively complicated formula, but we can make some “reasonableness” checks, by looking at special values of a.
  - If $a = 0$ the shaded region, in the figure at the beginning of this example, is exactly one quarter of a disk of radius r and so has area $\frac{1}{4}\pi r^2$. Substituting $a = 0$ into our answer does indeed give $\frac{1}{4}\pi r^2$.
  - At the other extreme, if $a = r$, the shaded region disappears completely and so has area 0. Subbing $a = r$ into our answer does indeed give 0, since $\arcsin 1 = \frac{\pi}{2}$.

Example 1.8.3

Example 1.8.4 $\left(\int_a^r x\sqrt{r^2-x^2}\,\mathrm{d}x\right)$

The integral $\int_a^r x\sqrt{r^2-x^2}\,\mathrm{d}x$ looks a lot like the integral we just did in the previous 3 examples. It can also be evaluated using the trigonometric substitution $x = r\sin u$ — but that is unnecessarily complicated. Just because you have now learned how to use trigonometric substitution[36] doesn't mean that you should forget everything you learned before.

Solution. This integral is *much* more easily evaluated using the simple substitution $u = r^2 - x^2$.

- Set $u = r^2 - x^2$. Then $\mathrm{d}u = -2x\,\mathrm{d}x$, and so

$$\begin{aligned}\int_a^r x\sqrt{r^2-x^2}\,\mathrm{d}x &= \int_{r^2-a^2}^{0} \sqrt{u}\,\frac{\mathrm{d}u}{-2}\\ &= -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{r^2-a^2}^{0}\\ &= \frac{1}{3}\left[r^2-a^2\right]^{3/2}\end{aligned}$$

Example 1.8.4

Enough sines and cosines — let us try a tangent substitution.

Example 1.8.5 $\left(\int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}}\right)$

Solution. As per our guidelines at the start of this section, the presence of the square root term $\sqrt{3^2+x^2}$ tells us to substitute $x = 3\tan u$.

- Substitute

$$x = 3\tan u \qquad\qquad \mathrm{d}x = 3\sec^2 u\,\mathrm{d}u$$

  This allows us to remove the square root:

$$\sqrt{9+x^2} = \sqrt{9+9\tan^2 u} = 3\sqrt{1+\tan^2 u} = 3\sqrt{\sec^2 u} = 3|\sec u|$$

- Hence our integral becomes

$$\int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}} = \int \frac{3\sec^2 u}{9\tan^2 u\cdot 3|\sec u|}\mathrm{d}u$$

---

[36] To paraphrase the Law of the Instrument, possibly Mark Twain and definitely some psychologists, when you have a shiny new hammer, everything looks like a nail.

- To remove the absolute value we must consider the range of values of u in the integral. Since $x = 3\tan u$ we have $u = \arctan(x/3)$. The range[37] of arctangent is $-\pi/2 \leqslant \arctan \leqslant \pi/2$ and so $u = \arctan(x/3)$ will always like between $-\pi/2$ and $+\pi/2$. Hence $\cos u$ will always be positive, which in turn implies that $|\sec u| = \sec u$.
- Using this fact our integral becomes:

$$\begin{aligned} \int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}} &= \int \frac{3\sec^2 u}{27\tan^2 u|\sec u|}\mathrm{d}u \\ &= \frac{1}{9}\int \frac{\sec u}{\tan^2 u}\mathrm{d}u && \text{since } \sec u > 0 \end{aligned}$$

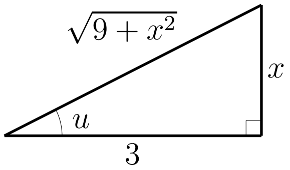
- Rewrite this in terms of sine and cosine

$$\begin{aligned} \int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}} &= \frac{1}{9}\int \frac{\sec u}{\tan^2 u}\mathrm{d}u && (1.8.1) \\ &= \frac{1}{9}\int \frac{1}{\cos u}\cdot\frac{\cos^2 u}{\sin^2 u}\mathrm{d}u = \frac{1}{9}\int \frac{\cos u}{\sin^2 u}\mathrm{d}u && (1.8.2) \end{aligned}$$

Now we can use the substitution rule with $y = \sin u$ and $\mathrm{d}y = \cos u \mathrm{d}u$

$$\begin{aligned} &= \frac{1}{9}\int \frac{\mathrm{d}y}{y^2} && (1.8.3) \\ &= -\frac{1}{9y} + C && (1.8.4) \\ &= -\frac{1}{9\sin u} + C && (1.8.5) \end{aligned}$$

- The original integral was a function of x, so we still have to rewrite $\sin u$ in terms of x. Remember that $x = 3\tan u$ or $u = \arctan(x/3)$. So u is the angle shown in the triangle below and we can read off the triangle that



$$\sin u = \frac{x}{\sqrt{9+x^2}}$$

$$\implies \int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}} = -\frac{\sqrt{9+x^2}}{9x} + C$$

Example 1.8.5

Example 1.8.6 $\left(\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x\right)$

Solution. This one requires a secant substitution, but otherwise is very similar to those above.

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[37] To be pedantic, we mean the range of the "standard" arctangent function or its "principal value". One can define other arctangent functions with different ranges.

- Set $x = \sec u$ and $\mathrm{d}x = \sec u \tan u \,\mathrm{d}u$. Then

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x &= \int \frac{\sec^2 u}{\sqrt{\sec^2 u - 1}} \sec u \tan u \,\mathrm{d}u \\ &= \int \sec^3 u \cdot \frac{\tan u}{\sqrt{\tan^2 u}}\mathrm{d}u && \text{since } \tan^2 u = \sec^2 u - 1 \\ &= \int \sec^3 u \cdot \frac{\tan u}{|\tan u|}\mathrm{d}u\end{aligned}$$

- As before we need to consider the range of u values in order to determine the sign of $\tan u$. Notice that the integrand is only defined when either $x < -1$ or $x > 1$; thus we should treat the cases $x < -1$ and $x > 1$ separately. Let us assume that $x > 1$ and we will come back to the case $x < -1$ at the end of the example.

  When $x > 1$, our $u = \operatorname{arcsec} x$ takes values in $(0, \pi/2)$. This follows since when $0 < u < \pi/2$, we have $0 < \cos u < 1$ and so $\sec u > 1$. Further, when $0 < u < \pi/2$, we have $\tan u > 0$. Thus $|\tan u| = \tan u$.

- Back to our integral, when $x > 1$:

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x &= \int \sec^3 u \cdot \frac{\tan u}{|\tan u|}\,\mathrm{d}u \\ &= \int \sec^3 u \mathrm{d}u && \text{since } \tan u \geqslant 0\end{aligned}$$

  We'll learn different ways to solve this in Examples 1.9.6 and 1.11.10.

$$= \frac{1}{2}\sec u \tan u + \frac{1}{2}\log|\sec u + \tan u| + C$$

- Since we started with a function of x we need to finish with one. We know that $\sec u = x$ and then we can use trig identities

  $\tan^2 u = \sec^2 u - 1 = x^2 - 1$ so $\tan u = \pm\sqrt{x^2-1}$, but we know $\tan u \geqslant 0$, so $\tan u = \sqrt{x^2-1}$

  Thus

$$\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log|x+\sqrt{x^2-1}| + C$$

- The above holds when $x > 1$. We can confirm that it is also true when $x < -1$ by showing the right-hand side is a valid antiderivative of the integrand. To do so we must differentiate our answer. Notice that we do not need to consider the sign of $x + \sqrt{x^2-1}$ when we differentiate since we have already seen that

$$\frac{\mathrm{d}}{\mathrm{d}x}\log|x| = \frac{1}{x}$$

when either $x < 0$ or $x > 0$. So the following computation applies to both $x > 1$ and $x < -1$. The expressions become quite long so we differentiate each term separately:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x\sqrt{x^2-1}\right] &= \left[\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}}\right] \\
&= \frac{1}{\sqrt{x^2-1}}\left[(x^2-1)+x^2\right] \\
\frac{\mathrm{d}}{\mathrm{d}x}\log\left|x+\sqrt{x^2-1}\right| &= \frac{1}{x+\sqrt{x^2-1}}\cdot\left[1+\frac{x}{\sqrt{x^2-1}}\right] \\
&= \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}$$

Putting things together then gives us

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}x\sqrt{x^2-1}+\frac{1}{2}\log|x+\sqrt{x^2-1}|+C\right] &= \frac{1}{2\sqrt{x^2-1}}\left[(x^2-1)+x^2+1\right]+0 \\
&= \frac{x^2}{\sqrt{x^2-1}}
\end{aligned}$$

This tells us that our answer for $x > 1$ is also valid when $x < -1$ and so

$$\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x = \frac{1}{2}x\sqrt{x^2-1}+\frac{1}{2}\log|x+\sqrt{x^2-1}|+C$$

when $x < -1$ and when $x > 1$.

In this example, we were lucky. The answer that we derived for $x > 1$ happened to be also valid for $x < -1$. This does not always happen with the $x = a\sec u$ substitution. When it doesn't, we have to apply separate $x > a$ and $x < -a$ analyses that are very similar to our $x > 1$ analysis above. Of course that doubles the tedium. So in the problem book, we will not pose questions that require separate $x > a$ and $x < -a$ computations.

Example 1.8.6

The method, as we have demonstrated it above, works when our integrand contains the square root of very specific families of quadratic polynomials. In fact, the same method works for more general quadratic polynomials — all we need to do is complete the square[38].

Example 1.8.7 $\left(\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\,\mathrm{d}x\right)$

This time we have an integral with a square root in the integrand, but the argument of the

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[38] If you have not heard of "completing the square" don't worry. It is not a difficult method and it will only take you a few moments to learn. It refers to rewriting a quadratic polynomial

$$P(x) = ax^2+bx+c \qquad \text{as} \qquad P(x) = a(x+h)^2+k$$

for new constants h, k.

square root, while a quadratic function of x, is not in one of the standard forms $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$. The reason that it is not in one of those forms is that the argument, x^2-2x-3, contains a term , namely $-2x$ that is of degree one in x. So we try to manipulate it into one of the standard forms by completing the square.

Solution.

- We first rewrite the quadratic polynomial x^2-2x-3 in the form $(x-a)^2+b$ for some constants a, b. The easiest way to do this is to expand both expressions and compare coefficients of x:

  $$x^2-2x-3=(x-a)^2+b=(x^2-2ax+a^2)+b$$

  So — if we choose $-2a=-2$ (so the coefficients of x^1 match) and $a^2+b=-3$ (so the coefficients of x^0 match), then both expressions are equal. Hence we set $a=1$ and $b=-4$. That is

  $$x^2-2x-3=(x-1)^2-4$$

  Many of you may have seen this method when learning to sketch parabolas.

- Once this is done we can convert the square root of the integrand into a standard form by making the simple substitution $y=x-1$. Here goes

  $$\begin{aligned}\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\,\mathrm{d}x &= \int_3^5 \frac{\sqrt{(x-1)^2-4}}{x-1}\,\mathrm{d}x\\ &= \int_2^4 \frac{\sqrt{y^2-4}}{y}\mathrm{d}y && \text{with } y=x-1, \mathrm{d}y=\mathrm{d}x\\ &= \int_0^{\pi/3} \frac{\sqrt{4\sec^2 u-4}}{2\sec u}\,2\sec u\tan u\,\mathrm{d}u && \text{with } y=2\sec u\\ & && \text{and } \mathrm{d}y=2\sec u\tan u\,\mathrm{d}u\end{aligned}$$

  Notice that we could also do this in fewer steps by setting $(x-1)=2\sec u, \mathrm{d}x=2\sec u\tan u\mathrm{d}u$.

- To get the limits of integration we used that
  - the value of u that corresponds to $y=2$ obeys $2=y=2\sec u=\frac{2}{\cos u}$ or $\cos u=1$, so that $u=0$ works and
  - the value of u that corresponds to $y=4$ obeys $4=y=2\sec u=\frac{2}{\cos u}$ or $\cos u=\frac{1}{2}$, so that $u={}^{\pi}/_{3}$ works.

- Now returning to the evaluation of the integral, we simplify and continue.

  $$\begin{aligned}\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\,\mathrm{d}x &= \int_0^{\pi/3} 2\sqrt{\sec^2 u-1}\,\tan u\,\mathrm{d}u\\ &= 2\int_0^{\pi/3} \tan^2 u\,\mathrm{d}u && \text{since } \sec^2 u=1+\tan^2 u\end{aligned}$$

In taking the square root of $\sec^2 u - 1 = \tan^2 u$ we used that $\tan u \geqslant 0$ on the range $0 \leqslant u \leqslant \frac{\pi}{3}$.

$$\begin{aligned} &= 2\int_0^{\pi/3} \big[\sec^2 u - 1\big]\, \mathrm{d}u && \text{since } \sec^2 u = 1+\tan^2 u\text{, again}\\ &= 2\Big[\tan u - u\Big]_0^{\pi/3} && \text{since } \frac{\mathrm{d}}{\mathrm{d}u}\tan u = \sec^2 u\\ &= 2\big[\sqrt{3} - \pi/3\big] \end{aligned}$$

Example 1.8.7

1.9▲ Partial fractions

Learning Objectives

- Use the method of partial fraction decomposition to evaluate the integrals of rational functions whose denominators can be factored so that there is at most one repeated linear root and at most one irreducible quadratic root.
- Memorize the antiderivative of secant.

Partial fractions is the name given to a technique of integration that may be used to integrate any rational function[39]. We already know how to integrate some simple rational functions.

$$\int \frac{1}{x}\,\mathrm{d}x = \log|x| + C \qquad \int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x) + C$$

Combining these with the substitution rule, we can integrate similar but more complicated rational functions.

$$\int \frac{1}{2x+3}\,\mathrm{d}x = \frac{1}{2}\log|2x+3| + C \qquad \int \frac{1}{3+4x^2}\,\mathrm{d}x = \frac{1}{2\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}\right) + C$$

By summing such terms together we can integrate yet more complicated forms.

$$\int \left[x + \frac{1}{x+1} + \frac{1}{x-1}\right]\mathrm{d}x = \frac{x^2}{2} + \log|x+1| + \log|x-1| + C$$

However, we are not (typically) presented with a rational function nicely decomposed into neat little pieces. It is far more likely that the rational function will be written as the ratio of two polynomials. For example:

$$\int \frac{x^3+x}{x^2-1}\mathrm{d}x$$

---

[39] Recall that a rational function is the ratio of two polynomials.

In this specific example it is not hard to confirm that

$$x+\frac{1}{x+1}+\frac{1}{x-1}=\frac{x(x+1)(x-1)+(x-1)+(x+1)}{(x+1)(x-1)}=\frac{x^3+x}{x^2-1}$$

and hence

$$\begin{aligned}\int \frac{x^3+x}{x^2-1}\mathrm{d}x &= \int \left[x+\frac{1}{x+1}+\frac{1}{x-1}\right]\mathrm{d}x\\ &= \frac{x^2}{2}+\log|x+1|+\log|x-1|+C\end{aligned}$$

Going in this direction (from a sum of terms to a single rational function) is straightforward. To be useful, we need to understand how to do this in reverse: decompose a given rational function into a sum of simpler pieces that we know how to integrate.

Suppose that $N(x)$ and $D(x)$ are polynomials. The basic strategy is to write $\frac{N(x)}{D(x)}$ as a sum of very simple, easy to integrate rational functions. These nice-to-integrate functions are:

(1) polynomials — we shall see below that these are needed when the degree[40] of $N(x)$ is equal to or strictly bigger than the degree of $D(x)$, and

(2) rational functions of the particularly simple form $\frac{A}{(ax+b)^n}$ and

(3) rational functions of the form $\frac{Ax+B}{(ax^2+bx+c)^m}$.

We already know how to integrate the first two forms, and we'll see how to integrate the third form in the near future.

To begin to explore this method of decomposition, let us go back to the example we just saw:

$$x+\frac{1}{x+1}+\frac{1}{x-1}=\frac{x(x+1)(x-1)+(x-1)+(x+1)}{(x+1)(x-1)}=\frac{x^3+x}{x^2-1}.$$

The technique that we will use is based on two observations.

(1) The denominators on the left-hand side are the factors of the denominator $x^2-1=(x-1)(x+1)$ on the right-hand side.

(2) Use $P(x)$ to denote the polynomial on the left-hand side, and then use $N(x)$ and $D(x)$ to denote the numerator and denominator of the right-hand side. That is

$$P(x)=x \qquad N(x)=x^3+x \qquad D(x)=x^2-1.$$

Then the degree of $N(x)$ is the sum of the degrees of $P(x)$ and $D(x)$. This is because the highest degree term in $N(x)$ is x^3, which comes from multiplying $P(x)$ by $D(x)$, as we see in

$$x+\frac{1}{x+1}+\frac{1}{x-1}=\frac{\overbrace{x}^{P(x)}\overbrace{(x+1)(x-1)}^{D(x)}+(x-1)+(x+1)}{(x+1)(x-1)}=\frac{x^3+x}{x^2-1}$$

---

[40] The degree of a polynomial is the largest power of x. For example, the degree of $2x^3+4x^2+6x+8$ is three.

More generally, the presence of a polynomial on the left-hand side is signalled on the right-hand side by the fact that the degree of the numerator is at least as large as the degree of the denominator.

1.9.1 ▸▸ Partial fraction decomposition examples

Rather than writing up the technique — known as the partial fraction decomposition — in full generality, we will instead illustrate it through a sequence of examples.

Example 1.9.1 $\left(\int \frac{x-3}{x^2-3x+2}\mathrm{d}x\right)$

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{x-3}{x^2-3x+2}$.

Solution.

- *Step 1.* We first check to see if a polynomial $P(x)$ is needed. To do so, we check to see if the degree of the numerator, $N(x)$, is strictly smaller than the degree of the denominator $D(x)$. In this example, the numerator, $x-3$, has degree one and that is indeed strictly smaller than the degree of the denominator, x^2-3x+2, which is two. In this case[41] we do not need to extract a polynomial $P(x)$ and we move on to step 2.

- *Step 2.* The second step is to factor the denominator

$$x^2 - 3x + 2 = (x-1)(x-2)$$

  In this example it is quite easy, but in future examples (and quite possibly in your homework and exams) you will have to work harder to factor the denominator. In Appendix .1 we have written up some simple tricks for factoring polynomials. (These should be review from high-school mathematics.) We will illustrate them in Example 1.9.3 below.

- *Step 3.* The third step is to write $\frac{x-3}{x^2-3x+2}$ in the form

$$\frac{x-3}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$$

  for some constants A and B. More generally, if the denominator consists of n different linear factors, then we decompose the ratio as

$$\text{rational function} = \frac{A_1}{\text{linear factor 1}} + \frac{A_2}{\text{linear factor 2}} + \cdots + \frac{A_n}{\text{linear factor n}}$$

  To proceed we need to determine the values of the constants A, B and there are several different methods to do so. Here are two methods

---

[41] We will soon get to an example (Example 1.9.2 in fact) in which the numerator degree is at least as large as the denominator degree — in that situation we have to extract a polynomial $P(x)$ before we can move on to step 2.

- *Step 3 – Algebra Method.* This approach has the benefit of being conceptually clearer and easier, but the downside is that it is more tedious.

  To determine the values of the constants A, B, we put[42] the right-hand side back over the common denominator $(x-1)(x-2)$.

  $$\frac{x-3}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

  The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

  $$x-3 = A(x-2)+B(x-1)$$

  Write the right-hand side as a polynomial in standard form (i.e. collect up all x terms and all constant terms)

  $$x-3 = (A+B)x + (-2A-B)$$

  For these two polynomials to be the same, the coefficient of x on the left-hand side and the coefficient of x on the right-hand side must be the same. Similarly the coefficients of x^0 (i.e. the constant terms) must match. This gives us a system of two equations.

  $$A+B=1 \qquad\qquad -2A-B=-3$$

  in the two unknowns A, B. We can solve this system by

  - using the first equation, namely $A+B=1$, to determine A in terms of B:

    $$A = 1-B$$

  - Substituting this into the remaining equation eliminates the A from second equation, leaving one equation in the one unknown B, which can then be solved for B:

    $$\begin{aligned} -2A-B &= -3 && \text{substitute } A=1-B \\ -2(1-B)-B &= -3 && \text{clean up} \\ -2+B &= -3 && \text{so } B=-1 \end{aligned}$$

  - Once we know B, we can substitute it back into $A=1-B$ to get A.

    $$A = 1-B = 1-(-1) = 2$$

  Hence

  $$\frac{x-3}{x^2-3x+2} = \frac{2}{x-1} - \frac{1}{x-2}$$

---

[42] That is, we take the decomposed form and sum it back together.

- *Step 3 – Sneaky Method.* This takes a little more work to understand, but it is more efficient than the algebra method.

  We wish to find A and B for which

  $$\frac{x-3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

  Note that the denominator on the left-hand side has been written in factored form.

  - To determine A, we multiply both sides of the equation by A's denominator, which is $x-1$,

    $$\frac{x-3}{x-2} = A + \frac{(x-1)B}{x-2}$$

    and then we completely eliminate B from the equation by evaluating at $x = 1$. This value of x is chosen to make $x-1 = 0$.

    $$\left.\frac{x-3}{x-2}\right|_{x=1} = A + \left.\frac{(x-1)B}{x-2}\right|_{x=1} \implies A = \frac{1-3}{1-2} = 2$$

  - To determine B, we multiply both sides of the equation by B's denominator, which is $x-2$,

    $$\frac{x-3}{x-1} = \frac{(x-2)A}{x-1} + B$$

    and then we completely eliminate A from the equation by evaluating at $x = 2$. This value of x is chosen to make $x-2 = 0$.

    $$\left.\frac{x-3}{x-1}\right|_{x=2} = \left.\frac{(x-2)A}{x-1}\right|_{x=2} + B \implies B = \frac{2-3}{2-1} = -1$$

  Hence we have (the thankfully consistent answer)

  $$\frac{x-3}{x^2-3x+2} = \frac{2}{x-1} - \frac{1}{x-2}$$

  Notice that no matter which method we use to find the constants we can easily check our answer by summing the terms back together:

  $$\frac{2}{x-1} - \frac{1}{x-2} = \frac{2(x-2)-(x-1)}{(x-2)(x-1)} = \frac{2x-4-x+1}{x^2-3x+2} = \frac{x-3}{x^2-3x+2}\checkmark$$

Step 4. The final step is to integrate.

$$\int \frac{x-3}{x^2-3x+2}\mathrm{d}x = \int \frac{2}{x-1}\mathrm{d}x + \int \frac{-1}{x-2}\mathrm{d}x = 2\log|x-1| - \log|x-2| + C$$

Example 1.9.1

Perhaps the first thing that you notice is that this process takes quite a few steps[43]. However no single step is all that complicated; it only takes practice. With that said, let's do another, slightly more complicated, one.

Example 1.9.2 $\left(\int \frac{3x^3-8x^2+4x-1}{x^2-3x+2}\mathrm{d}x\right)$

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{3x^3-8x^2+4x-1}{x^2-3x+2}$.

Solution.

- *Step 1.* We first check to see if the degree of the numerator $N(x)$ is strictly smaller than the degree of the denominator $D(x)$. In this example, the numerator, $3x^3 - 8x^2 + 4x - 1$, has degree three and the denominator, $x^2 - 3x + 2$, has degree two. As $3 \geqslant 2$, we have to implement the first step.

  The goal of the first step is to write $\frac{N(x)}{D(x)}$ in the form

  $$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

  with $P(x)$ being a polynomial and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. The right-hand side is $\frac{P(x)D(x)+R(x)}{D(x)}$, so we have to express the numerator in the form $N(x) = P(x)D(x) + R(x)$, with $P(x)$ and $R(x)$ being polynomials and with the degree of R being strictly smaller than the degree of D. $P(x)D(x)$ is a sum of expressions of the form $ax^nD(x)$. We want to pull as many expressions of this form as possible out of the numerator $N(x)$, leaving only a low degree remainder $R(x)$.

  We do this using long division — the same long division you learned in school, but with the base 10 replaced by x.

  - We start by observing that to get from the highest degree term in the denominator (x^2) to the highest degree term in the numerator ($3x^3$), we have to multiply it by $3x$. So we write,

    $$\begin{array}{r|l} & 3x \\ \cline{2-2} x^2-3x+2 & 3x^3-8x^2+4x-1 \end{array}$$

    In the above expression, the denominator is on the left, the numerator is on the right and $3x$ is written above the highest order term of the numerator. Always put lower powers of x to the right of higher powers of x — this mirrors how you do long division with numbers; lower powers of ten sit to the right of higher powers of ten.

  - Now we subtract $3x$ times the denominator, $x^2 - 3x + 2$, which is $3x^3 - 9x^2 + 6x$, from the numerator.

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[43] Though, in fairness, we did step 3 twice — and that is the most tedious bit... Actually — sometimes factoring the denominator can be quite challenging. We'll consider this issue in more detail shortly.

$$\begin{array}{rll} & 3x & \\ x^2-3x+2 & \big) \overline{3x^3 - 8x^2 + 4x - 1} & \\ & \underline{3x^3 - 9x^2 + 6x \phantom{{}-1}} & \longleftarrow 3x(x^2-3x+2) \\ & \phantom{3x^3-{}} x^2 - 2x - 1 & \end{array}$$

- This has left a remainder of $x^2 - 2x - 1$. To get from the highest degree term in the denominator (x^2) to the highest degree term in the remainder (x^2), we have to multiply by 1. So we write,

$$\begin{array}{rl} & 3x + \ 1 \\ x^2-3x+2 & \big) \overline{3x^3 - 8x^2 + 4x - 1} \\ & \underline{3x^3 - 9x^2 + 6x \phantom{{}-1}} \\ & \phantom{3x^3-{}} x^2 - 2x - 1 \end{array}$$

- Now we subtract 1 times the denominator, $x^2 - 3x + 2$, which is $x^2 - 3x + 2$, from the remainder.

$$\begin{array}{rll} & 3x + \ 1 & \\ x^2-3x+2 & \big) \overline{3x^3 - 8x^2 + 4x - 1} & \\ & \underline{3x^3 - 9x^2 + 6x \phantom{{}-1}} & \longleftarrow 3x(x^2-3x+2) \\ & \phantom{3x^3-{}} x^2 - 2x - 1 & \\ & \phantom{3x^3-{}} \underline{x^2 - 3x + 2} & \longleftarrow 1\,(x^2-3x+2) \\ & \phantom{3x^3-8x^2+{}} x - 3 & \end{array}$$

- This leaves a remainder of $x - 3$. Because the remainder has degree 1, which is smaller than the degree of the denominator (being degree 2), we stop.
- In this example, when we subtracted $3x(x^2 - 3x + 2)$ and $1(x^2 - 3x + 2)$ from $3x^3 - 8x^2 + 4x - 1$ we ended up with $x - 3$. That is,

$$3x^3 - 8x^2 + 4x - 1 \ - \ 3x(x^2 - 3x + 2) \ - \ 1(x^2 - 3x + 2) \ = \ x - 3$$

  or, collecting the two terms proportional to $(x^2 - 3x + 2)$

$$3x^3 - 8x^2 + 4x - 1 \ - \ (3x + 1)(x^2 - 3x + 2) \ = \ x - 3$$

  Moving the $(3x + 1)(x^2 - 3x + 2)$ to the right-hand side and dividing the whole equation by $x^2 - 3x + 2$ gives

$$\frac{3x^3 - 8x^2 + 4x - 1}{x^2 - 3x + 2} \ = \ 3x + 1 \ + \ \frac{x - 3}{x^2 - 3x + 2}$$

  And we can easily check this expression just by summing the two terms on the right-hand side.

We have written the integrand in the form $\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$, with the degree of $R(x)$ strictly smaller than the degree of $D(x)$, which is what we wanted. Observe that $R(x)$ is the final remainder of the long division procedure and $P(x)$ is at the top of the long division computation.

$$\begin{array}{r r l}
 & 3x+\ 1 & \longleftarrow P(x)\\
D(x)\rightarrow x^2-3x+2 & \overline{)\,3x^3-8x^2+4x-1} & \longleftarrow N(x)\\
 & 3x^3-9x^2+6x & \longleftarrow 3x\cdot D(x)\\ \cline{2-2}
 & x^2-2x-1 & \longleftarrow N(x)-3x\cdot D(x)\\
 & x^2-3x+2 & \longleftarrow 1\cdot D(x)\\ \cline{2-2}
 & x-3 & \longleftarrow R(x)=N(x)-(3x+1)D(x)
\end{array}$$

This is the end of Step 1. Oof! You should definitely practice this step.

- *Step 2.* The second step is to factor the denominator

  $$x^2-3x+2=(x-1)(x-2)$$

  We already did this in Example 1.9.1.

- *Step 3.* The third step is to write $\frac{x-3}{x^2-3x+2}$ in the form

  $$\frac{x-3}{x^2-3x+2}=\frac{A}{x-1}+\frac{B}{x-2}$$

  for some constants A and B. We already did this in Example 1.9.1. We found $A=2$ and $B=-1$.

- *Step 4.* The final step is to integrate.

  $$\begin{aligned}\int\frac{3x^3-8x^2+4x-1}{x^2-3x+2}\mathrm{d}x &= \int\big[3x+1\big]\mathrm{d}x+\int\frac{2}{x-1}\mathrm{d}x+\int\frac{-1}{x-2}\mathrm{d}x\\ &= \frac{3}{2}x^2+x+2\log|x-1|-\log|x-2|+C\end{aligned}$$

You can see that the integration step is quite quick — almost all the work is in preparing the integrand.

Example 1.9.2

Here is a very solid example. It is quite long and the steps are involved. However, please persist! No single step is too difficult.

Example 1.9.3 $\left(\int\frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5}\mathrm{d}x\right)$

In this example, we integrate $\frac{N(x)}{D(x)}=\frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5}$.

Solution.

- *Step 1.* Again, we start by comparing the degrees of the numerator and denominator. In this example, the numerator, $x^4+5x^3+16x^2+26x+22$, has degree four and the denominator, $x^3 + 3x^2 + 7x + 5$, has degree three. As $4 \geqslant 3$, we must execute the first step, which is to write $\frac{N(x)}{D(x)}$ in the form

$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

with $P(x)$ being a polynomial and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. This step is accomplished by long division, just as we did in Example 1.9.2. We'll go through the whole process in detail again.

Actually — before you read on ahead, please have a go at the long division. It is good practice.

- We start by observing that to get from the highest degree term in the denominator (x^3) to the highest degree term in the numerator (x^4), we have to multiply by x. So we write,

$$\begin{array}{r|l}
 & x \\ \cline{2-2}
x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22
\end{array}$$

- Now we subtract x times the denominator $x^3 + 3x^2 + 7x + 5$, which is $x^4 + 3x^3 + 7x^2 + 5x$, from the numerator.

$$\begin{array}{r|rl}
 & x & \\ \cline{2-2}
x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 & \\
 & x^4+3x^3+\ 7x^2+\ 5x & \longleftarrow x(x^3+3x^2+7x+5) \\ \cline{2-2}
 & 2x^3+\ 9x^2+21x+22 &
\end{array}$$

- The remainder was $2x^3 + 9x^2 + 21x + 22$. To get from the highest degree term in the denominator (x^3) to the highest degree term in the remainder ($2x^3$), we have to multiply by 2. So we write,

$$\begin{array}{r|r}
 & x+\ \ 2 \\ \cline{2-2}
x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 \\
 & x^4+3x^3+\ 7x^2+\ 5x \\ \cline{2-2}
 & 2x^3+\ 9x^2+21x+22
\end{array}$$

- Now we subtract 2 times the denominator $x^3 + 3x^2 + 7x + 5$, which is $2x^3 + 6x^2 + 14x + 10$, from the remainder.

$$\begin{array}{r|rl}
 & x+\ \ 2 & \\ \cline{2-2}
x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 & \\
 & x^4+3x^3+\ 7x^2+\ 5x & \longleftarrow x(x^3+3x^2+7x+5) \\ \cline{2-2}
 & 2x^3+\ 9x^2+21x+22 & \\
 & 2x^3+\ 6x^2+14x+10 & \longleftarrow 2(x^3+3x^2+7x+5) \\ \cline{2-2}
 & 3x^2+\ 7x+12 &
\end{array}$$

- This leaves a remainder of $3x^2 + 7x + 12$. Because the remainder has degree 2, which is smaller than the degree of the denominator, which is 3, we stop.

  - In this example, when we subtracted $x(x^3+3x^2+7x+5)$ and $2(x^3+3x^2+7x+5)$ from $x^4+5x^3+16x^2+26x+22$ we ended up with $3x^2+7x+12$. That is,

$$x^4+5x^3+16x^2+26x+22\ -\ x(x^3+3x^2+7x+5)\ -\ 2(x^3+3x^2+7x+5) \\ = 3x^2+7x+12$$

    or, collecting the two terms proportional to (x^3+3x^2+7x+5)

$$x^4+5x^3+16x^2+26x+22\ -\ (x+2)(x^3+3x^2+7x+5)\ =\ 3x^2+7x+12$$

    Moving the $(x+2)(x^3+3x^2+7x+5)$ to the right-hand side and dividing the whole equation by x^3+3x^2+7x+5 gives

$$\frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5} = x+2+\frac{3x^2+7x+12}{x^3+3x^2+7x+5}$$

This is of the form $\frac{N(x)}{D(x)} = P(x)+\frac{R(x)}{D(x)}$, with the degree of $R(x)$ strictly smaller than the degree of $D(x)$, which is what we wanted. Observe, once again, that $R(x)$ is the final remainder of the long division procedure and $P(x)$ is at the top of the long division computation.

$$\begin{array}{rr}
 & x+\ \ 2 \longleftarrow P(x) \\
x^3+3x^2+7x+5 & \big)\overline{x^4+5x^3+16x^2+26x+22} \\
 & \underline{x^4+3x^3+\ 7x^2+\ 5x} \\
 & 2x^3+\ 9x^2+21x+22 \\
 & \underline{2x^3+\ 6x^2+14x+10} \\
 & 3x^2+\ 7x+12 \longleftarrow R(x)
\end{array}$$

- *Step 2.* The second step is to factor the denominator $D(x) = x^3+3x^2+7x+5$. In the "real world" factorisation of polynomials is often very hard. Fortunately[44], this is not the "real world" and there is a trick available to help us find this factorisation. The reader should take some time to look at Appendix .1 before proceeding.
  - The trick exploits the fact that most polynomials that appear in homework assignments and on tests have integer coefficients and some integer roots. Any integer root of a polynomial that has integer coefficients, like $D(x) = x^3+3x^2+7x+5$, must divide the constant term of the polynomial exactly. Why this is true is explained[45] in Appendix .1.
  - So any integer root of x^3+3x^2+7x+5 must divide 5 exactly. Thus the only integers which can be roots of $D(x)$ are ± 1 and ± 5. Of course, not all of these give roots of the polynomial — in fact there is no guarantee that any of them will be. We have to test each one.

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[44] One does not typically think of mathematics assignments or exams as nice kind places… The polynomials that appear in the "real world" are not so forgiving. Nature, red in tooth and claw — to quote Tennyson inappropriately (especially when this author doesn't know any other words from the poem).

[45] Appendix .1 contains several simple tricks for factoring polynomials. We recommend that you have a look at them.

- To test if $+1$ is a root, we sub $x = 1$ into $D(x)$:

  $$D(1) = 1^3 + 3(1)^2 + 7(1) + 5 = 16$$

  As $D(1) \neq 0$, 1 is not a root of $D(x)$.
- To test if -1 is a root, we sub it into $D(x)$:

  $$D(-1) = (-1)^3 + 3(-1)^2 + 7(-1) + 5 = -1 + 3 - 7 + 5 = 0$$

  As $D(-1) = 0$, -1 is a root of $D(x)$. As -1 is a root of $D(x)$, $\big(x - (-1)\big) = (x+1)$ must factor $D(x)$ exactly. We can factor the $(x+1)$ out of $D(x) = x^3 + 3x^2 + 7x + 5$ by long division once again.
- Dividing $D(x)$ by $(x+1)$ gives:

  $$\begin{array}{rll} & x^2+2x+5 & \\ x+1 & \overline{)x^3+3x^2+7x+5} & \\ & \underline{x^3+x^2} & \longleftarrow x^2(x+1) \\ & 2x^2+7x+5 & \\ & \underline{2x^2+2x} & \longleftarrow 2x(x+1) \\ & 5x+5 & \\ & \underline{5x+5} & \longleftarrow 5(x+1) \\ & 0 & \end{array}$$

  This time, when we subtracted $x^2(x+1)$ and $2x(x+1)$ and $5(x+1)$ from $x^3 + 3x^2 + 7x + 5$ we ended up with 0 — as we knew would happen, because we knew that $x+1$ divides $x^3 + 3x^2 + 7x + 5$ exactly. Hence

  $$x^3 + 3x^2 + 7x + 5 \;-\; x^2(x+1) \;-\; 2x(x+1) \;-\; 5(x+1) \;=\; 0$$

  or

  $$x^3 + 3x^2 + 7x + 5 \;=\; x^2(x+1) \;+\; 2x(x+1) \;+\; 5(x+1)$$

  or

  $$x^3 + 3x^2 + 7x + 5 = (x^2 + 2x + 5)(x+1)$$
- It isn't quite time to stop yet; we should attempt to factor the quadratic factor, $x^2 + 2x + 5$. We can use the quadratic formula[46] to find the roots of $x^2 + 2x + 5$:

  $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

  Since this expression contains the square root of a negative number the equation $x^2 + 2x + 5 = 0$ has no real solutions; without the use of complex numbers, $x^2 + 2x + 5$ cannot be factored.

---

[46] To be precise, the quadratic equation $ax^2 + bx + c = 0$ has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The term $b^2 - 4ac$ is called the discriminant and it tells us about the number of solutions. If the discriminant is positive then there are two real solutions. When it is zero, there is a single solution. And if it is negative, there is no real solutions (you need complex numbers to say more than this).

We have reached the end of step 2. At this point we have

$$\frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5} = x+2+\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)}$$

- *Step 3.* The third step is to write $\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)}$ in the form

  $$\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+5}$$

  for some constants A, B and C.

  Note that the numerator, $Bx+C$ of the second term on the right-hand side is not just a constant. It is of degree one, which is exactly one smaller than the degree of the denominator, x^2+2x+5. More generally, if the denominator consists of n different linear factors and m different quadratic factors, then we decompose the ratio as

  $$\begin{aligned}\text{rational function} = {} & \frac{A_1}{\text{linear factor 1}}+\frac{A_2}{\text{linear factor 2}}+\cdots+\frac{A_n}{\text{linear factor n}} \\ & +\frac{B_1x+C_1}{\text{quadratic factor 1}}+\frac{B_2x+C_2}{\text{quadratic factor 2}}+\cdots+\frac{B_mx+C_m}{\text{quadratic factor m}}\end{aligned}$$

  To determine the values of the constants $A,\ B,\ C$, we put the right-hand side back over the common denominator $(x+1)(x^2+2x+5)$.

  $$\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+5} = \frac{A(x^2+2x+5)+(Bx+C)(x+1)}{(x+1)(x^2+2x+5)}$$

  The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

  $$3x^2+7x+12 = A(x^2+2x+5)+(Bx+C)(x+1)$$

  Again, as in Example 1.9.1, there are a couple of different ways to determine the values of A, B and C from this equation.

- *Step 3 – Algebra Method.* The conceptually clearest procedure is to write the right-hand side as a polynomial in standard form (i.e. collect up all x^2 terms, all x terms and all constant terms)

  $$3x^2+7x+12 = (A+B)x^2+(2A+B+C)x+(5A+C)$$

  For these two polynomials to be the same, the coefficient of x^2 on the left-hand side and the coefficient of x^2 on the right-hand side must be the same. Similarly the coefficients of x^1 must match and the coefficients of x^0 must match.

  This gives us a system of three equations

  $$A+B=3 \qquad 2A+B+C=7 \qquad 5A+C=12$$

  in the three unknowns A, B, C. We can solve this system by

- using the first equation, namely $A + B = 3$, to determine A in terms of B: $A = 3 - B$.
- Substituting this into the remaining two equations eliminates the A's from these two equations, leaving two equations in the two unknowns B and C.

$$\begin{array}{rrr} A = 3 - B & 2A + B + C = 7 & 5A + C = 12 \\ \Rightarrow & 2(3 - B) + B + C = 7 & 5(3 - B) + C = 12 \\ \Rightarrow & -B + C = 1 & -5B + C = -3 \end{array}$$

- Now we can use the equation $-B + C = 1$, to determine B in terms of C: $B = C - 1$.
- Substituting this into the remaining equation eliminates the B's leaving an equation in the one unknown C, which is easy to solve.

$$\begin{array}{rr} B = C - 1 & -5B + C = -3 \\ \Rightarrow & -5(C - 1) + C = -3 \\ \Rightarrow & -4C = -8 \end{array}$$

- So $C = 2$, and then $B = C - 1 = 1$, and then $A = 3 - B = 2$. Hence

$$\frac{3x^2 + 7x + 12}{(x + 1)(x^2 + 2x + 5)} = \frac{2}{x + 1} + \frac{x + 2}{x^2 + 2x + 5}$$

- *Step 3 – Sneaky Method.* While the above method is transparent, it is rather tedious. It is arguably better to use the second, sneakier and more efficient, procedure. In order for

$$3x^2 + 7x + 12 = A(x^2 + 2x + 5) + (Bx + C)(x + 1)$$

the equation must hold for all values of x.

  - In particular, it must be true for $x = -1$. When $x = -1$, the factor $(x + 1)$ multiplying $Bx + C$ is exactly zero. So B and C disappear from the equation, leaving us with an easy equation to solve for A:

$$3x^2 + 7x + 12\Big|_{x=-1} = \Big[A(x^2 + 2x + 5) + (Bx + C)(x + 1)\Big]_{x=-1} \Longrightarrow 8 = 4A \Longrightarrow A = 2$$

  - Sub this value of A back in and simplify.

$$\begin{aligned} 3x^2 + 7x + 12 &= 2(x^2 + 2x + 5) + (Bx + C)(x + 1) \\ x^2 + 3x + 2 &= (Bx + C)(x + 1) \end{aligned}$$

Since $(x + 1)$ is a factor on the right-hand side, it must also be a factor on the left-hand side.

$$(x + 2)(x + 1) = (Bx + C)(x + 1) \quad \Rightarrow \quad (x + 2) = (Bx + C) \quad \Rightarrow \quad B = 1,\ C = 2$$

So again we find that

$$\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)} = \frac{2}{x+1} + \frac{x+2}{x^2+2x+5}\checkmark$$

Thus our integrand can be written as

$$\frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5} = x+2+\frac{2}{x+1}+\frac{x+2}{x^2+2x+5}.$$

- *Step 4.* Now we can finally integrate! The first two pieces are easy.

  $$\int (x+2)\mathrm{d}x = \tfrac{1}{2}x^2+2x \qquad \int \frac{2}{x+1}\mathrm{d}x = 2\log|x+1|$$

  (We're leaving the arbitrary constant to the end of the computation.)

  The final piece is a little harder. The idea is to complete the square[47] in the denominator

  $$\frac{x+2}{x^2+2x+5} = \frac{x+2}{(x+1)^2+4}$$

  and then make a change of variables to make the fraction look like $\frac{ay+b}{y^2+1}$. In this case

  $$\frac{x+2}{(x+1)^2+4} = \frac{1}{4}\frac{x+2}{(\frac{x+1}{2})^2+1}$$

  so we make the change of variables $y = \frac{x+1}{2}, \mathrm{d}y = \frac{\mathrm{d}x}{2},\ x = 2y-1, \mathrm{d}x = 2\,\mathrm{d}y$

  $$\begin{aligned}\int \frac{x+2}{(x+1)^2+4}\mathrm{d}x &= \frac{1}{4}\int \frac{x+2}{\left(\frac{x+1}{2}\right)^2+1}\mathrm{d}x \\ &= \frac{1}{4}\int \frac{(2y-1)+2}{y^2+1}\,2\,\mathrm{d}y \ = \ \frac{1}{2}\int \frac{2y+1}{y^2+1}\,\mathrm{d}y \\ &= \int \frac{y}{y^2+1}\,\mathrm{d}y + \frac{1}{2}\int \frac{1}{y^2+1}\,\mathrm{d}y\end{aligned}$$

---

[47] This same idea arose in Section 1.8. Given a quadratic written as

$$Q(x) = ax^2+bx+c$$

rewrite it as

$$Q(x) = a(x+d)^2+e.$$

We can determine d and e by expanding and comparing coefficients of x:

$$ax^2+bx+c = a(x^2+2dx+d^2)+e = ax^2+2dax+(e+ad^2)$$

Hence $d = b/2a$ and $e = c-ad^2$.

Both integrals are easily evaluated, using the substitution $u = y^2 + 1$, $\mathrm{d}u = 2y\,\mathrm{d}y$ for the first.

$$\int \frac{y}{y^2+1}\,\mathrm{d}y = \int \frac{1}{u}\frac{\mathrm{d}u}{2} = \frac{1}{2}\log|u| = \frac{1}{2}\log(y^2+1) = \frac{1}{2}\log\left[\left(\frac{x+1}{2}\right)^2+1\right]$$

$$\frac{1}{2}\int \frac{1}{y^2+1}\,\mathrm{d}y = \frac{1}{2}\arctan y = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right)$$

That's finally it. Putting all of the pieces together

$$\begin{aligned}\int \frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5}\mathrm{d}x &= \frac{1}{2}x^2+2x+2\log|x+1| \\ &\quad + \frac{1}{2}\log\left[\left(\frac{x+1}{2}\right)^2+1\right] + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C\end{aligned}$$

Example 1.9.3

The best thing after working through a few a nice long examples is to do another nice long example — it is excellent practice[48]. We recommend that the reader attempt the problem before reading through our solution.

Example 1.9.4 $\left(\int \frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}\mathrm{d}x\right)$

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}$.

- *Step 1.* The degree of the numerator $N(x)$ is equal to the degree of the denominator $D(x)$, so the first step to write $\frac{N(x)}{D(x)}$ in the form

$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

with $P(x)$ being a polynomial (which should be of degree 0, i.e. just a constant) and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. By long division

$$\begin{array}{r|r} & 4 \\ \cline{2-2} x^3+5x^2+8x+4 & 4x^3+23x^2+45x+27 \\ & 4x^3+20x^2+32x+16 \\ \cline{2-2} & 3x^2+13x+11 \end{array}$$

---

[48] At the risk of quoting Nietzsche, "That which does not kill us makes us stronger." Though this author always preferred the logically equivalent contrapositive — "That which does not make us stronger will kill us." However no one is likely to be injured by practicing partial fractions or looking up quotes on Wikipedia. Its also a good excuse to remind yourself of what a contrapositive is — though we will likely look at them again when we get to sequences and series.

so

$$\frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4} = 4 + \frac{3x^2+13x+11}{x^3+5x^2+8x+4}$$

- *Step 2.* The second step is to factorise $D(x) = x^3+5x^2+8x+4$.
    - To start, we'll try and guess an integer root. Any integer root of $D(x)$ must divide the constant term, 4, exactly. Only ± 1, ± 2, ± 4 can be integer roots of x^3+5x^2+8x+4.
    - We test to see if ± 1 are roots.

      $$\begin{aligned} D(1) &= (1)^3+5(1)^2+8(1)+4 \neq 0 & \Rightarrow\quad & x=1 \text{ is not a root} \\ D(-1) &= (-1)^3+5(-1)^2+8(-1)+4 = 0 & \Rightarrow\quad & x=-1 \text{ is a root} \end{aligned}$$

      So $(x+1)$ must divide x^3+5x^2+8x+4 exactly.
    - By long division

      $$\begin{array}{rrrrr} & x^2 & +4x & +4 & \\ \cline{2-5} x+1 \big) & x^3 & +5x^2 & +8x & +4 \\ & x^3 & +x^2 & & \\ \cline{2-5} & & 4x^2 & +8x & +4 \\ & & 4x^2 & +4x & \\ \cline{3-4} & & & 4x & +4 \\ & & & 4x & +4 \\ \cline{4-5} & & & & 0 \end{array}$$

      so

      $$x^3+5x^2+8x+4 = (x+1)(x^2+4x+4) = (x+1)(x+2)(x+2)$$
    - Notice that we could have instead checked whether or not ± 2 are roots

      $$\begin{aligned} D(2) &= (2)^3+5(2)^2+8(2)+4 \neq 0 & \Rightarrow\quad & x=2 \text{ is not a root} \\ D(-2) &= (-2)^3+5(-2)^2+8(-2)+4 = 0 & \Rightarrow\quad & x=-2 \text{ is a root} \end{aligned}$$

      We now know that both -1 and -2 are roots of x^3+5x^2+8x+4 and hence both $(x+1)$ and $(x+2)$ are factors of x^3+5x^2+8x+4. Because x^3+5x^2+8x+4 is of degree three and the coefficient of x^3 is 1, we must have $x^3+5x^2+8x+4 = (x+1)(x+2)(x+a)$ for some constant a. Multiplying out the right-hand side shows that the constant term is $2a$. So $2a=4$ and $a=2$.

This is the end of step 2. We now know that

$$\frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4} \;=\; 4 + \frac{3x^2+13x+11}{(x+1)(x+2)^2}$$

- *Step 3.* The third step is to write $\frac{3x^2+13x+11}{(x+1)(x+2)^2}$ in the form

  $$\frac{3x^2+13x+11}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

  for some constants A, B and C.

  Note that there are two terms on the right-hand arising from the factor $(x+2)^2$. One has denominator $(x+2)$ and one has denominator $(x+2)^2$. More generally, for each factor $(x+a)^n$ in the denominator of the rational function on the left-hand side, we include

  $$\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_n}{(x+a)^n}$$

  in the partial fraction decomposition on the right-hand side[49].

  To determine the values of the constants A, B, C, we put the right-hand side back over the common denominator $(x+1)(x+2)^2$.

  $$\begin{aligned}\frac{3x^2+13x+11}{(x+1)(x+2)^2} &= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\\ &= \frac{A(x+2)^2 + B(x+1)(x+2) + C(x+1)}{(x+1)(x+2)^2}\end{aligned}$$

  The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

  $$3x^2+13x+11 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

  As in the previous examples, there are a couple of different ways to determine the values of A, B and C from this equation.

- *Step 3 – Algebra Method.* The conceptually clearest procedure is to write the right-hand side as a polynomial in standard form (i.e. collect up all x^2 terms, all x terms and all constant terms)

  $$3x^2+13x+11 = (A+B)x^2 + (4A+3B+C)x + (4A+2B+C)$$

  For these two polynomials to be the same, the coefficient of x^2 on the left-hand side and the coefficient of x^2 on the right-hand side must be the same. Similarly the coefficients of x^1 and the coefficients of x^0 (i.e. the constant terms) must match. This gives us a system of three equations,

  $$A+B=3 \qquad 4A+3B+C=13 \qquad 4A+2B+C=11$$

  in the three unknowns A, B, C. We can solve this system by

  - using the first equation, namely $A+B=3$, to determine A in terms of B: $A=3-B$.

---

[49] This is justified in the (optional) subsection "Justification of the Partial Fraction Decompositions" below.

  - Substituting this into the remaining equations eliminates the A, leaving two equations in the two unknown B, C.

$$4(3-B)+3B+C=13 \qquad 4(3-B)+2B+C=11$$

  or

$$-B+C=1 \qquad -2B+C=-1$$

  - We can now solve the first of these equations, namely $-B+C=1$, for B in terms of C, giving $B=C-1$.
  - Substituting this into the last equation, namely $-2B+C=-1$, gives $-2(C-1)+C=-1$ which is easily solved to give
  - $C=3$, and then $B=C-1=2$ and then $A=3-B=1$.

Hence

$$\frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}=4+\frac{3x^2+13x+11}{(x+1)(x+2)^2}=4+\frac{1}{x+1}+\frac{2}{x+2}+\frac{3}{(x+2)^2}$$

- *Step 3 – Sneaky Method.* The second, sneakier, method for finding A, B and C exploits the fact that $3x^2+13x+11=A(x+2)^2+B(x+1)(x+2)+C(x+1)$ must be true for all values of x. In particular, it must be true for $x=-1$. When $x=-1$, the factor $(x+1)$ multiplying B and C is exactly zero. So B and C disappear from the equation, leaving us with an easy equation to solve for A:

$$\begin{aligned} 3x^2+13x+11\Big|_{x=-1} &= \Big[A(x+2)^2+B(x+1)(x+2)+C(x+1)\Big]_{x=-1} \\ &\implies 1=A \end{aligned}$$

Sub this value of A back in and simplify.

$$\begin{aligned} 3x^2+13x+11 &= (1)(x+2)^2+B(x+1)(x+2)+C(x+1) \\ 2x^2+9x+7 &= B(x+1)(x+2)+C(x+1)=(xB+2B+C)(x+1) \end{aligned}$$

Since $(x+1)$ is a factor on the right-hand side, it must also be a factor on the left-hand side.

$$(2x+7)(x+1)=(xB+2B+C)(x+1) \quad \Rightarrow \quad (2x+7)=(xB+2B+C)$$

For the coefficients of x to match, B must be 2. For the constant terms to match, $2B+C$ must be 7, so C must be 3. Hence we again have

$$\frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}=4+\frac{3x^2+13x+11}{(x+1)(x+2)^2}=4+\frac{1}{x+1}+\frac{2}{x+2}+\frac{3}{(x+2)^2}$$

- *Step 4.* The final step is to integrate

$$\begin{aligned} \int\frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}\mathrm{d}x &= \int 4\mathrm{d}x+\int\frac{1}{x+1}\mathrm{d}x+\int\frac{2}{x+2}\mathrm{d}x+\int\frac{3}{(x+2)^2}\mathrm{d}x \\ &= 4x+\log|x+1|+2\log|x+2|-\frac{3}{x+2}+C \end{aligned}$$

Example 1.9.4

The method of partial fractions is not just confined to the problem of integrating rational functions. There are other integrals — such as $\int \sec x \, \mathrm{d}x$ and $\int \sec^3 x \, \mathrm{d}x$ — that can be transformed (via substitutions) into integrals of rational functions. We encountered both of these integrals in Sections 1.7 and 1.8 on trigonometric integrals and substitutions.

Example 1.9.5 ($\int \sec x \, \mathrm{d}x$)

Solution. In this example, we integrate $\sec x$. It is not yet clear what this integral has to do with partial fractions. To get to a partial fractions computation, we first make one of our old substitutions.

$$\begin{aligned}
\int \sec x \, \mathrm{d}x &= \int \frac{1}{\cos x} \mathrm{d}x && \text{massage the expression a little} \\
&= \int \frac{\cos x}{\cos^2 x} \mathrm{d}x && \text{substitute } u = \sin x,\ \mathrm{d}u = \cos x \, \mathrm{d}x \\
&= -\int \frac{\mathrm{d}u}{u^2 - 1} && \text{and use } \cos^2 x = 1 - \sin^2 x = 1 - u^2
\end{aligned}$$

So we now have to integrate $\frac{1}{u^2-1}$, which is a rational function of u, and so is perfect for partial fractions.

- *Step 1.* The degree of the numerator, 1, is zero, which is strictly smaller than the degree of the denominator, $u^2 - 1$, which is two. So the first step is skipped.
- *Step 2.* The second step is to factor the denominator:

  $$u^2 - 1 = (u-1)(u+1)$$

- *Step 3.* The third step is to write $\frac{1}{u^2-1}$ in the form

  $$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

  for some constants A and B.

- *Step 3 – Sneaky Method.*
  - Multiply through by the denominator to get

    $$1 = A(u+1) + B(u-1)$$

    This equation must be true for all u.
  - If we now set $u = 1$ then we eliminate B from the equation leaving us with

    $$1 = 2A \qquad \text{so } A = 1/2.$$

  - Similarly, if we set $u = -1$ then we eliminate A, leaving
$$1 = -2B \qquad \text{which implies } B = -1/2.$$

We have now found that $A = 1/2, B = -1/2$, so
$$\frac{1}{u^2-1} = \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right].$$

- It is always a good idea to check our work.
$$\frac{1/2}{u-1} + \frac{-1/2}{u+1} = \frac{1/2(u+1) - 1/2(u-1)}{(u-1)(u+1)} = \frac{1}{(u-1)(u+1)} \checkmark$$

- *Step 4.* The final step is to integrate.
$$\begin{aligned}
\int \sec x \, \mathrm{d}x &= -\int \frac{\mathrm{d}u}{u^2-1} && \text{after substitution}\\
&= -\frac{1}{2}\int \frac{\mathrm{d}u}{u-1} + \frac{1}{2}\int \frac{\mathrm{d}u}{u+1} && \text{partial fractions}\\
&= -\frac{1}{2}\log|u-1| + \frac{1}{2}\log|u+1| + C\\
&= -\frac{1}{2}\log|\sin(x)-1| + \frac{1}{2}\log|\sin(x)+1| + C && \text{rearrange a little}\\
&= \frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right| + C
\end{aligned}$$

Notice that since $-1 \leqslant \sin x \leqslant 1$, we are free to drop the absolute values in the last line if we wish.

Example 1.9.5

Another example in the same spirit, though a touch harder.

Example 1.9.6 $\left(\int \sec^3 x \, \mathrm{d}x\right)$

Solution.

- We'll start by converting it into the integral of a rational function using the substitution $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$.
$$\begin{aligned}
\int \sec^3 x \, \mathrm{d}x &= \int \frac{1}{\cos^3 x}\mathrm{d}x && \text{massage this a little}\\
&= \int \frac{\cos x}{\cos^4 x}\mathrm{d}x && \text{replace } \cos^2 x = 1 - \sin^2 x = 1 - u^2\\
&= \int \frac{\cos x \, \mathrm{d}x}{[1-\sin^2 x]^2}\\
&= \int \frac{\mathrm{d}u}{[1-u^2]^2}
\end{aligned}$$

- We could now find the partial fraction decomposition of the integrand $\frac{1}{[1-u^2]^2}$ by executing the usual four steps. But it is easier to use

  $$\frac{1}{u^2-1} = \frac{1}{2}\Big[\frac{1}{u-1} - \frac{1}{u+1}\Big]$$

  which we worked out in Example 1.9.5 above.

- Squaring this gives

  $$\begin{aligned}\frac{1}{[1-u^2]^2} &= \frac{1}{4}\Big[\frac{1}{u-1} - \frac{1}{u+1}\Big]^2\\ &= \frac{1}{4}\Big[\frac{1}{(u-1)^2} - \frac{2}{(u-1)(u+1)} + \frac{1}{(u+1)^2}\Big]\\ &= \frac{1}{4}\Big[\frac{1}{(u-1)^2} - \frac{1}{u-1} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\Big]\end{aligned}$$

  where we have again used $\frac{1}{u^2-1} = \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right]$ in the last step.

- It only remains to do the integrals and simplify.

$$\begin{aligned}\int \sec^3 x\,\mathrm{d}x &= \frac{1}{4}\int\Big[\frac{1}{(u-1)^2} - \frac{1}{u-1} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\Big]\mathrm{d}u\\ &= \frac{1}{4}\Big[-\frac{1}{u-1} - \log|u-1| + \log|u+1| - \frac{1}{u+1}\Big] + C && \text{group carefully}\\ &= \frac{-1}{4}\Big[\frac{1}{u-1} + \frac{1}{u+1}\Big] + \frac{1}{4}\Big[\log|u+1| - \log|u-1|\Big] + C && \text{sum carefully}\\ &= -\frac{1}{4}\frac{2u}{u^2-1} + \frac{1}{4}\log\Big|\frac{u+1}{u-1}\Big| + C && \text{clean up}\\ &= \frac{1}{2}\frac{u}{1-u^2} + \frac{1}{4}\log\Big|\frac{u+1}{u-1}\Big| + C && \text{put } u=\sin x\\ &= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{4}\log\Big|\frac{\sin x+1}{\sin x-1}\Big| + C\end{aligned}$$

Example 1.9.6

1.9.2 ▸▸ The form of partial fraction decompositions

In the examples above we used the partial fractions method to decompose rational functions into easily integrated pieces. Each of those examples was quite involved and we had to spend quite a bit of time factoring and doing long division. The key step in each of the computations was Step 3 — in that step we decomposed the rational function $\frac{N(x)}{D(x)}$ (or $\frac{R(x)}{D(x)}$), for which the degree of the numerator is strictly smaller than the degree of the denominator, into a sum of particularly simple rational functions, like $\frac{A}{x-a}$. We did not, however, give a systematic description of those decompositions.

In this subsection we fill that gap by describing the general[50] form of partial fraction decompositions. The justification of these forms is not part of the course, but the interested reader is invited to read the next (optional) subsection where such justification is given. In the following it is assumed that

- $N(x)$ and $D(x)$ are polynomials with the degree of $N(x)$ strictly smaller than the degree of $D(x)$.
- K is a constant.
- $a_1, a_2, \cdots, a_j$ are all different numbers.
- $m_1, m_2, \cdots, m_j$, and $n_1, n_2, \cdots, n_k$ are all strictly positive integers.
- $x^2+b_1x+c_1, x^2+b_2x+c_2, \ \cdots, \ x^2+b_kx+c_k$ are all different.

