Research Statement

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University of Illinois at Urbana-Champaign

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Disjoint Cycles and Equitable Coloring

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Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\delta(H) \geq 2k$. Then H contains k vertex-disjoint cycles.

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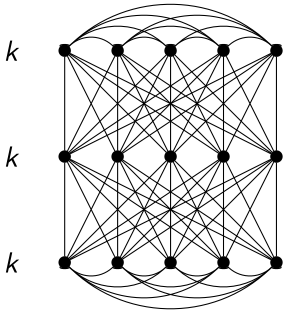

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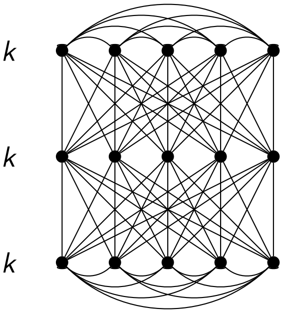

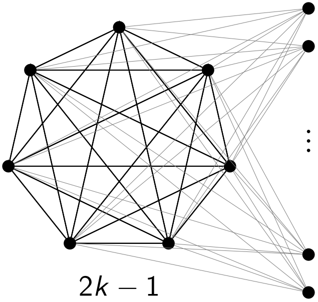

Enomoto, Wang: Ore Version

Definition

$\sigma_2(G) = \min_{xy \notin E(G)}\{d(x) + d(y)\}$

Theorem (Enomoto 1998; Wang 1999)

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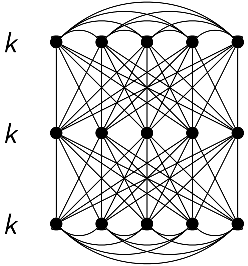

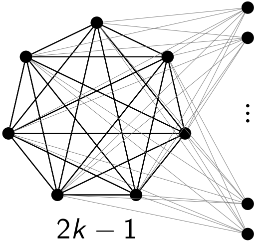

Idea: G should have k vertex-disjoint cycles if:

1. $|G| \geq 3k+1$
2. $\alpha(G) \leq n - 2k$
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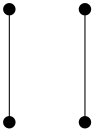

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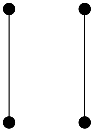

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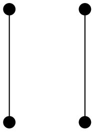

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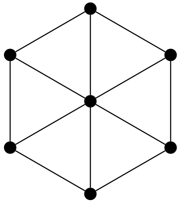

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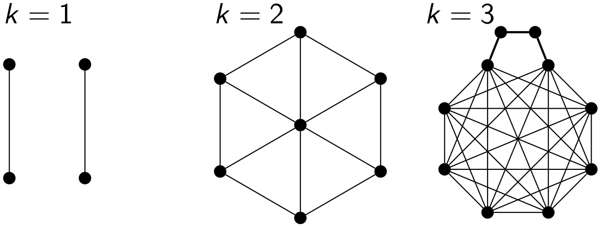

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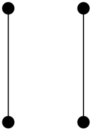

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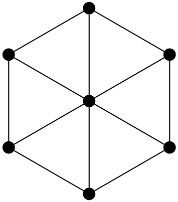

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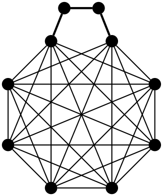

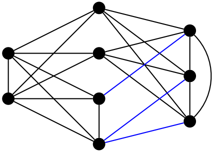

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Let $k \geq 3$, $n \geq 3k+1$, and let H be an n-vertex graph with $\delta(H) \geq 2k-1$ and $\alpha(H) \leq n-2k$. Then H contains k vertex-disjoint cycles.

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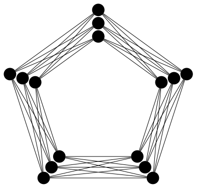

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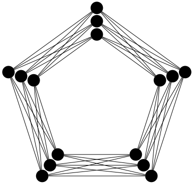

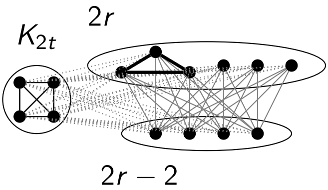

$n = 4(r+t) - (2t+2)$

Applications to Equitable Coloring

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An *equitable coloring* *of a graph* G *is a proper vertex coloring in which the sizes of any two color classes differ by at most one.*

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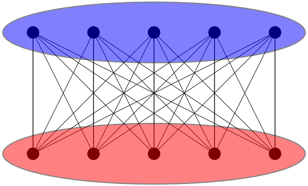

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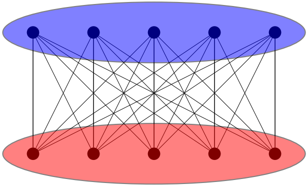

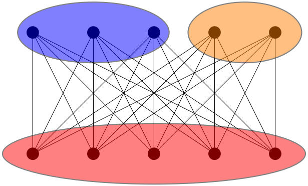

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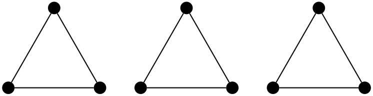

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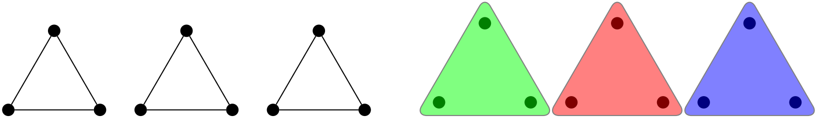

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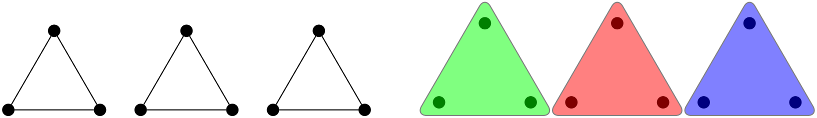

Theorem 1 (Hajnal-Szemerédi)

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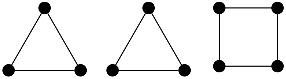

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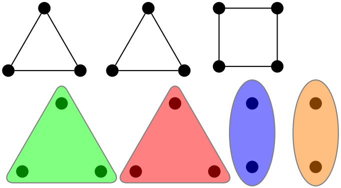

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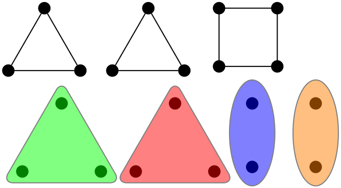

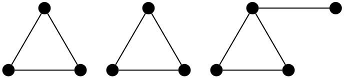

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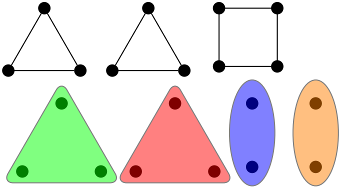