▸▸▸ Simple Linear Factor Case

If the denominator $D(x) = K(x-a_1)(x-a_2)\cdots(x-a_j)$ is a product of j different linear factors, then

Equation 1.9.7.

$$\frac{N(x)}{D(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_j}{x-a_j}$$

We can then integrate each term

$$\int \frac{A}{x-a}\mathrm{d}x = A\log|x-a| + C.$$

▸▸▸ General Linear Factor Case

If the denominator $D(x) = K(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j}$ then

Equation 1.9.8.

$$\begin{aligned}\frac{N(x)}{D(x)} = \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} \\ + \frac{A_{2,1}}{x-a_2} + \frac{A_{2,2}}{(x-a_2)^2} + \cdots + \frac{A_{2,m_2}}{(x-a_2)^{m_2}} + \cdots \\ + \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}}\end{aligned}$$

---

[50] Well — not the completely general form, in the sense that we are not allowing the use of complex numbers. As a result we have to use both linear and quadratic factors in the denominator. If we could use complex numbers we would be able to restrict ourselves to linear factors.

Notice that we could rewrite each line as

$$\begin{aligned}\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_m}{(x-a)^m} &= \frac{A_1(x-a)^{m-1}+A_2(x-a)^{m-2}+\cdots+A_m}{(x-a)^m}\\ &= \frac{B_1x^{m-1}+B_2x^{m-2}+\cdots+B_m}{(x-a)^m}\end{aligned}$$

which is a polynomial whose degree, $m-1$, is strictly smaller than that of the denominator $(x-a)^m$. But the form of Equation (1.9.8) is preferable because it is easier to integrate.

$$\begin{aligned}\int \frac{A}{x-a}\mathrm{d}x &= A\log|x-a|+C\\ \int \frac{A}{(x-a)^k}\mathrm{d}x &= -\frac{1}{k-1}\cdot\frac{A}{(x-a)^{k-1}} && \text{provided } k>1.\end{aligned}$$

▸▸▸ Simple Linear and Quadratic Factor Case

If $D(x)=K(x-a_1)\cdots(x-a_j)(x^2+b_1x+c_1)\cdots(x^2+b_kx+c_k)$ then

Equation 1.9.9.

$$\frac{N(x)}{D(x)}=\frac{A_1}{x-a_1}+\cdots+\frac{A_j}{x-a_j}+\frac{B_1x+C_1}{x^2+b_1x+c_1}+\cdots+\frac{B_kx+C_k}{x^2+b_kx+c_k}$$

Note that the numerator of each term on the right-hand side has degree one smaller than the degree of the denominator.

The quadratic terms $\frac{Bx+C}{x^2+bx+c}$ are integrated in a two-step process that is best illustrated with a simple example (see also Example 1.9.3 above).

Example 1.9.10 $\left(\int \frac{2x+7}{x^2+4x+13}\mathrm{d}x\right)$

Solution.

- Start by completing the square in the denominator:

$$\begin{aligned}x^2+4x+13 &= (x+2)^2+9 && \text{and thus}\\ \frac{2x+7}{x^2+4x+13} &= \frac{2x+7}{(x+2)^2+3^2}\end{aligned}$$

- Now set $y=(x+2)/3$, $\mathrm{d}y=\frac{1}{3}\mathrm{d}x$, or equivalently $x=3y-2$, $\mathrm{d}x=3\mathrm{d}y$:

$$\begin{aligned}\int \frac{2x+7}{x^2+4x+13}\mathrm{d}x &= \int \frac{2x+7}{(x+2)^2+3^2}\mathrm{d}x\\ &= \int \frac{6y-4+7}{3^2y^2+3^2}\cdot 3\mathrm{d}y\\ &= \int \frac{6y+3}{3(y^2+1)}\mathrm{d}y\\ &= \int \frac{2y+1}{y^2+1}\mathrm{d}y\end{aligned}$$

Notice that we chose 3 in $y = (x+2)/3$ precisely to transform the denominator into the form $y^2 + 1$.

- Now almost always the numerator will be a linear polynomial of y and we decompose as follows

$$\begin{aligned}\int \frac{2x+7}{x^2+4x+13}\mathrm{d}x &= \int \frac{2y+1}{y^2+1}\mathrm{d}y \\ &= \int \frac{2y}{y^2+1}\mathrm{d}y + \int \frac{1}{y^2+1}\mathrm{d}y \\ &= \log|y^2+1| + \arctan y + C \\ &= \log\left|\left(\frac{x+2}{3}\right)^2+1\right| + \arctan\left(\frac{x+2}{3}\right) + C\end{aligned}$$

Example 1.9.10

▸▸▸ Optional — General Linear and Quadratic Factor Case

If $D(x) = K(x-a_1)^{m_1}\cdots(x-a_j)^{m_j}(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k}$ then

Equation 1.9.11.

$$\begin{aligned}\frac{N(x)}{D(x)} &= \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} + \cdots \\ &+ \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}} \\ &+ \frac{B_{1,1}x+C_{1,1}}{x^2+b_1x+c_1} + \frac{B_{1,2}x+C_{1,2}}{(x^2+b_1x+c_1)^2} + \cdots + \frac{B_{1,n_1}x+C_{1,n_1}}{(x^2+b_1x+c_1)^{n_1}} + \cdots \\ &+ \frac{B_{k,1}x+C_{k,1}}{x^2+b_kx+c_k} + \frac{B_{k,2}x+C_{k,2}}{(x^2+b_kx+c_k)^2} + \cdots + \frac{B_{k,n_k}x+C_{k,n_k}}{(x^2+b_kx+c_k)^{n_k}}\end{aligned}$$

We have already seen how to integrate the simple and general linear terms, and the simple quadratic terms. Integrating general quadratic terms is not so straightforward.

1.9.3 ▸▸ Optional — justification of the partial fraction decompositions

We will now see the justification for the form of the partial fraction decompositions. We start by considering the case in which the denominator has only linear factors. Then we'll consider the case in which quadratic factors are allowed too[51].

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[51] In fact, quadratic factors are completely avoidable because, if we use complex numbers, then every polynomial can be written as a product of linear factors. This is the fundamental theorem of algebra.

▸▸▸ The Simple Linear Factor Case

In the most common partial fraction decomposition, we split up

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)}$$

into a sum of the form

$$\frac{A_1}{x-a_1}+\cdots+\frac{A_d}{x-a_d}$$

We now show that this decomposition can always be achieved, under the assumptions that the a_i's are all different and $N(x)$ is a polynomial of degree at most $d-1$. To do so, we shall repeatedly apply the following Lemma.

Lemma 1.9.12.

Let $N(x)$ and $D(x)$ be polynomials of degree n and d respectively, with $n \leqslant d$. Suppose that a is NOT a zero of $D(x)$. Then there is a polynomial $P(x)$ of degree $p < d$ and a number A such that

$$\frac{N(x)}{D(x)\,(x-a)} = \frac{P(x)}{D(x)} + \frac{A}{x-a}$$

Proof.

- To save writing, let $z = x - a$. We then write $\tilde{N}(z) = N(z+a)$ and $\tilde{D}(z) = D(z+a)$, which are again polynomials of degree n and d respectively. We also know that $\tilde{D}(0) = D(a) \neq 0$.
- In order to complete the proof we need to find a polynomial $\tilde{P}(z)$ of degree $p < d$ and a number A such that

  $$\frac{\tilde{N}(z)}{\tilde{D}(z)\,z} = \frac{\tilde{P}(z)}{\tilde{D}(z)} + \frac{A}{z} = \frac{\tilde{P}(z)z + A\tilde{D}(z)}{\tilde{D}(z)\,z}$$

  or equivalently, such that

  $$\tilde{P}(z)z + A\tilde{D}(z) = \tilde{N}(z).$$

- Now look at the polynomial on the left-hand side. Every term in $\tilde{P}(z)z$, has at least one power of z. So the constant term on the left-hand side is exactly the constant term in $A\tilde{D}(z)$, which is equal to $A\tilde{D}(0)$. The constant term on the right-hand side is equal to $\tilde{N}(0)$. So the constant terms on the left and right-hand sides are the same if we choose $A = \frac{\tilde{N}(0)}{\tilde{D}(0)}$. Recall that $\tilde{D}(0)$ cannot be zero, so A is well defined.
- Now move $A\tilde{D}(z)$ to the right-hand side.

  $$\tilde{P}(z)z = \tilde{N}(z) - A\tilde{D}(z)$$

  The constant terms in $\tilde{N}(z)$ and $A\tilde{D}(z)$ are the same, so the right-hand side contains no constant term and the right hand side is of the form $\tilde{N}_1(z)z$ for some polynomial $\tilde{N}_1(z)$.

- Since $\tilde{N}(z)$ is of degree at most d and $A\tilde{D}(z)$ is of degree exactly d, $\tilde{N}_1$ is a polynomial of degree $d-1$. It now suffices to choose $\tilde{P}(z) = \tilde{N}_1(z)$.

$\square$

Now back to

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)}$$

Apply Lemma 1.9.12, with $D(x) = (x-a_2)\times\cdots\times(x-a_d)$ and $a = a_1$. It says

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \frac{P(x)}{(x-a_2)\times\cdots\times(x-a_d)}$$

for some polynomial P of degree at most $d-2$ and some number A_1.

Apply Lemma 1.9.12 a second time, with $D(x) = (x-a_3)\times\cdots\times(x-a_d)$, $N(x) = P(x)$ and $a = a_2$. It says

$$\frac{P(x)}{(x-a_2)\times\cdots\times(x-a_d)} = \frac{A_2}{x-a_2} + \frac{Q(x)}{(x-a_3)\times\cdots\times(x-a_d)}$$

for some polynomial Q of degree at most $d-3$ and some number A_2.

At this stage, we know that

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{Q(x)}{(x-a_3)\times\cdots\times(x-a_d)}$$

If we just keep going, repeatedly applying Lemma 1, we eventually end up with

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_d}{x-a_d}$$

as required.

▸▸▸ The General Linear Factor Case

Now consider splitting

$$\frac{N(x)}{(x-a_1)^{n_1}\times\cdots\times(x-a_d)^{n_d}}$$

into a sum of the form[52]

$$\left[\frac{A_{1,1}}{x-a_1} + \cdots + \frac{A_{1,n_1}}{(x-a_1)^{n_1}}\right] + \cdots + \left[\frac{A_{d,1}}{x-a_d} + \cdots + \frac{A_{d,n_d}}{(x-a_d)^{n_d}}\right]$$

We now show that this decomposition can always be achieved, under the assumptions that the a_i's are all different and $N(x)$ is a polynomial of degree at most $n_1 + \cdots + n_d - 1$. To do so, we shall repeatedly apply the following Lemma.

---

[52] If we allow ourselves to use complex numbers as roots, this is the general case. We don't need to consider quadratic (or higher) factors since all polynomials can be written as products of linear factors with complex coefficients.

Lemma 1.9.13.

Let $N(x)$ and $D(x)$ be polynomials of degree n and d respectively, with $n < d + m$. Suppose that a is NOT a zero of $D(x)$. Then there is a polynomial $P(x)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that

$$\frac{N(x)}{D(x)\,(x-a)^m} = \frac{P(x)}{D(x)} + \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m}$$

Proof.

- As we did in the proof of the previous lemma, we write $z = x - a$. Then $\tilde{N}(z) = N(z+a)$ and $\tilde{D}(z) = D(z+a)$ are polynomials of degree n and d respectively, $\tilde{D}(0) = D(a) \neq 0$.
- In order to complete the proof we have to find a polynomial $\tilde{P}(z)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that

$$\begin{aligned}\frac{\tilde{N}(z)}{\tilde{D}(z)\,z^m} &= \frac{\tilde{P}(z)}{\tilde{D}(z)} + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_m}{z^m} \\ &= \frac{\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_m\tilde{D}(z)}{\tilde{D}(z)\,z^m}\end{aligned}$$

  or equivalently, such that

$$\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_{m-1}z\tilde{D}(z) + A_m\tilde{D}(z) = \tilde{N}(z)$$

- Now look at the polynomial on the left-hand side. Every single term on the left-hand side, except for the very last one, $A_m\tilde{D}(z)$, has at least one power of z. So the constant term on the left-hand side is exactly the constant term in $A_m\tilde{D}(z)$, which is equal to $A_m\tilde{D}(0)$. The constant term on the right-hand side is equal to $\tilde{N}(0)$. So the constant terms on the left and right-hand sides are the same if we choose $A_m = \frac{\tilde{N}(0)}{\tilde{D}(0)}$. Recall that $\tilde{D}(0) \neq 0$ so A_m is well defined.
- Now move $A_m\tilde{D}(z)$ to the right-hand side.

$$\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_{m-1}z\tilde{D}(z) = \tilde{N}(z) - A_m\tilde{D}(z)$$

  The constant terms in $\tilde{N}(z)$ and $A_m\tilde{D}(z)$ are the same, so the right-hand side contains no constant term and the right hand side is of the form $\tilde{N}_1(z)z$ with $\tilde{N}_1$ a polynomial of degree at most $d + m - 2$. (Recall that $\tilde{N}$ is of degree at most $d + m - 1$ and $\tilde{D}$ is of degree at most d.) Divide the whole equation by z to get

$$\tilde{P}(z)z^{m-1} + A_1 z^{m-2}\tilde{D}(z) + A_2 z^{m-3}\tilde{D}(z) + \cdots + A_{m-1}\tilde{D}(z) = \tilde{N}_1(z).$$

- Now, we can repeat the previous argument. The constant term on the left hand side, which is exactly equal to $A_{m-1}\tilde{D}(0)$ matches the constant term on the right-hand side, which is equal to $\tilde{N}_1(0)$ if we choose $A_{m-1} = \frac{\tilde{N}_1(0)}{\tilde{D}(0)}$. With this choice of A_{m-1}

$$\begin{aligned}\tilde{P}(z)z^{m-1} + A_1 z^{m-2}\tilde{D}(z) + A_2 z^{m-3}\tilde{D}(z) + \cdots + A_{m-2}z\tilde{D}(z)&\\ = \tilde{N}_1(z) - A_{m-1}\tilde{D}(z) = \tilde{N}_2(z)z&\end{aligned}$$

with $\tilde{N}_2$ a polynomial of degree at most $d + m - 3$. Divide by z and continue.

- After m steps like this, we end up with

$$\tilde{P}(z)z = \tilde{N}_{m-1}(z) - A_1\tilde{D}(z)$$

after having chosen $A_1 = \frac{\tilde{N}_{m-1}(0)}{\tilde{D}(0)}$.

- There is no constant term on the right side so that $\tilde{N}_{m-1}(z) - A_1\tilde{D}(z)$ is of the form $\tilde{N}_m(z)z$ with $\tilde{N}_m$ a polynomial of degree $d - 1$. Choosing $\tilde{P}(z) = \tilde{N}_m(z)$ completes the proof.

□

Now back to

$$\frac{N(x)}{(x - a_1)^{n_1} \times \cdots \times (x - a_d)^{n_d}}$$

Apply Lemma 1.9.13, with $D(x) = (x - a_2)^{n_2} \times \cdots \times (x - a_d)^{n_d}$, $m = n_1$ and $a = a_1$. It says

$$\begin{aligned}&\frac{N(x)}{(x - a_1)^{n_1} \times \cdots \times (x - a_d)^{n_d}}\\ &\qquad = \frac{A_{1,1}}{x - a_1} + \frac{A_{1,2}}{(x - a_1)^2} + \cdots + \frac{A_{1,n_1}}{(x - a)^{n_1}} + \frac{P(x)}{(x - a_2)^{n_2} \times \cdots \times (x - a_d)^{n_d}}\end{aligned}$$

Apply Lemma 1.9.13 a second time, with $D(x) = (x - a_3)^{n_3} \times \cdots \times (x - a_d)^{n_d}$, $N(x) = P(x)$, $m = n_2$ and $a = a_2$. And so on. Eventually, we end up with

$$\left[\frac{A_{1,1}}{x - a_1} + \cdots + \frac{A_{1,n_1}}{(x - a_1)^{n_1}}\right] + \cdots + \left[\frac{A_{d,1}}{x - a_d} + \cdots + \frac{A_{d,n_d}}{(x - a_d)^{n_d}}\right]$$

which is exactly what we were trying to show.

▸▸▸ Really Optional — The Fully General Case

We are now going to see that, in general, if $N(x)$ and $D(x)$ are polynomials with the degree of N being strictly smaller than the degree of D (which we'll denote $\deg(N) < \deg(D)$) and if

$$D(x) = K(x - a_1)^{m_1} \cdots (x - a_j)^{m_j}(x^2 + b_1 x + c_1)^{n_1} \cdots (x^2 + b_k x + c_k)^{n_k} \tag{E1}$$

(with $b_\ell^2 - 4c_\ell < 0$ for all $1 \leqslant \ell \leqslant k$ so that no quadratic factor can be written as a product of linear factors with real coefficients) then there are real numbers $A_{i,j}$, $B_{i,j}$, $C_{i,j}$ such that

$$\begin{aligned}\frac{N(x)}{D(x)} = {} & \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} + \cdots \\ & + \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}} \\ & + \frac{B_{1,1}x + C_{1,1}}{x^2+b_1x+c_1} + \frac{B_{1,2}x + C_{1,2}}{(x^2+b_1x+c_1)^2} + \cdots + \frac{B_{1,n_1}x + C_{1,n_1}}{(x^2+b_1x+c_1)^{n_1}} + \cdots \\ & + \frac{B_{k,1}x + C_{k,1}}{x^2+b_kx+c_k} + \frac{B_{k,2}x + C_{k,2}}{(x^2+b_kx+c_k)^2} + \cdots + \frac{B_{k,n_k}x + C_{1,n_k}}{(x^2+b_kx+c_k)^{n_k}}\end{aligned}$$

This was (1.9.11).

We start with two simpler results, that we'll use repeatedly to get (1.9.11). In the first simpler result, we consider the fraction $\frac{P(x)}{Q_1(x)\,Q_2(x)}$ with $P(x)$, $Q_1(x)$ and $Q_2(x)$ being polynomials with real coefficients and we are going to assume that when $P(x)$, $Q_1(x)$ and $Q_2(x)$ are factored as in (E1), no two of them have a common linear or quadratic factor. As an example, no two of

$$\begin{aligned}P(x) &= 2(x-3)(x-4)(x^2+3x+3) \\ Q_1(x) &= 2(x-1)(x^2+2x+2) \\ Q_2(x) &= 2(x-2)(x^2+2x+3)\end{aligned}$$

have such a common factor. But, for

$$\begin{aligned}P(x) &= 2(x-3)(x-4)(x^2+x+1) \\ Q_1(x) &= 2(x-1)(x^2+2x+2) \\ Q_2(x) &= 2(x-2)(x^2+x+1)\end{aligned}$$

$P(x)$ and $Q_2(x)$ have the common factor x^2+x+1.

Lemma 1.9.14.

Let $P(x)$, $Q_1(x)$ and $Q_2(x)$ be polynomials with real coefficients and with $\deg(P) < \deg(Q_1Q_2)$. Assume that no two of $P(x)$, $Q_1(x)$ and $Q_2(x)$ have a common linear or quadratic factor. Then there are polynomials P_1, P_2 with $\deg(P_1) < \deg(Q_1)$, $\deg(P_2) < \deg(Q_2)$, and

$$\frac{P(x)}{Q_1(x)\,Q_2(x)} = \frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)}$$

Proof. We are to find polynomials P_1 and P_2 that obey

$$P(x) = P_1(x)\,Q_2(x) + P_2(x)\,Q_1(x)$$

Actually, we are going to find polynomials p_1 and p_2 that obey

$$p_1(x)\,Q_1(x) + p_2(x)\,Q_2(x) = C \tag{E2}$$

for some nonzero constant C, and then just multiply (E2) by $\frac{P(x)}{C}$. To find p_1, p_2 and C we are going to use something called the Euclidean algorithm. It is an algorithm[53] that is used to efficiently find the greatest common divisors of two numbers. Because $Q_1(x)$ and $Q_2(x)$ have no common factors of degree 1 or 2, their "greatest common divisor" has degree 0, i.e. is a constant.

- The first step is to apply long division to $\frac{Q_1(x)}{Q_2(x)}$ to find polynomials $n_0(x)$ and $r_0(x)$ such that
$$\frac{Q_1(x)}{Q_2(x)} = n_0(x) + \frac{r_0(x)}{Q_2(x)} \qquad \text{with } \deg(r_0) < \deg(Q_2)$$
or, equivalently,
$$Q_1(x) = n_0(x)\,Q_2(x) + r_0(x) \qquad \text{with } \deg(r_0) < \deg(Q_2)$$
- The second step is to apply long division to $\frac{Q_2(x)}{r_0(x)}$ to find polynomials $n_1(x)$ and $r_1(x)$ such that
$$Q_2(x) = n_1(x)\,r_0(x) + r_1(x) \qquad \text{with } \deg(r_1) < \deg(r_0) \text{ or } r_1(x) = 0$$
- The third step (assuming that $r_1(x)$ was not zero) is to apply long division to $\frac{r_0(x)}{r_1(x)}$ to find polynomials $n_2(x)$ and $r_2(x)$ such that
$$r_0(x) = n_2(x)\,r_1(x) + r_2(x) \qquad \text{with } \deg(r_2) < \deg(r_1) \text{ or } r_2(x) = 0$$
- And so on.

As the degree of the remainder $r_i(x)$ decreases by at least one each time i is increased by one, the above iteration has to terminate with some $r_{\ell+1}(x) = 0$. That is, we choose ℓ to be index of the last nonzero remainder. Here is a summary of all of the long division steps.

$$\begin{aligned}
Q_1(x) &= n_0(x)\,Q_2(x) + r_0(x) && \text{with } \deg(r_0) < \deg(Q_2)\\
Q_2(x) &= n_1(x)\,r_0(x) + r_1(x) && \text{with } \deg(r_1) < \deg(r_0)\\
r_0(x) &= n_2(x)\,r_1(x) + r_2(x) && \text{with } \deg(r_2) < \deg(r_1)\\
r_1(x) &= n_3(x)\,r_2(x) + r_3(x) && \text{with } \deg(r_3) < \deg(r_2)\\
&\vdots\\
r_{\ell-2}(x) &= n_\ell(x)\,r_{\ell-1}(x) + r_\ell(x) && \text{with } \deg(r_\ell) < \deg(r_{\ell-1})\\
r_{\ell-1}(x) &= n_{\ell+1}(x)\,r_\ell(x) + r_{\ell+1}(x) && \text{with } r_{\ell+1} = 0
\end{aligned}$$

Now we are going to take a closer look at all of the different remainders that we have generated.

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[53] It appears in Euclid's Elements, which was written about 300 BC, and it was probably known even before that.

- From first long division step, namely $Q_1(x) = n_0(x)\,Q_2(x) + r_0(x)$ we have that the remainder
$$r_0(x) = Q_1(x) - n_0(x)\,Q_2(x)$$
- From the second long division step, namely $Q_2(x) = n_1(x)\,r_0(x) + r_1(x)$ we have that the remainder
$$\begin{aligned} r_1(x) &= Q_2(x) - n_1(x)\,r_0(x) = Q_2(x) - n_1(x)\big[Q_1(x) - n_0(x)\,Q_2(x)\big] \\ &= A_1(x)\,Q_1(x) + B_1(x)\,Q_2(x) \end{aligned}$$
with $A_1(x) = -n_1(x)$ and $B_1(x) = 1 + n_0(x)\,n_1(x)$.
- From the third long division step (assuming that $r_1(x)$ was not zero), namely $r_0(x) = n_2(x)\,r_1(x) + r_2(x)$, we have that the remainder
$$\begin{aligned} r_2(x) &= r_0(x) - n_2(x)\,r_1(x) \\ &= \big[Q_1(x) - n_0(x)\,Q_2(x)\big] - n_2(x)\big[A_1(x)\,Q_1(x) + B_1(x)\,Q_2(x)\big] \\ &= A_2(x)\,Q_1(x) + B_2(x)\,Q_2(x) \end{aligned}$$
with $A_2(x) = 1 - n_2(x)\,A_1(x)$ and $B_2(x) = -n_0(x) - n_2(x)\,B_1(x)$.
- And so on. Continuing in this way, we conclude that the final nonzero remainder $r_\ell(x) = A_\ell(x)\,Q_1(x) + B_\ell(x)\,Q_2(x)$ for some polynomials A_ℓ and B_ℓ.

Now the last nonzero remainder $r_\ell(x)$ has to be a nonzero constant C because

- ○ it is nonzero by the definition of $r_\ell(x)$ and
- ○ if $r_\ell(x)$ were a polynomial of degree at least one, then
  - $r_\ell(x)$ would be a factor of $r_{\ell-1}(x)$ because $r_{\ell-1}(x) = n_{\ell+1}(x)\,r_\ell(x)$ and
  - $r_\ell(x)$ would be a factor of $r_{\ell-2}(x)$ because $r_{\ell-2}(x) = n_\ell(x)\,r_{\ell-1}(x) + r_\ell(x)$ and
  - $r_\ell(x)$ would be a factor of $r_{\ell-3}(x)$ because $r_{\ell-3}(x) = n_{\ell-1}(x)\,r_{\ell-2}(x) + r_{\ell-1}(x)$ and
  - $\cdots$ and $\cdots$
  - $r_\ell(x)$ would be a factor of $r_1(x)$ because $r_1(x) = n_3(x)\,r_2(x) + r_3(x)$ and
  - $r_\ell(x)$ would be a factor of $r_0(x)$ because $r_0(x) = n_2(x)\,r_1(x) + r_2(x)$ and
  - $r_\ell(x)$ would be a factor of $Q_2(x)$ because $Q_2(x) = n_1(x)\,r_0(x) + r_1(x)$ and
  - $r_\ell(x)$ would be a factor of $Q_1(x)$ because $Q_1(x) = n_0(x)\,Q_2(x) + r_0(x)$
- ○ so that $r_\ell(x)$ would be a common factor for $Q_1(x)$ and $Q_2(x)$, in contradiction to the hypothesis that no two of $P(x)$, $Q_1(x)$ and $Q_2(x)$ have a common linear or quadratic factor.

We now have that $A_\ell(x)\,Q_1(x) + B_\ell(x)\,Q_2(x) = r_\ell(x) = C$. Multiplying by $\frac{P(x)}{C}$ gives
$$\tilde{P}_2(x)\,Q_1(x) + \tilde{P}_1(x)\,Q_2(x) = P(x) \quad \text{or} \quad \frac{\tilde{P}_1(x)}{Q_1(x)} + \frac{\tilde{P}_2(x)}{Q_2(x)} = \frac{P(x)}{Q_1(x)\,Q_2(x)}$$
with $\tilde{P}_2(x) = \frac{P(x)\,A_\ell(x)}{C}$ and $\tilde{P}_1(x) = \frac{P(x)\,B_\ell(x)}{C}$. We're not quite done, because there is still the danger that $\deg(\tilde{P}_1) \geqslant \deg(Q_1)$ or $\deg(\tilde{P}_2) \geqslant \deg(Q_2)$. To deal with that possibility, we long divide $\frac{\tilde{P}_1(x)}{Q_1(x)}$ and call the remainder $P_1(x)$.
$$\frac{\tilde{P}_1(x)}{Q_1(x)} = N(x) + \frac{P_1(x)}{Q_1(x)} \qquad \text{with } \deg(P_1) < \deg(Q_1)$$

Therefore we have that

$$\begin{aligned}\frac{P(x)}{Q_1(x)\,Q_2(x)} &= \frac{P_1(x)}{Q_1(x)} + N(x) + \frac{\tilde{P}_2(x)}{Q_2(x)}\\ &= \frac{P_1(x)}{Q_1(x)} + \frac{\tilde{P}_2(x) + N(x)Q_2(x)}{Q_2(x)}\end{aligned}$$

Denoting $P_2(x) = \tilde{P}_2(x) + N(x)Q_2(x)$ gives $\frac{P}{Q_1 Q_2} = \frac{P_1}{Q_1} + \frac{P_2}{Q_2}$ and since $\deg(P_1) < \deg(Q_1)$, the only thing left to prove is that $\deg(P_2) < \deg(Q_2)$.

We assume that $\deg(P_2) \geqslant \deg(Q_2)$ and look for a contradiction. We have

$$\begin{aligned}&\deg(P_2Q_1) \geqslant \deg(Q_1Q_2) > \deg(P_1Q_2)\\ \implies &\deg(P) = \deg(P_1Q_2 + P_2Q_1) = \deg(P_2Q_1) \geqslant \deg(Q_1Q_2)\end{aligned}$$

which contradicts the hypothesis that $\deg(P) < \deg(Q_1Q_2)$ and the proof is complete. ☐

For the second of the two simpler results, that we'll shortly use repeatedly to get (1.9.11), we consider $\frac{P(x)}{(x-a)^m}$ and $\frac{P(x)}{(x^2+bx+c)^m}$.

Lemma 1.9.15.

Let $m \geqslant 2$ be an integer, and let $Q(x)$ be either $x - a$ or $x^2 + bx + c$, with a, b and c being real numbers. Let $P(x)$ be a polynomial with real coefficients, which does not contain $Q(x)$ as a factor, and with $\deg(P) < \deg(Q^m) = m\deg(Q)$. Then, for each $1 \leqslant i \leqslant m$, there is a polynomial P_i with $\deg(P_i) < \deg(Q)$ or $P_i = 0$, such that

$$\frac{P(x)}{Q(x)^m} = \frac{P_1(x)}{Q(x)} + \frac{P_2(x)}{Q(x)^2} + \frac{P_3(x)}{Q(x)^3} + \cdots + \frac{P_{m-1}(x)}{Q(x)^{m-1}} + \frac{P_m(x)}{Q(x)^m}.$$

In particular, if $Q(x) = x - a$, then each $P_i(x)$ is just a constant A_i, and if $Q(x) = x^2 + bx + c$, then each $P_i(x)$ is a polynomial $B_ix + C_i$ of degree at most one.

Proof. We simply repeatedly use long divison to get

$$\begin{aligned}\frac{P(x)}{Q(x)^m} &= \frac{P(x)}{Q(x)}\,\frac{1}{Q(x)^{m-1}} = \left\{n_1(x) + \frac{r_1(x)}{Q(x)}\right\}\frac{1}{Q(x)^{m-1}}\\ &= \frac{r_1(x)}{Q(x)^m} + \frac{n_1(x)}{Q(x)}\,\frac{1}{Q(x)^{m-2}} = \frac{r_1(x)}{Q(x)^m} + \left\{n_2(x) + \frac{r_2(x)}{Q(x)}\right\}\frac{1}{Q(x)^{m-2}}\\ &= \frac{r_1(x)}{Q(x)^m} + \frac{r_2(x)}{Q(x)^{m-1}} + \frac{n_2(x)}{Q(x)}\,\frac{1}{Q(x)^{m-3}}\\ &\vdots\\ &= \frac{r_1(x)}{Q(x)^m} + \frac{r_2(x)}{Q(x)^{m-1}} + \cdots + \frac{r_{m-2}(x)}{Q(x)^3} + \frac{n_{m-2}(x)}{Q(x)}\,\frac{1}{Q(x)}\\ &= \frac{r_1(x)}{Q(x)^m} + \frac{r_2(x)}{Q(x)^{m-1}} + \cdots + \frac{r_{m-2}(x)}{Q(x)^3} + \left\{n_{m-1}(x) + \frac{r_{m-1}(x)}{Q(x)}\right\}\frac{1}{Q(x)}\\ &= \frac{r_1(x)}{Q(x)^m} + \frac{r_2(x)}{Q(x)^{m-1}} + \cdots + \frac{r_{m-2}(x)}{Q(x)^3} + \frac{r_{m-1}(x)}{Q(x)^2} + \frac{n_{m-1}(x)}{Q(x)}\end{aligned}$$

By the rules of long division every $\deg(r_i) < \deg(Q)$. It is also true that the final numerator, n_{m-1}, has $\deg(n_{m-1}) < \deg(Q)$ — that is, we kept dividing by Q until the degree of the quotient was less than the degree of Q. To see this, note that $\deg(P) < m\deg(Q)$ and

$$\begin{aligned}
\deg(n_1) &= \deg(P) - \deg(Q)\\
\deg(n_2) &= \deg(n_1) - \deg(Q) = \deg(P) - 2\deg(Q)\\
&\vdots\\
\deg(n_{m-1}) &= \deg(n_{m-2}) - \deg(Q) = \deg(P) - (m-1)\deg(Q)\\
&< m\deg(Q) - (m-1)\deg(Q)\\
&= \deg(Q)
\end{aligned}$$

So, if $\deg(Q) = 1$, then $r_1, r_2, \ldots, r_{m-1}, n_{m-1}$ are all real numbers, and if $\deg(Q) = 2$, then $r_1, r_2, \ldots, r_{m-1}, n_{m-1}$ all have degree at most one. ☐

We are now in a position to get (1.9.11). We use (E1) to factor[54] $D(x) = (x - a_1)^{m_1} Q_2(x)$ and use Lemma 1.9.14 to get

$$\frac{N(x)}{D(x)} = \frac{N(x)}{(x-a_1)^{m_1}Q_2(x)} = \frac{P_1(x)}{(x-a_1)^{m_1}} + \frac{P_2(x)}{Q_2(x)}$$

where $\deg(P_1) < m_1$, and $\deg(P_2) < \deg(Q_2)$. Then we use Lemma 1.9.15 to get

$$\frac{N(x)}{D(x)} = \frac{P_1(x)}{(x-a_1)^{m_1}} + \frac{P_2(x)}{Q_2(x)} = \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} + \frac{P_2(x)}{Q_2(x)}$$

We continue working on $\frac{P_2(x)}{Q_2(x)}$ in this way, pulling off of the denominator one $(x-a_i)^{m_i}$ or one $(x^2 + b_i x + c_i)^{n_i}$ at a time, until we exhaust all of the factors in the denominator $D(x)$.

1.10▲ Volumes

Learning Objectives

- Use an integral to represent the volume of a 3D object (providing it has some symmetry). Explain, using a picture, what each piece of the integral represents.
- Find the volume of surfaces of revolution using disks.
- Find volume by integrating over cross sectional areas.

Another simple[55] application of integration is computing volumes. We use the same strategy as we used to express areas of regions in two dimensions as integrals — approximate

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[54] This is assuming that there is at least one linear factor. If not, we factor $D(x) = (x^2 + b_1 x + c_1)^{n_1} Q_2(x)$ instead.

[55] Well — arguably the idea isn't too complicated and is a continuation of the idea used to compute areas in the previous section. In practice this can be quite tricky, as we shall see.

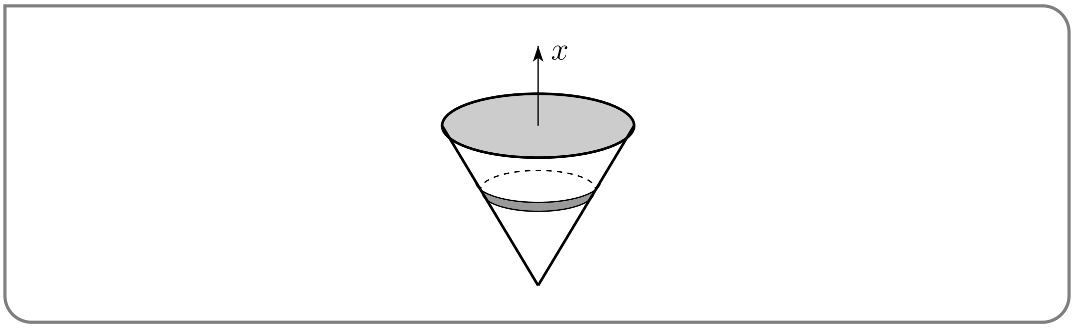
the region by a union of small, simple pieces whose volume we can compute and then then take the limit as the "piece size" tends to zero.

In many cases this will lead to "multivariable integrals" that are beyond our present scope[56]. But there are some special cases in which this leads to integrals that we can handle. Here are some examples.

Example 1.10.1 (Cone)

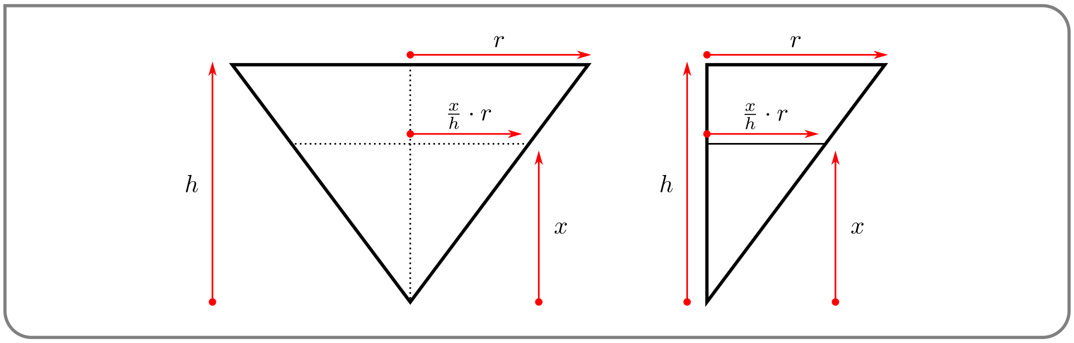
Find the volume of the circular cone of height h and radius r.

Solution. Here is a sketch of the cone. We have called the vertical axis x, just so that we



end up with a "dx" integral.

- In what follows we will slice the cone into thin horizontal "pancakes". In order to approximate the volume of those slices, we need to know the radius of the cone at a height x above its point. Consider the cross sections shown in the following figure.



  At full height h, the cone has radius r. If we cut the cone at height x, then by similar triangles (see the figure on the right) the radius will be $\frac{x}{h} \cdot r$.

- Now think of cutting the cone into n thin horizontal "pancakes". Each such pancake is approximately a squat cylinder of height $\Delta x = {}^{h}/_{n}$. This is very similar to how we approximated the area under a curve by n tall thin rectangles. Just as we approximated the area under the curve by summing these rectangles, we can approximate the volume of the cone by summing the volumes of these cylinders. Here is a side view of the cone and one of the cylinders.

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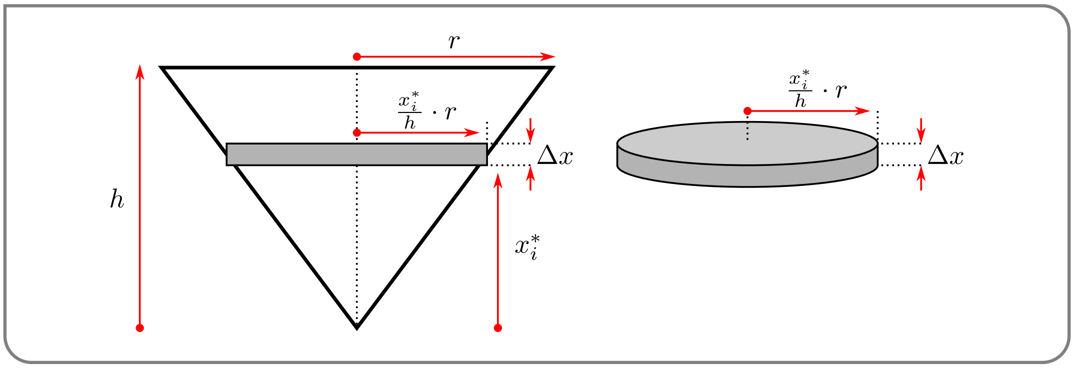
[56] Typically such integrals (and more) are covered in a third calculus course.



- We follow the method we used in Example 1.6.1, except that our slices are now pancakes instead of rectangles.
  - Pick a natural number n (that we will later send to infinity), then
  - subdivide the cone into n thin pancakes, each of width $\Delta x = \frac{h}{n}$.
  - For each $i = 1, 2, \ldots, n$, pancake number i runs from $x = x_{i-1} = (i-1) \cdot \Delta x$ to $x = x_i = i \cdot \Delta x$, and we approximate its volume by the volume of a squat cone. We pick a number x_i^* between x_{i-1} and x_i and approximate the pancake by a cylinder of height Δx and radius $\frac{x_i^*}{h} r$.
  - Thus the volume of pancake i is approximately $\pi \left(\frac{x_i^*}{h} r\right)^2 \Delta x$ (as shown in the figure above).
- So the Riemann sum approximation of the volume is

$$\text{Volume} \approx \sum_{i=1}^{n} \pi \left(\frac{x_i^*}{h} r\right)^2 \Delta x$$

- By taking the limit as $n \to \infty$ (i.e. taking the limit as the thickness of the pancakes goes to zero), we convert the Riemann sum into a definite integral (see Definition 1.1.10) and at the same time our approximation of the volume becomes the exact volume:

$$\int_0^h \pi \left(\frac{x}{h} r\right)^2 \mathrm{d}x$$

Our life[57] would be easier if we could avoid all this formal work with Riemann sums every time we encounter a new volume. So before we compute the above integral, let us redo the above calculation in a less formal manner.

- Start again from the picture of the cone and think of slicing it into thin pancakes,

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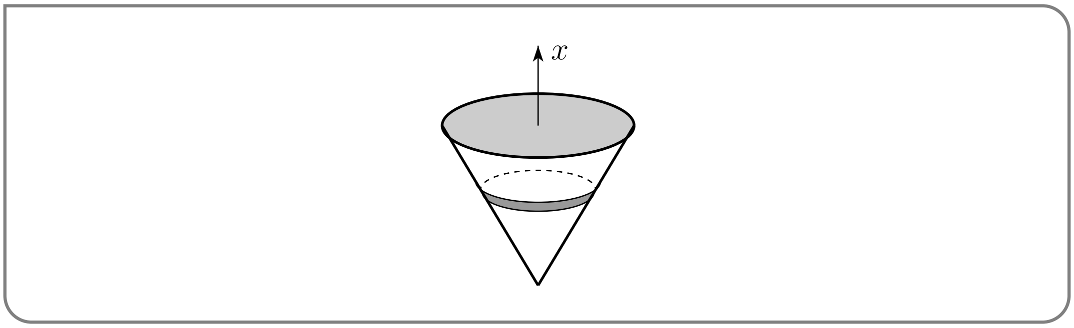
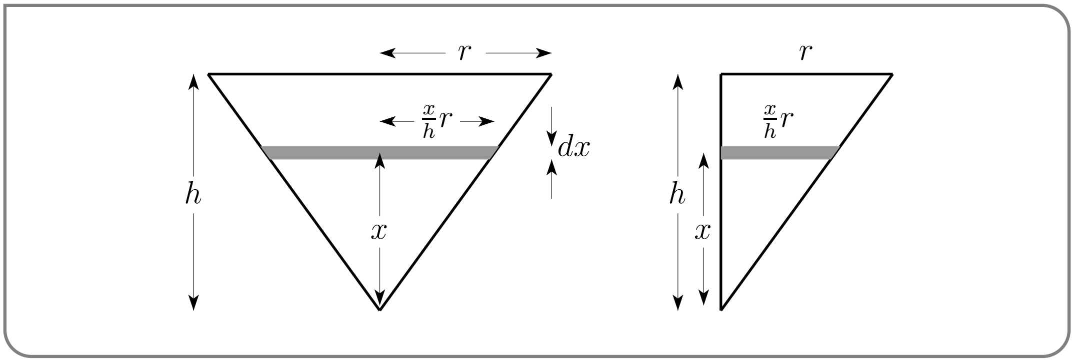
[57] At least the bits of it involving integrals.



each of width dx.



- The pancake at height x above the point of the cone (which is the fraction $\frac{x}{h}$ of the total height of the cone) has
  - radius $\frac{x}{h} \cdot r$ (the fraction $\frac{x}{h}$ of the full radius, r) and so
  - cross–sectional area $\pi\left(\frac{x}{h}r\right)^2$,
  - thickness dx — we have done something a little sneaky here, see the discussion below.
  - volume $\pi\left(\frac{x}{h}r\right)^2 \mathrm{d}x$

  As x runs from 0 to h, the total volume is

$$\begin{aligned}\int_0^h \pi\left(\frac{x}{h}r\right)^2 \mathrm{d}x &= \frac{\pi r^2}{h^2}\int_0^h x^2 \mathrm{d}x \\ &= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\ &= \frac{1}{3}\pi r^2 h\end{aligned}$$

In this second computation we are using a time-saving trick. As we saw in the formal computation above, what we really need to do is pick a natural number n, slice the cone into n pancakes each of thickness $\Delta x = h/n$ and then take the limit as $n \to \infty$. This led to the Riemann sum

$$\sum_{i=1}^{n} \pi\left(\frac{x_i^*}{h}r\right)^2 \Delta x \qquad \text{which becomes} \int_0^h \pi\left(\frac{x}{h}r\right)^2 \mathrm{d}x$$

So knowing that we will replace

$$\sum_{i=1}^{n} \longrightarrow \int_0^h$$
$$x_i^* \longrightarrow x$$
$$\Delta x \longrightarrow \mathrm{d}x$$

when we take the limit, we have just skipped the intermediate steps. While this is not entirely rigorous, it can be made so, and does save us a lot of algebra.

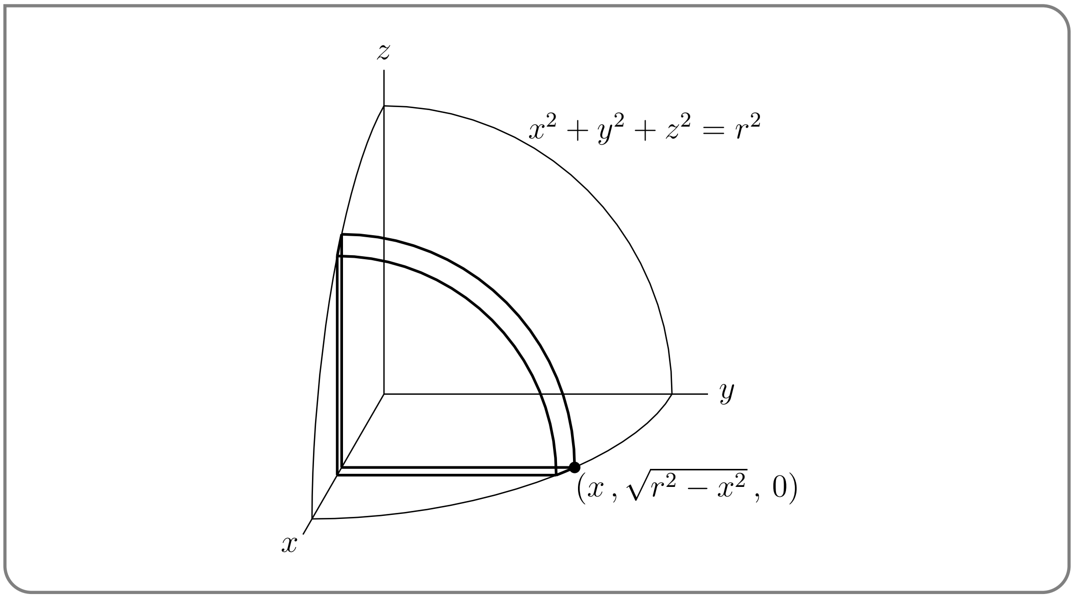
Example 1.10.1

Example 1.10.2 (Sphere)

Find the volume of the sphere of radius r.

Solution. We'll find the volume of the part of the sphere in the first octant[58], sketched below. Then we'll multiply by 8.

- To compute the volume, we slice it up into thin vertical "pancakes" (just as we did in the previous example).



- Each pancake is one quarter of a thin circular disk. The pancake a distance x from the yz–plane is shown in the sketch above. The radius of that pancake is the distance from the dot shown in the figure to the x–axis, i.e. the y–coordinate of the dot. To get the coordinates of the dot, observe that
  - it lies the xy–plane, and so has z–coordinate zero, and that

---

[58] The first octant is the set of all points (x, y, z) with $x \geqslant 0$, $y \geqslant 0$ and $z \geqslant 0$.

  - it also lies on the sphere, so that its coordinates obey $x^2 + y^2 + z^2 = r^2$. Since $z = 0$ and $y > 0$, $y = \sqrt{r^2 - x^2}$.
- So the pancake at distance x from the yz–plane has
  - thickness[59] $\mathrm{d}x$ and
  - radius $\sqrt{r^2 - x^2}$
  - cross–sectional area $\frac{1}{4}\pi\left(\sqrt{r^2 - x^2}\right)^2$ and hence
  - volume $\frac{\pi}{4}\left(r^2 - x^2\right)\mathrm{d}x$
- As x runs from 0 to r, the total volume of the part of the sphere in the first octant is

$$\int_0^r \frac{\pi}{4}\left(r^2 - x^2\right)\mathrm{d}x = \frac{\pi}{4}\left[r^2 x - \frac{x^3}{3}\right]_0^r = \frac{1}{6}\pi r^3$$

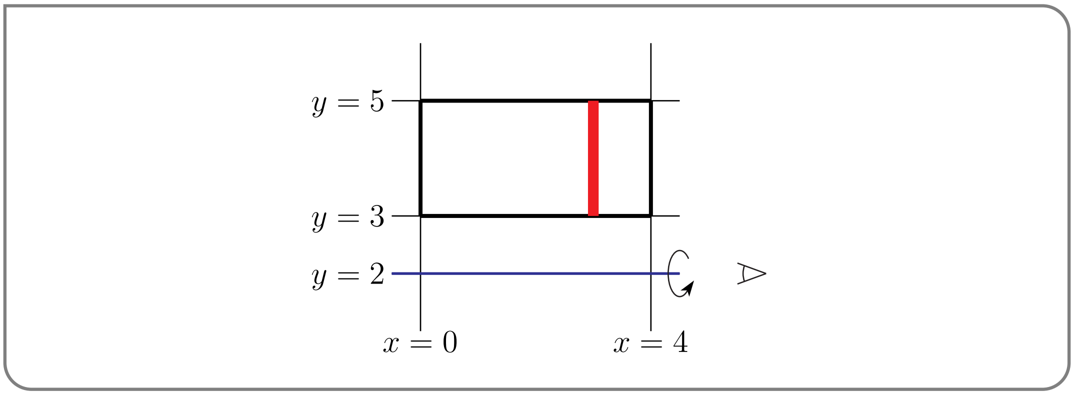
  and the total volume of the whole sphere is eight times that, which is $\frac{4}{3}\pi r^3$, as expected.

Example 1.10.2

Example 1.10.3 (Revolving a region)

The region between the lines $y = 3$, $y = 5$, $x = 0$ and $x = 4$ is rotated around the line $y = 2$. Find the volume of the region swept out.

Solution. As with most of these problems, we should start by sketching the problem.

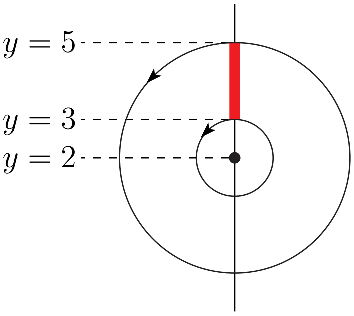


- Consider the region and slice it into thin vertical strips of width $\mathrm{d}x$.
- Now we are to rotate this region about the line $y = 2$. Imagine looking straight down the axis of rotation, $y = 2$, end on. The symbol in the figure above just to the right of the end the line $y = 2$ is supposed to represent your eye[60]. Here is what you see as the rotation takes place.

---

[59] Yet again what we really do is pick a natural number n, slice the octant of the sphere into n pancakes each of thickness $\Delta x = \frac{r}{n}$ and then take the limit $n \to \infty$. In the integral Δx is replaced by $\mathrm{d}x$. Knowing that this is what is going to happen, we again just skip a few steps.

[60] Okay okay…We missed the pupil. I'm sure there is a pun in there somewhere.



- Upon rotation about the line $y = 2$ our strip sweeps out a "washer"
  - whose cross–section is a disk of radius $5 - 2 = 3$ from which a disk of radius $3 - 2 = 1$ has been removed so that it has a
  - cross–sectional area of $\pi 3^2 - \pi 1^2 = 8\pi$ and a
  - thickness $\mathrm{d}x$ and hence a
  - volume $8\pi\,\mathrm{d}x$.
- As our leftmost strip is at $x = 0$ and our rightmost strip is at $x = 4$, the total

$$\text{Volume} = \int_0^4 8\pi\,\mathrm{d}x = (8\pi)(4) = 32\pi$$

Notice that we could also reach this answer by writing the volume as the difference of two cylinders.

- The outer cylinder has radius $(5 - 2)$ and length 4. This has volume

$$V_{outer} = \pi r^2 \ell = \pi \cdot 3^2 \cdot 4 = 36\pi.$$

- The inner cylinder has radius $(3 - 2)$ and length 4. This has volume

$$V_{inner} = \pi r^2 \ell = \pi \cdot 1^2 \cdot 4 = 4\pi.$$

- The volume we want is the difference of these two, namely

$$V = V_{outer} - V_{inner} = 32\pi.$$

Example 1.10.3

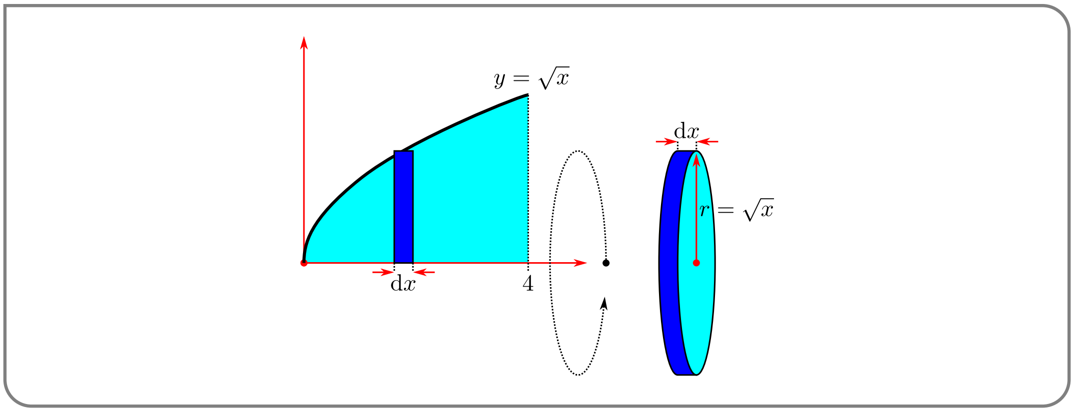
Let us turn up the difficulty a little on this last example.

Example 1.10.4 (Revolving again)

The region between the curve $y = \sqrt{x}$, and the lines $y = 0$, $x = 0$ and $x = 4$ is rotated around the line $y = 0$. Find the volume of the region swept out.

Solution. We can approach this in much the same way as the previous example.

- Consider the region and cut it into thin vertical strips of width dx.



- When we rotate the region about the line $y = 0$, each strip sweeps out a thin pancake
  - whose cross-section is a disk of radius $\sqrt{x}$ with a
  - cross-sectional area of $\pi(\sqrt{x})^2 = \pi x$ and a
  - thickness dx and hence a
  - volume $\pi x\,\mathrm{d}x$.
- As our leftmost strip is at $x = 0$ and our rightmost strip is at $x = 4$, the total

$$\text{Volume} = \int_0^4 \pi x\,\mathrm{d}x = \left[\frac{\pi}{2}x^2\right]_0^4 = 8\pi$$

Example 1.10.4

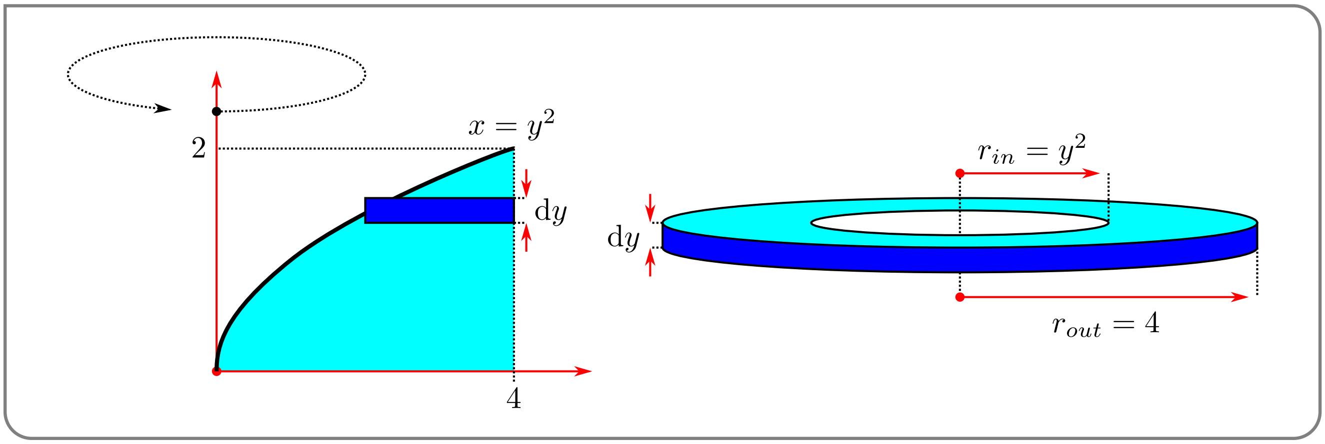
In the last example we considered rotating a region around the x-axis. Let us do the same but rotating around the y-axis.

Example 1.10.5 (Revolving yet again)

The region between the curve $y = \sqrt{x}$, and the lines $y = 0$, $x = 0$ and $x = 4$ is rotated around the line $x = 0$. Find the volume of the region swept out.

Solution.

- We will cut the region into horizontal slices, so we should write x as a function of y. That is, the region is bounded by $x = y^2$, $x = 4$, $y = 0$ and $y = 2$.
- Now slice the region into thin horizontal strips of width dy.



- When we rotate the region about y-axis, each strip sweeps out a thin washer
  - whose inner radius is y^2 and outer radius is 4, and
  - thickness is dy and hence
  - has volume $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi(16 - y^4)\mathrm{d}y$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 2$, the total

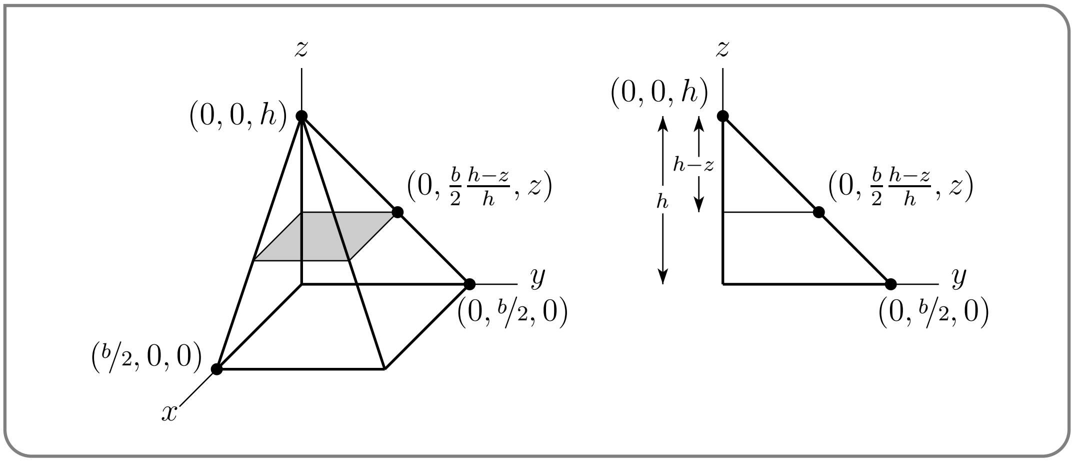
$$\text{Volume} = \int_0^2 \pi(16 - y^4)\mathrm{d}y = \left[16\pi y - \frac{\pi}{5}y^5\right]_0^2 = 32\pi - \frac{32\pi}{5} = \frac{128\pi}{5}.$$

Example 1.10.5

Example 1.10.6 (Pyramid)

Find the volume of the pyramid which has height h and whose base is a square of side b.

Solution. Here is a sketch of the part of the pyramid that is in the first octant; we display only this portion to make the diagrams simpler. Note that this diagram shows only 1



quarter of the whole pyramid.

- To compute its volume, we slice it up into thin horizontal "square pancakes". A typical pancake also appears in the sketch above.
  - The pancake at height z is the fraction $\frac{h-z}{h}$ of the distance from the peak of the pyramid to its base.
  - So the *full* pancake[61] at height z is a square of side $\frac{h-z}{h}b$. As a check, note that when $z = h$ the pancake has side $\frac{h-h}{h}b = 0$, and when $z = 0$ the pancake has side $\frac{h-0}{h}b = b$.
  - So the pancake has cross-sectional area $\left(\frac{h-z}{h}b\right)^2$ and thickness[62] $\mathrm{d}z$ and hence
  - volume $\left(\frac{h-z}{h}b\right)^2\mathrm{d}z$.
- The volume of the whole pyramid (not just the part of the pyramid in the first octant) is

$$\begin{aligned}
\int_0^h \left(\frac{h-z}{h}b\right)^2\mathrm{d}z &= \frac{b^2}{h^2}\int_0^h (h-z)^2\mathrm{d}z \\
&= \frac{b^2}{h^2}\int_h^0 -t^2\mathrm{d}t && \text{substitution rule with } t=(h-z), \mathrm{d}z \to -\mathrm{d}t \\
&= -\frac{b^2}{h^2}\left[\frac{t^3}{3}\right]_h^0 \\
&= -\frac{b^2}{h^2}\left[-\frac{h^3}{3}\right] \\
&= \frac{1}{3}b^2h
\end{aligned}$$

Example 1.10.6

Let's ramp up the difficulty a little.

Example 1.10.7 (Napkin Ring)

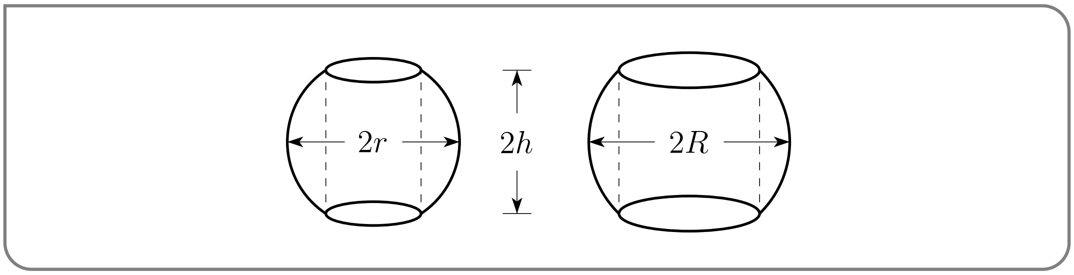
Suppose you make two napkin rings[63] by drilling holes with different diameters through two wooden balls. One ball has radius r and the other radius R with $r < R$. You choose the diameter of the holes so that both napkin rings have the same height, $2h$. See the figure below.

---

[61] Note that this is the full pancake, not just the part in the first octant.

[62] We are again using our Riemann sum avoiding trick.

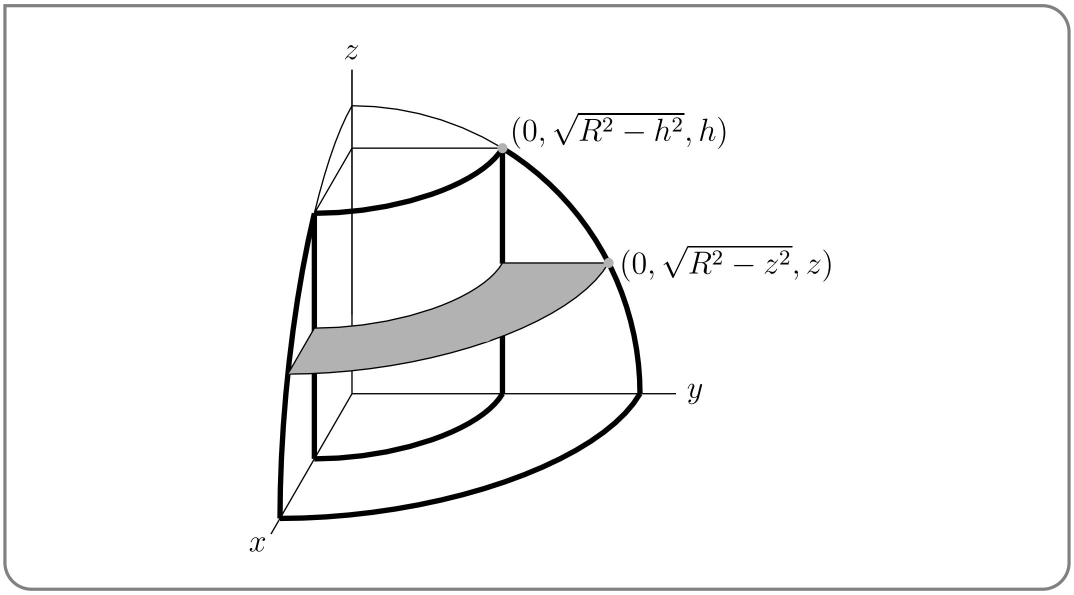
[63] Handy things to have (when combined with cloth napkins) if your parents are coming to dinner and you want to convince them that you are "taking care of yourself".



Which[64] ring has more wood in it?

Solution. We'll compute the volume of the napkin ring with radius R. We can then obtain the volume of the napkin ring of radius r, by just replacing $R \mapsto r$ in the result.

- To compute the volume of the napkin ring of radius R, we slice it up into thin horizontal "pancakes". Here is a sketch of the part of the napkin ring in the first octant showing a typical pancake.



- The coordinates of the two points marked in the yz–plane of that figure are found by remembering that
  - the equation of the sphere is $x^2 + y^2 + z^2 = R^2$.
  - The two points have $y > 0$ and are in the yz–plane, so that $x = 0$ for them. So $y = \sqrt{R^2 - z^2}$.
  - In particular, at the top of the napkin ring $z = h$, so that $y = \sqrt{R^2 - h^2}$.
- The pancake at height z, shown in the sketch, is a "washer" — a circular disk with a circular hole cut in its center.
  - The outer radius of the washer is $\sqrt{R^2 - z^2}$ and
  - the inner radius of the washer is $\sqrt{R^2 - h^2}$. So the

---

[64] A good question to ask to distract your parents from the fact you are serving frozen burritos.

  - cross–sectional area of the washer is

$$\pi\big(\sqrt{R^2-z^2}\,\big)^2-\pi\big(\sqrt{R^2-h^2}\,\big)^2=\pi(h^2-z^2)$$

- The pancake at height z
  - has thickness dz and
  - cross–sectional area $\pi(h^2-z^2)$ and hence
  - volume $\pi(h^2-z^2)\mathrm{d}z$.
- Since z runs from $-h$ to $+h$, the total volume of wood in the napkin ring of radius R is

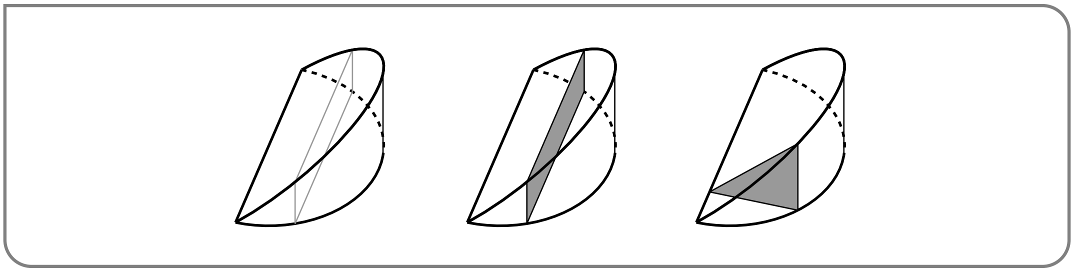
$$\begin{aligned}\int_{-h}^{h}\pi(h^2-z^2)\mathrm{d}z &= \pi\Big[h^2z-\frac{z^3}{3}\Big]_{-h}^{h}\\ &= \pi\Big[\Big(h^3-\frac{h^3}{3}\Big)-\Big((-h)^3-\frac{(-h)^3}{3}\Big)\Big]\\ &= \pi\Big[\frac{2}{3}h^3-\frac{2}{3}(-h)^3\Big]\\ &= \frac{4\pi}{3}h^3\end{aligned}$$

This volume is independent of R. Hence the napkin ring of radius r contains precisely the same volume of wood as the napkin ring of radius R!

Example 1.10.7

Example 1.10.8 (Notch)

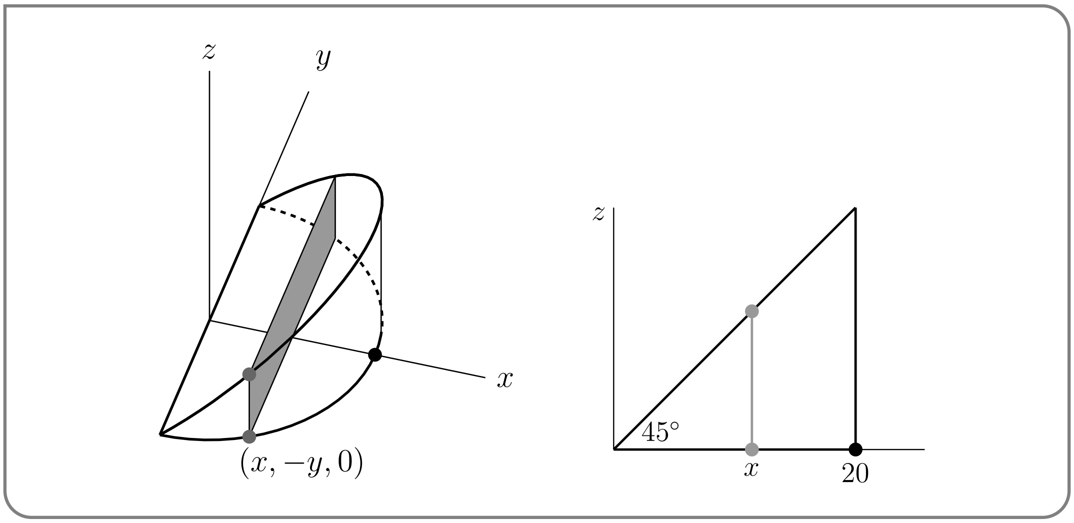
A 45° notch is cut to the centre of a cylindrical log having radius 20cm. One plane face of the notch is perpendicular to the axis of the log. See the sketch below. What volume of wood was removed?



Solution. We show two solutions to this problem which are of comparable difficulty. The difference lies in the shape of the pancakes we use to slice up the volume. In solution 1 we cut rectangular pancakes parallel to the yz–plane and in solution 2 we slice triangular pancakes parallel to the xz–plane.

Solution 1:

- Concentrate on the notch. Rotate it around so that the plane face lies in the xy–plane.
- Then slice the notch into vertical rectangles (parallel to the yz–plane) as in the figure on the left below.



- The cylindrical log had radius 20cm. So the circular part of the boundary of the base of the notch has equation $x^2 + y^2 = 20^2$. (We're putting the origin of the xy–plane at the centre of the circle.) If our coordinate system is such that x is constant on each slice, then
  - the base of the slice is the line segment from $(x, -y, 0)$ to $(x, +y, 0)$ where $y = \sqrt{20^2 - x^2}$ so that
  - the slice has width $2y = 2\sqrt{20^2 - x^2}$ and
  - height x (since the upper face of the notch is at 45° to the base — see the side view sketched in the figure on the right above).
  - So the slice has cross–sectional area $2x\sqrt{20^2 - x^2}$.
- On the base of the notch x runs from 0 to 20 so the volume of the notch is

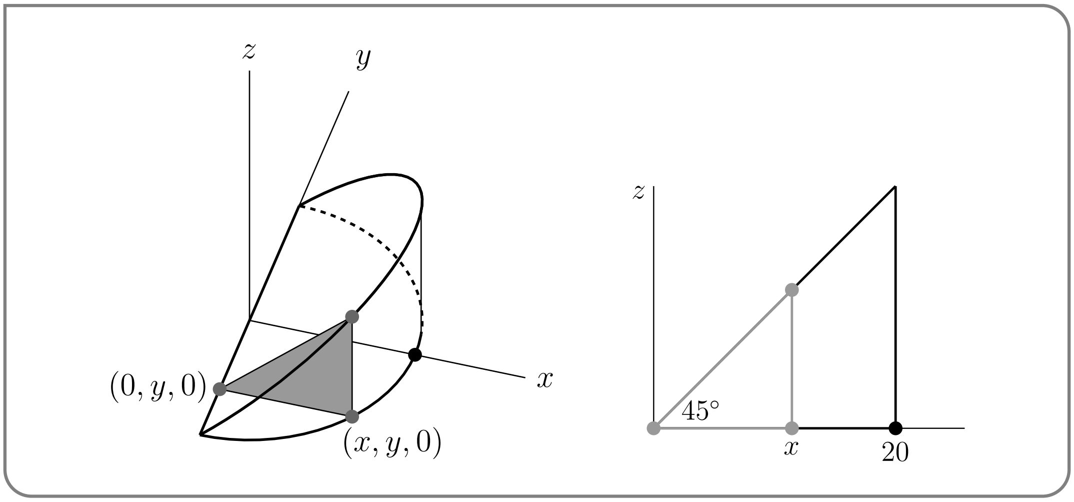
$$V = \int_0^{20} 2x\sqrt{20^2 - x^2}\mathrm{d}x$$

Make the change of variables $u = 20^2 - x^2$ (don't forget to change $\mathrm{d}x \to -\frac{1}{2x}\mathrm{d}u$):

$$\begin{aligned} V &= \int_{20^2}^{0} -\sqrt{u}\,\mathrm{d}u \\ &= \left[-\frac{u^{3/2}}{3/2}\right]_{20^2}^{0} \\ &= \frac{2}{3}20^3 = \frac{16,000}{3} \end{aligned}$$

Solution 2:

- Concentrate of the notch. Rotate it around so that its base lies in the xy–plane with the skinny edge along the y–axis.
- Slice the notch into triangles parallel to the xz–plane as in the figure on the left below. In the figure below, the triangle happens to lie in a plane where y is negative.



- The cylindrical log had radius 20cm. So the circular part of the boundary of the base of the notch has equation $x^2 + y^2 = 20^2$. Our coordinate system is such that y is constant on each slice, so that
  - the base of the triangle is the line segment from $(0, y, 0)$ to $(x, y, 0)$ where $x = \sqrt{20^2 - y^2}$ so that
  - the triangle has base $x = \sqrt{20^2 - y^2}$ and
  - height $x = \sqrt{20^2 - y^2}$ (since the upper face of the notch is at 45° to the base — see the side view sketched in the figure on the right above).
  - So the slice has cross–sectional area $\frac{1}{2}\left(\sqrt{20^2 - y^2}\right)^2$.
- On the base of the notch y runs from -20 to 20, so the volume of the notch is

$$\begin{aligned} V &= \frac{1}{2}\int_{-20}^{20} (20^2 - y^2)\mathrm{d}y \\ &= \int_0^{20} (20^2 - y^2)\mathrm{d}y \\ &= \left[20^2 y - \frac{y^3}{3}\right]_0^{20} \\ &= \frac{2}{3}20^3 = \frac{16,000}{3} \end{aligned}$$

Example 1.10.8

1.11▲ Integration by parts

Learning Objectives

- Explain how the product rule for derivatives corresponds to integration by parts for integrals.
- Use integration by parts to compute definite and indefinite integrals.
- Identify when integration by parts is an appropriate method to use.
- While performing integration by parts, identify which portion of the integral should be "u" and which part should be "$\mathrm{d}v$." This includes the case where $\mathrm{d}x = \mathrm{d}v$.
- Given an integral, identify which technique(s) from this course can be used to compute the integral.
- Use the integration techniques from this course flexibly and compute integrals that require more than one technique.

The fundamental theorem of calculus tells us that it is very easy to integrate a derivative. In particular, we know that

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)\right)\mathrm{d}x = F(x) + C$$

We can exploit this in order to develop another rule for integration — in particular a rule to help us integrate products of simpler function such as

$$\int xe^x \, \mathrm{d}x$$

In so doing we will arrive at a method called "integration by parts".

To do this we start with the product rule and integrate. Recall that the product rule says

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x)v(x) = u'(x)\,v(x) + u(x)\,v'(x)$$

Integrating this gives

$$\begin{aligned}\int \left[u'(x)\,v(x) + u(x)\,v'(x)\right]\mathrm{d}x &= \left[\text{a function whose derivative is } u'v + uv'\right] + C \\ &= u(x)v(x) + C\end{aligned}$$

Now this, by itself, is not terribly useful. In order to apply it we need to have a function whose integrand is a sum of products that is in exactly this form $u'(x)v(x) + u(x)v'(x)$. This is far too specialised.

However if we tease this apart a little:

$$\int \left[u'(x)\, v(x) + u(x)\, v'(x)\right] \mathrm{d}x = \int u'(x)\, v(x)\, \mathrm{d}x + \int u(x)\, v'(x)\, \mathrm{d}x$$

Bring one of the integrals to the left-hand side

$$u(x)v(x) - \int u'(x)\, v(x) \mathrm{d}x = \int u(x)\, v'(x)\, \mathrm{d}x$$

Swap left and right sides

$$\int u(x)\, v'(x)\, \mathrm{d}x = u(x)v(x) - \int u'(x)\, v(x) \mathrm{d}x$$

In this form we take the integral of one product and express it in terms of the integral of a different product. If we express it like that, it doesn't seem too useful. However, if the second integral is easier, then this process helps us.

Let us do a simple example before explaining this more generally.

Example 1.11.1 ($\int xe^x \,\mathrm{d}x$)

Compute the integral $\displaystyle\int xe^x \,\mathrm{d}x$.

Solution.

- We start by taking the equation above

$$\int u(x)\, v'(x)\, \mathrm{d}x = u(x)v(x) - \int u'(x)\, v(x) \mathrm{d}x$$

- Now set $u(x) = x$ and $v'(x) = e^x$. How did we know how to make this choice? We will explain some strategies later. For now, let us just accept this choice and keep going.
- In order to use the formula we need to know $u'(x)$ and $v(x)$. In this case it is quite straightforward: $u'(x) = 1$ and $v(x) = e^x$.
- Plug everything into the formula:

$$\int xe^x \,\mathrm{d}x = xe^x - \int e^x \,\mathrm{d}x$$

  So our original more difficult integral has been turned into a question of computing an easy one.

$$= xe^x - e^x + C$$

- We can check our answer by differentiating:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(xe^x - e^x + C\right) &= \underbrace{xe^x + 1 \cdot e^x}_{\text{by product rule}} - e^x + 0 \\ &= xe^x & \text{as required.} \end{aligned}$$

Example 1.11.1

The process we have used in the above example is called “integration by parts”. When our integrand is a product we try to write it as $u(x)v'(x)$ — we need to choose one factor to be $u(x)$ and the other to be $v'(x)$. We then compute $u'(x)$ and $v(x)$ and then apply the following theorem:

Theorem 1.11.2 (Integration by parts)**.**

Let $u(x)$ and $v(x)$ be continuously differentiable. Then

$$\int u(x)\,v'(x)\,\mathrm{d}x = u(x)\,v(x) - \int v(x)\,u'(x)\,\mathrm{d}x$$

If we write $\mathrm{d}v$ for $v'(x)\,\mathrm{d}x$ and $\mathrm{d}u$ for $u'(x)\,\mathrm{d}x$ (as the substitution rule suggests), then the formula becomes

$$\int u\mathrm{d}v = u\,v - \int v\mathrm{d}u$$

The application of this formula is known as integration by parts.
The corresponding statement for definite integrals is

$$\int_a^b u(x)\,v'(x)\,\mathrm{d}x = u(b)\,v(b) - u(a)\,v(a) - \int_a^b v(x)\,u'(x)\,\mathrm{d}x$$

Integration by parts is not as easy to apply as the product rule for derivatives. This is because it relies on us

(1) judiciously choosing $u(x)$ and $v'(x)$, then

(2) computing $u'(x)$ and $v(x)$ — which requires us to antidifferentiate $v'(x)$, and finally

(3) that the integral $\int u'(x)v(x)\,\mathrm{d}x$ is easier than the integral we started with.

Notice that any antiderivative of $v'(x)$ will do. All antiderivatives of $v'(x)$ are of the form $v(x) + A$ with A a constant. Putting this into the integration by parts formula gives

$$\begin{aligned}\int u(x)v'(x)\,\mathrm{d}x &= u(x)\,(v(x)+A) - \int u'(x)\,(v(x)+A)\,\mathrm{d}x\\ &= u(x)v(x) + Au(x) - \int u'(x)v(x)\mathrm{d}x - \underbrace{A\int u'(x)\mathrm{d}x}_{=Au(x)+C}\\ &= u(x)v(x) - \int u'(x)v(x)\,\mathrm{d}x + C\end{aligned}$$

So, that constant A will always cancel out. (That’s why we get to ignore it in this context.)

In most applications (but not all) our integrand will be a product of two factors so we have two choices for $u(x)$ and $v'(x)$. Typically one of these choices will be “good” (in that it results in a simpler integral) while the other will be "bad" (we cannot antidifferentiate our choice of $v'(x)$ or the resulting integral is harder). Let us illustrate what we mean by returning to our previous example.

Example 1.11.3 ($\int xe^x \, \mathrm{d}x$ — again)

Our integrand is the product of two factors

$$x \qquad \text{and} \qquad e^x$$

This gives us two obvious choices of u and v':

$$\begin{aligned} u(x) &= x & v'(x) &= e^x \\ &\text{or} && \\ u(x) &= e^x & v'(x) &= x \end{aligned}$$

We should explore both choices:

1. If take $u(x) = x$ and $v'(x) = e^x$. We then quickly compute

   $$u'(x) = 1 \qquad \text{and} \qquad v(x) = e^x$$

   which means we will need to integrate (in the right-hand side of the integration by parts formula)

   $$\int u'(x)v(x)\mathrm{d}x = \int 1 \cdot e^x \, \mathrm{d}x$$

   which looks straightforward. This is a good indication that this is the right choice of $u(x)$ and $v'(x)$.

2. But before we do that, we should also explore the other choice, namely $u(x) = e^x$ and $v'(x) = x$. This implies that

   $$u'(x) = e^x \qquad \text{and} \qquad v(x) = \frac{1}{2}x^2$$

   which means we need to integrate

   $$\int u'(x)v(x)\mathrm{d}x = \int \frac{1}{2}x^2 \cdot e^x \, \mathrm{d}x.$$

   This is at least as hard as the integral we started with. Hence we should try the first choice.

With our choice made, we integrate by parts to get

$$\begin{aligned} \int xe^x \, \mathrm{d}x &= xe^x - \int e^x \, \mathrm{d}x \\ &= xe^x - e^x + C. \end{aligned}$$

The above reasoning is a very typical workflow when using integration by parts.

Example 1.11.3

Integration by parts is often used

- to eliminate factors of x from an integrand like xe^x by using that $\frac{d}{dx}x = 1$ and
- to eliminate a $\log x$ from an integrand by using that $\frac{d}{dx}\log x = \frac{1}{x}$ and
- to eliminate inverse trig functions, like $\arctan x$, from an integrand by using that, for example, $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$.

Example 1.11.4 ($\int x \sin x \, dx$)

Solution.

- Again we have a product of two factors giving us two possible choices.
  (1) If we choose $u(x) = x$ and $v'(x) = \sin x$, then we get
  $$u'(x) = 1 \qquad \text{and} \qquad v(x) = -\cos x$$
  which is looking promising.
  (2) On the other hand if we choose $u(x) = \sin x$ and $v'(x) = x$, then we have
  $$u'(x) = \cos x \qquad \text{and} \qquad v(x) = \frac{1}{2}x^2$$
  which is looking worse — we'd need to integrate $\int \frac{1}{2}x^2 \cos x \, dx$.
- So we stick with the first choice. Plugging $u(x) = x$, $v(x) = -\cos x$ into integration by parts gives us
$$\begin{aligned}\int x \sin x \, dx &= -x \cos x - \int 1 \cdot (-\cos x) dx \\ &= -x\cos x + \sin x + C\end{aligned}$$
- Again we can check our answer by differentiating:
$$\begin{aligned}\frac{d}{dx}\left(-x\cos x + \sin x + C\right) &= -\cos x + x \sin x + \cos x + 0 \\ &= x \sin x \checkmark\end{aligned}$$

Once we have practised this a bit we do not really need to write as much. Let us solve it again, but showing only what we need to.

Solution.

- We use integration by parts to solve the integral.
- Set $u(x) = x$ and $v'(x) = \sin x$. Then $u'(x) = 1$ and $v(x) = -\cos x$, and
$$\begin{aligned}\int x \sin x \, dx &= -x\cos x + \int \cos x \, dx \\ &= -x\cos x + \sin x + C.\end{aligned}$$

Example 1.11.4

It is pretty standard practice to reduce the notation even further in these problems. As noted above, many people write the integration by parts formula as

$$\int u \mathrm{d}v = uv - \int v \mathrm{d}u$$

where $\mathrm{d}u, \mathrm{d}v$ are shorthand for $u'(x)\,\mathrm{d}x, v'(x)\,\mathrm{d}x$. Let us write up the previous example using this notation.

Example 1.11.5 ($\int x \sin x \,\mathrm{d}x$ yet again)

Solution. Using integration by parts, we set $u = x$ and $\mathrm{d}v = \sin x \,\mathrm{d}x$. This makes $\mathrm{d}u = 1\mathrm{d}x$ and $v = -\cos x$. Consequently

$$\begin{aligned}
\int x \sin x \,\mathrm{d}x &= \int u \mathrm{d}v \\
&= uv - \int v \mathrm{d}u \\
&= -x \cos x + \int \cos x \,\mathrm{d}x \\
&= -x \cos x + \sin x + C
\end{aligned}$$

You can see that this is a very neat way to write up these problems and we will continue using this shorthand in the examples that follow below.

Example 1.11.5

We can also use integration by parts to eliminate higher powers of x. We just need to apply the method more than once.

Example 1.11.6 ($\int x^2 e^x \,\mathrm{d}x$)

Solution.

- Let $u = x^2$ and $\mathrm{d}v = e^x \,\mathrm{d}x$. This then gives $\mathrm{d}u = 2x \,\mathrm{d}x$ and $v = e^x$, and

$$\int x^2 e^x \,\mathrm{d}x = x^2 e^x - \int 2x e^x \,\mathrm{d}x$$

- So we have reduced the problem of computing the original integral to one of integrating $2xe^x$. We know how to do this — just integrate by parts again:

$$\begin{aligned}
\int x^2 e^x \,\mathrm{d}x &= x^2 e^x - \int 2x e^x \,\mathrm{d}x && \text{set } u = 2x, \mathrm{d}v = e^x \,\mathrm{d}x \\
&= x^2 e^x - \left(2x e^x - \int 2e^x \,\mathrm{d}x \right) && \text{since } \mathrm{d}u = 2\mathrm{d}x, v = e^x \\
&= x^2 e^x - 2x e^x + 2e^x + C
\end{aligned}$$

- We can, if needed, check our answer by differentiating:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2e^x - 2xe^x + 2e^x + C\right) &= \left(x^2e^x + 2xe^x\right) - (2xe^x + 2e^x) + 2e^x + 0 \\ &= x^2e^x \checkmark\end{aligned}$$

A similar iterated application of integration by parts will work for integrals

$$\int P(x)\left(Ae^{ax} + B\sin(bx) + C\cos(cx)\right)\mathrm{d}x$$

where $P(x)$ is a polynomial and A, B, C, a, b, c are constants.

Example 1.11.6

Now let us look at integrands containing logarithms. We don't know the antiderivative of $\log x$, but we can eliminate $\log x$ from an integrand by using integration by parts with $u = \log x$. Remember $\log x = \log_e x = \ln x$.

Example 1.11.7 ($\int x \log x \,\mathrm{d}x$)

Solution.

- We have two choices for u and $\mathrm{d}v$.
  1. Set $u = x$ and $\mathrm{d}v = \log x \,\mathrm{d}x$. This gives $\mathrm{d}u = \mathrm{d}x$ but v is hard to compute — we haven't done it yet[65]. Before we go further along this path, we should look to see what happens with the other choice.
  2. Set $u = \log x$ and $\mathrm{d}v = x\,\mathrm{d}x$. This gives $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ and $v = \frac{1}{2}x^2$, and we have to integrate

$$\int v\,\mathrm{d}u = \int \frac{1}{x}\cdot\frac{1}{2}x^2\mathrm{d}x$$

     which is easy.

- So we proceed with the second choice.

$$\begin{aligned}\int x\log x\,\mathrm{d}x &= \frac{1}{2}x^2\log x - \int \frac{1}{2}x\,\mathrm{d}x \\ &= \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C\end{aligned}$$

- We can check our answer quickly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4} + C\right) = x\,\ln x + \frac{x^2}{2}\frac{1}{x} - \frac{x}{2} + 0 = x\,\ln x$$

---

[65] We will soon.

Example 1.11.7

Example 1.11.8 ($\int \log x \, \mathrm{d}x$)

It is not immediately obvious that one should use integration by parts to compute the integral

$$\int \log x \, \mathrm{d}x$$

since the integrand is not a product. But we should persevere — indeed this is a situation where our shorter notation helps to clarify how to proceed.

Solution.

- In the previous example we saw that we could remove the factor $\log x$ by setting $u = \log x$ and using integration by parts. Let us try repeating this. When we make this choice, we are then forced to take $\mathrm{d}v = \mathrm{d}x$ — that is we choose $v'(x) = 1$. Once we have made this sneaky move everything follows quite directly.
- We then have $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ and $v = x$, and the integration by parts formula gives us

$$\begin{aligned} \int \log x \, \mathrm{d}x &= x \log x - \int \frac{1}{x} \cdot x \, \mathrm{d}x \\ &= x \log x - \int 1 \mathrm{d}x \\ &= x \log x - x + C \end{aligned}$$

- As always, it is a good idea to check our result by verifying that the derivative of the answer really is the integrand.

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x \ln x - x + C\big) = \ln x + x\,\frac{1}{x} - 1 + 0 = \ln x$$

Example 1.11.8

The same method works almost exactly to compute the antiderivatives of $\arcsin(x)$ and $\arctan(x)$:

Example 1.11.9 ($\int \arctan(x)\, \mathrm{d}x$ and $\int \arcsin(x)\, \mathrm{d}x$)

Compute the antiderivatives of the inverse sine and inverse tangent functions.

Solution.

- Again neither of these integrands are products, but that is no impediment. In both cases we set $\mathrm{d}v = \mathrm{d}x$ (ie $v'(x) = 1$) and choose $v(x) = x$.

- For inverse tan we choose $u = \arctan(x)$, so $\mathrm{d}u = \frac{1}{1+x^2}\mathrm{d}x$:

$$\begin{aligned}
\int \arctan(x)\mathrm{d}x &= x\arctan(x) - \int x \cdot \frac{1}{1+x^2}\mathrm{d}x && \text{now use substitution rule}\\
&= x\arctan(x) - \int \frac{w'(x)}{2} \cdot \frac{1}{w}\mathrm{d}x && \text{with } w(x) = 1+x^2, w'(x) = 2x\\
&= x\arctan(x) - \frac{1}{2}\int \frac{1}{w}\mathrm{d}w\\
&= x\arctan(x) - \frac{1}{2}\log|w| + C\\
&= x\arctan(x) - \frac{1}{2}\log|1+x^2| + C && \text{but } 1+x^2 > 0, \text{so}\\
&= x\arctan(x) - \frac{1}{2}\log(1+x^2) + C
\end{aligned}$$

- Similarly for inverse sine we choose $u = \arcsin(x)$ so $\mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$:

$$\begin{aligned}
\int \arcsin(x)\mathrm{d}x &= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x && \text{now use substitution rule}\\
&= x\arcsin(x) - \int \frac{-w'(x)}{2} \cdot w^{-1/2}\mathrm{d}x && \text{with } w(x) = 1-x^2, w'(x) = -2x\\
&= x\arcsin(x) + \frac{1}{2}\int w^{-1/2}\mathrm{d}w\\
&= x\arcsin(x) + \frac{1}{2} \cdot 2w^{1/2} + C\\
&= x\arcsin(x) + \sqrt{1-x^2} + C
\end{aligned}$$

- Both can be checked quite quickly by differentiating — but we leave that as an exercise for the reader.

Example 1.11.9

Example 1.11.10 ($\int \sec^3 x\,\mathrm{d}x$ — by trickery)

Let's revisit the antiderivative of secant cubed, now that we know about integration by parts.

Solution.

- Set $u = \sec x$, $\mathrm{d}v = \sec^2 x\,\mathrm{d}x$. Hence $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$, $v = \tan x$ and

$$\begin{aligned}
\int \sec^3 x\,\mathrm{d}x &= \int \underbrace{\sec x}_{u}\ \underbrace{\sec^2 x\,\mathrm{d}x}_{\mathrm{d}v}\\
&= \underbrace{\sec x}_{u}\ \underbrace{\tan x}_{v} - \int \underbrace{\tan x}_{v}\ \underbrace{\sec x \tan x\,\mathrm{d}x}_{\mathrm{d}u}
\end{aligned}$$

- Since $\tan^2 x + 1 = \sec^2 x$, we have $\tan^2 x = \sec^2 x - 1$ and

$$\begin{aligned}\int \sec^3 x \, dx &= \sec x \ \tan x - \int [\sec^3 x - \sec x] dx \\ &= \sec x \ \tan x + \log|\sec x + \tan x| + C - \int \sec^3 x \, dx\end{aligned}$$

  where we used $\int \sec x \, dx = \log|\sec x + \tan x| + C$, which we saw in Example 1.9.5.

- Now moving the $\int \sec^3 x \, dx$ from the right-hand side to the left-hand side

$$\begin{aligned}2\int \sec^3 x \, dx &= \sec x \tan x + \log|\sec x + \tan x| + C && \text{and so} \\ \int \sec^3 x \, dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\log|\sec x + \tan x| + C\end{aligned}$$

  for a new arbitrary constant C (which is just one half the old one).

Example 1.11.10

There are many other examples we could do, but we'll finish with a tricky one. The learning objectives don't tell you to memorize this precise method, but it's a nice example of a very clever solution.

Example 1.11.11 (Optional: $\int e^x \sin x \, dx$)

Solution. Let us attempt this one a little naively and then we'll come back and do it more carefully (and successfully).

- We can choose either $u = e^x, dv = \sin x \, dx$ or the other way around.

  1. Let $u = e^x, dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$. This gives

$$\int e^x \sin x = -e^x \cos x + \int e^x \cos x \, dx$$

     So we are left with an integrand that is very similar to the one we started with. What about the other choice?

  2. Let $u = \sin x, dv = e^x \, dx$. Then $du = \cos x \, dx$ and $v = e^x$. This gives

$$\int e^x \sin x = e^x \sin x - \int e^x \cos x \, dx$$

     So we are again left with an integrand that is very similar to the one we started with.

- How do we proceed? — It turns out to be easier if you do both $\int e^x \sin x \, dx$ and $\int e^x \cos x \, dx$ simultaneously. We do so in the next example.

Example 1.11.11

Example 1.11.12 $\left(\text{Optional: } \int_a^b e^x \sin x\,\mathrm{d}x \text{ and } \int_a^b e^x \cos x\,\mathrm{d}x\right)$

This time we're going to do the two integrals

$$I_1 = \int_a^b e^x \sin x\,\mathrm{d}x \qquad I_2 = \int_a^b e^x \cos x\,\mathrm{d}x$$

at more or less the same time.

- First

$$\begin{aligned} I_1 = \int_a^b e^x \sin x\,\mathrm{d}x &= \int_a^b u\mathrm{d}v && \text{with } u = e^x, \mathrm{d}v = \sin x\,\mathrm{d}x \\ & && \text{so } v = -\cos x, \mathrm{d}u = e^x\,\mathrm{d}x \\ &= \Big[-e^x \cos x\Big]_a^b + \int_a^b e^x \cos x\,\mathrm{d}x \end{aligned}$$

We have not found I_1 but we have related it to I_2.

$$I_1 = \Big[-e^x \cos x\Big]_a^b + I_2$$

- Now start over with I_2.

$$\begin{aligned} I_2 = \int_a^b e^x \cos x\,\mathrm{d}x &= \int_a^b u\mathrm{d}v && \text{with } u = e^x, \mathrm{d}v = \cos x\,\mathrm{d}x \\ & && \text{so } v = \sin x, \mathrm{d}u = e^x\,\mathrm{d}x \\ &= \Big[e^x \sin x\Big]_a^b - \int_a^b e^x \sin x\,\mathrm{d}x \end{aligned}$$

Once again, we have not found I_2 but we have related it back to I_1.

$$I_2 = \Big[e^x \sin x\Big]_a^b - I_1$$

- So summarising, we have

$$I_1 = \Big[-e^x \cos x\Big]_a^b + I_2 \qquad\qquad I_2 = \Big[e^x \sin x\Big]_a^b - I_1$$

- So now, substitute the expression for I_2 from the second equation into the first equation to get

$$I_1 = \Big[-e^x \cos x + e^x \sin x\Big]_a^b - I_1 \quad \text{which implies} \quad I_1 = \frac{1}{2}\Big[e^x\big(\sin x - \cos x\big)\Big]_a^b$$

If we substitute the other way around we get

$$I_2 = \Big[e^x \sin x + e^x \cos x\Big]_a^b - I_2 \qquad \text{which implies} \qquad I_2 = \frac{1}{2}\Big[e^x\big(\sin x + \cos x\big)\Big]_a^b$$

That is,

$$\int_a^b e^x \sin x \,\mathrm{d}x = \frac{1}{2}\Big[e^x\big(\sin x - \cos x\big)\Big]_a^b \qquad \int_a^b e^x \cos x \,\mathrm{d}x = \frac{1}{2}\Big[e^x\big(\sin x + \cos x\big)\Big]_a^b$$

- This also says, for example, that $\frac{1}{2}e^x\big(\sin x - \cos x\big)$ is an antiderivative of $e^x \sin x$ so that

$$\int e^x \sin x \,\mathrm{d}x = \frac{1}{2}e^x\big(\sin x - \cos x\big) + C$$

- Note that we can always check whether or not this is correct. It is correct if and only if the derivative of the right-hand side is $e^x \sin x$. Here goes. By the product rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{1}{2}e^x\big(\sin x - \cos x\big) + C\Big] = \frac{1}{2}\Big[e^x\big(\sin x - \cos x\big) + e^x\big(\cos x + \sin x\big)\Big] = e^x \sin x$$

which is the desired derivative.

Example 1.11.12

1.12▲ Numerical integration

Learning Objectives

- Explain why we need numerical methods for integration, citing examples of problems we cannot solve with the fundamental theorem of calculus.
- Approximate integrals using right Riemann sums.
- Approximate integrals using the trapezoidal method.
- Explain how to derive the formula for trapezoid rule (and for right Riemann sums, which is review).
- Use Simpson's rule to approximate integrals. You are not required to reproduce the derivation of the formula, but you should be able to explain why n must be an even number.
- Given an integral to compute numerically with either the trapezoidal method or Simpson's rule, compute the max error given a particular n.
  Note: If you need the error formula on an exam, it will be provided to you.
- When computing an integral numerically with either the trapezoidal method

or Simpson's rule, determine a sufficient number of intervals, n that guarantees a desired level of accuracy.

- Use numerical integration to compute approximations to definite integrals where the function is not defined explicitly or where the indefinite integral cannot be represented using standard functions.
- Implement the three numerical methods learned so far both by hand and using a spreadsheet.

By now the reader will have come to appreciate that integration is generally quite a bit more difficult than differentiation. There are a great many simple-looking integrals, such as $\int e^{-x^2}\mathrm{d}x$, that are either very difficult or even impossible to express in terms of standard functions[66]. Such integrals are not merely mathematical curiosities, but arise very naturally in many contexts. For example, the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t$$

is extremely important in many areas of mathematics, and also in many practical applications of statistics.

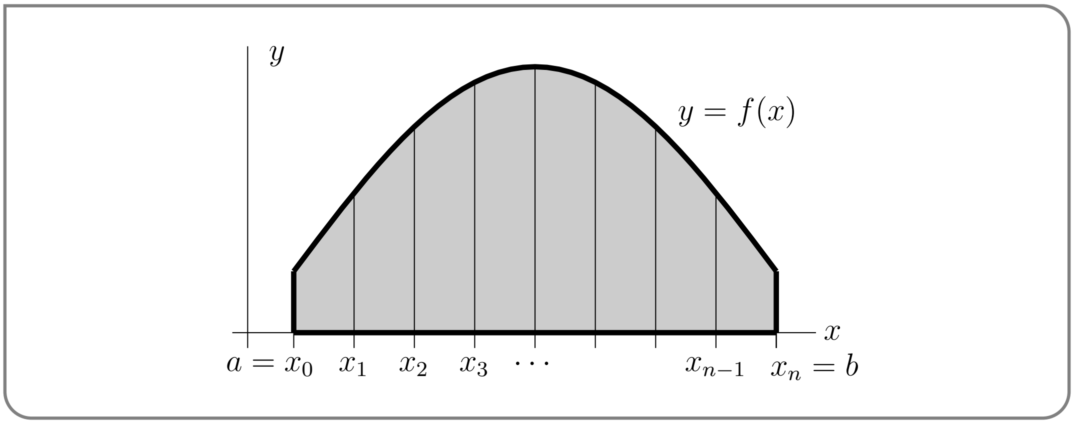
In such applications we need to be able to evaluate this integral (and many others) at a given numerical value of x. In this section we turn to the problem of how to find (approximate) numerical values for integrals, without having to evaluate them algebraically. To develop these methods we return to Riemann sums and our geometric interpretation of the definite integral as the signed area.

We start by describing (and applying) a few simple algorithms for generating, numerically, approximate values for the definite integral $\int_a^b f(x)\,\mathrm{d}x$. In each algorithm, we begin in much the same way as we approached Riemann sums.

- We first select an integer $n > 0$, called the "number of steps".
- We then divide the interval of integration, $a \leqslant x \leqslant b$, into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$. The first subinterval runs from $x_0 = a$ to $x_1 = a + \Delta x$. The second runs from x_1 to $x_2 = a + 2\Delta x$, and so on. The last runs from $x_{n-1} = b - \Delta x$ to $x_n = b$.

---

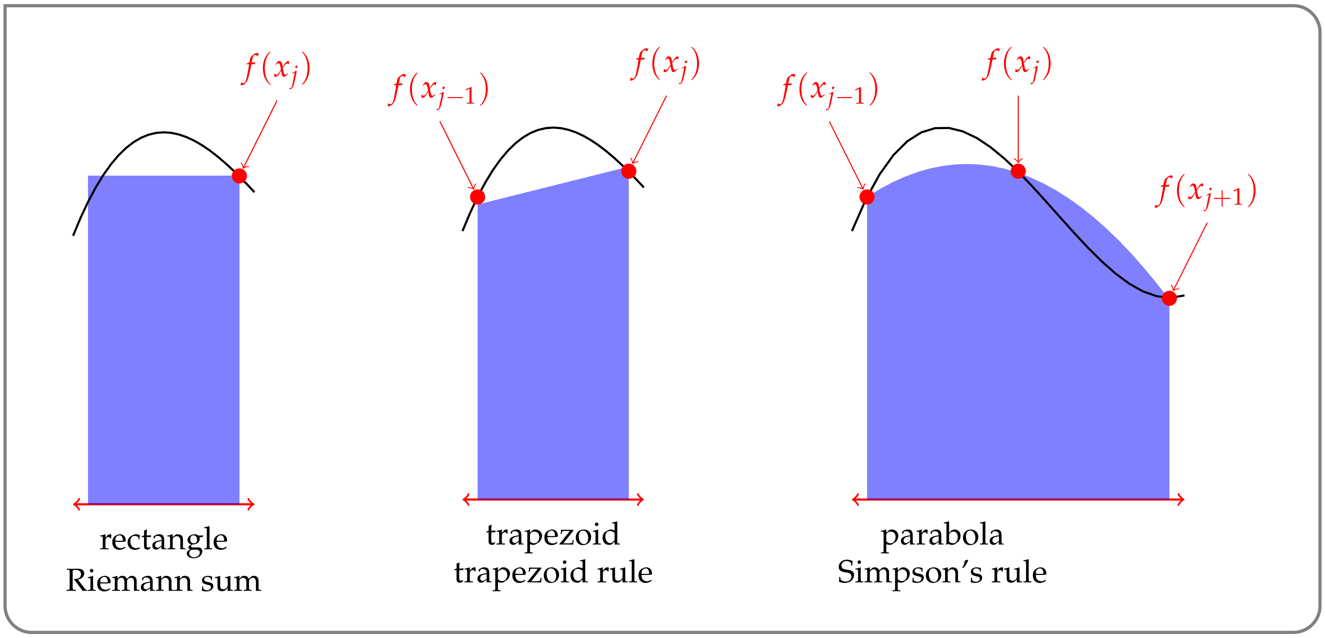
[66] We apologise for being a little sloppy here — but we just want to say that it can be very hard or even impossible to write some integrals as some finite sized expression involving polynomials, exponentials, logarithms and trigonometric functions. We don't want to get into a discussion of computability, though that is a very interesting topic.



This splits the original integral into n pieces:

$$\int_a^b f(x)\,\mathrm{d}x = \int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x$$

Each subintegral $\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x$ is approximated by the area of a simple geometric figure. The three algorithms we consider approximate the area by rectangles, trapezoids and parabolas (respectively).



We will explain these rules in detail below, but we give a brief overview here:

(1) A right Riemann sum approximates each subintegral by the area of a rectangle of height given by the value of the function at the right endpoint of the subinterval

$$\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x \approx f\left(x_j\right)\Delta x$$

This is illustrated in the leftmost figure above.

(2) The trapezoidal rule approximates each subintegral by the area of a trapezoid with vertices at $(x_{j-1}, 0), (x_{j-1}, f(x_{j-1})), (x_j, f(x_j)), (x_j, 0)$:

$$\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x \approx \frac{1}{2}\left[f(x_{j-1}) + f(x_j)\right]\Delta x$$

The trapezoid is illustrated in the middle figure above. We shall derive the formula for the area shortly.

(3) Simpson's rule approximates two adjacent subintegrals by the area under a parabola that passes through the points $(x_{j-1}, f(x_{j-1}))$, $(x_j, f(x_j))$ and $(x_{j+1}, f(x_{j+1}))$:

$$\int_{x_{j-1}}^{x_{j+1}} f(x)\,\mathrm{d}x \approx \frac{1}{3}\left[f(x_{j-1}) + 4f(x_j) + f(x_{j+1})\right]\Delta x$$

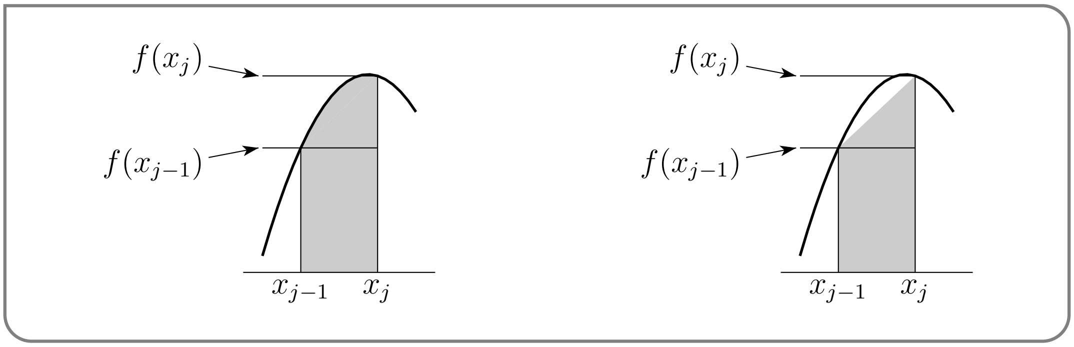
The parabola is illustrated in the right-hand figure above. We shall derive the formula for the area shortly.

Since you're already familiar with Riemann sums, we'll focus on the other two approximations.

1.12.1 ▸▸ The trapezoidal rule

Consider again the area represented by the integral $\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x$. The trapezoidal rule[67] (unsurprisingly) approximates this area by a trapezoid[68] whose vertices lie at

$$(x_{j-1}, 0), (x_{j-1}, f(x_{j-1})), (x_j, f(x_j)) \text{ and } (x_j, 0).$$



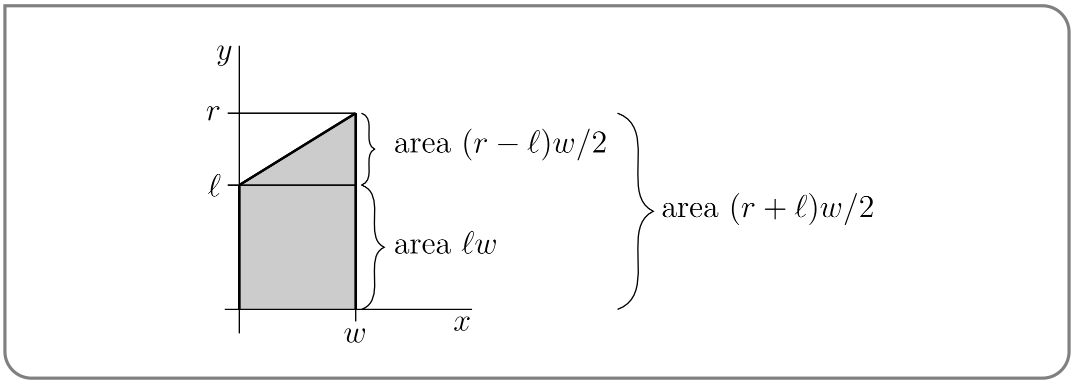
The trapezoidal approximation of the integral $\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x$ is the shaded region in the figure on the right above. It has width $x_j - x_{j-1} = \Delta x$. Its left-hand side has height $f(x_{j-1})$ and its right-hand side has height $f(x_j)$.

As the figure below shows, the area of a trapezoid is its width times its average height.

---

[67] This method is also called the "trapezoid rule" and "trapezium rule".

[68] A trapezoid is a four sided polygon, like a rectangle. But, unlike a rectangle, the top and bottom of a trapezoid need not be parallel.



So the trapezoidal rule approximates each subintegral by

$$\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x \approx \tfrac{f(x_{j-1})+f(x_j)}{2}\Delta x$$

Applying this approximation to each subinterval and then summing the result gives us the following approximation of the full integral

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \int_{x_0}^{x_1} f(x)\,\mathrm{d}x \;+ \int_{x_1}^{x_2} f(x)\,\mathrm{d}x \;+\cdots+ \int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x\\ &\approx \tfrac{f(x_0)+f(x_1)}{2}\Delta x + \tfrac{f(x_1)+f(x_2)}{2}\Delta x \;+\cdots+ \tfrac{f(x_{n-1})+f(x_n)}{2}\Delta x\\ &= \Big[\tfrac{1}{2}f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\tfrac{1}{2}f(x_n)\Big]\Delta x\end{aligned}$$

In summary:

Equation 1.12.1 (The trapezoidal rule).

The trapezoidal rule approximation is

$$\int_a^b f(x)\,\mathrm{d}x \approx \Big[\tfrac{1}{2}f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\tfrac{1}{2}f(x_n)\Big]\Delta x$$

where

$$\Delta x = \tfrac{b-a}{n},\quad x_0 = a,\quad x_1 = a+\Delta x,\quad x_2 = a+2\Delta x,\quad \cdots,\quad x_{n-1} = b-\Delta x,\quad x_n = b$$

Example 1.12.2 $\left(\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x\right)$

Solution. We'll use $n=8$ steps.

- We again have $f(x)=\frac{4}{1+x^2}$, $a=0$, $b=1$, $\Delta x=\frac{1}{8}$ and

$$x_0 = 0 \qquad x_1 = \tfrac{1}{8} \qquad x_2 = \tfrac{2}{8} \qquad \cdots \qquad x_7 = \tfrac{7}{8} \qquad x_8 = \tfrac{8}{8} = 1$$

- Applying the trapezoidal rule, Equation (1.12.1), gives

$$\begin{aligned}\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x &\approx \Bigg[\frac{1}{2}\overbrace{\frac{4}{1+x_0^2}}^{f(x_0)}+\overbrace{\frac{4}{1+x_1^2}}^{f(x_1)}+\cdots+\overbrace{\frac{4}{1+x_7^2}}^{f(x_{n-1})}+\frac{1}{2}\overbrace{\frac{4}{1+x_8^2}}^{f(x_n)}\Bigg]\Delta x\\ &=\Bigg[\frac{1}{2}\frac{4}{1+0^2}+\frac{4}{1+\frac{1}{8^2}}+\frac{4}{1+\frac{2^2}{8^2}}+\frac{4}{1+\frac{3^2}{8^2}}\\ &\qquad+\frac{4}{1+\frac{4^2}{8^2}}+\frac{4}{1+\frac{5^2}{8^2}}+\frac{4}{1+\frac{6^2}{8^2}}+\frac{4}{1+\frac{7^2}{8^2}}+\frac{1}{2}\frac{4}{1+\frac{8^2}{8^2}}\Bigg]\frac{1}{8}\\ &=\Bigg[\frac{1}{2}\times 4+3.939+3.765+3.507\\ &\qquad+3.2+2.876+2.56+2.266+\frac{1}{2}\times 2\Bigg]\frac{1}{8}\\ &=3.139\end{aligned}$$

  to three decimal places.

- The exact value of the integral is π. So the error in the approximation generated by eight steps of the trapezoidal rule is $|3.139-\pi| = 0.0026$, which is $100\frac{|3.139-\pi|}{\pi}\% = 0.08\%$ of the exact answer.

Example 1.12.2

Example 1.12.3 ($\int_0^\pi \sin x\,\mathrm{d}x$)

Solution. Again, let us use $n = 8$ steps.

- We have $a = 0$, $b = \pi$, $\Delta x = \frac{\pi}{8}$ and

$$x_0 = 0 \qquad x_1 = \tfrac{\pi}{8} \qquad x_2 = \tfrac{2\pi}{8} \qquad \cdots \qquad x_7 = \tfrac{7\pi}{8} \qquad x_8 = \tfrac{8\pi}{8} = \pi$$

- Applying the trapezoidal rule, Equation (1.12.1), gives

$$\begin{aligned}&\int_0^\pi \sin x\,\mathrm{d}x \approx \Big[\tfrac{1}{2}\sin(x_0)+\sin(x_1)+\cdots+\sin(x_7)+\tfrac{1}{2}\sin(x_8)\Big]\Delta x\\ &=\Big[\tfrac{1}{2}\sin 0+\sin\tfrac{\pi}{8}+\sin\tfrac{2\pi}{8}+\sin\tfrac{3\pi}{8}+\sin\tfrac{4\pi}{8}+\sin\tfrac{5\pi}{8}+\sin\tfrac{6\pi}{8}+\sin\tfrac{7\pi}{8}+\tfrac{1}{2}\sin\tfrac{8\pi}{8}\Big]\tfrac{\pi}{8}\\ &=\Big[\tfrac{1}{2}\times 0+0.3827+0.7071+0.9239+1.0000+0.9239+0.7071+0.3827+\tfrac{1}{2}\times 0\Big]\times 0.3927\\ &=5.0274\times 0.3927 = 1.974\end{aligned}$$

- The exact answer is $\int_0^\pi \sin x\,\mathrm{d}x = -\cos x\Big|_0^\pi = 2$. So with eight steps of the trapezoidal rule we achieved $100\frac{|1.974-2|}{2} = 1.3\%$ accuracy.

Example 1.12.3

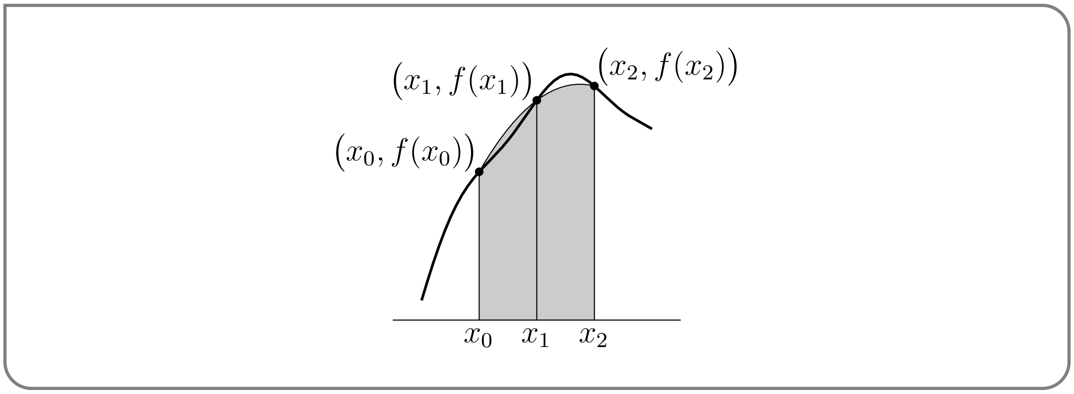
1.12.2 ▸▸ Simpson's rule

When we use the trapezoidal rule we approximate the area $\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x$ by the area between the x-axis and a straight line that runs from $(x_{j-1}, f(x_{j-1}))$ to $(x_j, f(x_j))$ — that is, we approximate the function $f(x)$ on this interval by a linear function that agrees with the function at each endpoint. An obvious way to extend this — just as we did when extending linear approximations to quadratic approximations in our differential calculus course — is to approximate the function with a quadratic. This is precisely what Simpson's[69] rule does.

Simpson's rule approximates the integral over two neighbouring subintervals by the area between a parabola and the x-axis. In order to describe this parabola we need 3 distinct points (which is why we approximate two subintegrals at a time). That is, we approximate

$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x = \int_{x_0}^{x_2} f(x)\,\mathrm{d}x$$

by the area bounded by the parabola that passes through the three points $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$, the x-axis and the vertical lines $x = x_0$ and $x = x_2$. We repeat this on the next pair of subintervals and approximate $\int_{x_2}^{x_4} f(x)\,\mathrm{d}x$ by the area between the x–axis and the part of a parabola with $x_2 \leqslant x \leqslant x_4$. This parabola passes through the three points $\big(x_2, f(x_2)\big)$, $\big(x_3, f(x_3)\big)$ and $\big(x_4, f(x_4)\big)$. And so on. Because Simpson's rule does the approximation two slices at a time, *n must be even*.



To derive Simpson's rule formula, we first find the equation of the parabola that passes through the three points $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$. Then we find the area between the x–axis and the part of that parabola with $x_0 \leqslant x \leqslant x_2$. To simplify this computation consider a parabola passing through the points $(-h, y_{-1})$, $(0, y_0)$ and (h, y_1).

Write the equation of the parabola as

$$y = Ax^2 + Bx + C$$

---

[69] Simpson's rule is named after the 18th century English mathematician Thomas Simpson, despite its use a century earlier by the German mathematician and astronomer Johannes Kepler. In many German texts the rule is often called Kepler's rule.

Then the area between it and the x-axis with x running from $-h$ to h is

$$\begin{aligned}\int_{-h}^{h} \left[Ax^2+Bx+C\right]\mathrm{d}x &= \left[\frac{A}{3}x^3+\frac{B}{2}x^2+Cx\right]_{-h}^{h}\\ &= \frac{2A}{3}h^3+2Ch \qquad \text{it is helpful to write it as}\\ &= \frac{h}{3}\left(2Ah^2+6C\right)\end{aligned}$$

Now, the three points $(-h,y_{-1}), (0,y_0)$ and (h,y_1) lie on this parabola if and only if

$$\begin{aligned}Ah^2-Bh+C&=y_{-1} && \text{at } (-h,y_{-1})\\ C&=y_0 && \text{at } (0,y_0)\\ Ah^2+Bh+C&=y_1 && \text{at } (h,y_1)\end{aligned}$$

Adding the first and third equations together gives us

$$2Ah^2+(B-B)h+2C=y_{-1}+y_1$$

To this we add four times the middle equation

$$2Ah^2+6C=y_{-1}+4y_0+y_1.$$

This means that

$$\begin{aligned}\text{area} &= \int_{-h}^{h}\left[Ax^2+Bx+C\right]\mathrm{d}x=\frac{h}{3}\left(2Ah^2+6C\right)\\ &= \frac{h}{3}\left(y_{-1}+4y_0+y_1\right)\end{aligned}$$

Note that here

- h is one half of the length of the x–interval under consideration
- y_{-1} is the height of the parabola at the left-hand end of the interval under consideration
- y_0 is the height of the parabola at the middle point of the interval under consideration
- y_1 is the height of the parabola at the right-hand end of the interval under consideration

So Simpson's rule approximates

$$\int_{x_0}^{x_2} f(x)\,\mathrm{d}x \approx \tfrac{1}{3}\Delta x\left[f(x_0)+4f(x_1)+f(x_2)\right]$$

and

$$\int_{x_2}^{x_4} f(x)\,\mathrm{d}x \approx \tfrac{1}{3}\Delta x\left[f(x_2)+4f(x_3)+f(x_4)\right]$$

and so on. Summing these all together gives:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \int_{x_0}^{x_2} f(x)\,\mathrm{d}x + \int_{x_2}^{x_4} f(x)\,\mathrm{d}x + \int_{x_4}^{x_6} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,\mathrm{d}x \\ &\approx \tfrac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \tfrac{\Delta x}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right] \\ &\quad + \tfrac{\Delta x}{3}\left[f(x_4) + 4f(x_5) + f(x_6)\right] + \cdots + \tfrac{\Delta x}{3}\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right] \\ &= \Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]\tfrac{\Delta x}{3}\end{aligned}$$

In summary

Equation 1.12.4 (Simpson's rule).

The Simpson's rule approximation is

$$\int_a^b f(x)\,\mathrm{d}x \approx \Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots \\ \cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]\tfrac{\Delta x}{3}$$

where n is even and

$$\Delta x = \tfrac{b-a}{n}, \quad x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \cdots, \quad x_{n-1} = b - \Delta x, \quad x_n = b$$

Notice that Simpson's rule requires essentially no more work than the trapezoidal rule. In both rules we must evaluate $f(x)$ at $x = x_0, x_1, \ldots, x_n$, but we add those terms multiplied by different constants[70].

Let's put it to work on our two running examples.

Example 1.12.5 $\left(\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x\right.$ — using Simpson's rule$\left.\right)$

Solution. We proceed almost identically to Example 1.12.2 and again use $n = 8$ steps.

- We have the same $\Delta, a, b, x_0, \ldots, x_n$ as Example 1.12.2.

---

[70] There is an easy generalisation of Simpson's rule that uses cubics instead of parabolas. It leads to the formula

$$\int_a^b f(x)\,\mathrm{d}x = \left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 2f(x_4) + 3f(x_5) + 3f(x_6) + 2f(x_7) + \cdots + f(x_n)\right]\frac{3\Delta x}{8}$$

where n is a multiple of 3. This result is known as Simpson's second rule and Simpson's 3/8 rule. While one can push this approach further (using quartics, quintics etc), it can sometimes lead to larger errors — the interested reader should look up Runge's phenomenon.

- Applying Equation 1.12.4 gives

  $$\begin{aligned}\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x &\approx \bigg[\frac{4}{1+0^2} + 4\frac{4}{1+\frac{1}{8^2}} + 2\frac{4}{1+\frac{2^2}{8^2}} + 4\frac{4}{1+\frac{3^2}{8^2}} \\ &\qquad + 2\frac{4}{1+\frac{4^2}{8^2}} + 4\frac{4}{1+\frac{5^2}{8^2}} + 2\frac{4}{1+\frac{6^2}{8^2}} + 4\frac{4}{1+\frac{7^2}{8^2}} + \frac{4}{1+\frac{8^2}{8^2}}\bigg]\frac{1}{8\times 3} \\ &= \Big[4 + 4\times 3.938461538 + 2\times 3.764705882 + 4\times 3.506849315 \\ &\qquad + 2\times 3.2 + 4\times 2.876404494 + 2\times 2.56 + 4\times 2.265486726 + 2\Big]\frac{1}{8\times 3} \\ &= 3.14159250\end{aligned}$$

  to eight decimal places.

- This agrees with π (the exact value of the integral) to six decimal places. So the error in the approximation generated by eight steps of Simpson's rule is $|3.14159250 - \pi| = 1.5\times 10^{-7}$, which is $100\frac{|3.14159250-\pi|}{\pi}\% = 5\times 10^{-6}\%$ of the exact answer.

Example 1.12.5

It is striking that the absolute error approximating with Simpson's rule is so much smaller than the error from the trapezoidal rules.

$$\begin{aligned}\text{trapezoid error} &= 0.0026 \\ \text{Simpson error} &= 0.00000015\end{aligned}$$

Buoyed by this success, we will also redo Example 1.12.3 using Simpson's rule.

Example 1.12.6 ($\int_0^\pi \sin x\,\mathrm{d}x$ — Simpson's rule)

Solution. We proceed almost identically to Example 1.12.3 and again use $n = 8$ steps.

- We have the same $\Delta, a, b, x_0, \ldots, x_n$ as Example 1.12.2.

- Applying Equation 1.12.4 gives

  $$\begin{aligned}\int_0^\pi \sin x\,\mathrm{d}x &\approx \Big[\sin(x_0) + 4\sin(x_1) + 2\sin(x_2) + \cdots + 4\sin(x_7) + \sin(x_8)\Big]\tfrac{\Delta x}{3} \\ &= \Big[\sin(0) + 4\sin(\tfrac{\pi}{8}) + 2\sin(\tfrac{2\pi}{8}) + 4\sin(\tfrac{3\pi}{8}) + 2\sin(\tfrac{4\pi}{8}) \\ &\qquad + 4\sin(\tfrac{5\pi}{8}) + 2\sin(\tfrac{6\pi}{8}) + 4\sin(\tfrac{7\pi}{8}) + \sin(\tfrac{8\pi}{8})\Big]\tfrac{\pi}{8\times 3} \\ &= \Big[0 + 4\times 0.382683 + 2\times 0.707107 + 4\times 0.923880 + 2\times 1.0 \\ &\qquad + 4\times 0.923880 + 2\times 0.707107 + 4\times 0.382683 + 0\Big]\tfrac{\pi}{8\times 3} \\ &= 15.280932\times 0.130900 \\ &= 2.00027\end{aligned}$$

- With only eight steps of Simpson's rule we achieved $100\frac{2.00027-2}{2} = 0.014\%$ accuracy.

Example 1.12.6

Again we contrast the error we achieved with the other two rules:

$$\begin{aligned} \text{trapezoid error} &= 0.026 \\ \text{Simpson error} &= 0.00027 \end{aligned}$$

This completes our derivation of the trapezoidal and Simpson's rules for approximating the values of definite integrals. So far we have not attempted to see how efficient and how accurate the algorithms are in general. That's our next task.

1.12.3 ▸▸ Two simple numerical integrators – error behaviour

Now we are armed with two new (relatively simple) methods for numerical integration, we should give thought to how practical they might be in the real world[71]. Two obvious considerations when deciding whether or not a given algorithm is of any practical value are

(a) the amount of computational effort required to execute the algorithm and

(b) the accuracy that this computational effort yields.

For algorithms like our simple integrators, the bulk of the computational effort usually goes into evaluating the function $f(x)$. The number of evaluations of $f(x)$ required for n steps of a Riemann sum is n, while the number required for n steps of the trapezoidal and Simpson's rules is $n+1$. So all three of our rules require essentially the same amount of effort – one evaluation of $f(x)$ per step.

To get a first impression of the error behaviour of these methods, we apply them to a problem whose answer we know exactly:

$$\int_0^{\pi} \sin x \, \mathrm{d}x = -\cos x\big|_0^{\pi} = 2.$$

To be a little more precise, we would like to understand how the errors of the two new methods change as we increase the effort we put in (as measured by the number of steps n). The following table lists the error in the approximate value for this number generated by our two rules applied with three different choices of n. It also lists the number of evaluations of f required to compute the approximation.

	Trapezoidal		**Simpson's**	
n	error	# evals	error	# evals
10	1.6×10^{-2}	11	1.1×10^{-4}	11
100	1.6×10^{-4}	101	1.1×10^{-8}	101
1000	1.6×10^{-6}	1001	1.1×10^{-12}	1001

---

[71] Indeed, even beyond the "real world" of many applications in first year calculus texts, some of the methods we have described are used by actual people (such as ship builders, engineers and surveyors) to estimate areas and volumes of actual objects!

Observe that

- Using 101 evaluations of f worth of Simpson's rule gives an error 145 times smaller than 1000 evaluations of f worth of the trapezoid rule.
- With the trapezoidal rule, increasing the number of steps by a factor of 10 appears to reduce the error by about a factor of $10^2 = n^2$.
- With Simpson's rule, increasing the number of steps by a factor of 10 appears to reduce the error by about a factor of $10^4 = n^4$.

So it looks like

$$\begin{aligned}\text{approx value of } \int_a^b f(x)\,\mathrm{d}x \text{ given by } n \text{ trapezoidal steps } &\approx \int_a^b f(x)\,\mathrm{d}x + K_T \cdot \frac{1}{n^2}\\ \text{approx value of } \int_a^b f(x)\,\mathrm{d}x \text{ given by } n \text{ Simpson's steps } &\approx \int_a^b f(x)\,\mathrm{d}x + K_S \cdot \frac{1}{n^4}\end{aligned}$$

with some constants K_M, K_T and K_S. It also seems that $K_T \approx 2K_M$.

The intuition about the error behaviour that we have just developed is, in fact, correct — provided the integrand $f(x)$ is reasonably smooth. To be more precise:

Theorem 1.12.7 (Numerical integration errors).

Assume that $|f''(x)| \leqslant M$ for all $a \leqslant x \leqslant b$. Then

$$\text{the total error introduced by the trapezoidal rule is bounded by } \frac{M}{12}\frac{(b-a)^3}{n^2}$$

when approximating $\int_a^b f(x)\,\mathrm{d}x$. Further, if $|f^{(4)}(x)| \leqslant L$ for all $a \leqslant x \leqslant b$, then

$$\text{the total error introduced by Simpson's rule is bounded by } \frac{L}{180}\frac{(b-a)^5}{n^4}.$$

Proving these error bounds is beyond the scope of Math 101 – for now, take our word for it.

In a typical application we would be asked to evaluate a given integral to some specified accuracy. For example, if you are manufacturer and your machinery can only cut materials to an accuracy of ${\frac{1}{10}}^{\text{th}}$ of a millimeter, there is no point in making design specifications more accurate than ${\frac{1}{10}}^{\text{th}}$ of a millimeter.

Example 1.12.8

Suppose, for example, that we wish to use the trapezoid rule to evaluate[72]

$$\int_0^1 e^{-x^2}\mathrm{d}x$$

---

[72] This is our favourite running example of an integral that cannot be evaluated algebraically — we need to use numerical methods.

to within an accuracy of 10^{-6}.

Solution.

- The integral has $a = 0$ and $b = 1$.
- The first two derivatives of the integrand are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} e^{-x^2} &= -2xe^{-x^2} && \text{and} \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{-x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(-2xe^{-x^2}\big) = -2e^{-x^2} + 4x^2e^{-x^2} = 2(2x^2-1)e^{-x^2} \end{aligned}$$

- As x runs from 0 to 1, $2x^2 - 1$ increases from -1 to 1, so that

$$0 \leqslant x \leqslant 1 \implies |2x^2-1| \leqslant 1,\ e^{-x^2} \leqslant 1 \implies \left|2(2x^2-1)e^{-x^2}\right| \leqslant 2$$

  So we take $M = 2$.

- The error introduced by the n-step trapezoid rule is at most

$$\begin{aligned} e_n &\leqslant \frac{M}{12}\frac{(b-a)^3}{n^2} \\ &\leqslant \frac{2}{12}\frac{(1-0)^3}{n^2} = \frac{1}{6n^2} \end{aligned}$$

- We need this error to be smaller than 10^{-6} so

$$\begin{aligned} e_n &\leqslant \frac{1}{6n^2} \leqslant 10^{-6} && \text{and so} \\ 6n^2 &\geqslant 10^6 && \text{clean up} \\ n^2 &\geqslant \frac{10^6}{6} = 166666.\overline{66} && \text{square root both sides} \\ n &\geqslant 408.25 \end{aligned}$$

  So 409 steps of the trapezoid rule will do the job.

Example 1.12.8

That seems like far too much work. So, we should look at Simpson's rule.

Example 1.12.9

Suppose now that we wish evaluate $\int_0^1 e^{-x^2}\,\mathrm{d}x$ to within an accuracy of 10^{-6} — but now using Simpson's rule. How many steps should we use?

Solution.

- Again we have $a = 0, b = 1$.

- We then need to bound $\frac{\mathrm{d}^4}{\mathrm{d}x^4}e^{-x^2}$ on the domain of integration, $0 \leqslant x \leqslant 1$.

$$\begin{aligned}\frac{\mathrm{d}^3}{\mathrm{d}x^3}e^{-x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\{2(2x^2-1)e^{-x^2}\} = 8xe^{-x^2} - 4x(2x^2-1)e^{-x^2}\\ &= 4(-2x^3+3x)e^{-x^2}\\ \frac{\mathrm{d}^4}{\mathrm{d}x^4}e^{-x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\{4(-2x^3+3x)e^{-x^2}\} = 4(-6x^2+3)e^{-x^2} - 8x(-2x^3+3x)e^{-x^2}\\ &= 4(4x^4-12x^2+3)e^{-x^2}\end{aligned}$$

- Now, for any x, $e^{-x^2} \leqslant 1$. Also, for $0 \leqslant x \leqslant 1$,

$$\begin{aligned} 0 \leqslant x^2, x^4 &\leqslant 1 && \text{so}\\ 3 \leqslant 4x^4+3 &\leqslant 7 && \text{and}\\ -12 \leqslant -12x^2 &\leqslant 0 && \text{adding these together gives}\\ -9 \leqslant 4x^4-12x^2+3 &\leqslant 7 \end{aligned}$$

  Consequently, $|4x^4-12x^2+3|$ is bounded by 9 and so

$$\left|\frac{\mathrm{d}^4}{\mathrm{d}x^4}e^{-x^2}\right| \leqslant 4\times 9 = 36$$

  So take $L = 36$.

- The error introduced by the n step Simpson's rule is at most

$$\begin{aligned} e_n &\leqslant \frac{L}{180}\frac{(b-a)^5}{n^4}\\ &\leqslant \frac{36}{180}\frac{(1-0)^5}{n^4} = \frac{1}{5n^4}\end{aligned}$$

- In order for this error to be no more than 10^{-6} we require n to satisfy

$$\begin{aligned} e_n \leqslant \frac{1}{5n^4} &\leqslant 10^{-6} && \text{and so}\\ 5n^4 &\geqslant 10^6\\ n^4 &\geqslant 200000 && \text{take fourth root}\\ n &\geqslant 21.15\end{aligned}$$

  So 22 steps of Simpson's rule will do the job.

- $n = 22$ steps actually results in an error of 3.5×10^{-8}. The reason that we get an error so much smaller than we need is that we have overestimated the number of steps required. This, in turn, occurred because we made quite a rough bound of $\left|\frac{\mathrm{d}^4}{\mathrm{d}x^4}f(x)\right| \leqslant 36$. If we are more careful then we will get a slightly smaller n. It actually turns out[73] that you only need $n = 10$ to approximate within 10^{-6}.

Example 1.12.9

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[73] The authors tested this empirically.

1.12.4 ▸▸ Using spreadsheets to compute numerical integrals

We saw in 1.1.5 that spreadsheets are natural tools for computing Riemann sums. They work nicely for trapezoid rule and Simpson's rule, as well.

Example 1.12.10

Using a spreadsheet, approximate

$$\int_0^{\pi} \frac{1}{1+x^2}\,\mathrm{d}x$$

using the three methods of this section, each with $n = 40$.

Solution. All three methods use mostly the same points, so we'll start by putting our x-values in column A and our function values in column B.

	A	B
1	`0`	`=1/(1+(A1)^2)`
2	`=A1+PI()/40`	↓

We copy the contents down the columns to row 41.

right Riemann sum The right Riemann sum approximation is

$$\frac{\pi}{40}\sum_{i=1}^{40} f(x_i),$$

where $f(x) = \frac{1}{1+x^2}$ and $x_i = i \cdot \frac{\pi}{40}$. The sum part of the expression centred above is the sum of cells B2 through B41. So, in a free cell, we write

`=PI()/40*SUM(B2:B41)`

and find $\int_0^{\pi} \frac{1}{1+x^2}\,\mathrm{d}x \approx 1.226942832$.

Trapezoid rule The trapezoid rule approximation is

$$\frac{\pi}{40}\left[\frac{1}{2}f(x_0) + \sum_{i=1}^{39} f(x_i) + \frac{1}{2}\cdot f(x_{40})\right],$$

using $x_i = i \cdot \frac{\pi}{40}$ and $f(x) = \frac{1}{1+x^2}$. So, in a free cell, we write

`=((B1+B41)/2+SUM(B2:B40))*PI()/40`

and find $\int_0^{\pi} \frac{1}{1+x^2}\,\mathrm{d}x \approx 1.262599921$.

Simpson's rule Simpson's rule has somewhat more complicated coefficients. We'll store them in column C. The first and last are 1, and don't fit the pattern of the others, so we'll put those in manually by just writing `1` in cells C1 and C41.

To generate the others, we *could* just go down the column writing "4,2,4,2,4,2,..." This will work for $n = 40$, but it would be tough for, say, $n = 4000$, so we'll demonstrate a way to get around entering them separately. We will set the entry in the nth row be $3 + (-1)^n$. Then when n is even, the entry is 4, and when n is odd, the entry is 2.

Once we have the coefficients in column C, we use column D to store the products, i.e. $1 \cdot f(0), 4 \cdot f(\pi/40)$, etc.

	A	B	C	D
1	`0`	`=1/(1+(A1)∧2)`	`=1`	`=B1*C1`
2	`=A1+PI()/40`	↓	`=3+(-1)∧ROW()`	↓
3	↓	↓	↓	↓
⋮	⋮	⋮	⋮	⋮
40	↓	↓	↓	↓
41	↓	↓	`=1`	↓

We want to compute

$$\frac{\pi/40}{3}\left[1\cdot f(x_0)+4\cdot f(x_1)+2\cdot f(x_2)+\cdots+4\cdot x_P39+1\cdot f(x_{40})\right]$$

so, in a free cell, we write

```
=PI()/120*SUM(D1:D41)
```

and find $\int_0^\pi \frac{1}{1+x^2}\,\mathrm{d}x \approx 1.262627246$.

Example 1.12.10

1.13▲ Improper integrals

Learning Objectives

- State the different ways an integral can be improper.
- Define what it means to *evaluate* an improper integral. In particular, use pictures to explain the area being computed and the limit being taken.
- Define what it means for an improper integral to converge or diverge.
- Demonstrate the convergence/divergence of $\int \frac{1}{x^p}\,\mathrm{d}x$ for general $p > 0$, with domains $(0,1]$ and $[1,\infty)$.
- Evaluate an improper integral (or prove it diverges) by explicitly writing and computing the appropriate limit.
- Use the comparison test to determine convergence/divergence for improper integrals without finding their antiderivatives.
- Use the limit comparison test to determine convergence/divergence of improper integrals without finding their antiderivatives.

1.13.1 ▸▸ Definitions

To this point we have only considered nicely behaved definite integrals $\int_a^b f(x)\,\mathrm{d}x$. Though the algebra involved in some of our examples was quite difficult, all the integrals had:

- finite limits of integration a and b, and
- a bounded integrand $f(x)$ (which was, in fact, continuous except possibly for finitely many jump discontinuities).

Not all integrals we need to study are quite so nice.

Definition 1.13.1.

An integral having either an infinite limit of integration or an unbounded integrand is called an improper integral.

Two examples are

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} \qquad \text{and} \qquad \int_0^1 \frac{\mathrm{d}x}{x}$$

The first has an infinite domain of integration and the integrand of the second tends to ∞ as x approaches the left end of the domain of integration. We'll start with an example that illustrates the traps that you can fall into if you treat such integrals sloppily. Then we'll see how to treat them carefully.

Example 1.13.2 $\left(\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x\right)$

Consider the integral

$$\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x$$

If we "do" this integral completely naively then we get

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x &= \frac{x^{-1}}{-1}\bigg|_{-1}^{1}\\ &= \frac{1}{-1} - \frac{-1}{-1}\\ &= -2\end{aligned}$$

which is $\underline{\textit{wrong}}$[74]. In fact, the answer is ridiculous. The integrand $\frac{1}{x^2} > 0$, so the integral has to be $\overline{\text{positive}}$.

The flaw in the argument is that the fundamental theorem of calculus, which says that

if $F'(x) = f(x)$ then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$

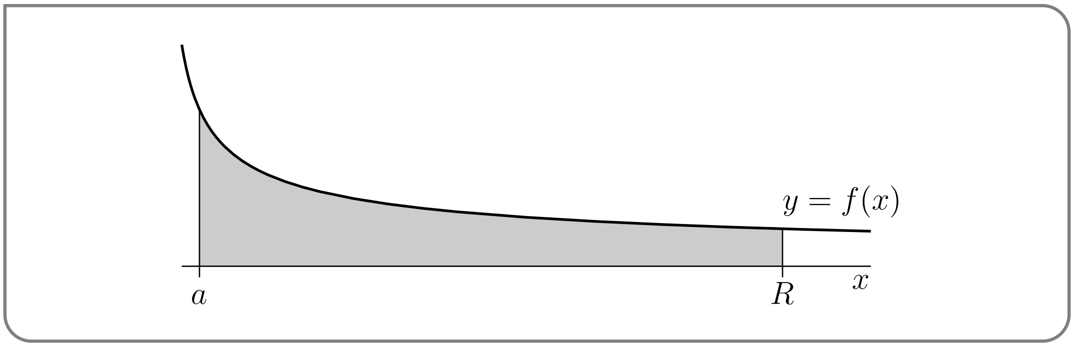
---

[74] Very wrong. But it is not an example of "not even wrong" — which is a phrase attributed to the physicist Wolfgang Pauli who was known for his harsh critiques of sloppy arguments. The phrase is typically used to describe arguments that are so incoherent that not only can one not prove they are true, but they lack enough coherence to be able to show they are false. The interested reader should do a little searchengineing and look at the concept of falisfyability.

is applicable only when $F'(x)$ exists and equals $f(x)$ for *all* $a \leqslant x \leqslant b$. In this case $F'(x) = \frac{1}{x^2}$ does not exist for $x = 0$. The given integral is improper. We'll see later that the correct answer is $+\infty$.

Example 1.13.2

Let us put this example to one side for a moment and turn to the integral $\int_a^\infty \frac{dx}{1+x^2}$. In this case, the integrand is bounded but the domain of integration extends to $+\infty$. We can evaluate this integral by sneaking up on it. We compute it on a bounded domain of integration, like $\int_a^R \frac{dx}{1+x^2}$, and then take the limit $R \to \infty$. Let us put this into practice:



Example 1.13.3 $\left(\int_a^\infty \frac{dx}{1+x^2}\right)$

Solution.

- Since the domain extends to $+\infty$ we first integrate on a finite domain

$$\begin{aligned}\int_a^R \frac{dx}{1+x^2} &= \arctan x\Big|_a^R \\ &= \arctan R - \arctan a\end{aligned}$$

- We then take the limit as $R \to +\infty$:

$$\begin{aligned}\int_a^\infty \frac{dx}{1+x^2} &= \lim_{R\to\infty} \int_a^R \frac{dx}{1+x^2} \\ &= \lim_{R\to\infty} \left[\arctan R - \arctan a\right] \\ &= \frac{\pi}{2} - \arctan a.\end{aligned}$$

Example 1.13.3

To be more precise, we actually formally *define* an integral with an infinite domain as the limit of the integral with a finite domain as we take one or more of the limits of integration to infinity.

Definition 1.13.4 (Improper integral with infinite domain of integration)**.**

(a) If the integral $\int_a^R f(x)\,\mathrm{d}x$ exists for all $R > a$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_a^R f(x)\,\mathrm{d}x$$

when the limit exists (and is finite).

(b) If the integral $\int_r^b f(x)\,\mathrm{d}x$ exists for all $r < b$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{r\to-\infty} \int_r^b f(x)\,\mathrm{d}x$$

when the limit exists (and is finite).

(c) If the integral $\int_r^R f(x)\,\mathrm{d}x$ exists for all $r < R$, then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{r\to-\infty} \int_r^c f(x)\,\mathrm{d}x + \lim_{R\to\infty} \int_c^R f(x)\,\mathrm{d}x$$

when both limits exist (and are finite). Any c can be used.

When the limit(s) exist, the integral is said to be convergent. Otherwise it is said to be divergent.

We must also be able to deal with an integral like $\int_0^1 \frac{\mathrm{d}x}{x}$ that has a finite domain of integration, but whose integrand is unbounded near one limit of integration[75]. Our approach is similar — we sneak up on the problem. We compute the integral on a smaller domain, such as $\int_t^1 \frac{\mathrm{d}x}{x}$, with $t > 0$, and then take the limit $t \to 0+$.

Example 1.13.5 $\left(\int_0^1 \frac{1}{x}\mathrm{d}x\right)$

Solution.

- Since the integrand is unbounded near $x = 0$, we integrate on the smaller domain $t \leqslant x \leqslant 1$ with $t > 0$:

$$\int_t^1 \frac{1}{x}\mathrm{d}x = \log|x|\Big|_t^1 = -\log|t|$$

- We then take the limit as $t \to 0^+$ to obtain

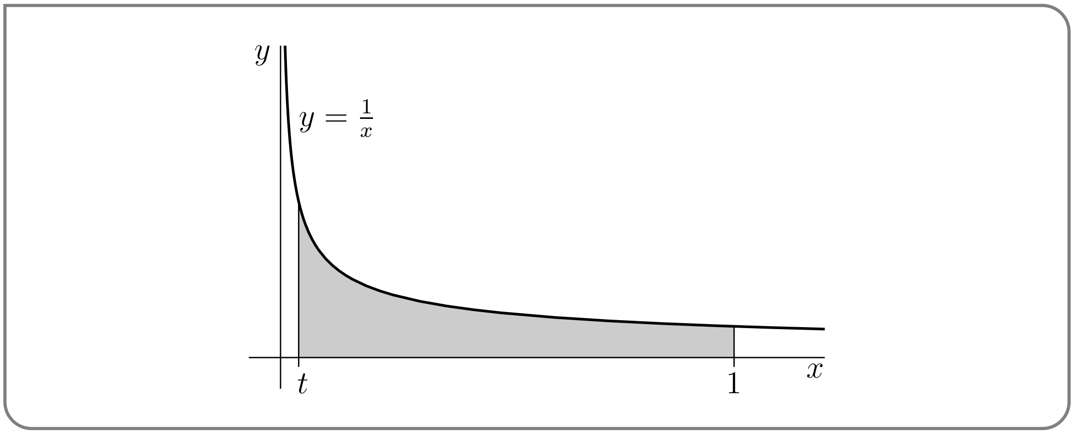
$$\int_0^1 \frac{1}{x}\mathrm{d}x = \lim_{t\to0^+} \int_t^1 \frac{1}{x}\mathrm{d}x = \lim_{t\to0^+} -\log|t| = +\infty$$

  Thus this integral diverges to $+\infty$.

---

[75] This will, in turn, allow us to deal with integrals whose integrand is unbounded somewhere inside the domain of integration.

Example 1.13.5



Indeed, we *define* integrals with unbounded integrands via this process:

Definition 1.13.6 (Improper integral with unbounded integrand)**.**

(a) If the integral $\int_t^b f(x)\,\mathrm{d}x$ exists for all $a < t < b$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a+} \int_t^b f(x)\,\mathrm{d}x$$

when the limit exists (and is finite).

(b) If the integral $\int_a^T f(x)\,\mathrm{d}x$ exists for all $a < T < b$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{T\to b-} \int_a^T f(x)\,\mathrm{d}x$$

when the limit exists (and is finite).

(c) Let $a < c < b$. If the integrals $\int_a^T f(x)\,\mathrm{d}x$ and $\int_t^b f(x)\,\mathrm{d}x$ exist for all $a < T < c$ and $c < t < b$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{T\to c-} \int_a^T f(x)\,\mathrm{d}x + \lim_{t\to c+} \int_t^b f(x)\,\mathrm{d}x$$

when both limit exist (and are finite).

When the limit(s) exist, the integral is said to be convergent. Otherwise it is said to be divergent.

Notice that (c) is used when the integrand is unbounded at some point in the middle of the domain of integration, such as was the case in our original example

$$\int_{-1}^{1} \frac{1}{x^2} \mathrm{d}x$$

A quick computation shows that this integral diverges to $+\infty$

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x &= \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{x^2}\mathrm{d}x + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x^2}\mathrm{d}x \\ &= \lim_{a\to 0^-} \left[1 - \frac{1}{a}\right] + \lim_{b\to 0^+} \left[\frac{1}{b} - 1\right] \\ &= +\infty\end{aligned}$$

More generally, if an integral has more than one "source of impropriety" (for example, an infinite domain of integration *and* an integrand with one or more infinite discontinuities, or an integrand with multiple infinite discontinuities) then you split it up into a sum of integrals with a single "source of impropriety" in each. For the integral, as a whole, to converge every term in that sum has to converge.

For example

Example 1.13.7 $\left(\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x-2)x^2}\right)$

Consider the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x-2)x^2}$$

- The domain of integration that extends to both $+\infty$ and $-\infty$.
- The integrand is singular (i.e. becomes infinite) at $x = 2$ and at $x = 0$.
- So we would write the integral as

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x-2)x^2} = \int_{-\infty}^{a} \frac{\mathrm{d}x}{(x-2)x^2} &+ \int_{a}^{0} \frac{\mathrm{d}x}{(x-2)x^2} + \int_{0}^{b} \frac{\mathrm{d}x}{(x-2)x^2} \\ &+ \int_{b}^{2} \frac{\mathrm{d}x}{(x-2)x^2} + \int_{2}^{c} \frac{\mathrm{d}x}{(x-2)x^2} + \int_{c}^{\infty} \frac{\mathrm{d}x}{(x-2)x^2}\end{aligned}$$

  where

  - a is any number strictly less than 0,
  - b is any number strictly between 0 and 2, and
  - c is any number strictly bigger than 2.

  So, for example, take $a = -1, b = 1, c = 3$.

- When we examine the right-hand side we see that
  - the first integral has domain of integration extending to $-\infty$

  - the second integral has an integrand that becomes unbounded as $x \to 0-$,
  - the third integral has an integrand that becomes unbounded as $x \to 0+$,
  - the fourth integral has an integrand that becomes unbounded as $x \to 2-$,
  - the fifth integral has an integrand that becomes unbounded as $x \to 2+$, and
  - the last integral has domain of integration extending to $+\infty$.
- Each of these integrals can then be expressed as a limit of an integral on a small domain.

Example 1.13.7

1.13.2 ▸▸ Examples

With the more formal definitions out of the way, we are now ready for some (important) examples.

Example 1.13.8 $\left(\int_1^\infty \frac{\mathrm{d}x}{x^p} \text{ with } p > 0\right)$

Solution.

- Fix any $p > 0$.
- The domain of the integral $\int_1^\infty \frac{\mathrm{d}x}{x^p}$ extends to $+\infty$ and the integrand $\frac{1}{x^p}$ is continuous and bounded on the whole domain.
- So we write this integral as the limit

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{R\to\infty} \int_1^R \frac{\mathrm{d}x}{x^p}$$

- The antiderivative of $1/x^p$ changes when $p = 1$, so we will split the problem into three cases, $p > 1$, $p = 1$ and $p < 1$.
- When $p > 1$,

$$\begin{aligned} \int_1^R \frac{\mathrm{d}x}{x^p} &= \frac{1}{1-p} x^{1-p} \Big|_1^R \\ &= \frac{R^{1-p} - 1}{1-p} \end{aligned}$$

  Taking the limit as $R \to \infty$ gives

$$\begin{aligned} \int_1^\infty \frac{\mathrm{d}x}{x^p} &= \lim_{R\to\infty} \int_1^R \frac{\mathrm{d}x}{x^p} \\ &= \lim_{R\to\infty} \frac{R^{1-p} - 1}{1-p} \\ &= \frac{-1}{1-p} = \frac{1}{p-1} \end{aligned}$$

  since $1 - p < 0$.

- Similarly when $p < 1$ we have

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{R\to\infty}\int_1^R \frac{\mathrm{d}x}{x^p} \qquad = \lim_{R\to\infty}\frac{R^{1-p}-1}{1-p}$$
$$= +\infty$$

  because $1-p > 0$ and the term R^{1-p} diverges to $+\infty$.

- Finally when $p = 1$

$$\int_1^R \frac{\mathrm{d}x}{x} = \log|R| - \log 1 = \log R$$

  Then taking the limit as $R \to \infty$ gives us

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{R\to\infty}\log|R| = +\infty.$$

- So summarising, we have

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \begin{cases}\text{divergent} & \text{if } p \leqslant 1\\ \frac{1}{p-1} & \text{if } p > 1\end{cases}$$

Example 1.13.8

Example 1.13.9 $\left(\int_0^1 \frac{\mathrm{d}x}{x^p} \text{ with } p > 0\right)$

Solution.

- Again fix any $p > 0$.

- The domain of integration of the integral $\int_0^1 \frac{\mathrm{d}x}{x^p}$ is finite, but the integrand $\frac{1}{x^p}$ becomes unbounded as x approaches the left end, 0, of the domain of integration.

- So we write this integral as

$$\int_0^1 \frac{\mathrm{d}x}{x^p} = \lim_{t\to 0+}\int_t^1 \frac{\mathrm{d}x}{x^p}$$

- Again, the antiderivative changes at $p = 1$, so we split the problem into three cases.

- When $p > 1$ we have

$$\int_t^1 \frac{\mathrm{d}x}{x^p} = \frac{1}{1-p}x^{1-p}\Big|_t^1$$
$$= \frac{1-t^{1-p}}{1-p}$$

Since $1-p<0$ when we take the limit as $t\to 0$ the term t^{1-p} diverges to $+\infty$ and we obtain

$$\int_0^1 \frac{\mathrm{d}x}{x^p} = \lim_{t\to 0^+} \frac{1-t^{1-p}}{1-p} = +\infty$$

- When $p=1$ we similarly obtain

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{x} &= \lim_{t\to 0+} \int_t^1 \frac{\mathrm{d}x}{x} \\ &= \lim_{t\to 0+} \big(-\log|t|\big) \\ &= +\infty\end{aligned}$$

- Finally, when $p<1$ we have

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{x^p} &= \lim_{t\to 0^+} \int_t^1 \frac{\mathrm{d}x}{x^p} \\ &= \lim_{t\to 0^+} \frac{1-t^{1-p}}{1-p} = \frac{1}{1-p}\end{aligned}$$

since $1-p>0$.

- In summary

$$\int_0^1 \frac{\mathrm{d}x}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p<1 \\ \text{divergent} & \text{if } p\geqslant 1\end{cases}$$

Example 1.13.9

Example 1.13.10 $\left(\int_0^\infty \frac{\mathrm{d}x}{x^p} \text{ with } p>0\right)$

Solution.

- Yet again fix $p>0$.
- This time the domain of integration of the integral $\int_0^\infty \frac{\mathrm{d}x}{x^p}$ extends to $+\infty$, and in addition the integrand $\frac{1}{x^p}$ becomes unbounded as x approaches the left end, 0, of the domain of integration.
- So we split the domain in two — given our last two examples, the obvious place to cut is at $x=1$:

$$\int_0^\infty \frac{\mathrm{d}x}{x^p} = \int_0^1 \frac{\mathrm{d}x}{x^p} + \int_1^\infty \frac{\mathrm{d}x}{x^p}$$

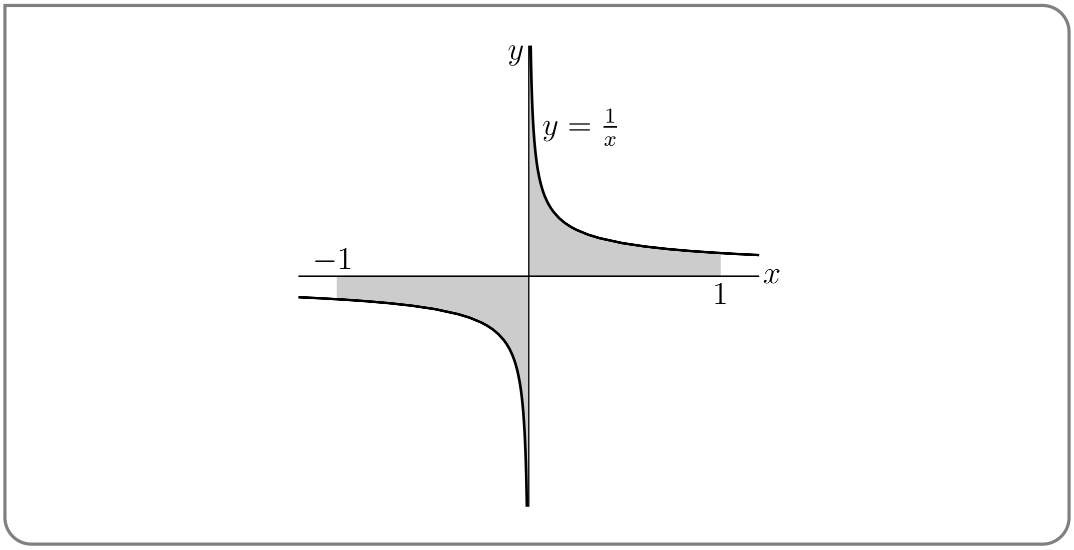
- We saw, in Example 1.13.9, that the first integral diverged whenever $p\geqslant 1$, and we also saw, in Example 1.13.8, that the second integral diverged whenever $p\leqslant 1$.

- So the integral $\int_0^\infty \frac{\mathrm{d}x}{x^p}$ diverges for all values of p.

Example 1.13.10

Example 1.13.11 $\left(\int_{-1}^1 \frac{\mathrm{d}x}{x}\right)$

This is a pretty subtle example. Look at the sketch below: This suggests that the signed area to the left of the y–axis should exactly cancel the area to the right of the y–axis making the value of the integral $\int_{-1}^1 \frac{\mathrm{d}x}{x}$ exactly zero.



But both of the integrals

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{x} &= \lim_{t\to0+}\int_t^1 \frac{\mathrm{d}x}{x} &&= \lim_{t\to0+}\Big[\log x\Big]_t^1 &&= \lim_{t\to0+}\log\frac{1}{t} &&= +\infty\\ \int_{-1}^0 \frac{\mathrm{d}x}{x} &= \lim_{T\to0-}\int_{-1}^T \frac{\mathrm{d}x}{x} &&= \lim_{T\to0-}\Big[\log|x|\Big]_{-1}^T &&= \lim_{T\to0-}\log|T| &&= -\infty\end{aligned}$$

diverge so $\int_{-1}^1 \frac{\mathrm{d}x}{x}$ diverges. Don't make the mistake of thinking that $\infty - \infty = 0$. *It is undefined*. And it is undefined for good reason.

For example, we have just seen that the area to the right of the y–axis is

$$\lim_{t\to0+}\int_t^1 \frac{\mathrm{d}x}{x} = +\infty$$

and that the area to the left of the y–axis is (substitute $-7t$ for T above)

$$\lim_{t\to0+}\int_{-1}^{-7t} \frac{\mathrm{d}x}{x} = -\infty$$

If $\infty - \infty = 0$, the following limit should be 0.

$$\begin{aligned}\lim_{t\to 0+}\left[\int_t^1 \frac{\mathrm{d}x}{x} + \int_{-1}^{-7t}\frac{\mathrm{d}x}{x}\right] &= \lim_{t\to 0+}\left[\log\frac{1}{t} + \log|-7t|\right]\\ &= \lim_{t\to 0+}\left[\log\frac{1}{t} + \log(7t)\right]\\ &= \lim_{t\to 0+}\left[-\log t + \log 7 + \log t\right] = \lim_{t\to 0+}\log 7\\ &= \log 7\end{aligned}$$

This appears to give $\infty - \infty = \log 7$. Of course the number 7 was picked at random. You can make $\infty - \infty$ be any number at all, by making a suitable replacement for 7.

Example 1.13.11

Example 1.13.12 (Example 1.13.2 revisited)

The careful computation of the integral of Example 1.13.2 is

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x^2}\,\mathrm{d}x &= \lim_{T\to 0-}\int_{-1}^{T}\frac{1}{x^2}\,\mathrm{d}x + \lim_{t\to 0+}\int_t^1\frac{1}{x^2}\,\mathrm{d}x\\ &= \lim_{T\to 0-}\left[-\frac{1}{x}\right]_{-1}^{T} + \lim_{t\to 0+}\left[-\frac{1}{x}\right]_t^1\\ &= \infty + \infty\end{aligned}$$

Hence the integral diverges to $+\infty$.

Example 1.13.12

Example 1.13.13 $\left(\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+x^2}\right)$

Since

$$\begin{aligned}\lim_{R\to\infty}\int_0^R\frac{\mathrm{d}x}{1+x^2} &= \lim_{R\to\infty}\Big[\arctan x\Big]_0^R &= \lim_{R\to\infty}\arctan R = \frac{\pi}{2}\\ \lim_{r\to -\infty}\int_r^0\frac{\mathrm{d}x}{1+x^2} &= \lim_{r\to-\infty}\Big[\arctan x\Big]_r^0 &= \lim_{r\to-\infty} -\arctan r = \frac{\pi}{2}\end{aligned}$$

The integral $\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+x^2}$ converges and takes the value π.

Example 1.13.13

Example 1.13.14

For what values of p does $\int_e^{\infty}\frac{\mathrm{d}x}{x(\log x)^p}$ converge?

Solution.

- For $x \geqslant e$, the denominator $x(\log x)^p$ is never zero. So the integrand is bounded on the entire domain of integration and this integral is improper only because the domain of integration extends to $+\infty$ and we proceed as usual.

- We have

$$\begin{aligned}
\int_e^\infty \frac{\mathrm{d}x}{x(\log x)^p} &= \lim_{R\to\infty}\int_e^R \frac{\mathrm{d}x}{x(\log x)^p} && \text{use substitution}\\
&= \lim_{R\to\infty}\int_1^{\log R} \frac{\mathrm{d}u}{u^p} && \text{with } u=\log x, \mathrm{d}u=\frac{\mathrm{d}x}{x}\\
&= \lim_{R\to\infty}\begin{cases}\frac{1}{1-p}\Big[(\log R)^{1-p}-1\Big] & \text{if } p\neq 1\\ \log(\log R) & \text{if } p=1\end{cases}\\
&= \begin{cases}\text{divergent} & \text{if } p\leqslant 1\\ \frac{1}{p-1} & \text{if } p>1\end{cases}
\end{aligned}$$

In this last step we have used similar logic that that used in Example 1.13.8, but with R replaced by $\log R$.

Example 1.13.14

Example 1.13.15 (the gamma function)

The gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,\mathrm{d}x$$

We shall now compute $\Gamma(n)$ for all natural numbers n.

- To get started, we'll compute

$$\Gamma(1) = \int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{R\to\infty}\int_0^R e^{-x}\,\mathrm{d}x = \lim_{R\to\infty}\Big[-e^{-x}\Big]_0^R = 1$$

- Then compute

$$\begin{aligned}
\Gamma(2) &= \int_0^\infty xe^{-x}\,\mathrm{d}x \\
&= \lim_{R\to\infty}\int_0^R xe^{-x}\,\mathrm{d}x && \text{use integration by parts with} \\
& && u = x, \mathrm{d}v = e^{-x}\mathrm{d}x, \\
& && v = -e^{-x}, \mathrm{d}u = \mathrm{d}x \\
&= \lim_{R\to\infty}\Big[-xe^{-x}\Big|_0^R + \int_0^R e^{-x}\,\mathrm{d}x\Big] \\
&= \lim_{R\to\infty}\Big[-xe^{-x} - e^{-x}\Big]_0^R \\
&= 1
\end{aligned}$$

  For the last equality, we used that $\lim\limits_{x\to\infty} xe^{-x} = 0$.

- Now we move on to general n, using the same type of computation as we just used to evaluate $\Gamma(2)$. For any natural number n,

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty x^n e^{-x}\,\mathrm{d}x \\
&= \lim_{R\to\infty}\int_0^R x^n e^{-x}\,\mathrm{d}x && \text{again integrate by parts with} \\
& && u = x^n, \mathrm{d}v = e^{-x}\mathrm{d}x, \\
& && v = -e^{-x}, \mathrm{d}u = nx^{n-1}\mathrm{d}x \\
&= \lim_{R\to\infty}\Big[-x^n e^{-x}\Big|_0^R + \int_0^R nx^{n-1}e^{-x}\,\mathrm{d}x\Big] \\
&= \lim_{R\to\infty} n\int_0^R x^{n-1}e^{-x}\,\mathrm{d}x \\
&= n\Gamma(n)
\end{aligned}$$

  To get to the third row, we used that $\lim\limits_{x\to\infty} x^n e^{-x} = 0$.

- Now that we know $\Gamma(2) = 1$ and $\Gamma(n+1) = n\Gamma(n)$, for all $n \in \mathbb{N}$, we can compute all of the $\Gamma(n)$'s.

$$\begin{aligned}
\Gamma(2) &= 1 \\
\Gamma(3) &= \Gamma(2+1) = 2\Gamma(2) = 2\cdot 1 \\
\Gamma(4) &= \Gamma(3+1) = 3\Gamma(3) = 3\cdot 2\cdot 1 \\
\Gamma(5) &= \Gamma(4+1) = 4\Gamma(4) = 4\cdot 3\cdot 2\cdot 1 \\
&\vdots \\
\Gamma(n) &= (n-1)\cdot(n-2)\cdots 4\cdot 3\cdot 2\cdot 1 = (n-1)!
\end{aligned}$$

That is, the factorial is just[76] the Gamma function shifted by one.

Example 1.13.15

1.13.3 ▸▸ Convergence tests for improper integrals

It is very common to encounter integrals that are too complicated to evaluate explicitly. Numerical approximation schemes, evaluated by computer, are often used instead (see Section 1.12). You want to be sure that the integral converges before feeding it into a computer[77]. Fortunately, it is usually possible to determine whether or not an improper integral converges even when you cannot evaluate it explicitly.

Remark 1.13.16. For pedagogical purposes, we are going to concentrate on the problem of determining whether or not an integral $\int_a^\infty f(x)\,\mathrm{d}x$ converges, when $f(x)$ has no singularities for $x \geqslant a$. Recall that the first step in analyzing any improper integral is to write it as a sum of integrals each of has only a single "source of impropriety" — either a domain of integration that extends to $+\infty$, or a domain of integration that extends to $-\infty$, or an integrand which is singular at one end of the domain of integration. So we are now going to consider only the first of these three possibilities. But the techniques that we are about to see have obvious analogues for the other two possibilities. And, at the end of this section, we'll give the analogous tools that are appropriate for use with an integrand which is singular at one end of a finite domain of integration.

Now let's start. Imagine that we have an improper integral $\int_a^\infty f(x)\,\mathrm{d}x$, that $f(x)$ has no singularities for $x \geqslant a$ and that $f(x)$ is complicated enough that we cannot evaluate the integral explicitly[78]. The idea is find another improper integral $\int_a^\infty g(x)\,\mathrm{d}x$

- with $g(x)$ simple enough that we can evaluate the integral $\int_a^\infty g(x)\,\mathrm{d}x$ explicitly, or at least determine easily whether or not $\int_a^\infty g(x)\,\mathrm{d}x$ converges, and
- with $g(x)$ behaving enough like $f(x)$ for large x that the integral $\int_a^\infty f(x)\,\mathrm{d}x$ converges if and only if $\int_a^\infty g(x)\,\mathrm{d}x$ converges.

So far, this is a pretty vague strategy. Here is a theorem which starts to make it more precise.

---

[76] The Gamma function is far more important than just a generalisation of the factorial. It appears all over mathematics, physics, statistics and beyond. It has all sorts of interesting properties and its definition can be extended from natural numbers n to all numbers excluding $0, -1, -2, -3, \ldots$. For example, one can show that

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}.$$

[77] Applying numerical integration methods to a divergent integral may result in perfectly reasonably looking but very wrong answers.

[78] You could, for example, think of something like our running example $\int_a^\infty e^{-t^2}\mathrm{d}t$.

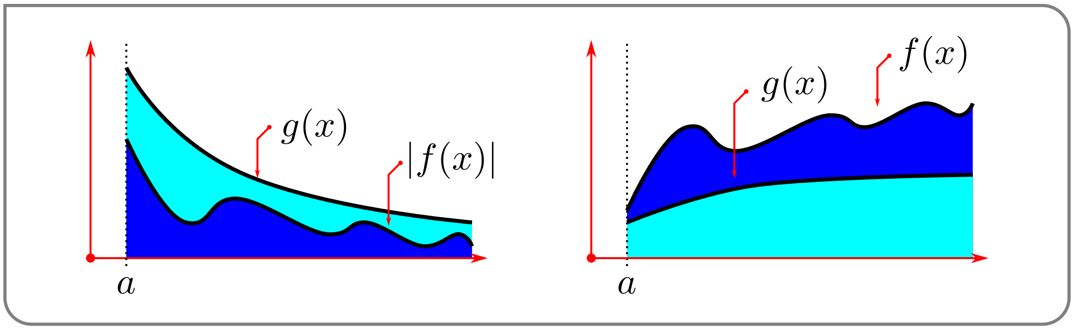
Theorem 1.13.17 (Comparison).

Let a be a real number. Let f and g be functions that are defined and continuous for all $x \geqslant a$ and assume that $g(x) \geqslant 0$ for all $x \geqslant a$.

(a) If $|f(x)| \leqslant g(x)$ for all $x \geqslant a$ and if $\int_a^\infty g(x)\,\mathrm{d}x$ converges then $\int_a^\infty f(x)\,\mathrm{d}x$ also converges.

(b) If $f(x) \geqslant g(x)$ for all $x \geqslant a$ and if $\int_a^\infty g(x)\,\mathrm{d}x$ diverges then $\int_a^\infty f(x)\,\mathrm{d}x$ also diverges.

We will not prove this theorem, but hopefully the following supporting arguments should at least appear reasonable to you. Consider the figure below:



- If $\int_a^\infty g(x)\,\mathrm{d}x$ converges, then the area of
$$\{\ (x,y)\ |\ x \geqslant a,\ 0 \leqslant y \leqslant g(x)\ \} \text{ is finite.}$$
When $|f(x)| \leqslant g(x)$, the region
$$\{\ (x,y)\ |\ x \geqslant a,\ 0 \leqslant y \leqslant |f(x)|\ \} \text{ is contained inside } \{\ (x,y)\ |\ x \geqslant a,\ 0 \leqslant y \leqslant g(x)\}$$
and so must also have finite area. Consequently the areas of both the regions
$$\{\ (x,y)\ |\ x \geqslant a,\ 0 \leqslant y \leqslant f(x)\ \} \text{ and } \{\ (x,y)\ |\ x \geqslant a,\ f(x) \leqslant y \leqslant 0\}$$
are finite too[79].

- If $\int_a^\infty g(x)\,\mathrm{d}x$ diverges, then the area of
$$\{\ (x,y)\ |\ x \geqslant a,\ 0 \leqslant y \leqslant g(x)\ \} \text{ is infinite.}$$
When $f(x) \geqslant g(x)$, the region
$$\{\ (x,y)\ |\ x \geqslant a,\ 0 \leqslant y \leqslant f(x)\ \} \text{ contains the region } \{\ (x,y)\ |\ x \geqslant a,\ 0 \leqslant y \leqslant g(x)\ \}$$
and so also has infinite area.

Example 1.13.18 $\left(\int_1^\infty e^{-x^2}\,\mathrm{d}x\right)$

We cannot evaluate the integral $\int_1^\infty e^{-x^2}\,\mathrm{d}x$ explicitly[80], however we would still like to un-

---

[79] We have separated the regions in which $f(x)$ is positive and negative, because the integral $\int_a^\infty f(x)\,\mathrm{d}x$ represents the signed area of the union of $\{\ (x,y)\ |\ x \geqslant a,\ 0 \leqslant y \leqslant f(x)\ \}$ and $\{\ (x,y)\ |\ x \geqslant a,\ f(x) \leqslant y \leqslant 0\ \}$.

[80] It has been the subject of many remarks and footnotes.

derstand if it is finite or not — does it converge or diverge?

Solution. We will use Theorem 1.13.17 to answer the question.

- So we want to find another integral that we can compute and that we can compare to $\int_1^\infty e^{-x^2}\,\mathrm{d}x$. To do so we pick an integrand that looks like e^{-x^2}, but whose indefinite integral we know — such as e^{-x}.
- When $x \geqslant 1$, we have $x^2 \geqslant x$ and hence $e^{-x^2} \leqslant e^{-x}$. Thus we can use Theorem 1.13.17 to compare

$$\int_1^\infty e^{-x^2}\mathrm{d}x \text{ with } \int_1^\infty e^{-x}\mathrm{d}x$$

- The integral

$$\begin{aligned}\int_1^\infty e^{-x}\,\mathrm{d}x &= \lim_{R\to\infty}\int_1^R e^{-x}\,\mathrm{d}x\\ &= \lim_{R\to\infty}\Big[-e^{-x}\Big]_1^R\\ &= \lim_{R\to\infty}\Big[e^{-1}-e^{-R}\Big] = e^{-1}\end{aligned}$$

converges.

- So, by Theorem 1.13.17, with $a = 1$, $f(x) = e^{-x^2}$ and $g(x) = e^{-x}$, the integral $\int_1^\infty e^{-x^2}\,\mathrm{d}x$ converges too (it is approximately equal to 0.1394).

Example 1.13.18

Example 1.13.19 $\left(\int_{1/2}^\infty e^{-x^2}\,\mathrm{d}x\right)$

Solution.

- The integral $\int_{1/2}^\infty e^{-x^2}\,\mathrm{d}x$ is quite similar to the integral $\int_1^\infty e^{-x^2}\,\mathrm{d}x$ of Example 1.13.18. But we cannot just repeat the argument of Example 1.13.18 because it is not true that $e^{-x^2} \leqslant e^{-x}$ when $0 < x < 1$.
- In fact, for $0 < x < 1$, $x^2 < x$ so that $e^{-x^2} > e^{-x}$.
- However the difference between the current example and Example 1.13.18 is

$$\int_{1/2}^\infty e^{-x^2}\,\mathrm{d}x - \int_1^\infty e^{-x^2}\,\mathrm{d}x = \int_{1/2}^1 e^{-x^2}\,\mathrm{d}x$$

which is clearly a well defined finite number (it's actually about 0.286). It is important to note that we are being a little sloppy by taking the difference of two integrals like this — we are assuming that both integrals converge. More on this below.

- So we would expect that $\int_{1/2}^{\infty} e^{-x^2}\,\mathrm{d}x$ should be the sum of the proper integral integral $\int_{1/2}^{1} e^{-x^2}\,\mathrm{d}x$ and the convergent integral $\int_{1}^{\infty} e^{-x^2}\,\mathrm{d}x$ and so should be a convergent integral. This is indeed the case. The theorem below provides the justification.

Example 1.13.19

Theorem 1.13.20.

Let a and c be real numbers with $a < c$ and let the function $f(x)$ be continuous for all $x \geqslant a$. Then the improper integral $\int_a^{\infty} f(x)\,\mathrm{d}x$ converges if and only if the improper integral $\int_c^{\infty} f(x)\,\mathrm{d}x$ converges.

Proof. By definition the improper integral $\int_a^{\infty} f(x)\,\mathrm{d}x$ converges if and only if the limit

$$\begin{aligned}\lim_{R\to\infty}\int_a^R f(x)\,\mathrm{d}x &= \lim_{R\to\infty}\left[\int_a^c f(x)\,\mathrm{d}x + \int_c^R f(x)\,\mathrm{d}x\right]\\ &= \int_a^c f(x)\,\mathrm{d}x + \lim_{R\to\infty}\int_c^R f(x)\,\mathrm{d}x\end{aligned}$$

exists and is finite. (Remember that, in computing the limit, $\int_a^c f(x)\,\mathrm{d}x$ is a finite constant independent of R and so can be pulled out of the limit.) But that is the case if and only if the limit $\lim_{R\to\infty}\int_c^R f(x)\,\mathrm{d}x$ exists and is finite, which in turn is the case if and only if the integral $\int_c^{\infty} f(x)\,\mathrm{d}x$ converges. ☐

Example 1.13.21

Does the integral $\int_1^{\infty} \frac{\sqrt{x}}{x^2+x}\,\mathrm{d}x$ converge or diverge?

Solution.

- Our first task is to identify the potential sources of impropriety for this integral.
- The domain of integration extends to $+\infty$, but we must also check to see if the integrand contains any singularities. On the domain of integration $x \geqslant 1$ so the denominator is never zero and the integrand is continuous. So the only problem is at $+\infty$.
- Our second task is to develop some intuition[81]. As the only problem is that the domain of integration extends to infinity, whether or not the integral converges will be determined by the behavior of the integrand for very large x.

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[81] This takes practice, practice and more practice. At the risk of alliteration — please perform plenty of practice problems.

- When x is very large, x^2 is much much larger than x (which we can write as $x^2 \gg x$) so that the denominator $x^2 + x \approx x^2$ and the integrand

$$\frac{\sqrt{x}}{x^2+x} \approx \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

- By Example 1.13.8, with $p = 3/2$, the integral $\int_1^\infty \frac{\mathrm{d}x}{x^{3/2}}$ converges. So we would expect that $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\,\mathrm{d}x$ converges too.

- Our final task is to verify that our intuition is correct. To do so, we want to apply part (a) of Theorem 1.13.17 with $f(x) = \frac{\sqrt{x}}{x^2+x}$ and $g(x)$ being $\frac{1}{x^{3/2}}$, or possibly some constant times $\frac{1}{x^{3/2}}$. That is, we need to show that for all $x \geqslant 1$ (i.e. on the domain of integration)

$$\frac{\sqrt{x}}{x^2+x} \leqslant \frac{A}{x^{3/2}}$$

  for some constant A. Let's try this.

- Since $x \geqslant 1$ we know that

$$x^2 + x > x^2$$

  Now take the reciprocal of both sides:

$$\frac{1}{x^2+x} < \frac{1}{x^2}$$

  Multiply both sides by $\sqrt{x}$ (which is always positive, so the sign of the inequality does not change)

$$\frac{\sqrt{x}}{x^2+x} < \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

- So Theorem 1.13.17(a) and Example 1.13.8, with $p = 3/2$ do indeed show that the integral $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\,\mathrm{d}x$ converges.

Example 1.13.21

Notice that in this last example we managed to show that the integral exists by finding an integrand that behaved the same way for large x. Our intuition then had to be bolstered with some careful inequalities to apply the comparison Theorem 1.13.17. It would be nice to avoid this last step and be able jump from the intuition to the conclusion without messing around with inequalities. Thankfully there is a variant of Theorem 1.13.17 that is often easier to apply and that also fits well with the sort of intuition that we developed to solve Example 1.13.21.

A key phrase in the previous paragraph is “behaves the same way for large x”. A good way to formalise this expression — “$f(x)$ behaves like $g(x)$ for large x” — is to require that the limit

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} \text{ exists and is a finite nonzero number.}$$

Suppose that this is the case and call the limit $L \neq 0$. Then

- the ratio $\frac{f(x)}{g(x)}$ must approach L as x tends to $+\infty$.
- So when x is very large — say $x > B$, for some big number B — we must have that
  $$\frac{1}{2}L \leqslant \frac{f(x)}{g(x)} \leqslant 2L \qquad\qquad \text{for all } x > B$$
  Equivalently, $f(x)$ lies between $\frac{L}{2}g(x)$ and $2Lg(x)$, for all $x \geqslant B$.
- Consequently, the integral of $f(x)$ converges if and only if the integral of $g(x)$ converges, by Theorems 1.13.17 and 1.13.20.

These considerations lead to the following variant of Theorem 1.13.17.

Theorem 1.13.22 (Limiting comparison).

Let $-\infty < a < \infty$. Let f and g be functions that are defined and continuous for all $x \geqslant a$ and assume that $g(x) \geqslant 0$ for all $x \geqslant a$.

(a) If $\int_a^\infty g(x)\,\mathrm{d}x$ converges and the limit

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

exists, then $\int_a^\infty f(x)\,\mathrm{d}x$ converges.

(b) If $\int_a^\infty g(x)\,\mathrm{d}x$ diverges and the limit

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

exists and is nonzero, then $\int_a^\infty f(x)\,\mathrm{d}x$ diverges.

Note that in (b) the limit must exist and be nonzero, while in (a) we only require that the limit exists. (It can be zero.)

Here is an example of how Theorem 1.13.22 is used.

Example 1.13.23 $\left(\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\,\mathrm{d}x\right)$

Does the integral $\displaystyle\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\,\mathrm{d}x$ converge or diverge?

Solution.

- Our first task is to identify the potential sources of impropriety for this integral.
- The domain of integration extends to $+\infty$. On the domain of integration the denominator is never zero so the integrand is continuous. Thus the only problem is at $+\infty$.
- Our second task is to develop some intuition about the behavior of the integrand for very large x. A good way to start is to think about the size of each term when x becomes big.
- When x is very large:
  - $e^{-x} \ll x^2$, so that the denominator $e^{-x} + x^2 \approx x^2$, and
  - $|\sin x| \leqslant 1 \ll x$, so that the numerator $x + \sin x \approx x$, and
  - the integrand $\frac{x+\sin x}{e^{-x}+x^2} \approx \frac{x}{x^2} = \frac{1}{x}$.

  Notice that we are using $A \ll B$ to mean that "A is much much smaller than B". Similarly $A \gg B$ means "A is much much bigger than B". We don't really need to be too precise about its meaning beyond this in the present context.
- Now, since $\int_1^\infty \frac{\mathrm{d}x}{x}$ diverges, we would expect $\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\,\mathrm{d}x$ to diverge too.
- Our final task is to verify that our intuition is correct. To do so, we set

$$f(x) = \frac{x+\sin x}{e^{-x}+x^2} \qquad\qquad g(x) = \frac{1}{x}$$

  and compute

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x+\sin x}{e^{-x}+x^2} \div \frac{1}{x} \\ &= \lim_{x\to\infty} \frac{(1+\sin x/x)x}{(e^{-x}/x^2+1)x^2} \times x \\ &= \lim_{x\to\infty} \frac{1+\sin x/x}{e^{-x}/x^2+1} \\ &= 1 \end{aligned}$$

- Since $\int_1^\infty g(x)\,\mathrm{d}x = \int_1^\infty \frac{\mathrm{d}x}{x}$ diverges, by Example 1.13.8 with $p = 1$, Theorem 1.13.22(b) now tells us that $\int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\,\mathrm{d}x$ diverges too.

Example 1.13.23

We finish this section by giving variants of Theorems 1.13.17 and 1.13.22 that are useful in dealing with an integrand that is singular at one end of a (finite) domain of integration. For the rest of this section, let $a < b$ and let $[a, b] = \{x \,|\, a \leqslant x \leqslant b\}$ be the domain of integration. We allow the integrand to be singular at one end of the domain of integration, which we'll call x_s. So

- if the integrand is singular at $x = a$, we set $x_s = a$ and write

  $$I = (a, b] = \{x \mid a < x \leqslant b\}$$

  and

- if the integrand is singular at $x = b$ we set $x_s = b$ and write

  $$I = [a, b) = \{x \mid a \leqslant x < b\}$$

Recall that “$x \in I$” means “x is an element of the set I”. Here is the analogue of Theorem 1.13.17

Theorem 1.13.24 (Comparison)**.**

Let f and g be functions that are defined and continuous for all $x \in I$ and assume that $g(x) \geqslant 0$ for all $x \in I$.

(a) If $|f(x)| \leqslant g(x)$ for all $x \in I$ and if $\int_a^b g(x)\,\mathrm{d}x$ converges then $\int_a^b f(x)\,\mathrm{d}x$ also converges.

(b) If $f(x) \geqslant g(x)$ for all $x \in I$ and if $\int_a^b g(x)\,\mathrm{d}x$ diverges then $\int_a^b f(x)\,\mathrm{d}x$ also diverges.

and here is the analogue of Theorem 1.13.22

Theorem 1.13.25 (Limiting comparison)**.**

Let f and g be functions that are defined and continuous for all $x \in I$ and assume that $g(x) \geqslant 0$ for all $x \in I$.

(a) If $\int_a^b g(x)\,\mathrm{d}x$ converges and the limit

$$\lim_{\substack{x \to x_s \\ x \in I}} \frac{f(x)}{g(x)}$$

exists, then $\int_a^b f(x)\,\mathrm{d}x$ converges.

(b) If $\int_a^b g(x)\,\mathrm{d}x$ diverges and the limit

$$\lim_{\substack{x \to x_s \\ x \in I}} \frac{f(x)}{g(x)}$$

exists and <u>is nonzero</u>, then $\int_a^b f(x)\,\mathrm{d}x$ diverges.

Note that in (b) the limit must exist and be nonzero, while in (a) we only require that the limit exists. (It can be zero.)

Chapter 2

PROBABILITY

2.1▲ Introduction

Learning Objectives

- Understand basic concepts, vocabulary, and notation related to probability

Before we start, a note. Most terms in this introductory section (probability, event, value) accord pretty well with their usage in everyday life. However, later on in the chapter we will introduce new vocabulary and notation (PDF, $\mathbb{E}$) whose interpretations are far less obvious. Keeping track of definitions will be key to understanding what's going on. Make flashcards if you have a hard time remembering different terms. If you read a term whose meaning you've forgotten, look it up! If we don't have the same vocabulary, then we aren't speaking the same language – so it will be difficult to explain things.

2.1.1 ▸▸ Foundational vocabulary and notation

Definition 2.1.1.

A **probability** is a number between 0 and 1. We interpret it as a likelihood.

If an outcome of an event has probability 1, it will certainly happen. If an outcome of an event has probability close to 0, then it will probably not happen. If an outcome has probability $\frac{1}{2}$, it has an equal chance of happening and not happening. If we have an event with an outcome with probability $\frac{1}{2}$ and the event happens a large number of times, we expect the outcome to occur in roughly half of those trials.

This important characteristic of probability experiments is known as the **law of large numbers**, which states that as the number of repetitions of an experiment is increased, the relative frequency obtained in the experiment tends to become closer and closer to the theoretical probability. Even though the outcomes do not happen according to any set pattern or order, overall, the long-term observed relative frequency will approach the theoretical probability. (The word **empirical** is often used instead of the word observed.)

A random variable is a lot like the ordinary variables you're used to using in functions, in that it is a kind of placeholder that can take on different values. The "random" part of the name explains that the values taken on are the result of an event – some experiment or random process.

Definition 2.1.2.

A **random variable** (or just **variable**) is a characteristic or measurement that can be determined for each outcome of some event.

Random variables are usually denoted with capital letters, like X or Y. When they correspond to a clear event, we may also give them names like "*Flip*" (for a coin flip) or "*Roll*" (for a dice roll).

Events result in **values**. The event of rolling a dice might result in the value 1; the event of flipping a coin might result in the value heads; and the event of choosing a person might result in the value Parham. We usually use lower-case letters as variables specifying values.

We'll mostly use events that result in numerical values, although coin flips are a handy experiment as well. Unless otherwise specified, you can assume values will be numbers. (Otherwise our formulas become quite abstract – we won't ask you to average people or integrate colours.)

Example 2.1.3

Let *Roll* be the random variable corresponding to the event of rolling a standard 6-sided dice[1]. *Roll* can result in any of the values 1, 2, 3, 4, 5, or 6.

Suppose we are playing a game and our points are determined by doubling the number rolled. We might write the following:

If $\mathit{Roll} = x$, the number of points earned is $2x$.

Example 2.1.3

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[1] In the interest of clarity, we'll use "dice" as its own singular (as is common in colloquial English), rather than "die" (which is more standard in academic English).

Notation 2.1.4.

We'll use the shorthand $Pr(A)$ to mean "the probability that A happens." For example:

$$Pr(E = x)$$

denotes "the probability that the event E results in the value x."

The equation $E = x$ can take some getting used to. Remember that E corresponds to the event (like rolling a dice), while x corresponds to the outcome of that event (e.g. 5).

Example 2.1.5

To express "the probability of a dice roll being 5 is 1/6," we write:

$$Pr(Roll = 5) = \frac{1}{6}$$

where *Roll* is the event (dice roll), 5 is the value, and $\frac{1}{6}$ is the probability.

Example 2.1.5

Example 2.1.6

If R is the roll of a fair dice, then

$$Pr(R = 1 \text{ or } R = 2) = \frac{1}{3}$$

Example 2.1.6

Example 2.1.7

Let F be the event of flipping a fair coin (that is, a coin that is equally likely to come up heads or tails), and let x be one of the values "heads" or "tails." Then:

$$Pr(F = x) = \frac{1}{2}$$

Example 2.1.7

Definition 2.1.8.

The **sample space** of an event is the set of all possible outcomes. We will use $\mathcal{S}$ to denote the sample space.

Example 2.1.9

If you roll a standard dice, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.

Example 2.1.9

Warning 2.1.10.

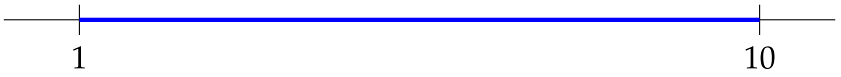
This seemingly straightforward definition can cause some confusion, especially when measured data is involved.

For example: suppose our random variable X is the mileage of a car picked at random out of a parking lot. If there are, say, 100 cars in that lot, then there are (at most) 100 values possible for X to take on. Unfortunately, we do not know those values.

When we use the context of a supposedly measured variable, we'll pretend that it could be anything theoretically sensible. In the case of the cars, we do know that all of those values will be nonnegative numbers. So, In this case, we could use the sample space $[0, \infty)$. Alternately we could say that each of those values is less than some arbitrarily huge number like 10^{12} km[2] and use the sample space $[0, 10^{12}]$.

Example 2.1.11

Let Y be the random variable corresponding to choosing *any* real number number in $[1, 10]$



$\mathcal{S} = [1, 10]$. There are infinitely many possible values, along a continuum, that could result.

Indeed, Y is an example of a *continuous random variable*. We won't define the *continuous* part of that designation here[3], but when we get out of this introductory section, this is the type of variable we'll be working with almost exclusively.

Example 2.1.11

Example 2.1.12 (A Probabilistic Model in Linguistics)

These introductory concepts are enough to start understanding probabilistic models in a wide array of fields. Here we'll consider a paragraph from a short paper about people's decisions to change the language they use over time.

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[2] that's farther than driving at 100 kph around the clock for one million years, so it's safe to say no car has more mileage than this

[3] The formal definition depends on something called the *cumulative distribution function*, which is no longer a part of Math 101.

The following quote is taken from the article: Abrams, D., Strogatz, S. Modelling the dynamics of language death. Nature 424, 900 (2003). DOI: `https://doi.org/10.1038/424900a`

> Consider a system of two competing languages, X and Y, in which the attractiveness of a language increases with both its number of speakers and its perceived status (a parameter that reflects the social or economic opportunities afforded to its speakers). Suppose an individual converts from Y to X with a probability, per unit of time, of $P_{yx}(x,s)$, where x is the fraction of the population speaking X, and $0 \leqslant s \leqslant 1$ is a measure of X's relative status. A minimal model for language change is therefore
>
> $$\frac{\mathrm{d}x}{\mathrm{d}t} = yP_{yx}(x,s) - xP_{xy}(x,s)$$
>
> where $y = 1 - x$ is the complementary fraction of the population speaking Y at time t.

Let's parse this quote in terms of vocabulary that is familiar to us.

- x is the fraction of a population speaking language X at a given time. So if everyone is speaking X, then $x = 1$; if half the population is speaking x, then $x = \frac{1}{2}$.
- y is the fraction of the population speaking language Y at a given time. Under the simplified assumptions of the model in the paper, everyone speaks either X or Y, but not both, at a particular time. So, $y = 1 - x$.
- $\frac{\mathrm{d}x}{\mathrm{d}t}$ is the rate of change of speakers of language X over time. So if $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive, then $\frac{\mathrm{d}x}{\mathrm{d}t}$ is increasing and so people are changing from language Y to language X; if $\frac{\mathrm{d}x}{\mathrm{d}t}$ is negative, then people are changing from language X to language Y.
- The random event in question is *a person changing their language*. The three values in its sample space are: person does not change; person changes from X to Y; and person changes from Y to X.
- The paper uses notation that is different from this textbook. They write P_{yx} for "the probability that a person changes from Y to X, and they write P_{xy} for "the probability that a person changes from X to Y.
- The probabilities come with arguments: $P_{yx}(x,s)$ and $P_{xy}(x,s)$. The variables inside the parentheses are function variables. How likely someone is to switch languages is not a fixed constant, but rather a function depending on how many people speak the language, and how much status that language is perceived to have. So, P_{yx} and P_{xy} are functions of multiple variables.

Now that we understand all the notation, we can figure out where the equation in the quote came from.

- x increases as people switch from speaking Y to speaking X. P_{yx} is the proportion of speakers of Y that we expect to change to X. The number of speakers of Y is y. So, we expect yP_{yx} people to change from Y to X.

- x decreases as people switch from speaking X to speaking Y. P_{xy} is the proportion of speakers of X that we expect to change to Y. The number of speakers of X is x. So, we expect xP_{xy} people to change from X to Y.
- All together, the change in x is (number of people coming to X from Y) minus (number of people going to Y from X), or

$$\frac{\mathrm{d}x}{\mathrm{d}t} = yP_{yx} - xP_{xy}$$

which is exactly the equation from the article.

Example 2.1.12

2.1.2 ▸▸ Combining events

Definition 2.1.13.

Two outcomes of an event are **disjoint** if no value in the sample space can be described by both outcomes.

For example, consider the event of rolling a dice, corresponding as usual to the random variable X. The outcomes $X = 1$ and $X = 2$ are disjoint, because no dice roll will result in both of them being true. On the other hand, the outcomes $X > 2$ and “X is even” are not disjoint, because a roll of 4 or 6 makes both of them true.

Example 2.1.14

Let X be a continuous random variable with sample space $[0, 10]$. For each collection of outcomes below, decide whether the outcomes are disjoint or not.

1. $X < 5$; $X \geqslant 5$
2. $X \geqslant 9$; $X \geqslant 8$
3. $1 < X < 2$; X even; X odd

Solution.

1. These are disjoint; no number is both less than five, and also greater than or equal to five.
2. These are not disjoint: $X = 9$, for example, makes both true. (So does $X = 9.5$, $X = 10$, etc.)
3. These are disjoint. If X is an integer, then it is even or odd but not both, and it is not in the interval $(1, 2)$. If X is in the interval $(1, 2)$, then it is not an integer, so it is not even and not odd.

Example 2.1.14

Theorem 2.1.15.

Suppose A and B represent disjoint outcomes of the same event. Then

$$Pr(A \text{ happens OR } B \text{ happens}) = Pr(A \text{ happens}) + Pr(B \text{ happens})$$

Warning 2.1.16.

In a mathematical context, "or" has a slightly different meaning from its colloquial use. When we say "A or B," we mean "A, B, *or both*."

An old joke is that a mathematician is told they may have a peanut butter cookie or a chocolate cookie, and takes one of each.

Proof. This comes from the interpretation of a probability as the proportion of trials where an outcome occurs. If A and B never occur at the same trial, then the proportion where one or the other occurs is simply the sum of the proportions where one occurs. ☐

Example 2.1.17

If X is the random variable corresponding to a dice throw, then $Pr(X \leqslant 3) = Pr(X = 1 \text{ OR } X = 2 \text{ OR } X = 3)$. Since the events $X = 1$, $X = 2$, and $X = 3$ are disjoint, this probability is equal to:

$$Pr(X = 1) + Pr(X = 2) + Pr(X = 3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

Example 2.1.17

Example 2.1.18

Suppose for the province of British Columbia, the probability that a randomly chosen adult resident will apply for employment insurance (EI) benefits this year is $\frac{3}{100}$, while the probability that a randomly chosen adult resident will be laid off from their job this year is $\frac{7}{100}$.

True or false: the probability that a randomly chosen adult resident will apply for EI or be laid off is $\frac{1}{10}$.

Solution. Not necessarily true (and almost certainly false). These are not disjoint events, so Theorem 2.1.15 does not apply.

Example 2.1.18

2.1.3 ▸▸ Equally likely outcomes

Definition 2.1.19.

Intuitively, a continuous random variable is **uniformly distributed** on an interval if the variable doesn't favour one region of the interval over any other region.

More formally:

Let X be a continuous random variable. X is **uniformly distributed** on the interval $[a, b]$ if there exists some constant c such that for any interval $[a_1, b_1]$ in $[a, b]$, $Pr(a_1 \leqslant X \leqslant b_1) = c(b_1 - a_1)$. That is, the probability that X is in a particular interval within $[a, b]$ depends only on the length of that interval.

Example 2.1.20

Suppose X is a continuous random variable that is **uniformly distributed** on the interval $[0, 10]$.

The intervals $[2, 5]$ and $[3, 6]$ have the same length, so $Pr(2 \leqslant X \leqslant 5) = Pr(3 \leqslant X \leqslant 6)$. The intervals $[2, 3]$, $[3, 4]$, and $[7, 8]$ have equal length, so $Pr(2 \leqslant X \leqslant 3) = Pr(3 \leqslant X \leqslant 4) = Pr(7 \leqslant X \leqslant 8)$. So, X is twice as likely to be in the interval $[2, 4]$ as it is to be in the interval $[7, 8]$.

Example 2.1.20

Corollary 2.1.21.

Suppose X is a continuous random variable that is uniformly distributed on its sample space, the interval $[a, b]$. Then for any interval $[a_1, b_1]$ with $a \leqslant a_1 \leqslant b_1 \leqslant b$,

$$Pr(a_1 \leqslant X \leqslant b_1) = \frac{b_1 - a_1}{b - a}$$

That is, the probability that X is in the interval $[a_1, b_1]$ is the ratio of the length of that interval to the sample space interval.

Proof. Since the sample space of X is $[a, b]$,

$$Pr(a \leqslant X \leqslant b) = 1$$

Since X is uniformly distributed on $[a, b]$, there exists a constant c such that $Pr(a_1 \leqslant X \leqslant b_1) = c(b_1 - a_1)$ for any interval $[a_1, b_1]$ inside the interval $[a, b]$. So,

$$1 = Pr(a \leqslant X \leqslant b) = c(b - a) \implies c = \frac{1}{b - a}$$

Then:

$$Pr(a_1 \leqslant X \leqslant b_1) = c(b_1 - a_1) = \frac{b_1 - a_1}{b - a}$$

□

Example 2.1.22

Let X be a continuous random variable that is uniformly distributed on its sample space, the interval $[0, 10]$. What is $Pr(7 \leqslant X \leqslant 9)$?

Solution.

The interval $[7, 9]$ has length 2; the sample space interval $[0, 10]$ has length 10. So,

$$Pr(7 \leqslant X \leqslant 9) = \frac{2}{10} = \frac{1}{5}$$

Example 2.1.22

Example 2.1.23

Let X be a continuous random variable that is uniformly distributed across its sample space $[-8, 17]$. Calculate the probabilities below.

1. $Pr(1 \leqslant X \leqslant 2)$
2. $Pr(-5 \leqslant X)$
3. $Pr(-10 \leqslant X \leqslant 10)$

Solution.

1. By Corollary 2.1.21, $Pr(1 \leqslant X \leqslant 2) = \frac{2-1}{17-(-8)} = \frac{1}{25}$
2. Since X only takes on values in its sample space $[-8, 17]$: $Pr(-5 \leqslant X) = Pr(-5 \leqslant X \leqslant 17)$. By Corollary 2.1.21, $Pr(-5 \leqslant X \leqslant 17) = \frac{17-(-5)}{17-(-8)} = \frac{22}{25}$
3. Since X only takes on values in its sample space $[-8, 27]$: $Pr(-10 \leqslant X \leqslant 10) = Pr(-8 \leqslant X \leqslant 10)$. Now the interval $[-8, 10]$ is inside our sample space, unlike the interval $[-10, 10]$, so we can apply Corollary 2.1.21.

   $Pr(-8 \leqslant X \leqslant 10) = \frac{10-(-8)}{17-(-8)} = \frac{18}{25}$

Example 2.1.23

Example 2.1.24

Suppose the continouous variable X is the age of a randomly chosen living person, measured in years with exact precision. Then X is more likely to be near 50 than it is to be near 110. So, X is *not* uniformly distributed.

Example 2.1.24

2.2▲ Probability density

Learning Objectives

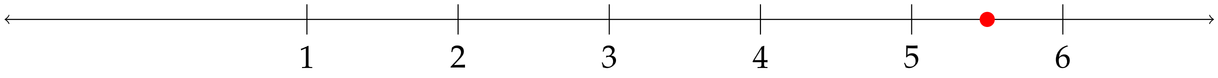
- Define Probability Density Function (PDF) informally as the function $f(t)$ such that $Pr(a \leqslant X \leqslant b) = \int_a^b f(t)\,\mathrm{d}t$.
- Interpret the PDF in terms of relative likelihoods of different regions.
- Use a PDF to compute probabilities.
- Learn properties of PDFs: $f(t) \geqslant 0$ and $\int_{-\infty}^{\infty} f(t)\mathrm{d}t = 1$.
- Use the properties of PDFs to find unknown parameters in its definition.

2.2.1 ▸▸ Density diagrams

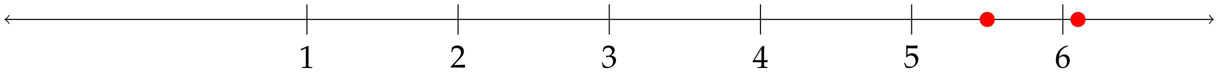
We're going to introduce a tool for visualizing random processes that will hopefully help topics in continuous random variables be more intuitive. We'll call that tool a *density diagram.*

Let X be some continuous random variable. If X is a process (like choosing a student at random and recording their height), we can imagine performing that process again and again and again. Suppose we do just that. Every time we get a new value of X, we put a mark on a number line. For example:

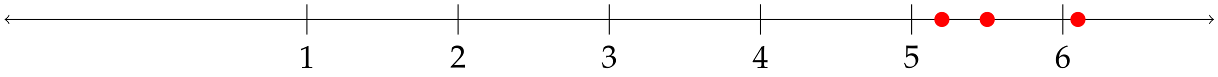
1. The first randomly-chosen student has height 5.5 feet:



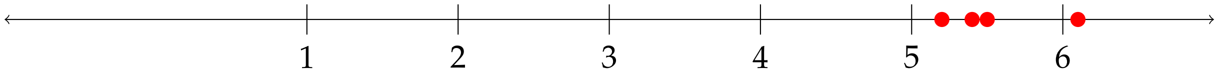
2. The second randomly-chosen student has height 6.1 feet:



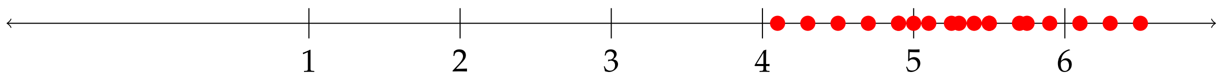
3. The third randomly-chosen student has height 5.2 feet:



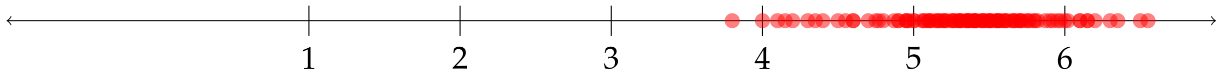
4. The third randomly-chosen student has height 5.4 feet:



5. After 20 choices, our results might look like this:



6. After 100 choices, our marks would start being so close together, they would be indistinguishable, so we might choose to make the marks slightly transparent. Then darker regions represent ranges where more heights have been chosen.



Example 2.2.1

Match the density diagrams to the variable descriptions so that every description corresponds to exactly one density diagram.

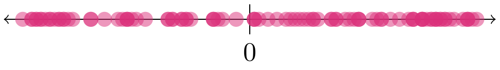
A. $Pr(X \leqslant 0) = Pr(X \geqslant 0)$.

B. X is uniformly distributed.

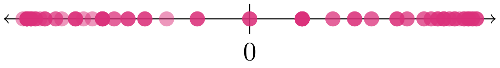
C. $Pr(X \leqslant 0) < Pr(X \geqslant 0)$.

D. $Pr(X \leqslant 0) > Pr(X \geqslant 0)$.

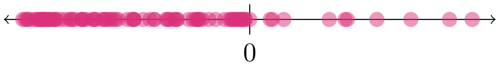
1.



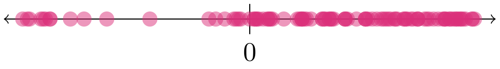
2.



3.



4.



Solution. In both 1 and 2, it seems like (roughly) the same number of trials resulted in positive and negative values of X. So in both cases, A holds. However, in 2, the distribution is not uniform: trials are more likely to have large absolute values than to be near 0. So, we match B to 1 and A to 2.

In 3, more trials gave $X \leqslant 0$ than $X \geqslant 0$, so we match that to D.

In 4, more trials gave $X \geqslant 0$ than $X \leqslant 0$, so we match that to C.

Example 2.2.1

2.2.2 ▸▸ Probability density function (PDF)

Informal Definition 2.2.2.

Let X be a continuous random variable, and suppose there is some function $f(x)$ with the following property: for all intervals (a, b) in the real numbers,

$$Pr(a \leqslant X \leqslant b) = \int_a^b f(x)\, \mathrm{d}x\ .$$

We call a function with this property the **probability density function (PDF)** of X.

(The formal definition of a PDF is the derivative of the cumulative distribution function (CDF), but we won't talk about CDFs in Math 101, so this informal definition will do.)

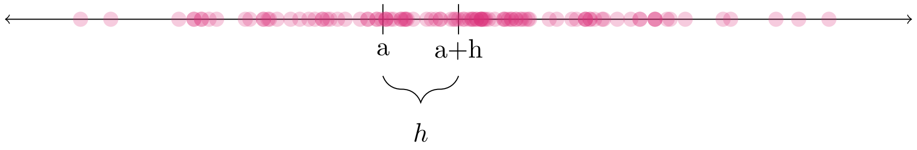
Intuitively: let $f(x)$ be the probability density function (PDF) of a continuous random variable X. If $f(x) > f(y)$, then values of X are likely to cluster more densely around x than around y.

Here's another way to think about probability density functions (PDFs). When looking at density diagrams, areas with more "hits" show up as having a higher *density* of marks. This idea will be central to this section: measuring the *density* of a continuous random variable.

A usual definition of density is something like

$$\frac{\text{how much stuff}}{\text{how much space}}\ .$$

Population density might be measured in people per square kilometre, liquid density might be measured in grams per mL, etc. Probability density follows a similar pattern: we'll measure *how likely a variable is to be in a given interval* and divide it by the size (length) of that interval.



Suppose the density diagram above represents some continuous random variable X, and we want to measure the probability density near the indicated point a. We start by defining a small interval around a. As is tradition, we take the interval between a and $a + h$, where h is some small[4] real number.

It doesn't make sense to count the marks is this interval, since the actual number will change as we repeat our trials, so instead we take the likeliness our random variable is to

---

[4] By "small," we mean $|h| \approx 0$. In the discussion that follows, we're considering the case $h > 0$; the case $h < 0$ proceeds in the same way.

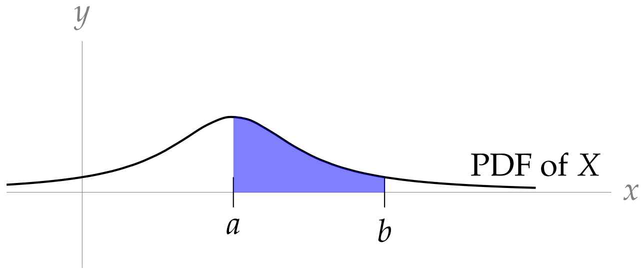
be in this interval: $Pr(a \leqslant X \leqslant a+h)$. The length of the interval is h. So, our probability density around a is about:

$$\frac{Pr(a \leqslant X \leqslant a+h)}{h}.$$

Let $f(x)$ be our PDF, so $f(a) \approx \frac{Pr(a \leqslant X \leqslant a+h)}{h}$ for a small h. For small h, the *area underneath the PDF* will be approximately:

$$\underbrace{f(a)}_{\text{height}} \cdot \underbrace{h}_{\text{width}} \approx \frac{Pr(a \leqslant X \leqslant a+h)}{h} \cdot h = Pr(A \leqslant X \leqslant a+h)$$

so the idea that $Pr(a \leqslant X \leqslant b) = \int_a^b f(x)\,\mathrm{d}x$ (area under PDF gives probability) fits.



shaded area: $Pr(a \leqslant X \leqslant b)$

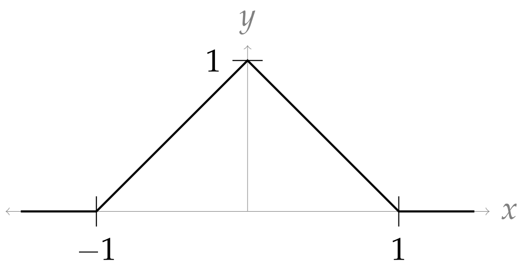
Example 2.2.3

Suppose X is a continuous random variable with probability density function (PDF)

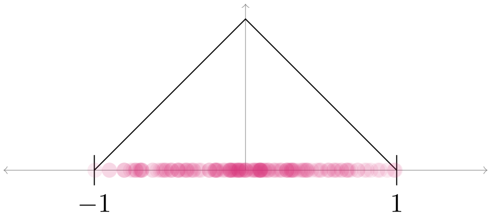
$$f(x) = \begin{cases} 1+x & -1 \leqslant x \leqslant 0 \\ 1-x & 0 < x \leqslant 1 \\ 0 & \text{else} \end{cases}.$$

1. Sketch $f(x)$.
2. On the same axes, sketch a density diagram for X.
3. Find $Pr(0 \leqslant X \leqslant \frac{1}{2})$.

Solution. We start with a sketch:



In our density diagram, the density is 0 outside the interval $[-1, 1]$, so there are no dots there. Inside the interval, the dots are densest near $x = 0$, and less dense near the endpoints. So, we imagine something like this:



Finally, we compute the desired probability:

$$\begin{aligned} Pr(0 \leqslant X \leqslant \frac{1}{2}) &= \int_0^{1/2} f(x)\,\mathrm{d}x \\ &= \int_0^{1/2} (1-x)\,\mathrm{d}x \\ &= \frac{3}{8} \end{aligned}$$

Example 2.2.3

Notation 2.2.4.

It is common to suppress the regions where a probability density function (PDF) is zero or doesn't exist. Instead of writing, say,

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{5000} & 0 \leqslant x < 100 \\ 0 & x > 100 \end{cases}$$

we may also write

$$f(x) = \begin{cases} \frac{x}{5000} & 0 \leqslant x < 100 \end{cases}$$

and it is understood that $f(x)$ is zero or doesn't exist when x *not* in the interval $[0, 100)$.

Another time-saving measure is to use the words "else" or "otherwise" in a piecewise-defined function. In the context of this function:

$$f(x) = \begin{cases} \frac{x}{5000} & 0 \leqslant x < 100 \\ 0 & \text{else} \end{cases}$$

"else" means "for all values of x *other than* the ones that have already been defined," i.e. for all values of x outside the interval $[0, 100)$.

Corollary 2.2.5.

Given a continuous random variable X with probability density function (PDF) $f(x)$:

1. $\int_{-\infty}^{\infty} f(x) = 1$.

   Furthermore,

2. $f(x) \geqslant 0$ for all real x in the domain of f.

Proof.

1. From Informal Definition 2.2.2, $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = Pr(-\infty \leqslant X \leqslant \infty)$, which must be 1, since *all* real numbers are between $-\infty$ and ∞.
2. From the informal definition, and the fact that probabilities are never negative, we can say that the area underneath $f(x)$ is never negative. This doesn't necessarily imply that $f(x) \geqslant 0$, so for this property, you'll have to take it from us that this property follows from the formal definition of a PDF.

☐

Example 2.2.6

A continuous random variable X has probability density function (PDF)

$$f(x) = \frac{a}{x^2+1}$$

for some constant a.

1. Find a.
2. Find $Pr(0 \leqslant X \leqslant 10)$.

Solution.

1. By Corollary 2.2.5,

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} \frac{a}{x^2+1}\mathrm{d}x = a\int_{-\infty}^{\infty} \frac{1}{x^2+1}\mathrm{d}x \\ &= a\left[\lim_{b\to-\infty}\int_{b}^{0} \frac{1}{x^2+1}\mathrm{d}x + \lim_{c\to\infty}\int_{0}^{c} \frac{1}{x^2+1}\mathrm{d}x\right] \\ &= a\left[\left(\lim_{b\to-\infty}(\arctan 0 - \arctan b)\right) + \left(\lim_{c\to\infty}(\arctan c - \arctan 0)\right)\right] \\ &= a\left[0 - \frac{-\pi}{2} + \frac{\pi}{2} + 0\right] = a\cdot\pi\end{aligned}$$

So, $a = \frac{1}{\pi}$.

2. By Informal Definition 2.2.2,

$$Pr(0 \leqslant X \leqslant 10) = \int_0^{10} f(x)\mathrm{d}x = \int_0^{10} \frac{1/\pi}{x^2+1}\mathrm{d}x = \frac{1}{\pi}[\arctan 10 - \arctan 0] = \frac{\arctan(10)}{\pi} \approx 0.47$$

Note: because $f(x)$ has even symmetry, we know $Pr(X \leqslant 0) = Pr(X \geqslant 0) = \frac{1}{2}$. Also, $Pr(0 \leqslant X \leqslant 10) \leqslant Pr(0 \leqslant X)$, so it stands to reason that our answer would be less than one-half.

Example 2.2.6

Corollary 2.2.7.

Let X be a continuous random variable. For any real number a,

$$Pr(X = a) = 0\,.$$

Furthermore,

$$Pr(X < a) = Pr(X \leqslant a) \qquad \text{and} \qquad Pr(X > a) = Pr(X \geqslant a)\,.$$

Proof. Let $f(x)$ be a PDF of X. Then $Pr(X = a) = Pr(a \leqslant X \leqslant a) = \int_a^a f(x)\,\mathrm{d}x = 0$. The other two properties follow directly from this. ☐

Warning 2.2.8.

Let $f(x)$ be the probability density function (PDF) of a continuous random variable. If $f(x) > f(y)$, it is not correct to say that x is more likely than y, because it is still the case that $Pr(X = x) = Pr(X = y) = 0$.

2.3▲ Expected value

Learning Objectives

- Explain what is meant by a "long-term average" and contrast this with the outcome of finitely many experiments.
- Define expected value for continuous systems.
- Compute the expected value for continuous systems.
- For an increasing or decreasing PDF use an intuitive argument to check whether the expected value is more or less than the halfway point of the space. Use

this to check expectation calculations.

2.3.1 ▸▸ Motivation: long-term average

Suppose I throw a 4-sided dice a large number of times, and record the number that comes up each time. What will the average (mean) of those numbers be?

To calculate the mean, I'll add up the results of my rolls and divide by the number of rolls I took.

$$\text{mean} = \frac{(\text{result of first roll}) + (\text{result of second roll}) + \cdots + (\text{result of last roll})}{\text{total number of rolls}}$$

The numerator will consist of the numbers 1 through 4, since these are the numbers resulting from a 4-sided dice roll. Let's regroup the numerator so we add up all the 1s first, then all the 2s second, etc.

$$= \frac{(1+1+\cdots)+(2+2+\cdots)+(3+3+\cdots)+(4+4+\cdots)}{\text{total number of rolls}}$$

$$= \frac{(1+1+\cdots)}{\text{total rolls}} + \frac{(2+2+\cdots)}{\text{total rolls}} + \frac{(3+3+\cdots)}{\text{total rolls}} + \frac{(4+4+\cdots)}{\text{total rolls}}$$

$$= \frac{1\cdot(\text{number of times 1 was rolled})}{\text{total rolls}} + \frac{2\cdot(\text{number of times 2 was rolled})}{\text{total rolls}}$$

$$+ \frac{3\cdot(\text{number of times 3 was rolled})}{\text{total rolls}} + \frac{4\cdot(\text{number of times 4 was rolled})}{\text{total rolls}}$$

$$= 1\cdot(\text{proportion of rolls resulting in 1}) + 2\cdot(\text{proportion of rolls resulting in 2})$$

$$+ 3\cdot(\text{proportion of rolls resulting in 3}) + 4\cdot(\text{proportion of rolls resulting in 4})$$

If we've rolled the dice a large number of times, we expect the proportion of rolls resulting in 1 to closely approximate $Pr(X=1)$, and so on.

$$\approx 1\cdot Pr(X=1) + 2\cdot Pr(X=2) + 3\cdot Pr(X=3) + 4\cdot Pr(X=4) = \sum_{x=1}^{4} x\cdot Pr(X=x)$$

This calculation, what we expect to have as our average if we perform the dice roll a large number of times, motivates Definition 2.3.1 below.

2.3.2 ▸▸ Definition and examples

The **expected value** or **expectation** of a random variables is often referred to as the **"long-term" average**. This means that over the long term of doing an experiment over and over, you would expect this average.

Definition 2.3.1.

Given a continuous random variable X with probability density function (PDF) $f(x)$, the expected value of X is given by

$$\int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x$$

Probability does not describe the short-term results of an experiment. It gives information about what can be expected *in the long term.* The **Law of Large Numbers** states that, as the number of trials in a probability experiment increases, the difference between the theoretical probability of an event and the relative frequency approaches zero (the theoretical probability and the relative frequency get closer and closer together).

Example 2.3.2

Let X be a continuous random variable with probability density function (PDF)

$$f(x) = ax^2(10 - x), \quad 0 \leqslant x \leqslant 10$$

where a is a constant.

Find a and $\mathbb{E}(X)$.

Solution.

From Corollary 2.2.5 part 1:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x) = 0 + \int_0^{10} ax^2(10-x)\mathrm{d}x = a\int_0^{10}(10x^2 - x^3)\mathrm{d}x \\ &= a\left[\frac{10}{3}x^3 - \frac{1}{4}x^4\right]_0^{10} = a\left[\frac{10^4}{3} - \frac{10^4}{4}\right] = a\frac{10^4}{12} \\ a &= \frac{12}{10^4} \end{aligned}$$

From Definition 2.3.1,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x$$

Note where $f(x) = 0$, we have $\int_a^b x \cdot f(x)\mathrm{d}x = \int_a^b 0\mathrm{d}x = 0$.

$$\begin{aligned} &= 0 + \int_0^{10} ax^3(10-x)\mathrm{d}x = a\int_0^{10}(10x^3 - x^4)\mathrm{d}x \\ &= a\left[\frac{10}{4}x^4 - \frac{1}{5}x^5\right]_0^{10} = a\left[\frac{10^5}{4} - \frac{10^5}{5}\right] = a\frac{10^5}{20} \\ &= \frac{12}{10^4} \cdot \frac{10^5}{20} = 6 \end{aligned}$$

Example 2.3.2

Example 2.3.3

Suppose Y is a continuous random variable with probability density function (PDF)

$$f(x) = e^x, \quad x \leqslant 0$$

Find $\mathbb{E}(Y)$.

Solution.

From Definition 2.3.1,

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} x \cdot f(x) \mathrm{d}x = 0 + \int_{-\infty}^{0} x \cdot e^x \mathrm{d}x = \lim_{a \to -\infty} \left[\int_a^0 x \cdot e^x \mathrm{d}x \right]$$

We use integration by parts with $u = x$, $\mathrm{d}v = e^x \mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = e^x$

$$= \lim_{a \to -\infty} \left[[xe^x]_a^0 - \int_a^0 e^x \mathrm{d}x \right] = \lim_{a \to -\infty} \left[-ae^a - [e^x]_a^0 \right] = \lim_{a \to -\infty} \left[-ae^a - 1 + e^a \right]$$

Note $\lim_{a \to -\infty} e^a = 0$, so $\lim_{a \to -\infty} -ae^a$ has the indeterminate form $0 \cdot \infty$. We use l'Hôpital's rule.

$$= \lim_{a \to -\infty} \left[\underbrace{\frac{-a}{e^{-a}}}_{\substack{\text{num}\to\infty \\ \text{den}\to\infty}} \right] - 1 + 0 = \lim_{a \to -\infty} \left[\frac{1}{e^{-a}} \right] - 1 = \lim_{a \to -\infty} [e^a] - 1 = -1$$

So, $\mathbb{E}(Y) = -1$.

Example 2.3.3

Example 2.3.4

Let Z be a continuous random variable with probability density function (PDF) $f(x) = \frac{1}{x^2}$, $x \geqslant 1$. Find $\mathbb{E}(Z)$.

Solution.

From Definition 2.3.1,

$$\begin{aligned} \mathbb{E}(Z) &= \int_{-\infty}^{\infty} x \cdot f(x) \mathrm{d}x = 0 + \int_1^{\infty} x \cdot x^{-2} \mathrm{d}x = \int_1^{\infty} x^{-1} \mathrm{d}x \\ &= \lim_{b \to \infty} \left[\int_1^b x^{-1} \mathrm{d}x \right] = \lim_{b \to \infty} [\log b] = \infty \end{aligned}$$

It is sometimes the case that the expectation of a continuous random variable is infinite. How should we interpret that?

A random variable Z with the given probability density function (PDF) has sample space $[1, \infty)$. It takes on finite values, but there is no limit to how large those values can be. (It is true that smaller values are more likely, since $f(x) = x^{-2}$ is a decreasing function. However, Z also takes on extremely large values from time to time.) $\mathbb{E}(Z) = \infty$ tells us that if we run our experiment Z a lot of times, over time the average will increase without bound.

Example 2.3.4

2.3.3 ▸▸ Checking your expectation calculation

The expectation of a random variable has several intuitive properties that can be used to quickly check that your answer is reasonable.

Theorem 2.3.5.

Let a, b be real numbers or $\pm\infty$ with $a < b$. Suppose a random variable X takes values from the interval $[a, b]$. Then $\mathbb{E}(X)$ will be some number in the interval $[a, b]$.

Proof. Suppose X is continuous, with probability density function (PDF) $f(x)$.

$$\mathbb{E}(X) = \int_a^b x f(x) \mathrm{d}x \leqslant \int_a^b b \cdot f(x) \mathrm{d}x = b \int_a^b f(x) \mathrm{d}x = b$$
$$\mathbb{E}(X) = \int_a^b x f(x) \mathrm{d}x \geqslant \int_a^b a \cdot f(x) \mathrm{d}x = a \int_a^b f(x) \mathrm{d}x = a$$

☐

Theorem 2.3.6.

Let a and b be real numbers with $a < b$. Suppose a continuous random variable X takes values from the interval $[a, b]$, and its probability density function (PDF) is *increasing* on the interval $[a, b]$. Then $\mathbb{E}(X) > \frac{a+b}{2}$.

Similarly, suppose a continuous random variable X takes values from the interval $[a, b]$, with $a < b$, and its probability density function (PDF) is *decreasing* on the interval $[a, b]$. Then $\mathbb{E}(X) < \frac{a+b}{2}$.

Proof. Intuitively, an increasing $f(x)$ means we have more high values than low values, so when we average them together, the average will be high. Similarly, decreasing $f(x)$ means we have more low values than high values, so when we average them together, the average will be low.

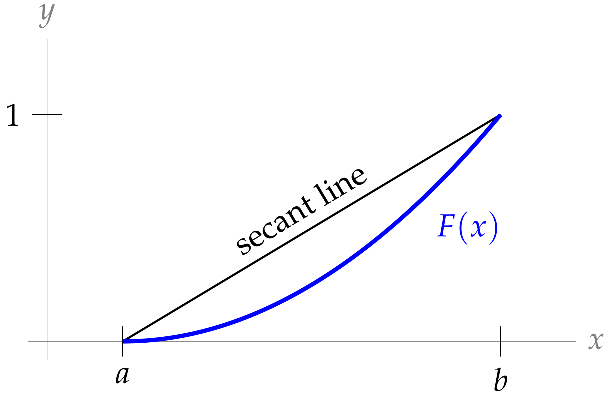
Let $F(x)$ be the cumulative distribution function (CDF) of $f(x)$, defined as $F(x) = \int_{-\infty}^x f(t)\, \mathrm{d}t$. Note the sample space of X is $[a, b]$, so for $x \in [a, b]$, $F(x) = \int_a^x f(t)\, \mathrm{d}t$. Also, $F(a) = 0$ and $F(b) = 1$.

We claim that $\int_a^b F(x)\,\mathrm{d}x < \frac{b-a}{2}$ when $f(x)$ is increasing, and $\int_a^b F(x)\,\mathrm{d}x > \frac{b-a}{2}$ when $f(x)$ is decreasing.

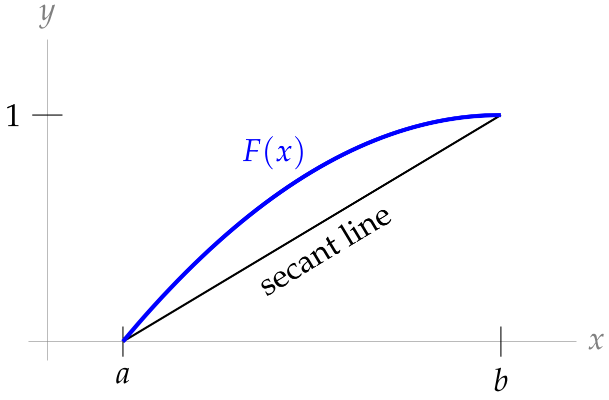
Proof of claim. If $f(x)$ is increasing on $[a,b]$, then $f'(x)$ is positive over $[a,b]$, i.e. $F''(x)$ is positive over $[a,b]$, i.e. $F(x)$ is concave up over $[a,b]$. Similarly, if $f(x)$ is a decreasing function on $[a,b]$, then $F(x)$ is concave down over $[a,b]$.

Concave-up functions lie below their secant lines, and concave-down functions lie above their secant lines. Note $F(a) = 0$ and $F(b) = 1$, as desired.

- If $f(x)$ is increasing on $[a,b]$, then $F(x)$ lies below the straight line from $(a,0)$ to $(b,1)$, so the area underneath $F(x)$ is less than the area underneath the secant line. That is, $\int_a^b F(x) < \frac{1}{2}(b-a)$.



- If $f(x)$ is decreasing on $[a,b]$, then $F(x)$ lies above the straight line from $(a,0)$ to $(b,1)$, so the area underneath $F(x)$ is greater than the area underneath the secant line. That is, $\int_a^b F(x) > \frac{1}{2}(b-a)$, as desired.



☐

With this claim in hand, we now consider the expected value of X. We'll use integration by parts with $u = x$, $\mathrm{d}v = f(x)\,\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = F(x)$.

$$\begin{aligned}\mathbb{E}(X) &= \int_a^b xf(x)\,\mathrm{d}x = [xF(x)]_a^b - \int_a^b F(x)\,\mathrm{d}x \\ &= b\underbrace{F(b)}_{1} - a\underbrace{F(a)}_{0} - \int_a^b F(x)\,\mathrm{d}x \\ &= b - \int_a^b F(x)\,\mathrm{d}x\end{aligned}$$

Using the claim,

$$\begin{cases}\mathbb{E}(X) > b - \frac{b-a}{2} & \text{if } f(x) \text{ is increasing} \\ \mathbb{E}(X) < b - \frac{b-a}{2} & \text{if } f(x) \text{ is decreasing.}\end{cases}$$

That is,

$$\begin{cases}\mathbb{E}(X) > \frac{b+a}{2} & \text{if } f(x) \text{ is increasing} \\ \mathbb{E}(X) < \frac{b+a}{2} & \text{if } f(x) \text{ is decreasing.}\end{cases}$$

☐

Example 2.3.7

Suppose X is a continuous random variable with probability density function (PDF)

$$f(x) = \begin{cases} e^{x-a} & \text{if } 1 \leqslant x \leqslant 5 \\ 0 & \text{else}\end{cases}$$

for some appropriate constant a. Using the two theorems in this section, give a range for $\mathbb{E}(X)$.

Solution. X only takes values from $[1,5]$, so by Theorem 2.3.5, $1 \leqslant \mathbb{E}(X) \leqslant 5$.

The probability density function (PDF) $f(x) = e^{x-a}$ is an increasing function, so by Theorem 2.3.6, $\mathbb{E}(X) > \frac{5+1}{2} = 3$.

So, $\mathbb{E}(X)$ is in the interval $(3,5]$.

Note: There is a unique value of a for which $f(x)$ is a probability density function (PDF). It is the value of a that satisfies the following equality:

$$1 = \int_1^5 e^{x-a}\mathrm{d}x$$

i.e. $a = \log\left(e^5 - e\right)$.

Example 2.3.7

Example 2.3.8

You calculate expected values for the various random variables described below. Which of the values can you immediately, with very little computation, say are wrong? Which seem reasonable?

1. W is a random variable that takes values from $[4, 5]$, and you calculate $\mathbb{E}(W) = 4.75$.
2. X is a random variable that takes values from $[-1, 0]$, and you calculate $\mathbb{E}(X) = 0.5$.
3. Y is a continuous random variable with probability density function (PDF)
$$f(x) = \begin{cases} \frac{1}{x} & 1 \leqslant x \leqslant e \\ 0 & \text{else} \end{cases}$$
and you calculate $\mathbb{E}(Y) = 1.9$.
4. Z is a continuous random variable with probability density function (PDF)
$$f(x) = \begin{cases} \frac{1}{x^2} & 1 \leqslant x \\ 0 & x < 1 \end{cases}$$
and you calculate $\mathbb{E}(Z) = -1$.
5. A is a continuous random variable with probability density function (PDF)
$$f(x) = \begin{cases} \frac{1}{x^3} & \frac{1}{\sqrt{2}} \leqslant x \\ 0 & x < \frac{1}{\sqrt{2}} \end{cases}$$
and you calculate $\mathbb{E}(A) = \sqrt{2}$.

Solution. From Theorem 2.3.5, $\mathbb{E}(X)$ and $\mathbb{E}(Z)$ are incorrect.

The PDF of Y is decreasing on $[1, e]$, so $\mathbb{E}(Y) < \frac{e+1}{2} \approx 3.72 = 1.85$ by Theorem 2.3.6. Therefore the result $\mathbb{E}(Y) = 1.9$ is incorrect.

For $\mathbb{E}(W)$, we don't have enough information to apply Theorem 2.3.6. However, it passes the test of Theorem 2.3.5. So $\mathbb{E}(W)$ is reasonable, though we have no way of knowing whether it its correct.

For $\mathbb{E}(A)$, Theorem 2.3.6 doesn't apply, since the values of A do not lie in a finite interval. However, it passes the test of Theorem 2.3.5. So $\mathbb{E}(W)$ is reasonable. (Indeed, if you go through the calculation, it is correct.)

Example 2.3.8

Example 2.3.9 (Conspiracy Theories)

The paper *On the Viability of Conspiratorial Beliefs*[5] investigates a probabilistic model[6] for

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[5] Grimes DR (2016) On the Viability of Conspiratorial Beliefs. PLoS ONE 11(1): e0147905. https://doi.org/10.1371/journal.pone.0147905

[6] The assumptions made that lead to this model are that every member of the conspiracy is equally likely to cause a leak (whether by negligence or on purpose); that leak events are independent of one another; and that the probability of a conspirator causing a leak in any given year is constant. The full derivation is beyond the scope of the text, but the interested reader may look up "Poisson distribution."
The paper goes on to approximate p using conspiracy theories that have been exposed. They also use demographic data to approximate $N(t)$. They apply the model to famous conspiracy theories (e.g. the moon landing being faked) to discuss whether such a plot could realistically remain secret until present day.

the length of time a conspiracy theory can remain secret. In particular, the author uses the formula

$$L(t) = 1 - e^{-t\left(1-(1-p)^{N(t)}\right)}$$

where $L(t)$ is the probability that, after t years, a leak has occurred that would cause the conspiracy to be exposed; $N(t)$ is the number of people involved in the conspiracy at time t; and p is the probability that a person involved will cause a leak in any particular year. (It is implied that $L(t) = 0$ for negative values of t.)

For this example, we'll only use a very basic version of the full model. Suppose there are 100 (immortal) people involved in a conspiracy, no new people are ever brought into the conspiracy, and each person has a 1% chance of causing a leak in one year.

(a) Using the model above, what is the expected amount of time it will take for a leak to occur?

(b) Using the model above, what is the probability that the conspiracy can survive without a leak for at least 5 years?

Solution.

(a) $L(t)$ is the probability that, at time t, at least one leak has occurred. Let T be the time that the first leak occurs. Then $L(t) = Pr(T \leqslant t)$. That is, if $f(x)$ is the PDF of T, then

$$L(t) = \int_0^t f(x)\,\mathrm{d}x$$

so, by the Fundamental Theorem of Calculus Part 1, $f(t) = L'(t)$.

Let's start by filling in our constants: $N(t) = 100$ and $p = \frac{1}{100}$.

$$L(t) = 1 - e^{-t\left(1-(1-p)^{N(t)}\right)} = 1 - e^{-t\left(1-0.99^{100}\right)} = 1 - e^{t\left(0.99^{100}-1\right)}$$

Note that $\left(0.99^{100} - 1\right)$ is a constant. In order to make the work below clearer, we'll replace it with c.

$$L(t) = 1 - e^{ct} \text{ where } c = \left(0.99^{100} - 1\right)$$

We find the PDF of T by differentiating $L(t)$.

$$L'(t) = -ce^{ct}$$

Now we use the definition of expected value, Definition 2.3.1

$$\mathbb{E}(T) = \int_{-\infty}^{\infty} t \cdot L'(t)\mathrm{d}t = \int_0^{\infty} t \cdot \left(-ce^{tc}\right)\mathrm{d}t = \lim_{b\to\infty}\int_0^b t \cdot \left(-ce^{ct}\right)\mathrm{d}t$$

We use integration by parts with $u = t$, $\mathrm{d}v = -ce^{ct}\mathrm{d}t$; $\mathrm{d}u = \mathrm{d}t$, $v = -e^{ct}$

$$\begin{aligned}
&= \lim_{b\to\infty}\left[\left[-te^{ct}\right]_0^b - \int_0^b -e^{ct}\mathrm{d}t\right]\\
&= \lim_{b\to\infty}\left[-be^{bc} + \left[\frac{1}{c}e^{ct}\right]_0^b\right]\\
&= \lim_{b\to\infty}\left[-be^{bc} + \frac{1}{c}e^{bc} - \frac{1}{c}\right]\\
&= \lim_{b\to\infty}\left[\left(\frac{1}{c} - b\right)e^{bc}\right] - \frac{1}{c}
\end{aligned}$$

Since $c < 0$, $\lim\limits_{b\to\infty} e^{bc} = 0$ (*). So, $\left(\frac{1}{c} - b\right)e^{bc}$ has the indeterminate form $-\infty \cdot 0$. We will re-write this in order to use l'Hôpital's rule.

$$\begin{aligned}
&= \lim_{b\to\infty}\left[\frac{\frac{1}{c} - b}{e^{-bc}}\right] - \frac{1}{c}\\
&= \lim_{b\to\infty}\left[\frac{\frac{\mathrm{d}}{\mathrm{d}b}\left[\frac{1}{c} - b\right]}{\frac{\mathrm{d}}{\mathrm{d}b}\left[e^{-bc}\right]}\right] - \frac{1}{c}\\
&= \lim_{b\to\infty}\left[\frac{-1}{-ce^{-bc}}\right] - \frac{1}{c}\\
&= \lim_{b\to\infty}\left[\frac{1}{c}e^{bc}\right] - \frac{1}{c}\\
&= 0 - \frac{1}{c} \qquad \text{using (*)}\\
&= -\frac{1}{.99^{100} - 1} = \frac{1}{1 - .99^{100}} \approx 1.58
\end{aligned}$$

So, the expected value of the time it would take for this conspiracy theory to be leaked is about 19 months.

(b) The probability that no leak has occurred at time $t = 5$ is $1 - L(5)$:

$$1 - L(5) = e^{5(.99^{100} - 1)} \approx 0.04$$

So, there's about a 4% chance that the conspiracy would survive at least 5 years without any leaks.

Example 2.3.9

2.4▴ Variance and standard deviation

Learning Objectives

- Define variance and standard deviation
- Explain in plain(ish) language what these quantities represent, in reference to their definitions.
- Compute standard deviation and variance using both the conventional definition and the alternative formulation $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.

2.4.1 ▸▸ Motivation: average difference from the average

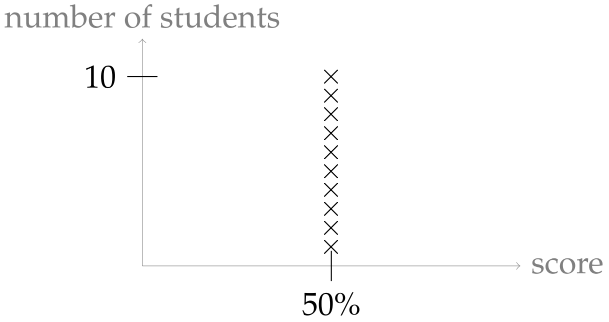
Imagine the average score on a test is 50%. You conduct the following experiment: pick a paper at random out of the stack of graded papers. Let X be the mark on that paper. Then "the average score is 50%" tells us that $\mathbb{E}(X) = 0.5$.

Now, contrast that with picking *your own* paper out of the stack. You're probably concerned with whether your paper is "usual" or "unusual." That is, if you hear the average is 50%, you'll wonder whether *your* paper is close to 50%, or far from it. You might want to quantify how "unusual" your own paper is by creating the variable $Y = |(\text{your score}) - 0.5|$.

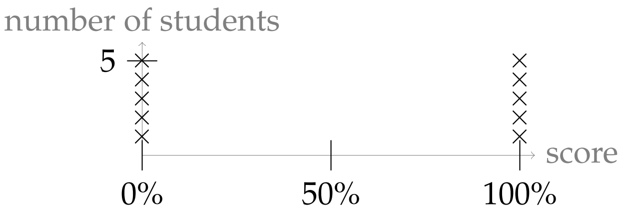
Now, let's zoom out to the whole class. They're all getting papers back, and all calculating their own values of Y. We might be concerned with how unusual their papers are, on average. So, we might compute $\mathbb{E}(Y)$. If $\mathbb{E}(Y)$ is quite small, then the average student sees a small difference between their exam and the class average of 0.5. If $\mathbb{E}(Y)$ is quite large, then the average student sees a large difference between their exam and the class average of 0.5.

More concretely, consider the scenarios below.

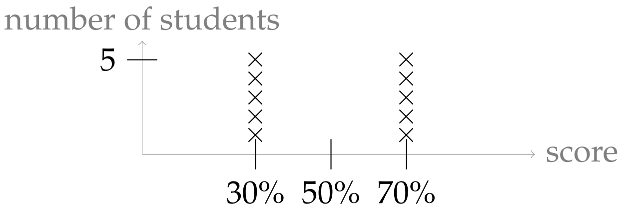
Scenario 1 Suppose there are 10 students in a class, and all of them get 50% on the exam. The class average is 50%, and as each student computes Y, they all find $Y = 0$. So, $\mathbb{E}(Y) = 0$.



Scenario 2 Suppose there are 10 students in a class. Half of them get 100% on the exam, and half get 0%. The class average is 50%, and as each student computes Y, they all find $Y = 0.5$. So, $\mathbb{E}(Y) = 0.5$.



Scenario 3 Suppose there are 10 students in class. Half of them get 70%, and half get 30%. The class average is 50%, and as each student computes Y, they all find $Y = 0.2$. So, $\mathbb{E}(Y) = 0.2$.



The scenario with lowest $\mathbb{E}(Y)$ is scenario 1, where every student got the same score, and so the difference between their score and the average is 0. The scenario with the highest $\mathbb{E}(Y)$ is scenario 2, where every student got a score that was as far away from the average as possible. The middle scenario, scenario 3, describes a class where nobody got the average score, but they aren't particularly close to or far from that average.

To generalize what we just discussed:

- X is a random variable
- $Y = |X - \mathbb{E}(X)|$ tells us how different X is from its expectation
- So, $\mathbb{E}(Y) = \mathbb{E}\Big(\big|X - \mathbb{E}(X)\big|\Big)$ is the expected difference from the expectation. We used this as a measure of how far off from $\mathbb{E}(X)$ our variable X could be and still be considered "usual". If $\mathbb{E}(Y)$ is a large number, then on average, $\mathbb{E}(X)$ is not a good indicator of what any individual X will be. If $\mathbb{E}(Y)$ is small, then we expect many values of X to actually be close to $\mathbb{E}(X)$.

2.4.2 ▸▸ Definitions and computations

Definition 2.4.1.

The **variance** of a random variable X, denoted $\mathrm{Var}(X)$, is:

$$\mathbb{E}\left[\Big(X - \mathbb{E}(X)\Big)^2\right].$$

The **standard deviation** of X, written $\sigma(X)$, is the square root of the variance:

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

Remark: Let X be a continuous random variable, with PDF $f(x)$. Then X^2 is a different continuous random variable, and it has its own PDF, say $g(x)$, which is probably different from $f(x)$. Then, if we're strictly following definitions, we need to know what $g(x)$ is before we can compute $\mathbb{E}(X^2)$. Luckily, however, $\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)$. Thanks to Ben Williams for pointing out that this is a special case of the *law of the unconscious statistician*.

We often consider values that are within one standard deviation of the expected value to be in the range of "usual" values. They may not be *exactly* average, but they're not unusually far away from average.

Example 2.4.2

One thousand students take a midterm, and we choose one student uniformly at random. X is the mark the student got on the midterm, out of 100. For this particular group of 1000 students, $\mathbb{E}(X) = 65$ and $\sigma(X) = 15$.

- In general, we think of students with grades from 50 to 80 as having a "usual" score. Those numbers come from $\mathbb{E}(X) - \sigma(X) = 50$ and $\mathbb{E}(X) + \sigma(X) = 80$.
- Suppose we select Student A, who earned 60 points. Although this is below the class average, it is *within one standard deviation of the expectation*. That is,

  $$|X - \mathbb{E}(X)| = 5 < 15 = \sigma(X).$$

  So this student is not below average in a really significant way.
- If we select Student B who scored 90, not only are they above the class average, they are *well above* the class average. The *difference* between X and $\mathbb{E}(X)$ is greater than usual.
- If we select Student C who scored 45, not only are they below the class average, they are *well below* the class average. The *difference* between X and $\mathbb{E}(X)$ is worse than usual.

Example 2.4.2

Corollary 2.4.3.

If X is a continuous random variable, then

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 \cdot f(x) \mathrm{d}x$$

where $f(x)$ is the probability density function (PDF) of X.

Note the similarities between $\mathrm{Var}(X)$ and $\mathbb{E}(Y)$ from the end of the last subsection, 2.4.1. Their interpretations are similar: $\mathrm{Var}(X)$ measures the expected *squared* difference between X and $\mathbb{E}(X)$[7].

---

[7] To explore why we need absolute values or squares, see Question 9 in Section 2.4 of the practice book.

One reason we replace $|X - \mathbb{E}(X)|$ with $(X - \mathbb{E}(X))^2$ is that $f(X) = |x - \mathbb{E}(X)|$ is not differentiable, while $f(x) = (x - \mathbb{E}(X))^2$ is differentiable. We want to be able to use calculus tools, so differentiability is desirable.

The variance of a random variable is often calculated in the manner below:

Corollary 2.4.4.

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \left[\mathbb{E}(X)\right]^2$$

Proof. From Corollary 2.4.3:

$$\begin{aligned}
\mathrm{Var}(X) &= \int_{-\infty}^{\infty} \left(x - \mathbb{E}(X)\right)^2 f(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(x^2 - 2x \cdot \mathbb{E}(X) + [\mathbb{E}(X)]^2\right) f(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x - 2 \cdot \mathbb{E}(X) \int_{-\infty}^{\infty} x f(x)\mathrm{d}x + [\mathbb{E}(X)]^2 \int_{-\infty}^{\infty} f(x)\mathrm{d}x
\end{aligned}$$

By the definition of $\mathbb{E}(X)$, Definition 2.3.1:

$$= \mathbb{E}(X^2) - 2 \cdot \mathbb{E}(X) \cdot \mathbb{E}(X) + [\mathbb{E}(X)]^2 \int_{-\infty}^{\infty} f(x)\mathrm{d}x$$

By property 1 of Corollary 2.2.5,

$$\begin{aligned}
&= \mathbb{E}(X^2) - 2 \cdot [\mathbb{E}(X)]^2 + [\mathbb{E}(X)]^2 \cdot 1 \\
&= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2
\end{aligned}$$

□

Example 2.4.5

Suppose X is a continuous random variable with probability density function (PDF)

$$f(x) = \begin{cases} \frac{x}{50} & \text{if } 0 \leqslant x \leqslant 10 \\ 0 & \text{else} \end{cases}$$

We will calculate $\mathrm{Var}(X)$ two ways.

- To calculate $\mathrm{Var}(X)$, we first need to know $\mathbb{E}(X)$.

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x = \int_0^{10} x \cdot \frac{x}{50}\mathrm{d}x \\
&= \int_0^{10} \frac{x^2}{50}\mathrm{d}x = \left.\frac{x^3}{150}\right|_0^{10} = \frac{10^3}{150} = \frac{20}{3}
\end{aligned}$$

- Using Corollary 2.4.3,

$$\begin{aligned}
\operatorname{Var}(X) &= \int_{-\infty}^{\infty} (x-\mathbb{E}(x))^2 \cdot f(x)\mathrm{d}x = \int_0^{10} \left(x - \frac{20}{3}\right)^2 \cdot \frac{x}{50}\mathrm{d}x \\
&= \int_0^{10} \left(x^2 - \frac{40}{3}x + \frac{400}{9}\right) \cdot \frac{x}{50}\mathrm{d}x = \frac{1}{50}\int_0^{10} \left(x^3 - \frac{40}{3}x^2 + \frac{400}{9}x\right)\mathrm{d}x \\
&= \frac{1}{50}\left(\frac{x^4}{4} - \frac{40x^3}{9} + \frac{200x^2}{9}\right)\Bigg|_0^{10} = \frac{1}{50}\left(\frac{10^4}{4} - \frac{40\cdot 10^3}{9} + \frac{200\cdot 10^2}{9}\right) \\
&= \frac{10^4}{50}\left(\frac{1}{4} - \frac{4}{9} + \frac{2}{9}\right) = \frac{50}{9}
\end{aligned}$$

- Using Corollary 2.4.4,

$$\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \\
&= \int_{-\infty}^{\infty} x^2 \cdot f(x)\mathrm{d}x - \left(\frac{20}{3}\right)^2 = \int_0^{10} x^2 \cdot \frac{x}{50}\mathrm{d}x - \frac{400}{9} \\
&= \frac{x^4}{50\cdot 4}\Bigg|_0^{10} - \frac{400}{9} = \frac{10^4}{200} - \frac{400}{9} = \frac{50}{9}
\end{aligned}$$

Example 2.4.5

2.4.3 ▸▸ Checking your standard deviation calculation

Corollary 2.4.6.

Let a, b be real numbers with $a < b$ and suppose a random variable X takes values from the interval $[a, b]$. Then

$$0 \leqslant \sigma(X) \leqslant \frac{b-a}{2}$$

Proof. First, consider what happens when we replace $\mathbb{E}(X)$ with $\frac{b+a}{2}$ (the midpoint of the sample space) in the definition of variance (Definition 2.4.1).

$$\begin{aligned}
\mathbb{E}\left(\left(X-\frac{b+a}{2}\right)^2\right) &= \int_a^b \left(x-\frac{b+a}{2}\right)^2 \cdot f(x)\mathrm{d}x \\
&= \int_a^b \left(x^2-(b+a)x+\left(\frac{b+a}{2}\right)^2\right) \cdot f(x)\mathrm{d}x \\
&= \int_a^b x^2 f(x)\mathrm{d}x - (b+a)\int_a^b xf(x)\mathrm{d}x + \left(\frac{b+a}{2}\right)^2 \int_a^b f(x)\mathrm{d}x \\
&= \mathbb{E}(X^2)-(b+a)\mathbb{E}(X)+\left(\frac{b+a}{2}\right)^2 \\
&= \mathbb{E}(X^2)-[\mathbb{E}(X)]^2+[\mathbb{E}(X)]^2-(b+a)\mathbb{E}(X)+\left(\frac{b+a}{2}\right)^2 \\
&= \mathbb{E}(X^2)-[\mathbb{E}(X)]^2+\left(\mathbb{E}(X)-\frac{b+a}{2}\right)^2 \\
&\geqslant \mathbb{E}(X^2)-[\mathbb{E}(X)]^2 = \mathrm{Var}(X) \qquad (*)
\end{aligned}$$

Since X takes values in the interval $[a,b]$:

$$\begin{aligned}
& a \leqslant X \leqslant b \\
\implies \quad & a-\frac{b+a}{2} \leqslant X-\frac{b+a}{2} \leqslant b-\frac{b+a}{2} \\
\implies \quad & -\frac{b-a}{2} \leqslant X-\frac{b+a}{2} \leqslant \frac{b-a}{2} \\
\implies \quad & 0 \leqslant \left(X-\frac{b+a}{2}\right)^2 \leqslant \left(\frac{b-a}{2}\right)^2
\end{aligned}$$

By Theorem 2.3.5,

$$0 \leqslant \mathbb{E}\left(\left(X-\frac{b+a}{2}\right)^2\right) \leqslant \left(\frac{b-a}{2}\right)^2$$

So, with our previous result (*),

$$\mathrm{Var}(X) \leqslant \mathbb{E}\left(\left(X-\frac{b+a}{2}\right)^2\right) \leqslant \left(\frac{b-a}{2}\right)^2$$

So,

$$\sigma(X) \leqslant \frac{b-a}{2}$$

□

Example 2.4.7

If the random variable X takes on values from the interval $[1,5]$, then $0 \leqslant \sigma(X) \leqslant 2$. Since $\sigma(X)=\sqrt{\mathrm{Var}(X)}$, then $0 \leqslant \mathrm{Var}(X) \leqslant 4$.

Example 2.4.7

Chapter 3

SEQUENCES AND SERIES

You have probably learned about Taylor polynomials[1] and, in particular, that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + E_n(x)$$

where $E_n(x)$ is the error introduced when you approximate e^x by its Taylor polynomial of degree n. You may have even seen a formula for $E_n(x)$. We are now going to ask what happens as n goes to infinity? Does the error go to zero, giving an exact formula for e^x? We shall later see that it does and that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

At this point we haven't defined, or developed, any understanding of this infinite sum. How do we compute the sum of an infinite number of terms? Indeed, when does a sum of an infinite number of terms even make sense? Clearly we need to build up foundations to deal with these ideas. Along the way we shall also see other functions for which the corresponding error obeys $\lim\limits_{n\to\infty} E_n(x) = 0$ for some values of x and not for other values of x.

To motivate the next section, consider using the above formula with $x = 1$ to compute the number e:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

As we stated above, we don't yet understand what to make of this infinite number of terms, but we might try to sneak up on it by thinking about what happens as we take

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[1] Now would be an excellent time to quickly read over your notes on the topic.

more and more terms.

1 term	$1 = 1$
2 terms	$1 + 1 = 2$
3 terms	$1 + 1 + \frac{1}{2} = 2.5$
4 terms	$1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.666666\ldots$
5 terms	$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708333\ldots$
6 terms	$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.716666\ldots$

By looking at the infinite sum in this way, we naturally obtain a sequence of numbers

$$\{\,1\,,\,2\,,\,2.5\,,\,2.666666\,,\ldots,\,2.708333\,,\ldots,\,2.716666\,,\ldots,\,\cdots\}.$$

The key to understanding the original infinite sum is to understand the behaviour of this sequence of numbers — in particularly, what do the numbers do as we go further and further? Does it settle down [2] to a given limit?

3.1▲ Sequences

Learning Objectives

- Find the limit of a sequence.

In the discussion above we used the term "sequence" without giving it a precise mathematical meaning. Let us rectify this now.

Definition 3.1.1.

A sequence is a list of infinitely[3] many numbers with a specified order. It is denoted

$$\left\{a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots\right\} \quad\text{or}\quad \left\{a_n\right\} \quad\text{or}\quad \left\{a_n\right\}_{n=1}^{\infty}$$

We will often specify a sequence by writing it more explicitly, like

$$\left\{a_n = f(n)\right\}_{n=1}^{\infty}$$

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[2] You will notice a great deal of similarity between the results of the next section and "limits at infinity" which was covered last term.

[3] For the more pedantic reader, here we mean a countably infinite list of numbers. The interested (pedantic or otherwise) reader should look up countable and uncountable sets.

where $f(n)$ is some function from the natural numbers to the real numbers.

Example 3.1.2

Here are three sequences.

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\} \qquad \text{or} \qquad \left\{a_n = \frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\left\{1, 2, 3, \cdots, n, \cdots\right\} \qquad \text{or} \qquad \left\{a_n = n\right\}_{n=1}^{\infty}$$

$$\left\{1, -1, 1, -1, \cdots, (-1)^{n-1}, \cdots\right\} \qquad \text{or} \qquad \left\{a_n = (-1)^{n-1}\right\}_{n=1}^{\infty}$$

It is not necessary that there be a simple explicit formula for the n^{th} term of a sequence. For example, the decimal digits of π is a perfectly good sequence

$$\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 6, 4, 3, 3, 8, \cdots\}$$

but there is no simple formula[4] for the n^{th} digit.

Example 3.1.2

Our primary concern with sequences will be the behaviour of a_n as n tends to infinity and, in particular, whether or not a_n "settles down" to some value as n tends to infinity.

Definition 3.1.3.

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to the limit A if a_n approaches A as n tends to infinity. If so, we write

$$\lim_{n\to\infty} a_n = A \qquad \text{or} \qquad a_n \to A \text{ as } n \to \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

The reader should immediately recognise the similarity with limits at infinity

$$\lim_{x\to\infty} f(x) = L \qquad \text{if} \qquad f(x) \to L \text{ as } x \to \infty$$

Example 3.1.4

Three of the four sequences in Example 3.1.2 diverge:

- The sequence $\{a_n = n\}_{n=1}^{\infty}$ diverges because a_n grows without bound, rather than approaching some finite value, as n tends to infinity.

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[4] There is, however, a remarkable result due to Bailey, Borwein and Plouffe that can be used to compute the n^{th} binary digit of π (i.e. writing π in base 2 rather than base 10) without having to work out the preceding digits.

- The sequence $\{a_n = (-1)^{n-1}\}_{n=1}^{\infty}$ diverges because a_n oscillates between $+1$ and -1 rather than approaching a single value as n tends to infinity.
- The sequence of the decimal digits of π also diverges, though the proof that this is the case is a bit beyond us right now[5].

The other sequence in Example 3.1.2 has $a_n = \frac{1}{n}$. As n tends to infinity, $\frac{1}{n}$ tends to zero. So

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Example 3.1.4

Example 3.1.5 $\left(\lim\limits_{n\to\infty} \frac{n}{2n+1}\right)$

Here is a little less trivial example. To study the behaviour of $\frac{n}{2n+1}$ as $n \to \infty$, it is a good idea to write it as

$$\frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

As $n \to \infty$, the $\frac{1}{n}$ in the denominator tends to zero, so that the denominator $2 + \frac{1}{n}$ tends to 2 and $\frac{1}{2+\frac{1}{n}}$ tends to $\frac{1}{2}$. So

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}$$

Example 3.1.5

Notice that in this last example, we are really using techniques that we used before to study infinite limits like $\lim\limits_{x\to\infty} f(x)$. This experience can be easily transferred to dealing with $\lim\limits_{n\to\infty} a_n$ limits by using the following result.

Theorem 3.1.6.

If

$$\lim_{x\to\infty} f(x) = L$$

and if $a_n = f(n)$ for all positive integers n, then

$$\lim_{n\to\infty} a_n = L$$

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[5] If the digits of π were to converge, then π would have to be a rational number. The irrationality of π (that it cannot be written as a fraction) was first proved by Lambert in 1761. Niven's 1947 proof is more accessible and we invite the interested reader to use their favourite search engine to find step–by–step guides to that proof.

Example 3.1.7 $\left(\lim_{n\to\infty} e^{-n}\right)$

Set $f(x) = e^{-x}$. Then $e^{-n} = f(n)$ and

$$\text{since } \lim_{x\to\infty} e^{-x} = 0 \qquad \text{we know that} \qquad \lim_{n\to\infty} e^{-n} = 0$$

Example 3.1.7

The bulk of the rules for the arithmetic of limits of functions that you already know also apply to the limits of sequences. That is, the rules you learned to work with limits such as $\lim_{x\to\infty} f(x)$ also apply to limits like $\lim_{n\to\infty} a_n$.

Theorem 3.1.8 (Arithmetic of limits).

Let A, B and C be real numbers and let the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to A and B respectively. That is, assume that

$$\lim_{n\to\infty} a_n = A \qquad \lim_{n\to\infty} b_n = B$$

Then the following limits hold.

(a) $\lim_{n\to\infty} \left[a_n + b_n\right] = A + B$
(The limit of the sum is the sum of the limits.)

(b) $\lim_{n\to\infty} \left[a_n - b_n\right] = A - B$
(The limit of the difference is the difference of the limits.)

(c) $\lim_{n\to\infty} Ca_n = CA$.

(d) $\lim_{n\to\infty} a_n\, b_n = A\, B$
(The limit of the product is the product of the limits.)

(e) If $B \neq 0$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$
(The limit of the quotient is the quotient of the limits *provided* the limit of the denominator is not zero.)

We use these rules to evaluate limits of more complicated sequences in terms of the limits of simpler sequences — just as we did for limits of functions.

Example 3.1.9

Combining Examples 3.1.5 and 3.1.7,

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{n}{2n+1}+7e^{-n}\right] &= \lim_{n\to\infty}\frac{n}{2n+1}+\lim_{n\to\infty}7e^{-n} && \text{by Theorem 3.1.8.a}\\ &= \lim_{n\to\infty}\frac{n}{2n+1}+7\lim_{n\to\infty}e^{-n} && \text{by Theorem 3.1.8.c}\\ &= \frac{1}{2}+7\cdot 0 && \text{by Examples 3.1.5 and 3.1.7}\\ &= \frac{1}{2}\end{aligned}$$

Example 3.1.9

There is also a squeeze theorem for sequences.

Theorem 3.1.10 (Squeeze theorem).

If $a_n \leqslant c_n \leqslant b_n$ for all natural numbers n, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L$$

Example 3.1.11

In this example we use the squeeze theorem to evaluate

$$\lim_{n\to\infty}\left[1+\frac{\pi_n}{n}\right]$$

where π_n is the n^{th} decimal digit of π. That is,

$$\pi_1 = 3 \quad \pi_2 = 1 \quad \pi_3 = 4 \quad \pi_4 = 1 \quad \pi_5 = 5 \quad \pi_6 = 9 \quad \cdots$$

We do not have a simple formula for π_n. But we do know that

$$0 \leqslant \pi_n \leqslant 9 \implies 0 \leqslant \frac{\pi_n}{n} \leqslant \frac{9}{n} \implies 1 \leqslant 1+\frac{\pi_n}{n} \leqslant 1+\frac{9}{n}$$

and we also know that

$$\lim_{n\to\infty} 1 = 1 \qquad \lim_{n\to\infty}\left[1+\frac{9}{n}\right] = 1$$

So the squeeze theorem with $a_n = 1$, $c_n = 1+\frac{\pi_n}{n}$, and $b_n = 1+\frac{9}{n}$ gives

$$\lim_{n\to\infty}\left[1+\frac{\pi_n}{n}\right] = 1$$

Example 3.1.11

Finally, recall that we can compute the limit of the composition of two functions using continuity. In the same way, we have the following result:

Theorem 3.1.12 (Continuous functions of limits).

If $\lim_{n\to\infty} a_n = L$ and if the function $g(x)$ is continuous at L, then

$$\lim_{n\to\infty} g(a_n) = g(L)$$

Example 3.1.13 $\left(\lim_{n\to\infty} \sin \frac{\pi n}{2n+1}\right)$

Write $\sin \frac{\pi n}{2n+1} = g\left(\frac{n}{2n+1}\right)$ with $g(x) = \sin(\pi x)$. We saw, in Example 3.1.5 that

$$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

Since $g(x) = \sin(\pi x)$ is continuous at $x = \frac{1}{2}$, which is the limit of $\frac{n}{2n+1}$, we have

$$\lim_{n\to\infty} \sin \frac{\pi n}{2n+1} = \lim_{n\to\infty} g\left(\frac{n}{2n+1}\right) = g\left(\frac{1}{2}\right) = \sin \frac{\pi}{2} = 1$$

Example 3.1.13

With this introduction to sequences and some tools to determine their limits, we can now return to the problem of understanding infinite sums.

3.2▲ Series

Learning Objectives

- Define sequences and series and, in particular, explain the difference between the two.
- Define partial sum.
- Explain what it means for a series to converge.
- Compute partial sums using spreadsheets.

A series is a sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

of infinitely many terms. In summation notation, it is written

$$\sum_{n=1}^{\infty} a_n$$

You already have a lot of experience with series, though you might not realise it. When you write a number using its decimal expansion you are really expressing it as a series. Perhaps the simplest example of this is the decimal expansion of $\frac{1}{3}$:

$$\frac{1}{3} = 0.3333\cdots$$

Recall that the expansion written in this way actually means

$$0.333333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n}$$

The summation index n is of course a dummy index. You can use any symbol you like (within reason) for the summation index.

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \sum_{i=1}^{\infty} \frac{3}{10^i} = \sum_{j=1}^{\infty} \frac{3}{10^j} = \sum_{\ell=1}^{\infty} \frac{3}{10^\ell}$$

A series can be expressed using summation notation in many different ways. For example the following expressions all represent the same series:

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \overbrace{\frac{3}{10}}^{n=1} + \overbrace{\frac{3}{100}}^{n=2} + \overbrace{\frac{3}{1000}}^{n=3} + \cdots$$

$$\sum_{j=2}^{\infty} \frac{3}{10^{j-1}} = \overbrace{\frac{3}{10}}^{j=2} + \overbrace{\frac{3}{100}}^{j=3} + \overbrace{\frac{3}{1000}}^{j=4} + \cdots$$

$$\sum_{\ell=0}^{\infty} \frac{3}{10^{\ell+1}} = \overbrace{\frac{3}{10}}^{\ell=0} + \overbrace{\frac{3}{100}}^{\ell=1} + \overbrace{\frac{3}{1000}}^{\ell=3} + \cdots$$

$$\frac{3}{10} + \sum_{n=2}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \overbrace{\frac{3}{100}}^{n=2} + \overbrace{\frac{3}{1000}}^{n=3} + \cdots$$

We can get from the first line to the second line by substituting $n = j - 1$ — don't forget to also change the limits of summation (so that $n = 1$ becomes $j - 1 = 1$ which is rewritten as $j = 2$). To get from the first line to the third line, substitute $n = \ell + 1$ everywhere, including in the limits of summation (so that $n = 1$ becomes $\ell + 1 = 1$ which is rewritten as $\ell = 0$).

Whenever you are in doubt as to what series a summation notation expression represents, it is a good habit to write out the first few terms, just as we did above.

Of course, at this point, it is not clear whether the sum of infinitely many terms adds up to a finite number or not. In order to make sense of this we will recast the problem in terms of the convergence of sequences (hence the discussion of the previous section). Before we proceed more formally let us illustrate the basic idea with a few simple examples.

Example 3.2.1 $\left(\sum\limits_{n=1}^{\infty} \frac{3}{10^n}\right)$

As we have just seen above the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$ is

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \overbrace{\frac{3}{10}}^{n=1} + \overbrace{\frac{3}{100}}^{n=2} + \overbrace{\frac{3}{1000}}^{n=3} + \cdots$$

Notice that the n^{th} term in that sum is

$$3 \times 10^{-n} = 0.\overbrace{00\cdots 0}^{n-1 \text{ zeroes}}3$$

So the sum of the first 5, 10, 15 and 20 terms in that series are

$$\sum_{n=1}^{5} \frac{3}{10^n} = 0.33333 \qquad \sum_{n=1}^{10} \frac{3}{10^n} = 0.3333333333$$

$$\sum_{n=1}^{15} \frac{3}{10^n} = 0.333333333333333 \qquad \sum_{n=1}^{20} \frac{3}{10^n} = 0.33333333333333333333$$

It sure looks like that, as we add more and more terms, we get closer and closer to $0.\dot{3} = \frac{1}{3}$. So it is very reasonable[6] to define $\sum_{n=1}^{\infty} \frac{3}{10^n}$ to be $\frac{1}{3}$.

Example 3.2.1

Example 3.2.2 $\left(\sum\limits_{n=1}^{\infty} 1 \text{ and } \sum_{n=1}^{\infty}(-1)^n\right)$

Every term in the series $\sum_{n=1}^{\infty} 1$ is exactly 1. So the sum of the first N terms is exactly N. As we add more and more terms this grows unboundedly. So it is very reasonable to say that the series $\sum_{n=1}^{\infty} 1$ diverges.

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[6] Of course we are free to define the series to be whatever we want. The hard part is defining it to be something that makes sense and doesn't lead to contradictions. We'll get to a more systematic definition shortly.

The series

$$\sum_{n=1}^{\infty}(-1)^n = \overbrace{(-1)}^{n=1} + \overbrace{1}^{n=2} + \overbrace{(-1)}^{n=3} + \overbrace{1}^{n=4} + \overbrace{(-1)}^{n=5} + \cdots$$

So the sum of the first N terms is 0 if N is even and -1 if N is odd. As we add more and more terms from the series, the sum alternates between 0 and -1 for ever and ever. So the sum of all infinitely many terms does not make any sense and it is again reasonable to say that the series $\sum_{n=1}^{\infty}(-1)^n$ diverges.

Example 3.2.2

In the above examples we have tried to understand the series by examining the sum of the first few terms and then extrapolating as we add in more and more terms. That is, we tried to sneak up on the infinite sum by looking at the limit of (partial) sums of the first few terms. This approach can be made into a more formal rigorous definition. More precisely, to define what is meant by the infinite sum $\sum_{n=1}^{\infty} a_n$, we approximate it by the sum of its first N terms and then take the limit as N tends to infinity.

Definition 3.2.3.

The N^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is the sum of its first N terms

$$S_N = \sum_{n=1}^{N} a_n.$$

Conversely, for each $N \geqslant 2$, $a_N = S_N - S_{N-1}$.
The partial sums form a sequence $\{S_N\}_{N=1}^{\infty}$. If this sequence of partial sums converges $S_N \to S$ as $N \to \infty$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

In the last definition, we introduced the notation S_N for the N^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ that has the smallest value of the index n being 1. It is sometimes convenient, as we shall see in Example 3.3.9, to express a series with the index having a smallest value other than one, like for example $\sum_{n=0}^{\infty} a_n$. Then there are a couple of different commonly used definitions for the partial sum notation S_N. One is $S_N = \sum_{n=0}^{N} a_n$, which is the sum of all terms having indices less than or equal to N. With this definition, we still have $a_N = S_N - S_{N-1}$. A second commonly used definition is the sum of the first N terms in the series, i.e. $S_N = \sum_{n=0}^{N-1} a_n$. Under this definition $a_N = S_{N+1} - S_N$. You may of course use whatever definition you like. But you should state clearly what definition you are using.

Example 3.2.4 (Computing partial sums with spreadsheets)

Spreadsheets can be a nice tool for finding partial sums. For practice, let's find the 50th partial sum S_{50} of the series:

$$\sum_{n=1}^{\infty} \frac{n}{e^n}.$$

To generate the *sequence* elements, a_n, we'll use the ROW() function. The ROW() function returns the row of the cell that it's in. (So, for example, if you write `=ROW()` into cell A25, you'll see the number 25.)

We can put the sequence elements into column A as follows:

	A
1	`=ROW()/(EXP(ROW()))`
2	↓

So a_1 is in cell A1, a_2 is in cell A2, etc. To compute S_{50}, into a blank cell, we write `=SUM(A1:A50)`, which results in the decimal approximation $S_{50} \approx 0.920673594$.

Example 3.2.4

As was the case for limits, differentiation and antidifferentiation, we can compute more complicated series in terms of simpler ones by understanding how series interact with the usual operations of arithmetic. It is, perhaps, not so surprising that there are simple rules for addition and subtraction of series and for multiplication of a series by a constant. Unfortunately there are no simple general rules for computing products or ratios of series.

Theorem 3.2.5 (Arithmetic of series).

Let C, S and T be real numbers and let the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to S and T respectively. That is, assume that

$$\sum_{n=1}^{\infty} a_n = S \qquad \sum_{n=1}^{\infty} b_n = T$$

Then the following hold.

(a) $\displaystyle\sum_{n=1}^{\infty} \left[a_n + b_n\right] = S + T$ and $\displaystyle\sum_{n=1}^{\infty} \left[a_n - b_n\right] = S - T$

(b) $\displaystyle\sum_{n=1}^{\infty} C a_n = CS.$

Example 3.2.6 (Arithmetic Progressions)

An arithmetic progression is a sequence for which the difference between successive terms is a constant. That is, there are constants a_1 and d such that the n^{th} term in the sequence is $a_n = a_1 + (n-1)d$. For example, the sequence $1, 2, 3, 4, \cdots$ has $a_1 = 1$, $d = 1$ and $a_n = n$. We will now compute the partial sum $S_N = \sum_{n=1}^{N} n$ of the corresponding series $\sum_{n=1}^{\infty} n$. To do so, we will use a trick that is often attributed to Carl Friedrich Gauss[7]. The trick consists of writing S_N twice, once in forward order and once in reverse order

$$\begin{aligned} S_N &= 1 + \quad 2 \quad + \quad 3 \quad + \cdots + N \\ S_N &= N + (N-1) + (N-2) + \cdots + 1 \end{aligned}$$

and then adding the two representations of S_N together.

$$\begin{aligned} 2S_N &= \overbrace{(N+1) + (N+1) + (N+1) + \cdots (N+1)}^{N \text{ terms}} \\ &= N(N+1) \\ \implies S_N &= \frac{N(N+1)}{2} \end{aligned}$$

Using this and a little arithmetic, we can compute the partial sums for the series that corresponds to the general arithmetic progression $a_n = a_1 + (n-1)d$. (Of course, you can get the same result by applying the above trick to $S_N = \sum_{n=1}^{N} \{a_1 + (n-1)d\}$.)

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \{a_1 + (n-1)d\} = \sum_{n=1}^{N} \{(a_1 - d) + nd\} \\ &= (a_1 - d) \sum_{n=1}^{N} 1 + d \sum_{n=1}^{N} n = (a_1 - d)N + d\tfrac{N(N+1)}{2} \\ &= \tfrac{N}{2}\Big\{a_1 + [a_1 + d(N-1)]\Big\} \\ &= \tfrac{N}{2}\{\text{first term} + \text{last term}\} \end{aligned}$$

Example 3.2.6

3.3▲ Convergence tests

It is very common to encounter series for which it is difficult, or even virtually impossible, to determine the sum exactly. Often you try to evaluate the sum approximately by truncating it, i.e. having the index run only up to some finite N, rather than infinity. But

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[7] Carl Friedrich Gauss (1777–1855) was a German mathematician and physicist who ranks among the world's most influential mathematicians. He is said to have reinvented this trick when he was in primary school. In fact, the trick was known many centuries before him. You can use your favourite search engine to learn more.

there is no point in doing so if the series diverges[8][9]. So you like to at least know if the series converges or diverges. Furthermore you would also like to know what error is introduced when you approximate $\sum_{n=1}^{\infty} a_n$ by the "truncated series" $\sum_{n=1}^{N} a_n$. That's called the truncation error. There are a number of "convergence tests" to help you with this.

Example 3.3.1 (Revisiting partial sums in spreadsheets)

If we're quite sure a series converges, then a high-order partial sum may help us approximate its value. But, without an error bound, it's hard to know which partial sum is accurate enough for the desired application. Furthermore, it can be tough to see whether a series converges or not just from its first few partial sums.

Example 1 Consider the series $\sum_{n=1}^{\infty} \frac{n^4}{e^n}$. A spreadsheet like this one:

	A	B
1	`=ROW()∧4/EXP(ROW())`	`=SUM(A$1:A1)`
2	↓	↓

gives us the following few partial sums in column B:

	B
1	0.367879441
2	2.533243973
3	6.565996511
4	11.25480007
5	15.46601694
6	18.67847976
7	20.86790836
8	22.24196328
9	23.05165501

At first glance, that looks like a divergent series. The entries in B (the partial sums) seem to be growing without approaching a single number. But, in fact, the series converges, and the limit of the partial sums is close to 24.

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[8] The authors should be a little more careful making such a blanket statement. While it is true that it is not wise to approximate a divergent series by taking N terms with N large, there are cases when one can get a very good approximation by taking N terms with N small! For example, the Taylor remainder theorem shows us that when the n^{th} derivative of a function $f(x)$ grows very quickly with n, Taylor polynomials of degree N, with N large, can give bad approximations of $f(x)$, while the Taylor polynomials of degree one or two can still provide very good approximations of $f(x)$ when x is very small. As an example of this, one of the triumphs of quantum electrodynamics, namely the computation of the anomalous magnetic moment of the electron, depends on precisely this. A number of important quantities were predicted using the first few terms of divergent power series. When those quantities were measured experimentally, the predictions turned out to be incredibly accurate.

[9] The field of asymptotic analysis often makes use of the first few terms of divergent series to generate approximate solutions to problems; this, along with numerical computations, is one of the most important techniques in applied mathematics. Indeed, there is a whole wonderful book (which, unfortunately, is too advanced for most Calculus 2 students) devoted to playing with divergent series called, unsurprisingly, "Divergent Series" by G.H. Hardy. This is not to be confused with the "Divergent" series by V. Roth set in a post-apocalyptic dystopian Chicago. That latter series diverges quite dramatically from mathematical topics, while the former does not have a film adaptation (yet).

Example 2 Consider the series $\sum_{n=1}^{\infty} \frac{1}{10n}$.

A spreadsheet like this one:

	A	B
1	`=1/(10*ROW())`	`=SUM($A$1:A1)`
2	↓	↓

gives us the following first few partial sums in column B:

	B
1	0.1
2	0.15
3	0.183333333
4	0.208333333
5	0.228333333
6	0.245
7	0.259285714
8	0.271785714
9	0.282896825
10	0.292896825
11	0.301987734
12	0.310321068
13	0.318013376
14	0.325156233
15	0.331822899
16	0.338072899

You might think that the partial sums are approching 0.3-something, but in fact, their limit is positive infinity.

The moral of these two stories is that we need good methods for determining whether a series converges or diverges, to be applied *before* we try to evaluate it.

Example 3.3.1

3.3.1 ▸▸ The divergence test

Learning Objectives

- Apply the divergence test to determine the divergence of applicable series.
- Explain in words why the divergence test works.
- Explain why and how the test can be inconclusive.

Our first test is very easy to apply, but it is also rarely useful. It just allows us to quickly reject some "trivially divergent" series. It is based on the observation that

- by definition, a series $\sum_{n=1}^{\infty} a_n$ converges to S when the partial sums $S_N = \sum_{n=1}^{N} a_n$ converge to S.
- Then, as $N \to \infty$, we have $S_N \to S$ and, because $N - 1 \to \infty$ too, we also have $S_{N-1} \to S$.
- So $a_N = S_N - S_{N-1} \to S - S = 0$.

This tells us that, if we already know that a given series $\sum a_n$ is convergent, then the n^{th} term of the series, a_n, must converge to 0 as n tends to infinity. In this form, the test is not so useful. However the contrapositive[10] of the statement is a useful test for *divergence*.

Theorem 3.3.2 (Divergence Test).

If the sequence $\{a_n\}_{n=1}^{\infty}$ fails to converge to zero as $n \to \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 3.3.3

Let $a_n = \frac{n}{n+1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1 + 1/n} = 1 \neq 0$$

So the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Example 3.3.3

Warning 3.3.4.

The divergence test is a "one way test". It tells us that if $\lim_{n\to\infty} a_n$ is nonzero, or fails to exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges. But it tells us *absolutely nothing* when $\lim_{n\to\infty} a_n = 0$. In particular, it is perfectly possible for a series $\sum_{n=1}^{\infty} a_n$ to *diverge* even though $\lim_{n\to\infty} a_n = 0$. An example is $\sum_{n=1}^{\infty} \frac{1}{n}$. We'll show in Example 3.3.7, below, that it diverges.

---

[10] Given a statement of the form "If A is true, then B is true" the contrapositive is "If B is not true, then A is not true". The two statements in quotation marks are logically equivalent — if one is true, then so is the other. In the present context we have

If $(\sum a_n$ converges$)$ then $(a_n$ converges to $0)$.

The contrapositive of this statement is then

If $(a_n$ does not converge to $0)$ then $(\sum a_n$ does not converge$)$.

Now while convergence or divergence of series like $\sum_{n=1}^{\infty} \frac{1}{n}$ can be determined using some clever tricks, it would be much better to have methods that are more systematic and rely less on being sneaky. Over the next subsections we will discuss several methods for testing series for convergence.

Note that while these tests will tell us whether or not a series converges, they do not (except in rare cases) tell us what the series adds up to. For example, the test we will see in the next subsection tells us quite immediately that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges. However it does not tell us its value[11].

3.3.2 ▸▸ The integral test

Learning Objectives

- State the conditions required to apply the integral test.
- Explain in words and with a picture why the integral test works.
- Use the integral test to to determine convergence or divergence of applicable series. In particular, use the integral test to derive the p-test.

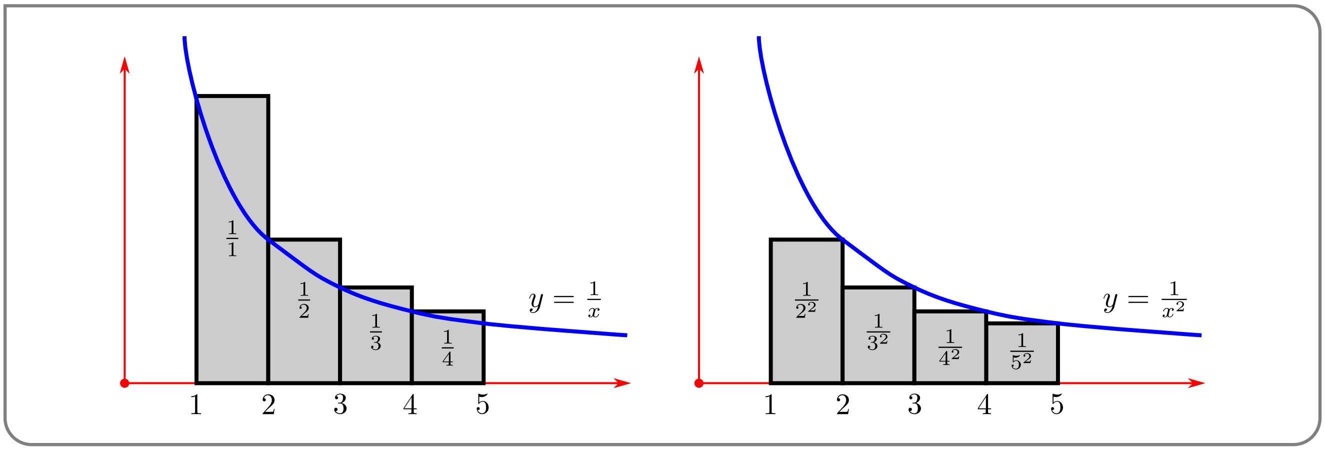
In the integral test, we think of a series $\sum_{n=1}^{\infty} a_n$, that we cannot evaluate explicitly, as the area of a union of rectangles, with a_n representing the area of a rectangle of width one and height a_n. Then we compare that area with the area represented by an integral, that we can evaluate explicitly, much as we did in Theorem 1.13.17, the comparison test for improper integrals. We'll start with a simple example, to illustrate the idea. Then we'll move on to a formulation of the test in general.

Example 3.3.5

Visualise the terms of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ as a bar graph — each term is a rectangle of height $\frac{1}{n}$ and width 1. The limit of the series is then the limiting area of this union of rectangles. Consider the sketch on the left below.

---

[11] This series converges to Apéry's constant 1.2020569031 . . . . The constant is named for Roger Apéry (1916–1994) who proved that this number must be irrational. This number appears in many contexts including the following cute fact — the reciprocal of Apéry's constant gives the probability that three positive integers, chosen at random, do not share a common prime factor.



It shows that the area of the shaded columns, $\sum_{n=1}^{4} \frac{1}{n}$, is bigger than the area under the curve $y = \frac{1}{x}$ with $1 \leqslant x \leqslant 5$. That is

$$\sum_{n=1}^{4} \frac{1}{n} \geqslant \int_1^5 \frac{1}{x} \mathrm{d}x$$

If we were to continue drawing the columns all the way out to infinity, then we would have

$$\sum_{n=1}^{\infty} \frac{1}{n} \geqslant \int_1^{\infty} \frac{1}{x} \mathrm{d}x$$

We are able to compute this improper integral exactly:

$$\int_1^{\infty} \frac{1}{x} \mathrm{d}x = \lim_{R\to\infty} \Big[\log |x| \Big]_1^R = +\infty$$

That is the area under the curve diverges to $+\infty$ and so the area represented by the columns must also diverge to $+\infty$.

It should be clear that the above argument can be quite easily generalised. For example the same argument holds *mutatis mutandis*[12] for the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Indeed we see from the sketch on the right above that

$$\sum_{n=2}^{N} \frac{1}{n^2} \leqslant \int_1^N \frac{1}{x^2} \mathrm{d}x$$

and hence

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leqslant \int_1^{\infty} \frac{1}{x^2} \mathrm{d}x$$

---

[12] Latin for "Once the necessary changes are made". This phrase still gets used a little, but these days mathematicians tend to write something equivalent in English. Indeed, English is pretty much the *lingua franca* for mathematical publishing. *Quidquid erit.*

This last improper integral is easy to evaluate:

$$\int_1^\infty \frac{1}{x^2}\mathrm{d}x = \lim_{R\to\infty}\left[-\frac{1}{x}\right]_1^R = \lim_{R\to\infty}\left(\frac{1}{1}-\frac{1}{R}\right) = 1$$

Thus we know that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \sum_{n=2}^{\infty}\frac{1}{n^2} \leqslant 2.$$

and so the series must converge.

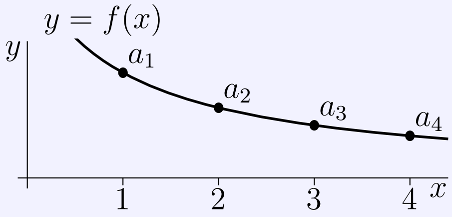
Example 3.3.5

The above arguments are formalised in the following theorem.

Theorem 3.3.6 (The Integral Test)**.**

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geqslant N_0$ and which obeys

(i) $f(x) \geqslant 0$ for all $x \geqslant N_0$ and
(ii) $f(x)$ decreases as x increases and
(iii) $f(n) = a_n$ for all $n \geqslant N_0$.



Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x)\,\mathrm{d}x \text{ converges}$$

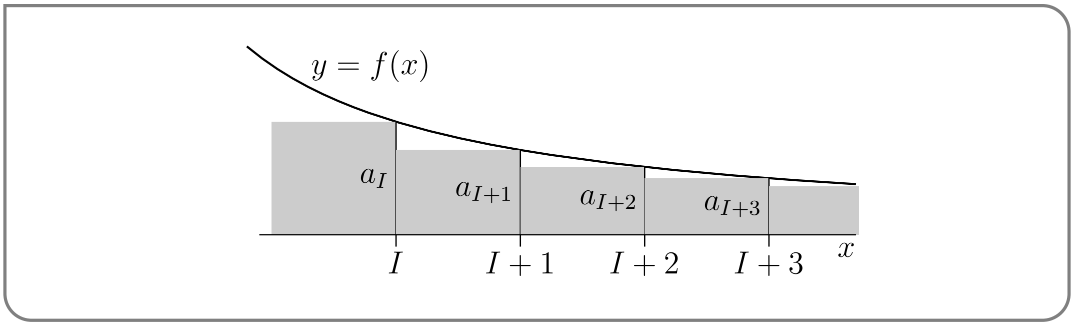
Furthermore, when the series converges, the truncation error

$$0 \leqslant \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \leqslant \int_N^\infty f(x)\,\mathrm{d}x \qquad \text{for all } N \geqslant N_0$$

Proof. Let I be any fixed integer with $I > N_0$. Then

- $\sum_{n=1}^\infty a_n$ converges if and only if $\sum_{n=I}^\infty a_n$ converges — removing a fixed finite number of terms from a series cannot impact whether or not it converges.

- Since $a_n \geqslant 0$ for all $n \geqslant I > N_0$, the sequence of partial sums $s_\ell = \sum_{n=I}^{\ell} a_n$ obeys $s_{\ell+1} = s_\ell + a_{n+1} \geqslant s_\ell$. That is, s_ℓ increases as ℓ increases.
- So $\{s_\ell\}$ must either converge to some finite number or increase to infinity. That is, either $\sum_{n=I}^{\infty} a_n$ converges to a finite number or it is $+\infty$.



Look at the figure above. The shaded area in the figure is $\sum_{n=I}^{\infty} a_n$ because

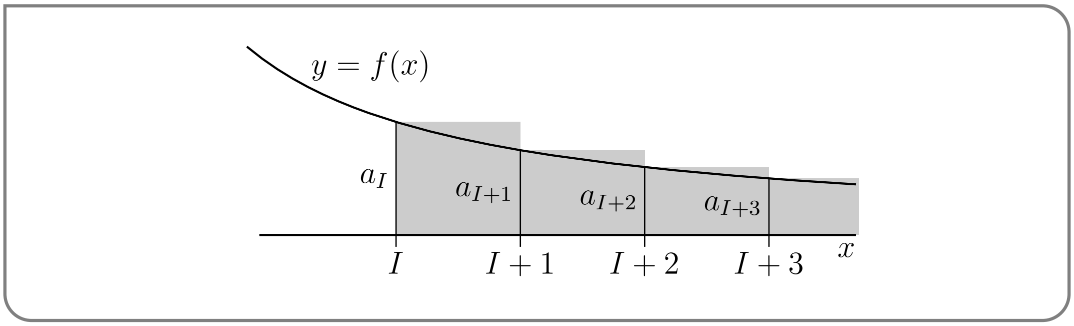
- the first shaded rectangle has height a_I and width 1, and hence area a_I and
- the second shaded rectangle has height a_{I+1} and width 1, and hence area a_{I+1}, and so on

This shaded area is smaller than the area under the curve $y = f(x)$ for $I - 1 \leqslant x < \infty$. So

$$0 \leqslant \sum_{n=I}^{\infty} a_n \leqslant \int_{I-1}^{\infty} f(x)\,\mathrm{d}x$$

and, if the integral is finite, the sum $\sum_{n=I}^{\infty} a_n$ is finite too. Furthermore, the desired bound on the truncation error is just the special case of this inequality with $I = N + 1$:

$$0 \leqslant \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n \leqslant \int_{N}^{\infty} f(x)\,\mathrm{d}x$$



For the "divergence case" look at the figure above. The (new) shaded area in the figure is again $\sum_{n=I}^{\infty} a_n$ because

- the first shaded rectangle has height a_I and width 1, and hence area a_I and
- the second shaded rectangle has height a_{I+1} and width 1, and hence area a_{I+1}, and so on

This time the shaded area is larger than the area under the curve $y = f(x)$ for $I \leqslant x < \infty$. So

$$\sum_{n=I}^{\infty} a_n \geqslant \int_I^{\infty} f(x)\,\mathrm{d}x$$

and, if the integral is infinite, the sum $\sum_{n=I}^{\infty} a_n$ is infinite too.

$\square$

Now that we have the integral test, it is straightforward to determine for which values of p the series[13]

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges.

Example 3.3.7 $\left(\text{The } p \text{ test: } \sum_{n=1}^{\infty} \frac{1}{n^p}\right)$

Let $p > 0$. We'll now use the integral test to determine whether or not the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (which is sometimes called the p–series) converges.

- To do so, we need a function $f(x)$ that obeys $f(n) = a_n = \frac{1}{n^p}$ for all n bigger than some N_0. Certainly $f(x) = \frac{1}{x^p}$ obeys $f(n) = \frac{1}{n^p}$ for all $n \geqslant 1$. So let's pick this f and try $N_0 = 1$. (We can always increase N_0 later if we need to.)
- This function also obeys the other two conditions of Theorem 3.3.6:
  (i) $f(x) > 0$ for all $x \geqslant N_0 = 1$ and
  (ii) $f(x)$ decreases as x increases because $f'(x) = -p\frac{1}{x^{p+1}} < 0$ for all $x \geqslant N_0 = 1$.
- So the integral test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if the integral $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges.
- We have already seen, in Example 1.13.8, that the integral $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges if and only if $p > 1$.

So we conclude that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. This is sometimes called the p–test.

---

[13] This series, viewed as a function of p, is called the Riemann zeta function, $\zeta(p)$, or the Euler-Riemann zeta function. It is extremely important because of its connections to prime numbers (among many other things). Indeed Euler proved that

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \prod_{\mathrm{P\ prime}} \left(1 - \mathrm{P}^{-p}\right)^{-1}$$

Riemann showed the connections between the zeros of this function (over complex numbers p) and the distribution of prime numbers. Arguably the most famous unsolved problem in mathematics, the Riemann hypothesis, concerns the locations of zeros of this function.

- In particular, the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is called the harmonic series, has $p = 1$ and so diverges. As we add more and more terms of this series together, the terms we add, namely $\frac{1}{n}$, get smaller and smaller and tend to zero, but they tend to zero so slowly that the full sum is still infinite.

- On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.000001}}$ has $p = 1.000001 > 1$ and so converges. This time as we add more and more terms of this series together, the terms we add, namely $\frac{1}{n^{1.000001}}$, tend to zero (just) fast enough that the full sum is finite. Mind you, for this example, the convergence takes place very slowly — you have to take a huge number of terms to get a decent approximation to the full sum. If we approximate $\sum_{n=1}^{\infty} \frac{1}{n^{1.000001}}$ by the truncated series $\sum_{n=1}^{N} \frac{1}{n^{1.000001}}$, we make an error of at most

$$\int_N^{\infty} \frac{\mathrm{d}x}{x^{1.000001}} = \lim_{R\to\infty} \int_N^R \frac{\mathrm{d}x}{x^{1.000001}} = \lim_{R\to\infty} -\frac{1}{0.000001}\left[\frac{1}{R^{0.000001}} - \frac{1}{N^{0.000001}}\right] = \frac{10^6}{N^{0.000001}}$$

This does tend to zero as $N \to \infty$, but really slowly.

Example 3.3.7

We now know that the dividing line between convergence and divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ occurs at $p = 1$. We can dig a little deeper and ask ourselves how much more quickly than $\frac{1}{n}$ the n^{th} term needs to shrink in order for the series to converge. We know that for large x, the function $\log x$ is smaller than x^a for any positive a — you can convince yourself of this with a quick application of L'Hôpital's rule. So it is not unreasonable to ask whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

converges. Notice that we sum from $n = 2$ because when $n = 1, n \log n = 0$. And we don't need to stop there[14]. We can analyse the convergence of this sum with any power of $\log n$.

Example 3.3.8 $\left(\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}\right)$

Let $p > 0$. We'll now use the integral test to determine whether or not the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

- As in the last example, we start by choosing a function that obeys $f(n) = a_n = \frac{1}{n(\log n)^p}$ for all n bigger than some N_0. Certainly $f(x) = \frac{1}{x(\log x)^p}$ obeys $f(n) = \frac{1}{n(\log n)^p}$ for all $n \geqslant 2$. So let's use that f and try $N_0 = 2$.

- Now let's check the other two conditions of Theorem 3.3.6:

  (i) Both x and $\log x$ are positive for all $x > 1$, so $f(x) > 0$ for all $x \geqslant N_0 = 2$.

---

[14] We could go even further and see what happens if we include powers of $\log(\log(n))$ and other more exotic slow growing functions.

(ii) As x increases both x and $\log x$ increase and so $x(\log x)^p$ increases and $f(x)$ decreases.

- So the integral test tells us that the series $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if the integral $\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^p}$ converges.
- To test the convergence of the integral, we make the substitution $u = \log x$, $\mathrm{d}u = \frac{\mathrm{d}x}{x}$.

$$\int_2^R \frac{\mathrm{d}x}{x(\log x)^p} = \int_{\log 2}^{\log R} \frac{\mathrm{d}u}{u^p}$$

  We already know that the integral $\int_1^\infty \frac{\mathrm{d}u}{u^p}$, and hence the integral $\int_2^R \frac{\mathrm{d}x}{x(\log x)^p}$, converges if and only if $p > 1$.

So we conclude that $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

Example 3.3.8

3.3.3 ▸▸ Geometric series

Learning Objectives

- Determine whether a given series is geometric.
- Given a geometric series, determine whether it converges or diverges.
- Evaluate convergent geometric series.

Example 3.3.9 (Geometric Series)

Let a and r be any two fixed real numbers with $a \neq 0$. The series

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n$$

is called the geometric series with first term a and ratio r.

Notice that we have chosen to start the summation index at $n = 0$. That's fine. The first[15] term is the $n = 0$ term, which is $ar^0 = a$. The second term is the $n = 1$ term, which is $ar^1 = ar$. And so on. We could have also written the series $\sum_{n=1}^{\infty} ar^{n-1}$. That's

---

[15] It is actually quite common in computer science to think of 0 as the first integer. In that context, the set of natural numbers is defined to contain 0:

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

exactly the same series — the first term is $ar^{n-1}\big|_{n=1} = ar^{1-1} = a$, the second term is $ar^{n-1}\big|_{n=2} = ar^{2-1} = ar$, and so on[16]. Regardless of how we write the geometric series, a is the first term and r is the ratio between successive terms.

Geometric series have the extremely useful property that there is a very simple formula for their partial sums. Denote the partial sum by

$$S_N = \sum_{n=0}^{N} ar^n = a + ar + ar^2 + \cdots + ar^N.$$

The secret to evaluating this sum is to see what happens when we multiply it by r:

$$\begin{aligned} rS_N &= r\big(a + ar + ar^2 + \cdots + ar^N\big) \\ &= ar + ar^2 + ar^3 + \cdots + ar^{N+1} \end{aligned}$$

Notice that this is almost the same[17] as S_N. The only differences are that the first term, a, is missing and one additional term, ar^{N+1}, has been tacked on the end. So

$$\begin{aligned} S_N &= a + ar + ar^2 + \cdots + ar^N \\ rS_N &= \phantom{a +{}} ar + ar^2 + \cdots + ar^N + ar^{N+1} \end{aligned}$$

Hence taking the difference of these expressions cancels almost all the terms:

$$(1-r)S_N = a - ar^{N+1} = a(1 - r^{N+1})$$

Provided $r \neq 1$ we can divide both side by $1 - r$ to isolate S_N:

$$S_N = a \cdot \frac{1 - r^{N+1}}{1 - r}.$$

On the other hand, if $r = 1$, then

$$S_N = \underbrace{a + a + \cdots + a}_{N+1 \text{ terms}} = a(N+1)$$

So in summary:

$$S_N = \begin{cases} a\frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases} \tag{3.3.1}$$

---

while the notation

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

is used to denote the (strictly) positive integers. Remember that in this text, as is more standard in mathematics, we define the set of natural numbers to be the set of (strictly) positive integers.

[16] This reminds the authors of the paradox of Hilbert's hotel. The hotel with an infinite number of rooms is completely full, but can always accommodate one more guest. The interested reader should use their favourite search engine to find more information on this.

[17] One can find similar properties of other special series, that allow us, with some work, to cancel many terms in the partial sums. The interested reader should look up "creative telescoping" to see how this idea might be used more generally, though it is somewhat beyond this course.

Now that we have this expression we can determine whether or not the series converges. If $|r| < 1$, then r^{N+1} tends to zero as $N \to \infty$, so that S_N converges to $\frac{a}{1-r}$ as $N \to \infty$ and

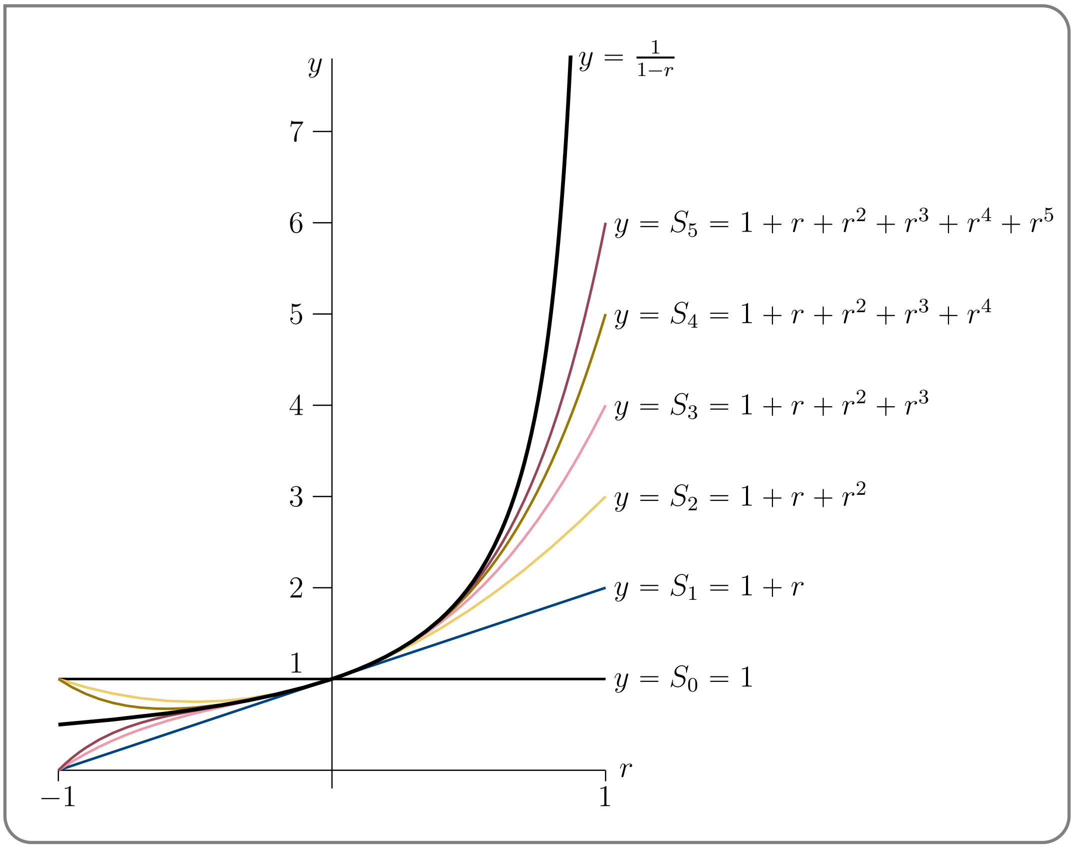
$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1. \tag{3.3.2}$$

On the other hand if $|r| \geqslant 1$, S_N diverges. To understand this divergence, consider the following 4 cases:

- If $r > 1$, then r^N grows to ∞ as $N \to \infty$.
- If $r < -1$, then the magnitude of r^N grows to ∞, and the sign of r^N oscillates between $+$ and $-$, as $N \to \infty$.
- If $r = +1$, then $N + 1$ grows to ∞ as $N \to \infty$.
- If $r = -1$, then r^N just oscillates between $+1$ and -1 as $N \to \infty$.

In each case the sequence of partial sums does not converge and so the series does not converge.

Here are some sketches of the graphs of $\frac{1}{1-r}$ and S_N, $0 \leqslant N \leqslant 5$, for $a = 1$ and $-1 \leqslant r < 1$.



In these sketches we see that

- when $0 < r < 1$, and also when $-1 < r < 0$ with N odd, we have $S_N = \frac{1-r^{N+1}}{1-r} < \frac{1}{1-r}$. On the other hand, when $-1 < r < 0$ with N even, we have $S_N = \frac{1-r^{N+1}}{1-r} > \frac{1}{1-r}$.

- When $0 < |r| < 1$, $S_N = \frac{1-r^{N+1}}{1-r}$ gets closer and closer to $\frac{1}{1-r}$ as N increases.
- When $r = -1$, S_N just alternates between 0, when N is odd, and 1, when N is even.

Example 3.3.9

Now that we know how to handle geometric series let's return to Example 3.2.1.

Example 3.3.10 (Decimal Expansions)

The decimal expansion

$$0.3333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n}$$

is a geometric series with the first term $a = \frac{3}{10}$ and the ratio $r = \frac{1}{10}$. So, by Example 3.3.9,

$$0.3333\cdots = \sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3/10}{1 - 1/10} = \frac{3/10}{9/10} = \frac{1}{3}$$

just as we would have expected.

We can push this idea further. Consider the repeating decimal expansion:

$$0.16161616\cdots = \frac{16}{100} + \frac{16}{10000} + \frac{16}{1000000} + \cdots$$

This is another geometric series with the first term $a = \frac{16}{100}$ and the ratio $r = \frac{1}{100}$. So, by Example 3.3.9,

$$0.16161616\cdots = \sum_{n=1}^{\infty} \frac{16}{100^n} = \frac{16/100}{1 - 1/100} = \frac{16/100}{99/100} = \frac{16}{99}$$

again, as expected. In this way any periodic decimal expansion converges to a ratio of two integers — that is, to a rational number.

Here is another more complicated example.

$$\begin{aligned} 0.1234343434\cdots &= \frac{12}{100} + \frac{34}{10000} + \frac{34}{1000000} + \cdots \\ &= \frac{12}{100} + \sum_{n=2}^{\infty} \frac{34}{100^n} \\ &= \frac{12}{100} + \frac{34}{10000}\frac{1}{1 - 1/100} \qquad \text{by Example 3.3.9 with } a = \frac{34}{100^2} \text{ and } r = \frac{1}{100} \\ &= \frac{12}{100} + \frac{34}{10000}\frac{100}{99} \\ &= \frac{1222}{9900} \end{aligned}$$

Example 3.3.10

Typically, it is quite difficult to write down a neat closed form expression for the partial sums of a series. Geometric series are very notable exceptions to this.

3.3.4 ▸▸ The comparison test

Learning Objectives

- State the comparison test and explain why it works.
- Given a series, decide if the comparison test is appropriate. If so, determine a good series to use as a comparison.
- Apply the comparison test to determine the convergence or divergence of series.
- State the limit comparison test and explain why it follows naturally from the comparison test.
- Use the limit comparison test to determine whether a series converges or diverges. Supply good candidate series for comparison.
- Understand how to use both comparison tests to completely justify that a series converges or diverges. (Note what must be written explicitly.)

Our next convergence test is the comparison test. It is much like the comparison test for improper integrals (see Theorem 1.13.17) and is true for much the same reasons. The rough idea is quite simple. A sum of larger terms must be bigger than a sum of smaller terms. So if we know the big sum converges, then the small sum must converge too. On the other hand, if we know the small sum diverges, then the big sum must also diverge. Formalising this idea gives the following theorem.

Theorem 3.3.11 (The Comparison Test)**.**

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \leqslant Kc_n$ for all $n \geqslant N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \geqslant Kd_n \geqslant 0$ for all $n \geqslant N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

"Proof". We will not prove this theorem here. We'll just observe that it is very reasonable. That's why there are quotation marks around "Proof".

(a) If $\sum_{n=0}^{\infty} c_n$ converges to a finite number and if the terms in $\sum_{n=0}^{\infty} a_n$ are smaller than the terms in $\sum_{n=0}^{\infty} c_n$, then it is no surprise that $\sum_{n=0}^{\infty} a_n$ converges too.

(b) If $\sum\limits_{n=0}^{\infty} d_n$ diverges (i.e. adds up to ∞) and if the terms in $\sum\limits_{n=0}^{\infty} a_n$ are larger than the terms in $\sum\limits_{n=0}^{\infty} d_n$, then of course $\sum\limits_{n=0}^{\infty} a_n$ adds up to ∞, and so diverges, too.

☐

The comparison test for series is also used in much the same way as is the comparison test for improper integrals. Of course, one needs a good series to compare against, and often the series $\sum n^{-p}$ (from Example 3.3.7), for some $p > 0$, turns out to be just what is needed.

Example 3.3.12 $\left(\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}\right)$

We could determine whether or not the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$ converges by applying the integral test. But it is not worth the effort[18]. Whether or not any series converges is determined by the behaviour of the summand[19] for very large n. So the first step in tackling such a problem is to develop some intuition about the behaviour of a_n when n is very large.

- *Step 1: Develop intuition.* In this case, when n is very large[20] $n^2 \gg 2n \gg 3$ so that $\frac{1}{n^2+2n+3} \approx \frac{1}{n^2}$. We already know, from Example 3.3.7, that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. So $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which has $p = 2$, converges, and we would expect that $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$ converges too.
- *Step 2: Verify intuition.* We can use the comparison test to confirm that this is indeed the case. For any $n \geqslant 1$, $n^2 + 2n + 3 > n^2$, so that $\frac{1}{n^2+2n+3} \leqslant \frac{1}{n^2}$. So the comparison test, Theorem 3.3.11, with $a_n = \frac{1}{n^2+2n+3}$ and $c_n = \frac{1}{n^2}$, tells us that $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$ converges.

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[18] Go back and quickly scan Theorem 3.3.6; to apply it we need to show that $\frac{1}{n^2+2n+3}$ is positive and decreasing (it is), and then we need to integrate $\int \frac{1}{x^2+2x+3}\mathrm{d}x$. To do that we reread the notes on partial fractions, then rewrite $x^2 + 2x + 3 = (x + 1)^2 + 2$ and so

$$\int_1^{\infty} \frac{1}{x^2+2x+3}\mathrm{d}x = \int_1^{\infty} \frac{1}{(x+1)^2+2}\mathrm{d}x \cdots$$

and then arctangent appears, etc etc. Urgh. Okay — let's go back to the text now and see how to avoid this.

[19] To understand this consider any series $\sum_{n=1}^{\infty} a_n$. We can always cut such a series into two parts — pick some huge number like 10^6. Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{10^6} a_n + \sum_{n=10^6+1}^{\infty} a_n$$

The first sum, though it could be humongous, is finite. So the left hand side, $\sum_{n=1}^{\infty} a_n$, is a well–defined finite number if and only if $\sum_{n=10^6+1}^{\infty} a_n$, is a well–defined finite number. The convergence or divergence of the series is determined by the second sum, which only contains a_n for "large" n.

[20] The symbol "$\gg$" means "much larger than". Similarly, the symbol "$\ll$" means "much less than". Good shorthand symbols can be quite expressive.

Example 3.3.12

Of course the previous example was "rigged" to give an easy application of the comparison test. It is often relatively easy, using arguments like those in Example 3.3.12, to find a "simple" series $\sum_{n=1}^\infty b_n$ with b_n almost the same as a_n when n is large. However it is pretty rare that $a_n \leqslant b_n$ for all n. It is much more common that $a_n \leqslant Kb_n$ for some constant K. This is enough to allow application of the comparison test. Here is an example.

Example 3.3.13 $\left(\sum_{n=1}^\infty \frac{n+\cos n}{n^3-1/3}\right)$

As in the previous example, the first step is to develop some intuition about the behaviour of a_n when n is very large.

- *Step 1: Develop intuition.* When n is very large,
  - $n \gg |\cos n|$ so that the numerator $n + \cos n \approx n$ and
  - $n^3 \gg 1/3$ so that the denominator $n^3 - 1/3 \approx n^3$.

  So when n is very large

$$a_n = \frac{n+\cos n}{n^3 - 1/3} \approx \frac{n}{n^3} = \frac{1}{n^2}$$

  We already know from Example 3.3.7, with $p = 2$, that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges, so we would expect that $\sum_{n=1}^\infty \frac{n+\cos n}{n^3-1/3}$ converges too.

- *Step 2: Verify intuition.* We can use the comparison test to confirm that this is indeed the case. To do so we need to find a constant K such that $|a_n| = \frac{|n+\cos n|}{n^3-1/3} = \frac{n+\cos n}{n^3-1/3}$ is smaller than $\frac{K}{n^2}$ for all n. A good way[21] to do that is to factor the dominant term (in this case n) out of the numerator and also factor the dominant term (in this case n^3) out of the denominator.

$$a_n = \frac{n+\cos n}{n^3 - 1/3} = \frac{n}{n^3}\,\frac{1+\frac{\cos n}{n}}{1-\frac{1}{3n^3}} = \frac{1}{n^2}\,\frac{1+\frac{\cos n}{n}}{1-\frac{1}{3n^3}}$$

  So now we need to find a constant K such that $\frac{1+(\cos n)/n}{1-1/3n^3}$ is smaller than K for all $n \geqslant 1$.

  - First consider the numerator $1 + (\cos n)\frac{1}{n}$. For all $n \geqslant 1$
    - $\frac{1}{n} \leqslant 1$ and
    - $|\cos n| \leqslant 1$

    So the numerator $1 + (\cos n)\frac{1}{n}$ is always smaller than $1 + (1)\frac{1}{1} = 2$.
  - Next consider the denominator $1 - 1/3n^3$.
    - When $n \geqslant 1$, $\frac{1}{3n^3}$ lies between $\frac{1}{3}$ and 0 so that
    - $1 - \frac{1}{3n^3}$ is between $\frac{2}{3}$ and 1 and consequently

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[21] This is very similar to how we computed limits at infinity way way back near the beginning of the last semester.

    - * $\frac{1}{1-1/3n^3}$ is between $\frac{3}{2}$ and 1.
  - ○ As the numerator $1+(\cos n)\frac{1}{n}$ is always smaller than 2 and $\frac{1}{1-1/3n^3}$ is always smaller than $\frac{3}{2}$, the fraction

$$\frac{1+\frac{\cos n}{n}}{1-\frac{1}{3n^3}} \leqslant 2\left(\frac{3}{2}\right) = 3$$

We now know that

$$|a_n| = \frac{1}{n^2}\,\frac{1+2/n}{1-1/3n^3} \leqslant \frac{3}{n^2}$$

and, since we know $\sum_{n=1}^{\infty} n^{-2}$ converges, the comparison test tells us that $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3-1/3}$ converges.

Example 3.3.13

The last example was actually a relatively simple application of the comparison theorem — finding a suitable constant K can be *really* tedious. Fortunately, there is a variant of the comparison test that completely eliminates the need to explicitly find K.

The idea behind this isn't too complicated. We have already seen that the convergence or divergence of a series depends not on its first few terms, but just on what happens when n is really large. Consequently, if we can work out how the series terms behave for really big n then we can work out if the series converges. So instead of comparing the terms of our series for all n, just compare them when n is big.

Theorem 3.3.14 (Limit Comparison Theorem)**.**

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists.

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

(b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Proof. (a) Because we are told that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, we know that,

- when n is large, $\frac{a_n}{b_n}$ is very close to L, so that $\left|\frac{a_n}{b_n}\right|$ is very close to $|L|$.
- In particular, there is some natural number N_0 so that $\left|\frac{a_n}{b_n}\right| \leqslant |L|+1$, for all $n \geqslant N_0$, and hence

- $|a_n| \leqslant Kb_n$ with $K = |L| + 1$, for all $n \geqslant N_0$.
- The comparison Theorem 3.3.11 now implies that $\sum_{n=1}^{\infty} a_n$ converges.

(b) Let's suppose that $L > 0$. (If $L < 0$, just replace a_n with $-a_n$.) Because we are told that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, we know that,

- when n is large, $\frac{a_n}{b_n}$ is very close to L.
- In particular, there is some natural number N so that $\frac{a_n}{b_n} \geqslant \frac{L}{2}$, and hence
- $a_n \geqslant Kb_n$ with $K = \frac{L}{2} > 0$, for all $n \geqslant N$.
- The comparison Theorem 3.3.11 now implies that $\sum_{n=1}^{\infty} a_n$ diverges.

□

The next two examples illustrate how much of an improvement the above theorem is over the straight comparison test (though, of course, we needed the comparison test to develop the limit comparison test).

Example 3.3.15 $\left(\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2-2n+3}\right)$

Set $a_n = \frac{\sqrt{n+1}}{n^2-2n+3}$. We first try to develop some intuition about the behaviour of a_n for large n and then we confirm that our intuition was correct.

- *Step 1: Develop intuition.* When $n \gg 1$, the numerator $\sqrt{n+1} \approx \sqrt{n}$, and the denominator $n^2 - 2n + 3 \approx n^2$ so that $a_n \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ and it looks like our series should converge by Example 3.3.7 with $p = \frac{3}{2}$.
- *Step 2: Verify intuition.* To confirm our intuition we set $b_n = \frac{1}{n^{3/2}}$ and compute the limit

  $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n^2-2n+3}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n+1}}{n^2-2n+3}$$

  Again it is a good idea to factor the dominant term out of the numerator and the dominant term out of the denominator.

  $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2\sqrt{1+1/n}}{n^2\left(1-2/n+3/n^2\right)} = \lim_{n\to\infty} \frac{\sqrt{1+1/n}}{1-2/n+3/n^2} = 1$$

  We already know that the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by Example 3.3.7 with $p = \frac{3}{2}$. So our series converges by the limit comparison test, Theorem 3.3.14.

Example 3.3.15

Example 3.3.16 $\left(\sum_{n=1}^\infty \frac{\sqrt{n+1}}{n^2-2n+3}\text{, again}\right)$

We can also try to deal with the series of Example 3.3.15, using the comparison test directly. But that requires us to find K so that

$$\frac{\sqrt{n+1}}{n^2-2n+3} \leqslant \frac{K}{n^{3/2}}$$

We might do this by examining the numerator and denominator separately:

- The numerator isn't too bad since for all $n \geqslant 1$:

$$\begin{aligned} n+1 &\leqslant 2n \qquad \text{and so} \\ \sqrt{n+1} &\leqslant \sqrt{2n} \end{aligned}$$

- The denominator is quite a bit more tricky, since we need a *lower* bound, rather than an upper bound, and we cannot just write $|n^2-2n+3| \geqslant n^2$, which is false. Instead we have to make a more careful argument. In particular, we'd like to find N_0 and K' so that $n^2-2n+3 \geqslant K'n^2$, i.e. $\frac{1}{n^2-2n+3} \leqslant \frac{1}{K'n^2}$ for all $n \geqslant N_0$. For $n \geqslant 4$, we have $2n = \frac{1}{2}4n \leqslant \frac{1}{2}n \cdot n = \frac{1}{2}n^2$. So for $n \geqslant 4$,

$$n^2-2n+3 \geqslant n^2 - \frac{1}{2}n^2 + 3 \geqslant \frac{1}{2}n^2$$

Putting the numerator and denominator back together we have

$$\frac{\sqrt{n+1}}{n^2-2n+3} \leqslant \frac{\sqrt{2n}}{n^2/2} = 2\sqrt{2}\frac{1}{n^{3/2}} \qquad \text{for all } n \geqslant 4$$

and the comparison test then tells us that our series converges. It is pretty clear that the approach of Example 3.3.15 was much more straightforward.

Example 3.3.16

3.3.5 ▸▸ The alternating series test

Learning Objectives

- Use the alternating series test to determine convergence of series.
- Give a heuristic explanation to justify the alternating series test.
- Given an alternating series that converges, use partial sums to find an interval containing the value of the series.

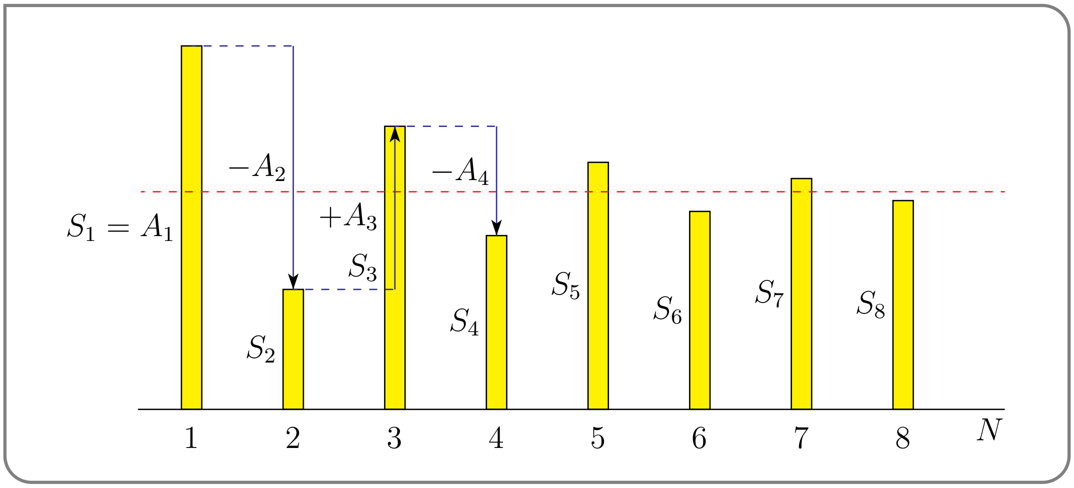
When the signs of successive terms in a series alternate between $+$ and $-$, like for example in $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, the series is called an *alternating series*. More generally, the series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

is alternating if every $A_n \geqslant 0$. Often (but not always) the terms in alternating series get successively smaller. That is, then $A_1 \geqslant A_2 \geqslant A_3 \geqslant \cdots$. In this case:

- The first partial sum is $S_1 = A_1$.
- The second partial sum, $S_2 = A_1 - A_2$, is smaller than S_1 by A_2.
- The third partial sum, $S_3 = S_2 + A_3$, is bigger than S_2 by A_3, but because $A_3 \leqslant A_2$, S_3 remains smaller than S_1. See the figure below.
- The fourth partial sum, $S_4 = S_3 - A_4$, is smaller than S_3 by A_4, but because $A_4 \leqslant A_3$, S_4 remains bigger than S_2. Again, see the figure below.
- And so on.

So the successive partial sums oscillate, but with ever decreasing amplitude. If, in addition, A_n tends to 0 as n tends to ∞, the amplitude of oscillation tends to zero and the sequence $S_1, S_2, S_3, \cdots$ converges to some limit S. This is illustrated in the figure



Here is a convergence test for alternating series that exploits this structure, and that is really easy to apply.

Theorem 3.3.17 (Alternating Series Test)**.**

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $A_n \geqslant 0$ for all $n \geqslant 1$ and
(ii) $A_{n+1} \leqslant A_n$ for all $n \geqslant 1$ (i.e. the sequence is monotone decreasing) and
(iii) $\lim_{n\to\infty} A_n = 0$.

Then

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N A_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^{N} (-1)^{n-1} A_n$.

"Proof". We shall only give part of the proof here. We shall fix any natural number N and concentrate on the last statement, which gives a bound on the truncation error (which is the error introduced when you approximate the full series by the partial sum S_N)

$$E_N = S - S_N = \sum_{n=N+1}^{\infty} (-1)^{n-1} A_n = (-1)^N \Big[A_{N+1} - A_{N+2} + A_{N+3} - A_{N+4} + \cdots \Big]$$

This is of course another series. We're going to study the partial sums

$$S_{N,\ell} = \sum_{n=N+1}^{\ell} (-1)^{n-1} A_n = (-1)^N \sum_{m=1}^{\ell-N} (-1)^{m-1} A_{N+m}$$

for that series.

- If $\ell' > N+1$, with $\ell' - N$ even,

$$(-1)^N S_{N,\ell'} = \overbrace{(A_{N+1} - A_{N+2})}^{\geqslant 0} + \overbrace{(A_{N+3} - A_{N+4})}^{\geqslant 0} + \cdots + \overbrace{(A_{\ell'-1} - A_{\ell'})}^{\geqslant 0} \geqslant 0 \quad \text{and}$$
$$(-1)^N S_{N,\ell'+1} = \overbrace{(-1)^N S_{N,\ell'}}^{\geqslant 0} + \overbrace{A_{\ell'+1}}^{\geqslant 0} \geqslant 0$$

  This tells us that $(-1)^N S_{N,\ell} \geqslant 0$ for all $\ell > N+1$, both even and odd.

- Similarly, if $\ell' > N+1$, with $\ell' - N$ odd,

$$(-1)^N S_{N,\ell'} = A_{N+1} - (\overbrace{A_{N+2} - A_{N+3}}^{\geqslant 0}) - (\overbrace{A_{N+4} - A_{N+5}}^{\geqslant 0}) - \cdots - (\overbrace{A_{\ell'-1} - A_{\ell'}}^{\geqslant 0}) \leqslant A_{N+1}$$
$$(-1)^N S_{N,\ell'+1} = \overbrace{(-1)^N S_{N,\ell'}}^{\leqslant A_{N+1}} - \overbrace{A_{\ell'+1}}^{\geqslant 0} \leqslant A_{N+1}$$

  This tells us that $(-1)^N S_{N,\ell} \leqslant A_{N+1}$ for all for all $\ell > N+1$, both even and odd.

So we now know that $S_{N,\ell}$ lies between its first term, $(-1)^N A_{N+1}$, and 0 for all $\ell > N+1$. While we are not going to prove it here, this implies that, since $A_{N+1} \to 0$ as $N \to \infty$, the series converges and that

$$S - S_N = \lim_{\ell\to\infty} S_{N,\ell}$$

lies between $(-1)^N A_{N+1}$ and 0.

□

Example 3.3.18

We have already seen, in Example 3.3.7, that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. On the other hand, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges by the alternating series test with $A_n = \frac{1}{n}$. Note that

(i) $A_n = \frac{1}{n} \geqslant 0$ for all $n \geqslant 1$, so that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ really is an alternating series, and
(ii) $A_n = \frac{1}{n}$ decreases as n increases, and
(iii) $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

so that all of the hypotheses of the alternating series test, i.e. of Theorem 3.3.17, are satisfied. We shall see, in Example 3.5.18, that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2.$$

Example 3.3.18

Example 3.3.19 (e)

You may already know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. In any event, we shall prove this in Example 3.6.3, below. In particular

$$\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

is an alternating series and satisfies all of the conditions of the alternating series test, Theorem 3.3.17a:

(i) The terms in the series alternate in sign.
(ii) The magnitude of the n^{th} term in the series decreases monotonically as n increases.
(iii) The n^{th} term in the series converges to zero as $n \to \infty$.

So the alternating series test guarantees that, if we approximate, for example,

$$\frac{1}{e} \approx \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!}$$

then the error in this approximation lies between 0 and the next term in the series, which is $\frac{1}{10!}$. That is

$$\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}-\frac{1}{9!}\leqslant\frac{1}{e}\leqslant\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}-\frac{1}{9!}+\frac{1}{10!}$$

so that

$$\frac{1}{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}-\frac{1}{9!}+\frac{1}{10!}}\leqslant e\leqslant\frac{1}{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}-\frac{1}{9!}}$$

which, to seven decimal places says

$$2.7182816\leqslant e\leqslant 2.7182837$$

(To seven decimal places $e = 2.7182818$.)

The alternating series test tells us that, for any natural number N, the error that we make when we approximate $\frac{1}{e}$ by the partial sum $S_N = \sum_{n=0}^{N}\frac{(-1)^n}{n!}$ has magnitude no larger than $\frac{1}{(N+1)!}$. This tends to zero spectacularly quickly as N increases, simply because $(N+1)!$ increases spectacularly quickly as N increases[22]. For example $20! \approx 2.4\times 10^{27}$.

Example 3.3.19

Example 3.3.20

We will shortly see, in Example 3.5.18, that if $-1 < x \leqslant 1$, then

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Suppose that we have to compute $\log\frac{11}{10}$ to within an accuracy of 10^{-12}. Since $\frac{11}{10}=1+\frac{1}{10}$, we can get $\log\frac{11}{10}$ by evaluating $\log(1+x)$ at $x=\frac{1}{10}$, so that

$$\log\frac{11}{10}=\log\left(1+\frac{1}{10}\right)=\frac{1}{10}-\frac{1}{2\times10^2}+\frac{1}{3\times10^3}-\frac{1}{4\times10^4}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n\times10^n}$$

By the alternating series test, this series converges. Also by the alternating series test, approximating $\log\frac{11}{10}$ by throwing away all but the first N terms

$$\log\frac{11}{10}\approx\frac{1}{10}-\frac{1}{2\times10^2}+\frac{1}{3\times10^3}-\frac{1}{4\times10^4}+\cdots+(-1)^{N-1}\frac{1}{N\times10^N}=\sum_{n=1}^{N}(-1)^{n-1}\frac{1}{n\times10^n}$$

introduces an error whose magnitude is no more than the magnitude of the first term that we threw away.

$$\text{error}\leqslant\frac{1}{(N+1)\times10^{N+1}}$$

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[22] The interested reader may wish to check out "Stirling's approximation", which says that $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

To achieve an error that is no more than 10^{-12}, we have to choose N so that

$$\frac{1}{(N+1)\times 10^{N+1}} \leqslant 10^{-12}$$

The best way to do so is simply to guess — we are not going to be able to manipulate the inequality $\frac{1}{(N+1)\times 10^{N+1}} \leqslant \frac{1}{10^{12}}$ into the form $N \leqslant \cdots$, and even if we could, it would not be worth the effort. We need to choose N so that the denominator $(N+1)\times 10^{N+1}$ is at least 10^{12}. That is easy, because the denominator contains the factor 10^{N+1} which is at least 10^{12} whenever $N+1 \geqslant 12$, i.e. whenever $N \geqslant 11$. So we will achieve an error of less than 10^{-12} if we choose $N = 11$.

$$\left.\frac{1}{(N+1)\times 10^{N+1}}\right|_{N=11} = \frac{1}{12\times 10^{12}} < \frac{1}{10^{12}}$$

This is not the smallest possible choice of N, but in practice that just doesn't matter — your computer is not going to care whether or not you ask it to compute a few extra terms. If you really need the smallest N that obeys $\frac{1}{(N+1)\times 10^{N+1}} \leqslant \frac{1}{10^{12}}$, you can next just try $N = 10$, then $N = 9$, and so on.

$$\begin{aligned}
\left.\frac{1}{(N+1)\times 10^{N+1}}\right|_{N=11} &= \frac{1}{12\times 10^{12}} < \frac{1}{10^{12}}\\
\left.\frac{1}{(N+1)\times 10^{N+1}}\right|_{N=10} &= \frac{1}{11\times 10^{11}} < \frac{1}{10\times 10^{11}} = \frac{1}{10^{12}}\\
\left.\frac{1}{(N+1)\times 10^{N+1}}\right|_{N=9} &= \frac{1}{10\times 10^{10}} = \frac{1}{10^{11}} > \frac{1}{10^{12}}
\end{aligned}$$

So in this problem, the smallest acceptable $N = 10$.

Example 3.3.20

3.3.6 ▸▸ The ratio test

Learning Objectives

- State the ratio test and explain its connection with geometric series.
- Apply the ratio test to series when appropriate. In particular, to series involving factorials and/or exponentials.
- State when the ratio test is inconclusive and explain what that means.

The idea behind the ratio test comes from a reexamination of the geometric series. Recall that the geometric series

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} ar^n$$

converges when $|r| < 1$ and diverges otherwise. So the convergence of this series is completely determined by the number r. This number is just the ratio of successive terms — that is, $r = a_{n+1}/a_n$.

In general the ratio of successive terms of a series, $\frac{a_{n+1}}{a_n}$, is not constant, but depends on n. However, as we have noted above, the convergence of a series $\sum a_n$ is determined by the behaviour of its terms when n is large. In this way, the behaviour of this ratio when n is small tells us nothing about the convergence of the series, but the limit of the ratio as $n \to \infty$ does. This is the basis of the ratio test.

Theorem 3.3.21 (Ratio Test).

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geqslant N$.

(a) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Warning 3.3.22.

Beware that the ratio test provides absolutely no conclusion about the convergence or divergence of the series $\sum\limits_{n=1}^{\infty} a_n$ if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. See Example 3.3.25, below.

Proof. (a) Pick any number R obeying $L < R < 1$. We are assuming that $\left|\frac{a_{n+1}}{a_n}\right|$ approaches L as $n \to \infty$. In particular there must be some natural number M so that $\left|\frac{a_{n+1}}{a_n}\right| \leqslant R$ for all $n \geqslant M$. So $|a_{n+1}| \leqslant R|a_n|$ for all $n \geqslant M$. In particular

$$\begin{aligned}
|a_{M+1}| &\leqslant R\,|a_M| \\
|a_{M+2}| &\leqslant R\,|a_{M+1}| \leqslant R^2\,|a_M| \\
|a_{M+3}| &\leqslant R\,|a_{M+2}| \leqslant R^3\,|a_M| \\
&\vdots \\
|a_{M+\ell}| &\leqslant R^\ell\,|a_M|
\end{aligned}$$

for all $\ell \geqslant 0$. The series $\sum_{\ell=0}^{\infty} R^\ell\,|a_M|$ is a geometric series with ratio R smaller than one in magnitude and so converges. Consequently, by the comparison test with a_n replaced by $A_\ell = a_{n+\ell}$ and c_n replaced by $C_\ell = R^\ell\,|a_M|$, the series $\sum\limits_{\ell=1}^{\infty} a_{M+\ell} = \sum\limits_{n=M+1}^{\infty} a_n$ converges. So the series $\sum\limits_{n=1}^{\infty} a_n$ converges too.

(b) We are assuming that $\left|\frac{a_{n+1}}{a_n}\right|$ approaches $L > 1$ as $n \to \infty$. In particular there must be some natural number $M > N$ so that $\left|\frac{a_{n+1}}{a_n}\right| \geqslant 1$ for all $n \geqslant M$. So $|a_{n+1}| \geqslant |a_n|$ for all $n \geqslant M$. That is, $|a_n|$ increases as n increases as long as $n \geqslant M$. So $|a_n| \geqslant |a_M|$ for all $n \geqslant M$ and a_n cannot converge to zero as $n \to \infty$. So the series diverges by the divergence test.

□

Example 3.3.23 ($\sum_{n=0}^{\infty} anx^{n-1}$)

Fix any two nonzero real numbers a and x. We have already seen in Example 3.3.9 — we have just renamed r to x — that the geometric series $\sum_{n=0}^{\infty} ax^n$ converges when $|x| < 1$ and diverges when $|x| \geqslant 1$. We are now going to consider a new series, constructed by differentiating[23] each term in the geometric series $\sum_{n=0}^{\infty} ax^n$. This new series is

$$\sum_{n=0}^{\infty} a_n \qquad \text{with} \quad a_n = a\,n\,x^{n-1}$$

Let's apply the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a\,(n+1)\,x^n}{a\,n\,x^{n-1}}\right| = \frac{n+1}{n}|x| = \left(1+\frac{1}{n}\right)|x| \to L = |x| \quad \text{as } n \to \infty$$

The ratio test now tells us that the series $\sum_{n=0}^{\infty} a\,n\,x^{n-1}$ converges if $|x| < 1$ and diverges if $|x| > 1$. It says nothing about the cases $x = \pm 1$. But in both of those cases $a_n = a\,n\,(\pm 1)^n$ does not converge to zero as $n \to \infty$ and the series diverges by the divergence test.

Example 3.3.23

Notice that in the above example, we had to apply another convergence test in addition to the ratio test. This will be commonplace when we reach power series and Taylor series — the ratio test will tell us something like

> The series converges for $|x| < R$ and diverges for $|x| > R$.

Of course, we may still have to to determine what happens when $x = +R, -R$. To determine convergence or divergence in those cases we will need to use one of the other tests we have seen.

Example 3.3.24 ($\sum_{n=0}^{\infty} \frac{a}{n+1} X^{n+1}$, $|X| \neq 1$)

Once again, fix any two nonzero real numbers a and X. We again start with the geometric series $\sum_{n=0}^{\infty} ax^n$ but this time we construct a new series by integrating[24] each term, ax^n,

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[23] We shall see later, in Theorem 3.5.11, that the function $\sum_{n=0}^{\infty} anx^{n-1}$ is indeed the derivative of the function $\sum_{n=0}^{\infty} ax^n$. Of course, such a statement only makes sense where these series converge — how can you differentiate a divergent series? (This is not an allusion to a popular series of dystopian novels.) Actually, there is quite a bit of interesting and useful mathematics involving divergent series, but it is well beyond the scope of this course.

[24] We shall also see later, in Theorem 3.5.11, that the function $\sum_{n=0}^{\infty} \frac{a}{n+1} x^{n+1}$ is indeed an antiderivative of the function $\sum_{n=0}^{\infty} ax^n$.

from $x = 0$ to $x = X$ giving $\frac{a}{n+1}X^{n+1}$. The resulting new series is

$$\sum_{n=0}^{\infty} a_n \qquad \text{with } a_n = \frac{a}{n+1}X^{n+1}$$

To apply the ratio test we need to compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{a}{n+2}X^{n+2}}{\frac{a}{n+1}X^{n+1}}\right| = \frac{n+1}{n+2}|X| = \frac{1+\frac{1}{n}}{1+\frac{2}{n}}|X| \to L = |X| \quad \text{as } n \to \infty$$

The ratio test now tells us that the series $\sum_{n=0}^{\infty} \frac{a}{n+1}X^{n+1}$ converges if $|X| < 1$ and diverges if $|X| > 1$. (It says nothing about the cases $X = \pm 1$.)

If $X = 1$, the series reduces to

$$\sum_{n=0}^{\infty} \frac{a}{n+1}X^{n+1}\bigg|_{X=1} = \sum_{n=0}^{\infty} \frac{a}{n+1} = a\sum_{m=1}^{\infty} \frac{1}{m} \qquad \text{with } m = n+1$$

which is just a times the harmonic series, which we know diverges, by Example 3.3.7.

If $X = -1$, the series reduces to

$$\sum_{n=0}^{\infty} \frac{a}{n+1}X^{n+1}\bigg|_{X=-1} = \sum_{n=0}^{\infty} (-1)^{n+1}\frac{a}{n+1}$$

which converges by the alternating series test. See Example 3.3.18.

In conclusion, the series $\sum_{n=0}^{\infty} \frac{a}{n+1}X^{n+1}$ converges if and only if $-1 \leqslant X < 1$.

Example 3.3.24

The ratio test is often quite easy to apply, but one must always be careful when the limit of the ratio is 1. The next example illustrates this.

Example 3.3.25 ($L = 1$)

In this example, we are going to see three different series that all have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. One is going to diverge and the other two are going to converge.

- The first series is the harmonic series

  $$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = \frac{1}{n}$$

  We have already seen, in Example 3.3.7, that this series diverges. It has

  $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to L = 1 \quad \text{as } n \to \infty$$

- The second series is the alternating harmonic series

$$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = (-1)^{n-1}\frac{1}{n}$$

We have already seen, in Example 3.3.18, that this series converges. But it also has

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^n \frac{1}{n+1}}{(-1)^{n-1}\frac{1}{n}}\right| = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to L = 1 \quad \text{as } n \to \infty$$

- The third series is

$$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = \frac{1}{n^2}$$

We have already seen, in Example 3.3.7 with $p = 2$, that this series converges. But it also has

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \to L = 1 \quad \text{as } n \to \infty$$

Example 3.3.25

Let's do a somewhat artificial example that forces us to combine a few of the techniques we have seen.

Example 3.3.26 $\left(\sum_{n=1}^{\infty} \frac{(-3)^n\sqrt{n+1}}{2n+3}x^n,\ |x| \neq \frac{1}{3}\right)$

Again, the convergence of this series will depend on x.

- Let us start with the ratio test — so we compute

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-3)^{n+1}\sqrt{n+2}(2n+3)x^{n+1}}{(-3)^n\sqrt{n+1}(2n+5)x^n}\right| \\ &= |-3| \cdot \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{2n+3}{2n+5} \cdot |x|\end{aligned}$$

So in the limit as $n \to \infty$ we are left with

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3|x|$$

- The ratio test then tells us that if $3|x| > 1$ the series diverges, while when $3|x| < 1$ the series converges.

- This leaves us with the cases $x = +\frac{1}{3}$ and $-\frac{1}{3}$.

- Setting $x = \frac{1}{3}$ gives the series

  $$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n+1}}{2n+3}$$

  The fact that the terms alternate here suggests that we use the alternating series test. That will show that this series converges provided $\frac{\sqrt{n+1}}{2n+3}$ decreases as n increases. So we define the function

  $$f(t) = \frac{\sqrt{t+1}}{2t+3}$$

  (which is constructed by replacing the n in $\frac{\sqrt{n+1}}{2n+3}$ with t) and verify that $f(t)$ is a decreasing function of t. To prove that, it suffices to show its derivative is negative when $t \geqslant 1$:

  $$\begin{aligned} f'(t) &= \frac{(2t+3) \cdot \frac{1}{2} \cdot (t+1)^{-1/2} - 2\sqrt{t+1}}{(2t+3)^2} \\ &= \frac{(2t+3) - 4(t+1)}{2\sqrt{t+1}(2t+3)^2} \\ &= \frac{-2t-1}{2\sqrt{t+1}(2t+3)^2} \end{aligned}$$

  When $t \geqslant 1$ this is negative and so $f(t)$ is a decreasing function. Thus we can apply the alternating series test to show that the series converges when $x = \frac{1}{3}$.

- When $x = -\frac{1}{3}$ the series becomes

  $$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3}.$$

  Notice that when n is large, the summand is approximately $\frac{\sqrt{n}}{2n}$ which suggests that the series will diverge by comparison with $\sum n^{-1/2}$. To formalise this, we can use the limit comparison theorem:

  $$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{n+1}}{2n+3} \frac{1}{n^{-1/2}} &= \lim_{n\to\infty} \frac{\sqrt{n} \cdot \sqrt{1+1/n}}{n(2+3/n)} \cdot n^{1/2} \\ &= \lim_{n\to\infty} \frac{n \cdot \sqrt{1+1/n}}{n(2+3/n)} \\ &= \frac{1}{2} \end{aligned}$$

  So since this ratio has a finite limit and the series $\sum n^{-1/2}$ diverges, we know that our series also diverges.

So in summary the series converges when $-\frac{1}{3} < x \leqslant \frac{1}{3}$ and diverges otherwise.

Example 3.3.26

3.3.7 ▸▸ Convergence test list

Learning Objectives

- Given a series, determine which test to use to test for convergence/divergence.

We now have a nice toolbox of convergence tests.

- *Special types of series*
  - for geometric series, we know when they converge, and can even find out what they converge *to*
  - we know when p-series converge; they are often our go-to choices for comparison series
- *Divergence test*
  - works well when the n^{th} term in the series *fails* to converge to zero as n tends to infinity
  - *cannot* be used to conclude that a series converges
- *Alternating Series Test*
  - works well when successive terms in the series alternate in sign
  - don't forget to check that successive terms decrease in magnitude and tend to zero as n tends to infinity
- *Integral test*
  - works well when, if you substitute x for n in the n^{th} term, you get a function $f(x)$ that you can integrate
  - don't forget to check that $f(x) \geqslant 0$ and that $f(x)$ decreases as x increases
- *Ratio test*
  - works well when $\frac{a_{n+1}}{a_n}$ simplifies enough that you can easily compute $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$
  - this often happens when a_n contains exponential functions, like 7^n, or factorials, like $n!$
  - don't forget that $L = 1$ tells you nothing about the convergence/divergence of the series
- *Comparison test and limit comparison test*
  - works well when, for very large n, the n^{th} term a_n is approximately the same as a simpler term b_n (see Example 3.3.13) and it is easy to determine whether or not $\sum_{n=1}^{\infty} b_n$ converges

  - don't forget to check that $b_n \geqslant 0$
  - usually the limit comparison test is easier to apply than the comparison test

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3.4▲ Absolute and conditional convergence

Learning Objectives

- Define both absolute convergence and conditional convergence.
- Use absolute convergence to determine the convergence of some series.
- Explain why "absolute convergence implies conditional convergence" only works one way.

We have now seen examples of series that converge and of series that diverge. But we haven't really discussed how robust the convergence of series is — that is, can we tweak the coefficients in some way while leaving the convergence unchanged? A good example of this is the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

This is a simple geometric series and we know it converges. We have also seen, as examples 3.3.23 and 3.3.24 showed us, that we can multiply or divide the n^{th} term by n and it will still converge. We can even multiply the n^{th} term by $(-1)^n$, and it will still converge. Pretty robust.

On the other hand, we have explored the Harmonic series and its relatives quite a lot and we know it is much more delicate. While

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, we also know the following two series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.00000001}} \qquad\qquad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$$

This suggests that the divergence of the Harmonic series is much more delicate. In this section, we discuss one way to characterise this sort of delicate convergence — especially in the presence of changes of sign.

3.4.1 ▸▸ Definitions

Definition 3.4.1 (Absolute and conditional convergence)**.**

(a) A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges we say that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

If you consider these definitions for a moment, it should be clear that absolute convergence is a stronger condition than just simple convergence. All the terms in $\sum_n |a_n|$ are forced to be positive (by the absolute value signs), so that $\sum_n |a_n|$ must be bigger than $\sum_n a_n$— making it easier for $\sum_n |a_n|$ to diverge. This is formalised by the following theorem, which is an immediate consequence of the comparison test, Theorem 3.3.11.a, with $c_n = |a_n|$.

Theorem 3.4.2 (Absolute convergence implies convergence)**.**

If the series $\sum_{n=1}^{\infty} |a_n|$ converges then the series $\sum_{n=1}^{\infty} a_n$ also converges. That is, absolute convergence implies convergence.

Recall that some of our convergence tests (for example, the integral test) may only be applied to series with positive terms. Theorem 3.4.2 opens up the possibility of applying "positive only" convergence tests to series whose terms are not all positive, by checking for "absolute convergence" rather than for plain "convergence".

Example 3.4.3 $\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\right)$

The *alternating harmonic series* $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ of Example 3.3.18 converges (by the alternating series test). But the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ of Example 3.3.7 diverges (by the integral test). So the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ converges conditionally.

Example 3.4.3

Example 3.4.4 $\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}\right)$

Because the series $\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ of Example 3.3.7 converges (by the integral test), the series $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^2}$ converges absolutely, and hence converges.

Example 3.4.4

Example 3.4.5 (random signs)

Imagine flipping a coin infinitely many times. Set $\sigma_n = +1$ if the n^{th} flip comes up heads and $\sigma_n = 0$ if the n^{th} flip comes up tails. We know that the series $\sum_{n=1}^{\infty} \left|(-1)^{\sigma_n}\frac{1}{n^2}\right| = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges. So $\sum_{n=1}^{\infty}(-1)^{\sigma_n}\frac{1}{n^2}$ converges absolutely, and hence converges.

Example 3.4.5

3.5▲ Power series

Learning Objectives

- Define power series for a function centred at a point.
- Explain what is meant by "radius of convergence."
- Compute the radius of convergence of a given power series.
- Translate the radius of convergence and the centre of a power series into the largest open interval over which the series converges.
- Given a power series, determine which test to use to test for convergence/divergence.
- Perform operations on power series as per Theorems 3.5.11 and 3.5.16, keeping in mind the radius of convergence.

Let's return to the simple geometric series

$$\sum_{n=0}^{\infty} x^n$$

where x is some real number. As we have seen (back in Example 3.3.9), when $|x| < 1$ this series converges, while for $|x| \geqslant 1$ the series diverges. Consequently we can consider this series to be a function of x:

$$f(x) = \sum_{n=0}^{\infty} x^n \qquad \text{on the domain } |x| < 1.$$

Furthermore (also from Example 3.3.9) we know what the function is:

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Hence we can consider the series $\sum_{n=0}^{\infty} x^n$ as a new way of representing the function $\frac{1}{1-x}$ when $|x| < 1$. This series is an example of a power series.

Of course, representing a function as simple as $\frac{1}{1-x}$ by a series doesn't seem like it is going to make life easier. However, the idea of representing a function by a series turns out to be extremely helpful. Power series are robust mathematical objects and interact very nicely with not only standard arithmetic operations, but also with differentiation and integration (see Theorem 3.5.11). This means, for example, that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n && \text{provided } |x| < 1\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}x^n && \text{just differentiate term by term}\\
&= \sum_{n=0}^{\infty} nx^{n-1}
\end{aligned}$$

and in a very similar way

$$\begin{aligned}
\int \frac{1}{1-x}\mathrm{d}x &= \int \sum_{n=0}^{\infty} x^n \mathrm{d}x && \text{provided } |x| < 1\\
&= \sum_{n=0}^{\infty}\int x^n \mathrm{d}x && \text{just integrate term by term}\\
&= C + \sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}
\end{aligned}$$

We are hiding some mathematics under the word "just" in the work above, but you can see that once we have a power series representation of a function, differentiation and integration become very straightforward.

Our goal for this section is the development of machinery to define and understand power series. This will allow us to answer questions[25] like

$$\text{Is } e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}\ ?$$

Our starting point (now that we have equipped ourselves with basic ideas about series), is the definition of power series.

---

[25] Recall that $n! = 1 \times 2 \times 3 \times \cdots \times n$ is called "n factorial". By convention, $0! = 1$.

3.5.1 ▸▸ Radius of convergence

Definition 3.5.1.

A series of the form

$$A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots = \sum_{n=0}^{\infty} A_n (x-c)^n$$

is called a *power series in* $(x-c)$ or a *power series centered on* c. The numbers A_n are called the coefficients of the power series.

One often considers power series centered on $c = 0$ and then the series reduces to

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots = \sum_{n=0}^{\infty} A_n x^n$$

For example $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is the power series with $c = 0$ and $A_n = \frac{1}{n!}$. Typically, as in that case, the coefficients A_n are given fixed numbers, but the “x” is to be thought of as a variable. Thus each power series is really a whole family of series — a different series for each value of x.

One possible value of x is $x = c$ and then the series reduces[26] to

$$\begin{aligned} \sum_{n=0}^{\infty} A_n (x-c)^n \Big|_{x=c} &= \sum_{n=0}^{\infty} A_n (c-c)^n \\ &= \underbrace{A_0}_{n=0} + \underbrace{0}_{n=1} + \underbrace{0}_{n=2} + \underbrace{0}_{n=3} + \cdots \end{aligned}$$

and so simply converges to A_0.

We now know that a power series converges when $x = c$. We can now use our convergence tests to determine for what other values of x the series converges. Perhaps most straightforward is the ratio test. The n^{th} term in the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ is $a_n = A_n(x-c)^n$. To apply the ratio test we need to compute the limit

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{A_{n+1}(x-c)^{n+1}}{A_n(x-c)^n}\right| \\ &= \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| \cdot |x-c| \\ &= |x-c| \cdot \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right|. \end{aligned}$$

When we do so there are several possible outcomes.

---

[26] *By convention*, when the term $(x-c)^0$ appears in a power series, it has value 1 for all values of x, even $x = c$.

- If the limit of ratios exists and is non-zero

  $$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = A \neq 0,$$

  then the ratio test says that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$

  - converges when $A \cdot |x-c| < 1$, i.e. when $|x-c| < {}^1/_A$, and
  - diverges when $A \cdot |x-c| > 1$, i.e. when $|x-c| > {}^1/_A$.

  Because of this, when the limit exists, the quantity

  Equation 3.5.2.

  $$R = \frac{1}{A} = \left[\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|\right]^{-1}$$

  is called the *radius of convergence* of the series[27].

- If the limit of ratios exists and is zero

  $$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = 0$$

  then $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right||x-c| = 0$ for every x and the ratio test tells us that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for every number x. In this case we say that the series has an infinite radius of convergence.

- If the limit of ratios diverges to $+\infty$

  $$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = +\infty$$

  then $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right||x-c| = +\infty$ for every $x \neq c$. The ratio test then tells us that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ diverges for every number $x \neq c$. As we have seen above, when $x = c$, the series reduces to $A_0 + 0 + 0 + 0 + 0 + \cdots$, which converges. In this case we say that the series has radius of convergence zero.

- If $\left|\frac{A_{n+1}}{A_n}\right|$ does not approach a limit as $n \to \infty$, then we learn nothing from the ratio test and we must use other tools to understand the convergence of the series.

All of these possibilities occur in nature. We give an example of each below. But first, the concept of “radius of convergence” is important enough to warrant a formal definition.

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[27] The use of the word “radius” might seem a little odd here, since we are really describing the interval in the real line where the series converges. However, when one starts to consider power series over complex numbers, the radius of convergence does describe a circle inside the complex plane and so “radius” is a more natural descriptor.

Definition 3.5.3.

(a) Let $0 < R < \infty$. If $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for $|x-c| < R$, and diverges for $|x-c| > R$, then we say that the series has radius of convergence R.

(b) If $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for every number x, we say that the series has an infinite radius of convergence.

(c) If $\sum_{n=0}^{\infty} A_n(x-c)^n$ diverges for every $x \neq c$, we say that the series has radius of convergence zero.

Example 3.5.4 (Finite nonzero radius of convergence)

We already know that, if $a \neq 0$, the geometric series $\sum\limits_{n=0}^{\infty} ax^n$ converges when $|x| < 1$ and diverges when $|x| \geqslant 1$. So, in the terminology of Definition 3.5.3, the geometric series has radius of convergence $R = 1$. As a consistency check, we can also compute R using (3.5.2). The series $\sum\limits_{n=0}^{\infty} ax^n$ has $A_n = a$. So

$$R = \left[\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|\right]^{-1} = \left[\lim_{n\to\infty} 1\right]^{-1} = 1$$

as expected.

Example 3.5.4

Example 3.5.5 (Radius of convergence = $+\infty$)

The series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ has $A_n = \frac{1}{n!}$. So

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| &= \lim_{n\to\infty}\frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1\times 2\times 3\times\cdots\times n}{1\times 2\times 3\times\cdots\times n\times(n+1)}\\ &= \lim_{n\to\infty}\frac{1}{n+1}\\ &= 0\end{aligned}$$

and $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence ∞. It converges for every x.

Example 3.5.5

Example 3.5.6 (Radius of convergence = 0)

The series $\sum_{n=0}^{\infty} n!x^n$ has $A_n = n!$. So

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}\frac{1\times 2\times 3\times 4\times\cdots\times n\times(n+1)}{1\times 2\times 3\times 4\times\cdots\times n}\\ &= \lim_{n\to\infty}(n+1)\\ &= +\infty\end{aligned}$$

and $\sum_{n=0}^{\infty} n!x^n$ has radius of convergence zero[28]. It converges only for $x = 0$, where it takes the value $0! = 1$.

Example 3.5.6

Example 3.5.7

Comparing the series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \cdots$$

to

$$\sum_{n=0}^{\infty} A_n x^n = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 + \cdots$$

we see that

$$A_0 = 1 \quad A_1 = 2 \qquad A_2 = 1 \qquad A_3 = 2 \qquad A_4 = 1 \qquad A_5 = 2 \qquad \cdots$$

so that

$$\frac{A_1}{A_0} = 2 \qquad \frac{A_2}{A_1} = \frac{1}{2} \qquad \frac{A_3}{A_2} = 2 \qquad \frac{A_4}{A_3} = \frac{1}{2} \qquad \frac{A_5}{A_4} = 2 \qquad \cdots$$

and $\frac{A_{n+1}}{A_n}$ does not converge as $n \to \infty$. Since the limit of the ratios does not exist, we cannot tell anything from the ratio test. Nonetheless, we can still figure out for which x's our power series converges.

- Because every coefficient A_n is either 1 or 2, the n^{th} term in our series obeys

  $$\left|A_n x^n\right| \leqslant 2|x|^n$$

  and so is smaller than the n^{th} term in the geometric series $\sum_{n=0}^{\infty} 2|x|^n$. This geometric series converges if $|x| < 1$. So, by the comparison test, our series converges for $|x| < 1$ too.

---

[28] Because of this, it might seem that such a series is fairly pointless. However there are all sorts of mathematical games that can be played with them without worrying about their convergence. Such "formal" power series can still impart useful information and the interested reader is invited to look up "generating functions" with their preferred search engine.

- Since every A_n is at least one, the n^{th} term in our series obeys

  $$\left|A_n x^n\right| \geqslant |x|^n$$

  If $|x| \geqslant 1$, this $a_n = A_n x^n$ cannot converge to zero as $n \to \infty$, and our series diverges by the divergence test.

In conclusion, our series converges if and only if $|x| < 1$, and so has radius of convergence 1.

Example 3.5.7

Example 3.5.8

Lets construct a series from the digits of π. Now to avoid dividing by zero, let us set

$$A_n = 1 + \text{the } n^{\text{th}} \text{ digit of } \pi$$

Since $\pi = 3.141591\ldots$

$$A_0 = 4 \qquad A_1 = 2 \qquad A_2 = 5 \qquad A_3 = 2 \qquad A_4 = 6 \qquad A_5 = 10 \qquad A_6 = 2 \qquad \cdots$$

Consequently every A_n is an integer between 1 and 10 and gives us the series

$$\sum_{n=0}^{\infty} A_n x^n = 4 + 2x + 5x^2 + 2x^3 + 6x^4 + 10x^5 + \cdots$$

The number π is irrational and consequently the ratio $\frac{A_{n+1}}{A_n}$ cannot have a limit as $n \to \infty$. If you do not understand why this is the case then don't worry too much about it[29]. As in the last example, the limit of the ratios does not exist and we cannot tell anything from the ratio test. But we can still figure out for which x's it converges.

- Because every coefficient A_n is no bigger (in magnitude) than 10, the n^{th} term in our series obeys

  $$\left|A_n x^n\right| \leqslant 10|x|^n$$

  and so is smaller than the n^{th} term in the geometric series $\sum_{n=0}^{\infty} 10|x|^n$. This geometric series converges if $|x| < 1$. So, by the comparison test, our series converges for $|x| < 1$ too.

- Since every A_n is at least one, the n^{th} term in our series obeys

  $$\left|A_n x^n\right| \geqslant |x|^n$$

  If $|x| \geqslant 1$, this $a_n = A_n x^n$ cannot converge to zero as $n \to \infty$, and our series diverges by the divergence test.

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[29] This is a little beyond the scope of the course. Roughly speaking, think about what would happen if the limit of the ratios did exist. If the limit were smaller than 1, then it would tell you that the terms of our series must be getting smaller and smaller and smaller — which is impossible because they are all integers between 1 and 10. Similarly if the limit existed and were bigger than 1 then the terms of the series would have to get bigger and bigger and bigger — also impossible. Hence if the ratio exists then it must be equal to 1 — but in that case because the terms are integers, they would have to be all equal when n became big enough. But that means that the expansion of π would be eventually periodic — something that only rational numbers do.

In conclusion, our series converges if and only if $|x| < 1$, and so has radius of convergence 1.

Example 3.5.8

Though we won't prove it, it is true that every power series has a radius of convergence, whether or not the limit $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$ exists.

Theorem 3.5.9.

Let $\sum_{n=0}^{\infty} A_n(x-c)^n$ be a power series. Then one of the following alternatives must hold.

(a) The power series converges for every number x. In this case we say that the radius of convergence is ∞.

(b) There is a number $0 < R < \infty$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$. Then R is called the radius of convergence.

(c) The series converges for $x = c$ and diverges for all $x \neq c$. In this case, we say that the radius of convergence is 0.

Suppose that the power series $\sum_{n=0}^{\infty} A_n(x-c)^n$ has radius of convergence R. Then from Theorem 3.5.9, we have that:

- if $R = \infty$, then it converges for all real numbers, which is also denoted $(-\infty, \infty)$, and
- if $R = 0$, then it converges only when $x = c$, and
- if $0 < R < \infty$, then we know that the series converges for any x which obeys

$$\begin{aligned} |x-c| < R \quad &\text{or equivalently} \quad -R < x-c < R \\ &\text{or equivalently} \quad c-R < x < c+R \end{aligned}$$

  But we do not (yet) know whether or not the series converges at the two end points of that interval. We do know, however, that the interval over which it converges must be one of the four options below:

  - ◦ $c - R < x < c + R$, which is also denoted $(c - R\,,\, c + R)$, or
  - ◦ $c - R \leqslant x < c + R$, which is also denoted $[c - R\,,\, c + R)$, or
  - ◦ $c - R < x \leqslant c + R$, which is also denoted $(c - R\,,\, c + R]$, or
  - ◦ $c - R \leqslant x \leqslant c + R$, which is also denoted $[c - R\,,\, c + R]$.

To reiterate — while a finite radius convergence R tells us that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$, it does not (by itself) tell us whether or not the series converges when $|x-c| = R$, i.e. when $x = c \pm R$.

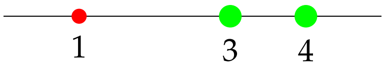
Example 3.5.10

We are told that a certain power series with centre $c = 3$, converges at $x = 4$ and diverges at $x = 1$. What else can we say about the convergence or divergence of the series for other values of x?

We are told that the series is centred at 3, so its terms are all powers of $(x-3)$ and it is of the form

$$\sum_{n\geqslant 0} A_n (x-3)^n.$$

A good way to summarise the convergence data we are given is with a figure like the one below. Large green dots mark the values of x where the series is known to converge. (Recall that every power series converges at its centre.) The small red dot marks the value of x where the series is known to diverge.



Can we say more about the convergence and/or divergence of the series for other values of x? Yes!

Let us think about the radius of convergence, R, of the series. We know that it must exist and the information we have been given allows us to bound R. Recall that

- the series converges at x provided that $|x-3| < R$ and
- the series diverges at x if $|x-3| > R$.

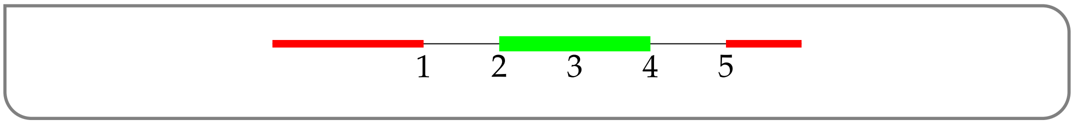
We have been told that

- the series converges when $x = 4$, which tells us that
  - $x = 4$ cannot obey $|x-3| > R$ so
  - $x = 4$ must obey $|x-3| \leqslant R$, i.e. $|4-3| \leqslant R$, i.e. $R \geqslant 1$
- the series diverges when $x = 1$ so we also know that
  - $x = 1$ cannot obey $|x-3| < R$ so
  - $x = 1$ must obey $|x-3| \geqslant R$, i.e. $|1-3| \geqslant R$, i.e. $R \leqslant 2$

We still don't know R exactly. But we do know that $1 \leqslant R \leqslant 2$. Consequently,

- since 1 is the smallest that R could be, the series certainly converges at x if $|x-3| < 1$, i.e. if $2 < x < 4$ and
- since 2 is the largest that R could be, the series certainly diverges at x if $|x-3| > 2$, i.e. if $x > 5$ or if $x < 1$.

The following figure provides a resume of all of this convergence data — there is convergence at green x's (marked with a thick line) and divergence at red x's (thin lines).



Notice that from the data given we cannot say anything about the convergence or divergence of the series on the intervals $(1,2]$ and $(4,5]$.

One lesson that we can derive from this example is:

- if a series has centre c and converges at a,

- then it also converges at all points between c and a, as well as at all points of distance strictly less than $|a-c|$ from c on the other side of c from a.

Example 3.5.10

3.5.2 ▸▸ Working with power series

Just as we have done previously with limits, differentiation and integration, we can construct power series representations of more complicated functions by using those of simpler functions. Here is a theorem that helps us to do so.

Theorem 3.5.11 (Operations on Power Series).

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n(x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n(x-c)^n$$

for all x obeying $|x-c| < R$. In particular, we are assuming that both power series have radius of convergence at least R. Also let K be a constant. Then

$$\begin{aligned}
f(x)+g(x) &= \sum_{n=0}^{\infty} [A_n + B_n]\,(x-c)^n \\
Kf(x) &= \sum_{n=0}^{\infty} K\,A_n\,(x-c)^n \\
(x-c)^N f(x) &= \sum_{n=0}^{\infty} A_n\,(x-c)^{n+N} \quad \text{for any integer } N \geqslant 1 \\
&= \sum_{k=N}^{\infty} A_{k-N}\,(x-c)^k \quad \text{where } k = n+N \\
f'(x) &= \sum_{n=0}^{\infty} A_n\,n\,(x-c)^{n-1} = \sum_{n=1}^{\infty} A_n\,n\,(x-c)^{n-1} \\
\int_c^x f(t)\,\mathrm{d}t &= \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1} \\
\int f(x)\,\mathrm{d}x &= \left[\sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1}\right] + C \quad \text{with } C \text{ an arbitrary constant}
\end{aligned}$$

for all x obeying $|x-c| < R$.
In particular the radius of convergence of each of the six power series on the right hand sides is at least R. In fact, if R is the radius of convergence of $\sum_{n=0}^{\infty} A_n(x-c)^n$, then R is also the radius of convergence of all of the above right hand sides, with the possible exceptions of $\sum_{n=0}^{\infty} [A_n+B_n]\,(x-c)^n$ and $\sum_{n=0}^{\infty} KA_n\,(x-c)^n$ when $K=0$.

Example 3.5.12

The last statement of Theorem 3.5.11 might seem a little odd, but consider the following two power series centred at 0:

$$\sum_{n=0}^{\infty} 2^n x^n \text{ and } \sum_{n=0}^{\infty} (1-2^n)x^n.$$

The ratio test tells us that they both have radius of convergence $R = \frac{1}{2}$. However their sum is

$$\sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} (1-2^n) x^n = \sum_{n=0}^{\infty} x^n$$

which has the larger radius of convergence 1.

A more extreme example of the same phenomenon is supplied by the two series

$$\sum_{n=0}^{\infty} 2^n x^n \text{ and } \sum_{n=0}^{\infty} (-2^n) x^n.$$

They are both geometric series with radius of convergence $R = \frac{1}{2}$. But their sum is

$$\sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} (-2^n) x^n = \sum_{n=0}^{\infty} (0) x^n$$

which has radius of convergence $+\infty$.

Example 3.5.12

We'll now use this theorem to build power series representations for a bunch of functions out of the one simple power series representation that we know — the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for all } |x| < 1 .$$

Example 3.5.13 $\left(\frac{1}{1-x^2}\right)$

Find a power series representation for $\frac{1}{1-x^2}$.

Solution. The secret to finding power series representations for a good many functions is to manipulate them into a form in which $\frac{1}{1-y}$ appears and use the geometric series representation $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$. We have deliberately renamed the variable to y here — it does not have to be x. We can use that strategy to find a power series expansion for $\frac{1}{1-x^2}$ — we just have to recognize that $\frac{1}{1-x^2}$ is the same as $\frac{1}{1-y}$ if we set y to x^2.

$$\begin{aligned}
\frac{1}{1-x^2} &= \left.\frac{1}{1-y}\right|_{y=x^2} &&= \left[\sum_{n=0}^{\infty} y^n\right]_{y=x^2} \qquad \text{if } |y| < 1, \text{ i.e. } |x| < 1 \\
&= \sum_{n=0}^{\infty} \left(x^2\right)^n &&= \sum_{n=0}^{\infty} x^{2n} \\
&= 1 + x^2 + x^4 + x^6 + \cdots
\end{aligned}$$

This is a perfectly good power series. There is nothing wrong with the power of x being $2n$. (This just means that the coefficients of all odd powers of x are zero.) In fact, you should try to always write power series in forms that are as easy to understand as possible. The geometric series that we used at the end of the first line converges for

$$|y| < 1 \iff \left|x^2\right| < 1 \iff |x| < 1$$

So our power series has radius of convergence 1.

Example 3.5.13

Example 3.5.14 $\left(\frac{x}{2+x^2}\right)$

Find a power series representation for $\frac{x}{2+x^2}$.

Solution. This example is just a more algebraically involved variant of the last one. Again, the strategy is to manipulate $\frac{x}{2+x^2}$ into a form in which $\frac{1}{1-y}$ appears.

$$\begin{aligned}
\frac{x}{2+x^2} &= \frac{x}{2}\,\frac{1}{1+x^2/2} = \frac{x}{2}\,\frac{1}{1-\left(-x^2/2\right)} \qquad \text{set } -\frac{x^2}{2} = y \\
&= \frac{x}{2}\,\frac{1}{1-y}\bigg|_{y=-\frac{x^2}{2}} = \frac{x}{2}\left[\sum_{n=0}^{\infty} y^n\right]_{y=-\frac{x^2}{2}} \qquad \text{if } |y| < 1 \\
&= \frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{2}\right)^n = \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n+1} \qquad \text{by Theorem 3.5.11, twice} \\
&= \frac{x}{2} - \frac{x^3}{4} + \frac{x^5}{8} - \frac{x^7}{16} + \cdots
\end{aligned}$$

The geometric series that we used in the second line converges when

$$|y| < 1 \iff \left|-x^2/2\right| < 1 \iff |x|^2 < 2 \iff |x| < \sqrt{2}$$

So the given power series has radius of convergence $\sqrt{2}$.

Example 3.5.14

Example 3.5.15 (Nonzero centre)

Find a power series representation for $\frac{1}{5-x}$ with centre 3.

Solution. The new wrinkle in this example is the requirement that the centre be 3. That the centre is to be 3 means that we need a power series in powers of $x - c$, with $c = 3$. So we are looking for a power series of the form $\sum_{n=0}^{\infty} A_n (x-3)^n$. The easy way to find such a series is to force an $x - 3$ to appear by adding and subtracting a 3.

$$\frac{1}{5-x} = \frac{1}{5-(x-3)-3} = \frac{1}{2-(x-3)}$$

Now we continue, as in the last example, by manipulating $\frac{1}{2-(x-3)}$ into a form in which $\frac{1}{1-y}$ appears.

$$\begin{aligned}
\frac{1}{5-x} = \frac{1}{2-(x-3)} &= \frac{1}{2}\,\frac{1}{1-\frac{x-3}{2}} \qquad \text{set } \frac{x-3}{2} = y \\
&= \frac{1}{2}\,\frac{1}{1-y}\bigg|_{y=\frac{x-3}{2}} = \frac{1}{2}\left[\sum_{n=0}^{\infty} y^n\right]_{y=\frac{x-3}{2}} \qquad \text{if } |y| < 1 \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(x-3)^n}{2^{n+1}} \\
&= \frac{1}{2} + \frac{x-3}{4} + \frac{(x-3)^2}{8} + \frac{(x-3)^3}{16} + \cdots
\end{aligned}$$

The geometric series that we used in the second line converges when

$$|y| < 1 \iff \left|\frac{x-3}{2}\right| < 1 \iff |x-3| < 2 \iff -2 < x-3 < 2$$

So the power series has radius of convergence 2.

Example 3.5.15

In the previous two examples, to construct a new series from an existing series, we replaced x by a simple function. The following theorem gives us some more (but certainly not all) commonly used substitutions.

Theorem 3.5.16 (Substituting in a Power Series)**.**

Assume that the function $f(x)$ is given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n x^n$$

for all x in the interval I. Also let K and k be real constants. Then

$$f\left(Kx^k\right) = \sum_{n=0}^{\infty} A_n K^n\, x^{kn}$$

whenever Kx^k is in I. In particular, if $\sum_{n=0}^{\infty} A_n x^n$ has radius of convergence R, K is nonzero and k is a natural number, then $\sum_{n=0}^{\infty} A_n K^n\, x^{kn}$ has radius of convergence $\sqrt[k]{R/|K|}$.

Example 3.5.17 $\left(\frac{1}{(1-x)^2}\right)$

Find a power series representation for $\frac{1}{(1-x)^2}$.

Solution. Once again the trick is to express $\frac{1}{(1-x)^2}$ in terms of $\frac{1}{1-x}$. Notice that

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty} x^n\right\} \\ &= \sum_{n=1}^{\infty} n x^{n-1} \qquad \text{by Theorem 3.5.11}\end{aligned}$$

Note that the $n = 0$ term has disappeared because, for $n = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = \frac{\mathrm{d}}{\mathrm{d}x}x^0 = \frac{\mathrm{d}}{\mathrm{d}x}1 = 0$$

Also note that the radius of convergence of this series is one. We can see this via Theorem 3.5.11. That theorem tells us that the radius of convergence of a power series is not changed by differentiation — and since $\sum_{n=0}^{\infty} x^n$ has radius of convergence one, so too does its derivative.

Example 3.5.17

Example 3.5.18 ($\log(1 + x)$)

Find a power series representation for $\log(1 + x)$.

Solution. Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\log(1+x) = \frac{1}{1+x}$ so that $\log(1+t)$ is an antiderivative of $\frac{1}{1+t}$ and

$$\begin{aligned}\log(1+x) &= \int_0^x \frac{\mathrm{d}t}{1+t} = \int_0^x \Big[\sum_{n=0}^{\infty}(-t)^n\Big]\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty}\int_0^x (-t)^n\,\mathrm{d}t \qquad \text{by Theorem 3.5.11} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\end{aligned}$$

Theorem 3.5.11 guarantees that the radius of convergence is exactly one (the radius of convergence of the geometric series $\sum_{n=0}^{\infty}(-t)^n$) and that

$$\log(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} \qquad \text{for all} \quad -1 < x < 1$$

Example 3.5.18

Example 3.5.19 ($\arctan x$)

Find a power series representation for $\arctan x$.

Solution. Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$ so that $\arctan t$ is an antiderivative of $\frac{1}{1+t^2}$ and

$$\begin{aligned}\arctan x &= \int_0^x \frac{\mathrm{d}t}{1+t^2} = \int_0^x \Big[\sum_{n=0}^\infty (-t^2)^n\Big]\,\mathrm{d}t = \sum_{n=0}^\infty \int_0^x (-1)^n t^{2n}\,\mathrm{d}t \\ &= \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\end{aligned}$$

Theorem 3.5.11 guarantees that the radius of convergence is exactly one (the radius of convergence of the geometric series $\sum_{n=0}^\infty (-t^2)^n$) and that

$$\arctan x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{for all } -1 < x < 1$$

Example 3.5.19

The operations on power series dealt with in Theorem 3.5.11 are fairly easy to apply. Unfortunately, Theorem 3.5.11 alone will not get us power series representations of many of our standard functions (like e^x and $\sin x$). Fortunately we can find such representations by extending Taylor polynomials[30] to Taylor series.

3.6▲ Taylor series

Learning Objectives

- Define Taylor series
- Recognize Taylor series of classical functions (Theorem 3.6.5), and their radii of convergence. (These will be provided on the final exam, so rote memorization is not necessary.)
- Find Taylor series of common functions via the definition.
- Manipulate known series (for example, the geometric series) to derive power series for difficult-to-evaluate functions (for example, the logarithm) possibly using variable substitution.
- Find Taylor series by manipulating Taylor series for known functions (operations on power series as per Theorems 3.5.11 and 3.5.16, keeping in mind the

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[30] Now is a good time to review your notes from last term, though we'll give you a whirlwind review over the next page or two.

radius of convergence)

- Explain the utility of representing complicated functions $\left(\text{eg. } \arctan x \text{ or } \int_0^x \sin t^2\, \mathrm{d}t\right)$ as an infinite sum of polynomials.
- Bound the error in approximating a function by finitely many terms in its series representation.

3.6.1 ▸▸ Extending Taylor polynomials

Recall[31] that Taylor polynomials provide a hierarchy of approximations to a given function $f(x)$ near a given point a. Typically, the quality of these approximations improves as we move up the hierarchy.

- The crudest approximation is the constant approximation $f(x) \approx f(a)$.
- Then comes the linear, or tangent line, approximation $f(x) \approx f(a) + f'(a)\,(x-a)$.
- Then comes the quadratic approximation
$$f(x) \approx f(a) + f'(a)\,(x-a) + \frac{1}{2}f''(a)\,(x-a)^2$$
- In general, the Taylor polynomial of degree n, for the function $f(x)$, about the expansion point a, is the polynomial, $T_n(x)$, determined by the requirements that $f^{(k)}(a) = T_n^{(k)}(a)$ for all $0 \leqslant k \leqslant n$. That is, f and T_n have the same derivatives at a, up to order n. Explicitly,
$$\begin{aligned} f(x) \approx T_n(x) &= f(a) + f'(a)\,(x-a) + \frac{1}{2}f''(a)\,(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)\,(x-a)^n \\ &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)\,(x-a)^k \end{aligned}$$

These are, of course, approximations — often very good approximations near $x = a$ — but still just approximations. One might hope that if we let the degree, n, of the approximation go to infinity then the error in the approximation might go to zero. If that is the case then the "infinite" Taylor polynomial would be an exact representation of the function. Let's see how this might work.

Fix a real number a and suppose that all derivatives of the function $f(x)$ exist. Then, as you may have seen in 9.6.5 of the Math 100 text, for any natural number n,

Equation 3.6.1.

$$f(x) = T_n(x) + E_n(x)$$

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[31] Please review your notes from last term if this material is feeling a little unfamiliar.

where $T_n(x)$ is the Taylor polynomial of degree n for the function $f(x)$ expanded about a, and $E_n(x) = f(x) - T_n(x)$ is the error in the approximation $f(x) \approx T_n(x)$. The Taylor polynomial[32] is given by the formula

Equation 3.6.1-a

$$T_n(x) = f(a) + f'(a)\,(x-a) + \cdots + \tfrac{1}{n!}f^{(n)}(a)\,(x-a)^n$$

while the error satisfies[33]

Equation 3.6.1-b

$$E_n(x) = \tfrac{1}{(n+1)!}f^{(n+1)}(c)\,(x-a)^{n+1}$$

for some c strictly between a and x. Note that we typically do not know the value of c in the formula for the error. Instead we use the bounds on c to find bounds on $f^{(n+1)}(c)$ and so bound the error.

In order for our Taylor polynomial to be an exact representation of the function $f(x)$ we need the error $E_n(x)$ to be zero. This will not happen when n is finite unless $f(x)$ is a polynomial. However it can happen in the limit as $n \to \infty$, and in that case we can write $f(x)$ as the limit

$$f(x) = \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \tfrac{1}{k!}f^{(k)}(a)\,(x-a)^k$$

This is really a limit of partial sums, and so we can write

$$f(x) = \sum_{k=0}^{\infty} \tfrac{1}{k!}f^{(k)}(a)\,(x-a)^k$$

which is a power series representation of the function. Let us formalise this in a definition.

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[32] Did you take a quick look at your notes?

[33] This is probably the most commonly used formula for the error, but there are other equivalent ways of writing it.

Definition 3.6.2 (Taylor series).

The Taylor series for the function $f(x)$ expanded around a is the power series

$$\sum_{n=0}^{\infty} \tfrac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

When $a = 0$ it is also called the Maclaurin series of $f(x)$. If $\lim_{n\to\infty} E_n(x) = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

Demonstrating that $\lim_{n\to\infty} E_n(x) = 0$ is sometimes pretty difficult. But, for many of the standard functions you are used to dealing with, it turns out to be pretty easy. Let's compute a few Taylor series and see how we do it.

Example 3.6.3 (Exponential Series)

Find the Maclaurin series for $f(x) = e^x$.

Solution. Just as was the case for computing Taylor polynomials, we need to compute the derivatives of the function at the particular choice of a. Since we are asked for a Maclaurin series, $a = 0$. So now we just need to find $f^{(k)}(0)$ for all integers $k \geqslant 0$.

We know that $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$ and so

$$\begin{aligned} e^x = f(x) = f'(x) = f''(x) = \cdots = f^{(k)}(x) = \cdots &\qquad \text{which gives} \\ 1 = f(0) = f'(0) = f''(0) = \cdots = f^{(k)}(0) = \cdots. & \end{aligned}$$

Equations (3.6.1) and (3.6.1-a) then give us

$$e^x = f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + E_n(x)$$

We shall see, in the optional Example 3.6.6 below, that, for any fixed x, $\lim\limits_{n\to\infty} E_n(x) = 0$. Consequently, for all x,

$$e^x = \lim_{n\to\infty}\left[1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n\right] = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

Example 3.6.3

We have now seen power series representations for the functions

$$\frac{1}{1-x} \qquad \frac{1}{(1-x)^2} \qquad \log(1+x) \qquad \arctan(x) \qquad e^x.$$

We do not think that you, the reader, will be terribly surprised to see that we develop series for sine and cosine next.

Example 3.6.4 (Sine and Cosine Series)

The trigonometric functions $\sin x$ and $\cos x$ also have widely used Maclaurin series expansions (i.e. Taylor series expansions about $a = 0$). To find them, we first compute all derivatives at general x.

$$\begin{array}{llllll} f(x)=\sin x & f'(x)=\cos x & f''(x)=-\sin x & f^{(3)}(x)=-\cos x & f^{(4)}(x)=\sin x & \cdots \\ g(x)=\cos x & g'(x)=-\sin x & g''(x)=-\cos x & g^{(3)}(x)=\sin x & g^{(4)}(x)=\cos x & \cdots \end{array}$$

Now set $x = a = 0$.

$$\begin{array}{lllllll} f(x)=\sin x & f(0)=0 & f'(0)=1 & f''(0)=0 & f^{(3)}(0)=-1 & f^{(4)}(0)=0 & \cdots \\ g(x)=\cos x & g(0)=1 & g'(0)=0 & g''(0)=-1 & g^{(3)}(0)=0 & g^{(4)}(0)=1 & \cdots \end{array}$$

For $\sin x$, all even numbered derivatives (at $x = 0$) are zero, while the odd numbered derivatives alternate between 1 and -1. Very similarly, for $\cos x$, all odd numbered derivatives (at $x = 0$) are zero, while the even numbered derivatives alternate between 1 and -1. So, the Taylor polynomials that best approximate $\sin x$ and $\cos x$ near $x = a = 0$ are

$$\begin{aligned} \sin x &\approx x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots \\ \cos x &\approx 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots \end{aligned}$$

We shall see, in the optional Example 3.6.7 below, that, for both $\sin x$ and $\cos x$, we have $\lim\limits_{n\to\infty} E_n(x) = 0$ so that

$$\begin{aligned} f(x) &= \lim_{n\to\infty}\Big[f(0) + f'(0)\,x + \cdots + \tfrac{1}{n!}f^{(n)}(0)\,x^n\Big] \\ g(x) &= \lim_{n\to\infty}\Big[g(0) + g'(0)\,x + \cdots + \tfrac{1}{n!}g^{(n)}(0)\,x^n\Big] \end{aligned}$$

Reviewing the patterns we found in the derivatives, we conclude that, for all x,

$$\begin{aligned} \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n\tfrac{1}{(2n+1)!}x^{2n+1} \\ \cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n\tfrac{1}{(2n)!}x^{2n} \end{aligned}$$

and, in particular, both of the series on the right hand sides converge for all x.

We could also test for convergence of the series using the ratio test. Computing the ratios of successive terms in these two series gives us

$$\begin{aligned} \left|\frac{A_{n+1}}{A_n}\right| &= \frac{|x|^{2n+3}/(2n+3)!}{|x|^{2n+1}/(2n+1)!} = \frac{|x|^2}{(2n+3)(2n+2)} \\ \left|\frac{A_{n+1}}{A_n}\right| &= \frac{|x|^{2n+2}/(2n+2)!}{|x|^{2n}/(2n)!} = \frac{|x|^2}{(2n+2)(2n+1)} \end{aligned}$$

for sine and cosine respectively. Hence as $n \to \infty$ these ratios go to zero and consequently both series are convergent for all x. (This is very similar to what was observed in Example 3.5.5.)

Example 3.6.4

We have developed power series representations for a number of important functions[34] . Here is a theorem that summarizes them.

Theorem 3.6.5.

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots && \text{for all } -\infty < x < \infty \\
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} &&= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots && \text{for all } -\infty < x < \infty \\
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n} &&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots && \text{for all } -\infty < x < \infty \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \cdots && \text{for all } -1 < x < 1 \\
\log(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} &&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots && \text{for all } -1 < x \leqslant 1 \\
\arctan x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} &&= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots && \text{for all } -1 \leqslant x \leqslant 1
\end{aligned}$$

Notice that the series for sine and cosine sum to something that looks very similar to

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[34] The reader might ask whether or not we will give the series for other trigonometric functions or their inverses. While the tangent function has a perfectly well defined series, its coefficients are not as simple as those of the series we have seen — they form a sequence of numbers known (perhaps unsurprisingly) as the "tangent numbers". They, and the related Bernoulli numbers, have many interesting properties, links to which the interested reader can find with their favourite search engine. The Maclaurin series for inverse sine is

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{4^{-n}}{2n+1} \frac{(2n)!}{(n!)^2} x^{2n+1}$$

which is quite tidy, but proving it is beyond the scope of the course.

the series for e^x:

$$\begin{aligned}\sin(x)+\cos(x) &= \left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right)+\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots\right)\\ &= 1+x-\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5-\cdots\\ e^x &= 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\cdots\end{aligned}$$

So both series have coefficients with the same absolute value (namely $\frac{1}{n!}$), but there are differences in sign[35].

Example 3.6.6 $\left(\text{Optional — Why } \sum_{n=0}^{\infty}\frac{1}{n!}x^n \text{ is } e^x.\right)$

We have already seen, in Example 3.6.3, that

$$e^x = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+E_n(x)$$

By (3.6.1-b)

$$E_n(x)=\frac{1}{(n+1)!}e^c x^{n+1}$$

for some (unknown) c between 0 and x. Fix any real number x. We'll now show that $E_n(x)$ converges to zero as $n\to\infty$.

To do this we need get bound the size of e^c, and to do this, consider what happens if x is positive or negative.

- If $x<0$ then $x\leqslant c\leqslant 0$ and hence $e^x\leqslant e^c\leqslant e^0=1$.
- On the other hand, if $x\geqslant 0$ then $0\leqslant c\leqslant x$ and so $1=e^0\leqslant e^c\leqslant e^x$.

In either case we have that $0\leqslant e^c\leqslant 1+e^x$. Because of this the error term

$$|E_n(x)|=\left|\frac{e^c}{(n+1)!}x^{n+1}\right|\leqslant[e^x+1]\frac{|x|^{n+1}}{(n+1)!}$$

We claim that this upper bound, and hence the error $E_n(x)$, quickly shrinks to zero as $n\to\infty$.

Call the upper bound (except for the factor e^x+1, which is independent of n) $e_n(x)=\frac{|x|^{n+1}}{(n+1)!}$. To show that this shrinks to zero as $n\to\infty$, let's write it as follows.

$$e_n(x)=\frac{|x|^{n+1}}{(n+1)!}=\overbrace{\frac{|x|}{1}\cdot\frac{|x|}{2}\cdot\frac{|x|}{3}\cdots\frac{|x|}{n}\cdot\frac{|x|}{|n+1|}}^{n+1\text{ factors}}$$

---

[35] Warning: antique sign–sine pun. No doubt the reader first saw it many years syne.

Now let k be an integer bigger than $|x|$. We can split the product

$$\begin{aligned} e_n(x) &= \overbrace{\left(\frac{|x|}{1}\cdot\frac{|x|}{2}\cdot\frac{|x|}{3}\cdots\frac{|x|}{k}\right)}^{k \text{ factors}}\cdot\left(\frac{|x|}{k+1}\cdots\frac{|x|}{|n+1|}\right)\\ &\leqslant \underbrace{\left(\frac{|x|}{1}\cdot\frac{|x|}{2}\cdot\frac{|x|}{3}\cdots\frac{|x|}{k}\right)}_{=Q(x)}\cdot\left(\frac{|x|}{k+1}\right)^{n+1-k}\\ &= Q(x)\cdot\left(\frac{|x|}{k+1}\right)^{n+1-k}\end{aligned}$$

Since k does not depend not n (though it does depend on x), the function $Q(x)$ does not change as we increase n. Additionally, we know that $|x| < k+1$ and so $\frac{|x|}{k+1} < 1$. Hence as we let $n \to \infty$ the above bound must go to zero.

Alternatively, compare $e_n(x)$ and $e_{n+1}(x)$.

$$\frac{e_{n+1}(x)}{e_n(x)} = \frac{\frac{|x|^{n+2}}{(n+2)!}}{\frac{|x|^{n+1}}{(n+1)!}} = \frac{|x|}{n+2}$$

When n is bigger than, for example $2|x|$, we have $\frac{e_{n+1}(x)}{e_n(x)} < \frac{1}{2}$. That is, increasing the index on $e_n(x)$ by one decreases the size of $e_n(x)$ by a factor of at least two. As a result $e_n(x)$ must tend to zero as $n \to \infty$.

Consequently, for all x, $\lim_{n\to\infty} E_n(x) = 0$, as claimed, and we really have

$$e^x = \lim_{n\to\infty}\left[1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n\right] = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

Example 3.6.6

We can show that the error terms in Maclaurin polynomials for sine and cosine go to zero as $n \to \infty$ using very much the same approach as in Example 3.6.6.

Example 3.6.7 $\left(\text{Optional — Why } \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sin x \text{ and } \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = \cos x\right)$

Let $f(x)$ be either $\sin x$ or $\cos x$. We know that every derivative of $f(x)$ will be one of $\pm\sin(x)$ or $\pm\cos(x)$. Consequently, when we compute the error term using equation (3.6.1-b) we always have $\left|f^{(n+1)}(c)\right| \leqslant 1$ and hence

$$|E_n(x)| \leqslant \frac{|x|^{n+1}}{(n+1)!}.$$

In Example 3.6.3, we showed that $\frac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$ — so all the hard work is already done. Since the error term shrinks to zero for both $f(x) = \sin x$ and $f(x) = \cos x$, and

$$f(x) = \lim_{n\to\infty}\left[f(0) + f'(0)\,x + \cdots + \tfrac{1}{n!}f^{(n)}(0)\,x^n\right]$$

as required.

Example 3.6.7

3.6.2 ▸▸ Computing with Taylor series

Taylor series have a great many applications. (Hence their place in this course.) One of the most immediate of these is that they give us an alternate way of computing many functions. For example, the first definition we see for the sine and cosine functions is in terms of triangles. Those definitions, however, do not lend themselves to computing sine and cosine except at very special angles. Armed with power series representations, however, we can compute them to very high precision at any angle. To illustrate this, consider the computation of π — a problem that dates back to the Babylonians.

Example 3.6.8 (Computing the number π)

There are numerous methods for computing π to any desired degree of accuracy[36]. Many of them use the Maclaurin expansion

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

of Theorem 3.6.5. Since $\arctan(1) = \frac{\pi}{4}$, the series gives us a very pretty formula for π:

$$\begin{aligned} \frac{\pi}{4} &= \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \\ \pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) \end{aligned}$$

Unfortunately, this series is not very useful for computing π because it converges so slowly. If we approximate the series by its N^{th} partial sum, then the alternating series test (Theorem 3.3.17) tells us that the error is bounded by the first term we drop. To guarantee that we have 2 decimal digits of π correct, we need to sum about the first 200 terms!

A much better way to compute π using this series is to take advantage of the fact that $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$:

$$\begin{aligned} \pi &= 6\arctan\left(\frac{1}{\sqrt{3}}\right) = 6\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \frac{1}{(\sqrt{3})^{2n+1}} \\ &= 2\sqrt{3}\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \frac{1}{3^n} \\ &= 2\sqrt{3}\Big(1 - \frac{1}{3\times 3} + \frac{1}{5\times 9} - \frac{1}{7\times 27} + \frac{1}{9\times 81} - \frac{1}{11\times 243} + \cdots\Big) \end{aligned}$$

---

[36] The computation of π has a very, very long history and your favourite search engine will turn up many sites that explore the topic. For a more comprehensive history one can turn to books such as "A history of Pi" by Petr Beckmann and "The joy of π" by David Blatner.

Again, this is an alternating series and so (via Theorem 3.3.17) the error we introduce by truncating it is bounded by the first term dropped. For example, if we keep ten terms, stopping at $n = 9$, we get $\pi = 3.141591$ (to 6 decimal places) with an error between zero and

$$\frac{2\sqrt{3}}{21 \times 3^{10}} < 3 \times 10^{-6}$$

In 1699, the English astronomer/mathematician Abraham Sharp (1653–1742) used 150 terms of this series to compute 72 digits of π — by hand!

This is just one of very many ways to compute π. Another one, which still uses the Maclaurin expansion of $\arctan x$, but is much more efficient, is

$$\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$$

This formula was used by John Machin in 1706 to compute π to 100 decimal digits — again, by hand.

Example 3.6.8

Power series also give us access to new functions which might not be easily expressed in terms of the functions we have been introduced to so far. The following is a good example of this.

Example 3.6.9 (Error function)

The *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \,\mathrm{d}t$$

is used in computing "bell curve" probabilities. The indefinite integral of the integrand e^{-t^2} cannot be expressed in terms of standard functions. But we can still evaluate the integral to within any desired degree of accuracy by using the Taylor expansion of the exponential. Start with the Maclaurin series for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

and then substitute $x = -t^2$ into this:

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$$

We can then apply Theorem 3.5.11 to integrate term-by-term:

$$\begin{aligned}\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x \left[\sum_{n=0}^{\infty} \frac{\left(-t^2\right)^n}{n!} \right] \mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}\end{aligned}$$

For example, for the bell curve, the probability of being within one standard deviation of the mean is

$$\begin{aligned}\operatorname{erf}\left(1/\sqrt{2}\right) &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\left(1/\sqrt{2}\right)^{2n+1}}{(2n+1)n!} = \frac{2}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)2^n n!}\\ &= \sqrt{\frac{2}{\pi}}\Big(1-\frac{1}{3\times 2}+\frac{1}{5\times 2^2\times 2}-\frac{1}{7\times 2^3\times 3!}+\frac{1}{9\times 2^4\times 4!}-\cdots\Big)\end{aligned}$$

This is yet another alternating series. If we keep five terms, stopping at $n = 4$, we get 0.68271 (to 5 decimal places) with, by Theorem 3.3.17 again, an error between zero and the first dropped term, which is minus

$$\sqrt{\frac{2}{\pi}}\,\frac{1}{11\times 2^5\times 5!} < 2\times 10^{-5}$$

Example 3.6.9

Example 3.6.10

Evaluate

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n3^n} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1}{n3^n}$$

Solution. There are not very many series that can be easily evaluated exactly. But occasionally one encounters a series that can be evaluated simply by realizing that it is exactly one of the series in Theorem 3.6.5, just with a specific value of x. The left hand given series is

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,\frac{1}{3^n} = \frac{1}{3}-\frac{1}{2}\,\frac{1}{3^2}+\frac{1}{3}\,\frac{1}{3^3}-\frac{1}{4}\,\frac{1}{3^4}+\cdots$$

The series in Theorem 3.6.5 that this most closely resembles is

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-\cdots$$

Indeed

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,\frac{1}{3^n} &= \frac{1}{3}-\frac{1}{2}\,\frac{1}{3^2}+\frac{1}{3}\,\frac{1}{3^3}-\frac{1}{4}\,\frac{1}{3^4}+\cdots\\ &= \left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right]_{x=\frac{1}{3}}\\ &= \Big[\log(1+x)\Big]_{x=\frac{1}{3}}\\ &= \log\frac{4}{3}\end{aligned}$$

We were asked to evaluate two series. The signs of the first alternate, while those of the right-hand series do not. Otherwise, the terms are the same. We can flip every second sign in a power series just by using a negative x.

$$\begin{aligned}\Big[\log(1+x)\Big]_{x=-\frac{1}{3}} &= \left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right]_{x=-\frac{1}{3}}\\ &= -\frac{1}{3}-\frac{1}{2}\frac{1}{3^2}-\frac{1}{3}\frac{1}{3^3}-\frac{1}{4}\frac{1}{3^4}+\cdots\end{aligned}$$

which is exactly minus the desired remaining series. So

$$\sum_{n=1}^{\infty}\frac{1}{n3^n}=-\Big[\log(1+x)\Big]_{x=-\frac{1}{3}}=-\log\frac{2}{3}=\log\frac{3}{2}.$$

Example 3.6.10

Example 3.6.11

Let $f(x)=\sin(2x^3)$. Find $f^{(15)}(0)$, the fifteenth derivative of f at $x=0$.

Solution. This is a bit of a trick question. We could of course use the product and chain rules to directly apply fifteen derivatives and then set $x=0$, but that would be extremely tedious[37]. There is a much more efficient approach that exploits two pieces of knowledge that we have.

- From equation (3.6.1-a), we see that the coefficient of $(x-a)^n$ in the Taylor series of $f(x)$ with expansion point a is exactly $\frac{1}{n!}f^{(n)}(a)$. So $f^{(n)}(a)$ is exactly $n!$ times the coefficient of $(x-a)^n$ in the Taylor series of $f(x)$ with expansion point a.
- We know, or at least can easily find, the Taylor series for $\sin(2x^3)$.

Let's apply that strategy.

- First, we know that, for all y,

$$\sin y = y-\frac{1}{3!}y^3+\frac{1}{5!}y^5-\cdots$$

- Just substituting $y=2x^3$, we have

$$\begin{aligned}\sin(2x^3) &= 2x^3-\frac{1}{3!}\left(2x^3\right)^3+\frac{1}{5!}\left(2x^3\right)^5-\cdots\\ &= 2x^3-\frac{8}{3!}x^9+\frac{2^5}{5!}x^{15}-\cdots\end{aligned}$$

---

[37] We could get a computer algebra system to do it for us without much difficulty — but we wouldn't learn much in the process. The point of this example is to illustrate that one can do more than just represent a function with Taylor series. More on this in the next section.

- So the coefficient of x^{15} in the Taylor series of $f(x) = \sin(2x^3)$ with expansion point $a = 0$ is $\frac{2^5}{5!}$

and we have

$$f^{(15)}(0) = 15! \times \frac{2^5}{5!} = 348{,}713{,}164{,}800$$

Example 3.6.11

Example 3.6.12 (Computing the number e)

Back in Example 3.6.6, we saw that

$$e^x = 1 + x + \tfrac{x^2}{2!} + \cdots + \tfrac{x^n}{n!} + \tfrac{1}{(n+1)!} e^c x^{n+1}$$

for some (unknown) c between 0 and x. This can be used to approximate the number e, with any desired degree of accuracy. Setting $x = 1$ in this equation gives

$$e = 1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{n!} + \tfrac{1}{(n+1)!} e^c$$

for some c between 0 and 1. Even though we don't know c exactly, we can bound that term quite readily. We do know that e^c in an increasing function[38] of c, and so $1 = e^0 \leqslant e^c \leqslant e^1 = e$. Thus we know that

$$\frac{1}{(n+1)!} \leqslant e - \Big(1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{n!}\Big) \leqslant \frac{e}{(n+1)!}$$

So we have a lower bound on the error, but our upper bound involves the e — precisely the quantity we are trying to get a handle on.

But all is not lost. Let's look a little more closely at the right-hand inequality when $n = 1$:

$$\begin{aligned} e - (1+1) &\leqslant \frac{e}{2} && \text{move the } e\text{'s to one side} \\ \frac{e}{2} &\leqslant 2 && \text{and clean it up} \\ e &\leqslant 4. \end{aligned}$$

Now this is a pretty crude bound[39] but it isn't hard to improve. Try this again with $n = 2$:

$$\begin{aligned} e - (1 + 1 + \frac{1}{2}) &\leqslant \frac{e}{6} && \text{move } e\text{'s to one side} \\ \frac{5e}{6} &\leqslant \frac{5}{2} \\ e &\leqslant 3. \end{aligned}$$

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[38] Check the derivative!

[39] The authors hope that by now we all "know" that e is between 2 and 3, but maybe we don't know how to prove it.

Better. Now we can rewrite our bound:

$$\frac{1}{(n+1)!} \leqslant e - \left(1+1+\tfrac{1}{2!}+\cdots+\tfrac{1}{n!}\right) \leqslant \frac{e}{(n+1)!} \leqslant \frac{3}{(n+1)!}$$

If we set $n = 4$ in this we get

$$\frac{1}{120} = \frac{1}{5!} \leqslant e - \left(1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}\right) \leqslant \frac{3}{120}$$

So the error is between $\frac{1}{120}$ and $\frac{3}{120} = \frac{1}{40}$ — this approximation isn't guaranteed to give us the first 2 decimal places. If we ramp n up to 9 however, we get

$$\frac{1}{10!} \leqslant e - \left(1+1+\frac{1}{2}+\cdots+\frac{1}{9!}\right) \leqslant \frac{3}{10!}$$

Since $10! = 3628800$, the upper bound on the error is $\frac{3}{3628800} < \frac{3}{3000000} = 10^{-6}$, and we can approximate e by

$$\begin{aligned}
&1+1+\tfrac{1}{2!}+\tfrac{1}{3!}+\tfrac{1}{4!}+\tfrac{1}{5!}+\tfrac{1}{6!}+\tfrac{1}{7!}+\tfrac{1}{8!}+\tfrac{1}{9!}\\
=&1+1+0.5+0.1\dot{6}+0.041\dot{6}+0.008\dot{3}+0.0013\dot{8}+0.0001984+0.0000248+0.0000028\\
=&2.718282
\end{aligned}$$

and it is correct to six decimal places.

Example 3.6.12

3.6.3 ▸▸ Evaluating limits using Taylor expansions

Taylor polynomials provide a good way to understand the behaviour of a function near a specified point and so are useful for evaluating complicated limits. Here are some examples.

Example 3.6.13

In this example, we'll start with a relatively simple limit, namely

$$\lim_{x\to 0} \frac{\sin x}{x}$$

The first thing to notice about this limit is that, as x tends to zero, both the numerator, $\sin x$, and the denominator, x, tend to 0. So we may not evaluate the limit of the ratio by simply dividing the limits of the numerator and denominator. To find the limit, or show that it does not exist, we are going to have to exhibit a cancellation between the numerator and the denominator. Let's start by taking a closer look at the numerator. By Example 3.6.4,

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

Consequently[40]

$$\frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots$$

Every term in this series, except for the very first term, is proportional to a strictly positive power of x. Consequently, as x tends to zero, all terms in this series, except for the very first term, tend to zero. In fact the sum of all terms, starting with the second term, also tends to zero. That is,

$$\lim_{x\to 0}\Big[-\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots\Big] = 0$$

(We won't justify that statement here.) So,

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x} &= \lim_{x\to 0}\Big[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots\Big]\\ &= 1 + \lim_{x\to 0}\Big[-\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots\Big]\\ &= 1\end{aligned}$$

Example 3.6.13

The limit in the previous example can also be evaluated relatively easily using l'Hôpital's rule[41]. While the following limit can also, in principal, be evaluated using l'Hôpital's rule, it is much more efficient to use Taylor series[42].

Example 3.6.14

In this example we evaluate

$$\lim_{x\to 0}\frac{\arctan x - x}{\sin x - x}$$

Once again, the first thing to notice about this limit is that, as x tends to zero, the numerator tends to $\arctan 0 - 0$, which is 0, and the denominator tends to $\sin 0 - 0$, which is also 0. So we may not evaluate the limit of the ratio by simply dividing the limits of the numerator and denominator. Again, to find the limit, or show that it does not exist, we are

---

[40] We are hiding some mathematics behind this "consequently". What we are really using is our knowledge of Taylor polynomials to write

$$f(x) = \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + E_5(x)$$

where $E_5(x) = \frac{f^{(6)}(c)}{6!}x^6$ and c is between 0 and x. We are effectively hiding "$E_5(x)$" inside the "$\cdots$". Now we can divide both sides by x (assuming $x \neq 0$):

$$\frac{\sin(x)}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \frac{E_5(x)}{x}.$$

and everything is fine provided the term $\frac{E_5(x)}{x}$ stays well behaved.

[41] Many of you learned about l'Hôpital's rule in school and all of you should have seen it last term in your differential calculus course.

[42] It takes 3 applications of l'Hôpital's rule and some careful cleaning up of the intermediate expressions. Oof!

going to have to exhibit a cancellation between the numerator and the denominator. To get a more detailed understanding of the behaviour of the numerator and denominator near $x = 0$, we find their Taylor expansions. By Example 3.5.19,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

so the numerator

$$\arctan x - x = -\frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

By Example 3.6.4,

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

so the denominator

$$\sin x - x = -\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

and the ratio

$$\frac{\arctan x - x}{\sin x - x} = \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \cdots}{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots}$$

Notice that every term in both the numerator and the denominator contains a common factor of x^3, which we can cancel out.

$$\frac{\arctan x - x}{\sin x - x} = \frac{-\frac{1}{3} + \frac{x^2}{5} - \cdots}{-\frac{1}{3!} + \frac{1}{5!}x^2 - \cdots}$$

As x tends to zero,

- the numerator tends to $-\frac{1}{3}$, which is not 0, and
- the denominator tends to $-\frac{1}{3!} = -\frac{1}{6}$, which is also not 0.

so we may now legitimately evaluate the limit of the ratio by simply dividing the limits of the numerator and denominator.

$$\begin{aligned}
\lim_{x\to 0}\frac{\arctan x - x}{\sin x - x} &= \lim_{x\to 0}\frac{-\frac{1}{3} + \frac{x^2}{5} - \cdots}{-\frac{1}{3!} + \frac{1}{5!}x^2 - \cdots} \\
&= \frac{\lim_{x\to 0}\big[-\frac{1}{3} + \frac{x^2}{5} - \cdots\big]}{\lim_{x\to 0}\big[-\frac{1}{3!} + \frac{1}{5!}x^2 - \cdots\big]} \\
&= \frac{-1/3}{-1/3!} \\
&= 2
\end{aligned}$$

Example 3.6.14

Part

APPENDIX

.1▲ Roots of Polynomials

Being able to factor polynomials is a very important part of many of the computations in this course. Related to this is the process of finding roots (or zeros) of polynomials. That is, given a polynomial $P(x)$, find all numbers r so that $P(r) = 0$.

In the case of a quadratic $P(x) = ax^2 + bx + c$, we can use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The corresponding formulas for cubics and quartics[43] are extremely cumbersome, and no such formula exists for polynomials of degree 5 and higher[44].

Despite this there are many tricks[45] for finding roots of polynomials that work well in some situations but not all. Here we describe approaches that will help you find integer and rational roots of polynomials that will work well on exams, quizzes and homework assignments.

Consider the quadratic equation $x^2 - 5x + 6 = 0$. We could[46] solve this using the quadratic formula

$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 6}}{2} = \frac{5 \pm 1}{2} = 2, 3.$$

Hence $x^2 - 5x + 6$ has roots $x = 2, 3$ and so it factors as $(x-3)(x-2)$. Notice[47] that the numbers 2 and 3 divide the constant term of the polynomial, 6. This happens in general and forms the basis of our first trick.

Trick .1.1 (A very useful trick)**.**

If r or $-r$ is an integer root of a polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with integer coefficients, then r is a factor of the constant term a_0.

Proof. If r is a root of the polynomial we know that $P(r) = 0$. Hence

$$a_n \cdot r^n + \cdots + a_1 \cdot r + a_0 = 0$$

If we isolate a_0 in this expression we get

$$a_0 = -\left[a_n r^n + \cdots + a_1 r\right]$$

---

[43] The method for cubics was developed in the 15th century by del Ferro, Cardano and Ferrari (Cardano's student). Ferrari then went on to discover a formula for the roots of a quartic. His formula requires the solution of an associated cubic polynomial.

[44] This is the famous Abel-Ruffini theorem.

[45] There is actually a large body of mathematics devoted to developing methods for factoring polynomials. Polynomial factorisation is a fundamental problem for most computer algebra systems. The interested reader should make use of their favourite search engine to find out more.

[46] We probably shouldn't do it this way for such a simple polynomial, but for pedagogical purposes we do here.

[47] Many of you may have been taught this approach in highschool.

We can see that r divides every term on the right-hand side. This means that the right-hand side is an integer times r. Thus the left-hand side, being a_0, is an integer times r, as required. The argument for when $-r$ is a root is almost identical. ☐

Let us put this observation to work.

Example .1.1

Find the integer roots of $P(x) = x^3 - x^2 + 2$.

Solution.

- The constant term in this polynomial is 2.
- The only divisors of 2 are $1, 2$. So the only candidates for integer roots are $\pm 1, \pm 2$.
- Trying each in turn

$$\begin{aligned} P(1) &= 2 & P(-1) &= 0 \\ P(2) &= 6 & P(-2) &= -10 \end{aligned}$$

- Thus the only integer root is -1.

Example .1.1

Example .1.2

Find the integer roots of $P(x) = 3x^3 + 8x^2 - 5x - 6$.

Solution.

- The constant term is -6.
- The divisors of 6 are $1, 2, 3, 6$. So the only candidates for integer roots are $\pm 1, \pm 2, \pm 3, \pm 6$.
- We try each in turn (it is tedious but not difficult):

$$\begin{aligned} P(1) &= 0 & P(-1) &= 4 \\ P(2) &= 40 & P(-2) &= 12 \\ P(3) &= 132 & P(-3) &= 0 \\ P(6) &= 900 & P(-6) &= -336 \end{aligned}$$

- Thus the only integer roots are 1 and -3.

Example .1.2

We can generalise this approach in order to find rational roots. Consider the polynomial $6x^2 - x - 2$. We can find its zeros using the quadratic formula:

$$x = \frac{1 \pm \sqrt{1+48}}{12} = \frac{1 \pm 7}{12} = -\frac{1}{2}, \frac{2}{3}.$$

Notice now that the numerators, 1 and 2, both divide the constant term of the polynomial (being 2). Similarly, the denominators, 2 and 3, both divide the coefficient of the highest power of x (being 6). This is quite general.

Trick .1.2 (Another nice trick).

If b/d or $-b/d$ is a rational root in lowest terms (i.e. b and d are integers with no common factors) of a polynomial $Q(x) = a_n x^n + \cdots + a_1 x + a_0$ with integer coefficients, then the numerator b is a factor of the constant term a_0 and the denominator d is a factor of a_n.

Proof. Since ${}^{b}/_{d}$ is a root of $P(x)$ we know that

$$a_n(b/d)^n + \cdots + a_1(b/d) + a_0 = 0$$

Multiply this equation through by d^n to get

$$a_n b^n + \cdots + a_1 b d^{n-1} + a_0 d^n = 0$$

Move terms around to isolate $a_0 d^n$:

$$a_0 d^n = -\left[a_n b^n + \cdots + a_1 b d^{n-1}\right]$$

Now every term on the right-hand side is some integer times b. Thus the left-hand side must also be an integer times b. We know that d does not contain any factors of b, hence a_0 must be some integer times b (as required).

Similarly we can isolate the term $a_n b^n$:

$$a_n b^n = -\left[a_{n-1} b^{n-1} d + \cdots + a_1 b d^{n-1} + a_0 d^n\right]$$

Now every term on the right-hand side is some integer times d. Thus the left-hand side must also be an integer times d. We know that b does not contain any factors of d, hence a_n must be some integer times d (as required).

The argument when $-{}^{b}/_{d}$ is a root is nearly identical. □

We should put this to work:

Example .1.3

$P(x) = 2x^2 - x - 3.$

Solution.

- The constant term in this polynomial is $3 = 1 \times 3$ and the coefficient of the highest power of x is $2 = 1 \times 2$.
- Thus the only candidates for integer roots are $\pm 1,\ \pm 3$.
- By our newest trick, the only candidates for fractional roots are $\pm\frac{1}{2},\ \pm\frac{3}{2}$.

- We try each in turn[48]

$$\begin{aligned} P(1) &= -2 & P(-1) &= 0 \\ P(3) &= 12 & P(-3) &= 18 \\ P\left(\tfrac{1}{2}\right) &= -3 & P\left(-\tfrac{1}{2}\right) &= -2 \\ P\left(\tfrac{3}{2}\right) &= 0 & P\left(-\tfrac{3}{2}\right) &= 3 \end{aligned}$$

so the roots are -1 and $\frac{3}{2}$.

Example .1.3

The tricks above help us to find integer and rational roots of polynomials. With a little extra work we can extend those methods to help us factor polynomials. Say we have a polynomial $P(x)$ of degree p and have established that r is one of its roots. That is, we know $P(r) = 0$. Then we can factor $(x - r)$ out from $P(x)$ — it is always possible to find a polynomial $Q(x)$ of degree $p - 1$ so that

$$P(x) = (x - r)Q(x)$$

In sufficiently simple cases, you can probably do this factoring by inspection. For example, $P(x) = x^2 - 4$ has $r = 2$ as a root because $P(2) = 2^2 - 4 = 0$. In this case, $P(x) = (x - 2)(x + 2)$ so that $Q(x) = (x + 2)$. As another example, $P(x) = x^2 - 2x - 3$ has $r = -1$ as a root because $P(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$. In this case, $P(x) = (x + 1)(x - 3)$ so that $Q(x) = (x - 3)$.

For higher degree polynomials we need to use something more systematic — long divison.

Trick .1.3 (Long Division).

Once you have found a root r of a polynomial, even if you cannot factor $(x - r)$ out of the polynomial by inspection, you can find $Q(x)$ by dividing $P(x)$ by $x - r$, using the long division algorithm you learned[49] in school, but with 10 replaced by x.

Example .1.4

Factor $P(x) = x^3 - x^2 + 2$.

Solution.

- We can go hunting for integer roots of the polynomial by looking at the divisors of the constant term. This tells us to try $x = \pm 1, \pm 2$.

---

[48] Again, this is a little tedious, but not difficult. Its actually pretty easy to code up for a computer to do. Modern polynomial factoring algorithms do more sophisticated things, but these are a pretty good way to start.

[49] This is a standard part of most highschool mathematics curricula, but perhaps not all. You should revise this carefully.

- A quick computation shows that $P(-1) = 0$ while $P(1), P(-2), P(2) \neq 0$. Hence $x = -1$ is a root of the polynomial and so $x + 1$ must be a factor.

- So we divide $\frac{x^3-x^2+2}{x+1}$. The first term, x^2, in the quotient is chosen so that when you multiply it by the denominator, $x^2(x+1) = x^3 + x^2$, the leading term, x^3, matches the leading term in the numerator, $x^3 - x^2 + 2$, exactly.

$$\begin{array}{r|llll}
\multicolumn{1}{r}{} & x^2 & & & \\ \cline{2-4}
x+1 & x^3- & x^2+ & 2 & \\
\multicolumn{1}{r}{} & x^3+ & x^2 & & \longleftarrow x^2(x+1)
\end{array}$$

- When you subtract $x^2(x+1) = x^3 + x^2$ from the numerator $x^3 - x^2 + 2$ you get the remainder $-2x^2 + 2$. Just like in public school, the 2 is not normally "brought down" until it is actually needed.

$$\begin{array}{r|llll}
\multicolumn{1}{r}{} & x^2 & & & \\ \cline{2-4}
x+1 & x^3- & x^2+ & 2 & \\
\multicolumn{1}{r}{} & x^3+ & x^2 & & \longleftarrow x^2(x+1) \\ \cline{2-4}
\multicolumn{1}{r}{} & & -2x^2 & &
\end{array}$$

- The next term, $-2x$, in the quotient is chosen so that when you multiply it by the denominator, $-2x(x+1) = -2x^2 - 2x$, the leading term $-2x^2$ matches the leading term in the remainder exactly.

$$\begin{array}{r|llll}
\multicolumn{1}{r}{} & x^2 - 2x & & & \\ \cline{2-4}
x+1 & x^3- & x^2+ & 2 & \\
\multicolumn{1}{r}{} & x^3+ & x^2 & & \longleftarrow x^2(x+1) \\ \cline{2-4}
\multicolumn{1}{r}{} & & -2x^2 & & \\
\multicolumn{1}{r}{} & & -2x^2 - 2x & & \longleftarrow -2x(x+1)
\end{array}$$

And so on.

$$\begin{array}{r|llll}
\multicolumn{1}{r}{} & x^2 - 2x + 2 & & & \\ \cline{2-4}
x+1 & x^3- & x^2+ & 2 & \\
\multicolumn{1}{r}{} & x^3+ & x^2 & & \longleftarrow x^2(x+1) \\ \cline{2-4}
\multicolumn{1}{r}{} & & -2x^2 & & \\
\multicolumn{1}{r}{} & & -2x^2 - 2x & & \longleftarrow -2x(x+1) \\ \cline{3-3}
\multicolumn{1}{r}{} & & & 2x+2 & \\
\multicolumn{1}{r}{} & & & 2x+2 & \longleftarrow 2(x+1) \\ \cline{4-4}
\multicolumn{1}{r}{} & & & 0 &
\end{array}$$

- Note that we finally end up with a remainder 0. A nonzero remainder would have signalled a computational error, since we know that the denominator $x - (-1)$ must divide the numerator $x^3 - x^2 + 2$ exactly.

- We conclude that

  $$(x+1)(x^2-2x+2) = x^3 - x^2 + 2$$

  To check this, just multiply out the left hand side explicitly.

- Applying the high school quadratic root formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ to x^2-2x+2 tells us that it has no real roots and that we cannot factor it further[50].

Example .1.4

We finish by describing an alternative to long division. The approach is roughly equivalent, but is perhaps more straightforward at the expense of requiring more algebra.

Example .1.5

Factor $P(x) = x^3 - x^2 + 2$, again.

Solution. Let us do this again but avoid long division.

- From the previous example, we know that $\frac{x^3-x^2+2}{x+1}$ must be a polynomial (since -1 is a root of the numerator) of degree 2. So write

  $$\frac{x^3-x^2+2}{x+1} = ax^2+bx+c$$

  for some, as yet unknown, coefficients $a,\ b$ and c.

- Cross multiplying and simplifying gives us

  $$\begin{aligned} x^3-x^2+2 &= (ax^2+bx+c)(x+1) \\ &= ax^3+(a+b)x^2+(b+c)x+c \end{aligned}$$

- Now matching coefficients of the various powers of x on the left and right hand sides

  $$\begin{aligned} &\text{coefficient of } x^3\text{:} & a &= 1 \\ &\text{coefficient of } x^2\text{:} & a+b &= -1 \\ &\text{coefficient of } x^1\text{:} & b+c &= 0 \\ &\text{coefficient of } x^0\text{:} & c &= 2 \end{aligned}$$

- This gives us a system of equations that we can solve quite directly. Indeed it tells us immediately that that $a=1$ and $c=2$. Subbing $a=1$ into $a+b=-1$ tells us that $1+b=-1$ and hence $b=-2$.

- Thus

  $$x^3-x^2+2 = (x+1)(x^2-2x+2).$$

Example .1.5

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[50] Because we are not permitted to use complex numbers.

Appendix A

LIST OF LEARNING OBJECTIVES

Chapter 1: Integration

1.1: Definition of the integral

- Interpret the definite integral $\int_a^b f(x)\,\mathrm{d}x$ as signed area when $a < b$.
- Understand why the definite integral sometimes gives negative numbers, even though areas cannot be negative.
- Evaluate certain definite integrals using geometry and the interpretation of definite integral as "area under the curve."
- Understand that the areas of curved shapes can be approximated by cutting up those shapes into many small rectangles and/or triangles.
- Understand what an area function of the form $\int_a^x f(t)\,\mathrm{d}t$ is, and compute them for simple functions using geometry
- Explain using a picture how to approximate area using left- and right Riemann sums both theoretically (for unknown "n") and concretely for a small number of rectangles.
- Express sums using sigma notation.
- Manipulate sums using arithmetic properties: constant sums, factoring and addition.
- Write right Riemann sums in sigma notation.
- Understand the definition of a definite integral as the limit of a Riemann sum.
- Compute Riemann sums with a spreadsheet.

1.2: Basic properties of the definite integral

- Explain (using pictures, words, equations, and inequalities) the arithmetic of integrals, as well as properties of definite integrals.
- Understand that antidifferentiation is linear, in the same way that differentiation is linear.

1.3: The fundamental theorem of calculus

- Understand what an area function of the form $\int_a^x f(t)\,\mathrm{d}t$ is, and compute them for simple functions using geometry.
- Given a function, sketch the area function $A(x)$. Pay special attention to where the slope of $A(x)$ is relatively large and relatively small.
- Produce a compelling argument that $A(x) = \int_a^x f(t)\,\mathrm{d}t$ should satisfy $A'(x) = f(x)$ if f is continuous at x; illustrate what can go wrong if f has a simple jump discontinuity at x.
- State the fundamental theorem of calculus, part 1 (FTC1).
- Use FTC1 to differentiate a function defined as a definite integral (area function).
- Use FTC1 to prove the fundamental theorem of calculus, part 2 (FTC2).
- Use FTC2 to compute definite integrals.
- Define the indefinite integral and explain how it differs from the definite integral.
- Explain why antiderivatives are not unique.
- Understand what an antiderivative (also called indefinite integral) is.
- Find antiderivatives of basic functions by inspection. (In particular, those important integrals listed in Theorem 1.3.16.)
- Find antiderivatives of polynomials using the power rule.

1.4: Interpretations of the definite integral

- Apply knowledge of integration (approximation via rectangles, antidifferentiation, FTC) in context (i.e. word problems).

1.5: Substitution

- Explain how the chain rule for derivatives corresponds to the substitution method for antiderivatives.
- Use a given substitution to evaluate an indefinite integral.
- Show how a given substitution affects the bounds of integration when used with a definite integral.
- Recognize when a substitution will simplify a given integral (definite or indefinite), and determine the form of an effective substitution.
- Compute integrals where the integrand requires manipulation to reveal an effective substitution.
- Compute integrals using a sequence of substitutions. For example, the integral $\int \sin^2(x^2)\cos(x^2)[2x]\,\mathrm{d}x$.

1.6: Area between curves

- Set up and compute the area between two curves (perhaps with other geometric boundaries). This includes the case where the two curves intersect or where we integrate along the y axis.
- Determine whether it is more advantageous to compute an area by integrating in x or by integrating in y.

1.7: Trigonometric integrals

- Memorize the derivative of tangent.
- Compute integrals involving powers of sine and cosine by utilizing an appropriate substitution.
- Use trigonometric identities to compute integrals involving even powers of sine and cosine. You must know: $\sin^2 x + \cos^2 x = 1$, $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, $\cos^2 x = \frac{1}{2}(1+\cos(2x))$, $\sin(2x) = 2\sin x\cos x$, and $\tan^2 x + 1 = \sec^2 x$.
- Use the definitions of different trigonometric functions to convert integrals into an easier form, where appropriate. This includes antidifferentiating products of various trig functions by converting them into sines and cosines.

1.8: Trigonometric substitution

- Recognize when it's appropriate to use the method of trigonometric substitution when computing an integral.
- Identify which substitution and which trig identity is required during trig substitution. In particular, remember the Pythagorean identity three ways: $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $\sec^2\theta - 1 = \tan^2\theta$.
- Compute integrals using trig substitution.
- Reduce compositions like $\sin(\cos^{-1}(x))$ to radical functions of x, where no trigonometric functions appear. Extend this skill to simplify any composition of the form $f(g(x))$, where $f(\theta)$ is one of $\sin\theta$, $\cos\theta$, etc., and $g(x)$ is one of $\tan^{-1}x$, $\sec^{-1}x$, or $\sin^{-1}x$.
- Memorize (or be able to independently derive) exact values of $\sin\theta$ and $\cos\theta$ for all θ that are integer multiples of $\frac{\pi}{4}$ or $\frac{\pi}{6}$, and use them to simplify definite integrals.

1.9: Partial fractions

- Use the method of partial fraction decomposition to evaluate the integrals of rational functions whose denominators can be factored so that there is at most one repeated linear root and at most one irreducible quadratic root.
- Memorize the antiderivative of secant.

1.10: Volumes

- Use an integral to represent the volume of a 3D object (providing it has some symmetry). Explain, using a picture, what each piece of the integral represents.
- Find the volume of surfaces of revolution using disks.
- Find volume by integrating over cross sectional areas.

1.11: Integration by parts

- Explain how the product rule for derivatives corresponds to integration by parts for integrals.
- Use integration by parts to compute definite and indefinite integrals.
- Identify when integration by parts is an appropriate method to use.
- While performing integration by parts, identify which portion of the integral should be "u" and which part should be "$\mathrm{d}v$." This includes the case where $\mathrm{d}x = \mathrm{d}v$.
- Given an integral, identify which technique(s) from this course can be used to compute the integral.
- Use the integration techniques from this course flexibly and compute integrals that require more than one technique.

1.12: Numerical integration

- Explain why we need numerical methods for integration, citing examples of problems we cannot solve with the fundamental theorem of calculus.
- Approximate integrals using right Riemann sums.
- Approximate integrals using the trapezoidal method.
- Explain how to derive the formula for trapezoid rule (and for right Riemann sums, which is review).
- Use Simpson's rule to approximate integrals. You are not required to reproduce the derivation of the formula, but you should be able to explain why n must be an even number.
- Given an integral to compute numerically with either the trapezoidal method or Simpson's rule, compute the max error given a particular n.
  Note: If you need the error formula on an exam, it will be provided to you.
- When computing an integral numerically with either the trapezoidal method or Simpson's rule, determine a sufficient number of intervals, n that guarantees a desired level of accuracy.
- Use numerical integration to compute approximations to definite integrals where the function is not defined explicitly or where the indefinite integral cannot be represented using standard functions.
- Implement the three numerical methods learned so far both by hand and using a spreadsheet.

1.13: Improper integrals

- State the different ways an integral can be improper.
- Define what it means to *evaluate* an improper integral. In particular, use pictures to explain the area being computed and the limit being taken.
- Define what it means for an improper integral to converge or diverge.
- Demonstrate the convergence/divergence of $\int \frac{1}{x^p}\,\mathrm{d}x$ for general $p > 0$, with domains $(0, 1]$ and $[1, \infty)$.
- Evaluate an improper integral (or prove it diverges) by explicitly writing and computing the appropriate limit.
- Use the comparison test to determine convergence/divergence for improper integrals without finding their antiderivatives.
- Use the limit comparison test to determine convergence/divergence of improper integrals without finding their antiderivatives.

Chapter 2: Probability

2.1: Introduction

- Understand basic concepts, vocabulary, and notation related to probability

2.2: Probability density

- Define Probability Density Function (PDF) informally as the function $f(t)$ such that $Pr(a \leqslant X \leqslant b) = \int_a^b f(t)\,\mathrm{d}t$.
- Interpret the PDF in terms of relative likelihoods of different regions.
- Use a PDF to compute probabilities.
- Learn properties of PDFs: $f(t) \geqslant 0$ and $\int_{-\infty}^{\infty} f(t)\mathrm{d}t = 1$.
- Use the properties of PDFs to find unknown parameters in its definition.

2.3: Expected value

- Explain what is meant by a "long-term average" and contrast this with the outcome of finitely many experiments.
- Define expected value for continuous systems.
- Compute the expected value for continuous systems.
- For an increasing or decreasing PDF use an intuitive argument to check whether the expected value is more or less than the halfway point of the space. Use this to check expectation calculations.

2.4: Variance and standard deviation

- Define variance and standard deviation
- Explain in plain(ish) language what these quantities represent, in reference to their definitions.
- Compute standard deviation and variance using both the conventional definition and the alternative formulation $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.

Chapter 3: Sequences and Series

3.1: Sequences

- Find the limit of a sequence.

3.2: Series

- Define sequences and series and, in particular, explain the difference between the two.
- Define partial sum.
- Explain what it means for a series to converge.
- Compute partial sums using spreadsheets.

3.3: Convergence tests

- Apply the divergence test to determine the divergence of applicable series.
- Explain in words why the divergence test works.
- Explain why and how the test can be inconclusive.
- State the conditions required to apply the integral test.
- Explain in words and with a picture why the integral test works.
- Use the integral test to to determine convergence or divergence of applicable series. In particular, use the integral test to derive the p-test.
- Determine whether a given series is geometric.
- Given a geometric series, determine whether it converges or diverges.
- Evaluate convergent geometric series.
- State the comparison test and explain why it works.
- Given a series, decide if the comparison test is appropriate. If so, determine a good series to use as a comparison.
- Apply the comparison test to determine the convergence or divergence of series.
- State the limit comparison test and explain why it follows naturally from the comparison test.
- Use the limit comparison test to determine whether a series converges or diverges. Supply good candidate series for comparison.
- Understand how to use both comparison tests to completely justify that a series converges or diverges. (Note what must be written explicitly.)
- Use the alternating series test to determine convergence of series.
- Give a heuristic explanation to justify the alternating series test.
- Given an alternating series that converges, use partial sums to find an interval containing the value of the series.

- State the ratio test and explain its connection with geometric series.
- Apply the ratio test to series when appropriate. In particular, to series involving factorials and/or exponentials.
- State when the ratio test is inconclusive and explain what that means.
- Given a series, determine which test to use to test for convergence/divergence.

3.4: Absolute and conditional convergence

- Define both absolute convergence and conditional convergence.
- Use absolute convergence to determine the convergence of some series.
- Explain why "absolute convergence implies conditional convergence" only works one way.

3.5: Power series

- Define power series for a function centred at a point.
- Explain what is meant by "radius of convergence."
- Compute the radius of convergence of a given power series.
- Translate the radius of convergence and the centre of a power series into the largest open interval over which the series converges.
- Given a power series, determine which test to use to test for convergence/divergence.
- Perform operations on power series as per Theorems 3.5.11 and 3.5.16, keeping in mind the radius of convergence.

3.6: Taylor series

- Define Taylor series
- Recognize Taylor series of classical functions (Theorem 3.6.5), and their radii of convergence. (These will be provided on the final exam, so rote memorization is not necessary.)
- Find Taylor series of common functions via the definition.
- Manipulate known series (for example, the geometric series) to derive power series for difficult-to-evaluate functions (for example, the logarithm) possibly using variable substitution.
- Find Taylor series by manipulating Taylor series for known functions (operations on power series as per Theorems 3.5.11 and 3.5.16, keeping in mind the radius of convergence)

- Explain the utility of representing complicated functions $\left(\text{eg. } \arctan x \text{ or } \int_0^x \sin t^2 \, \mathrm{d}t\right)$ as an infinite sum of polynomials.
- Bound the error in approximating a function by finitely many terms in its series representation.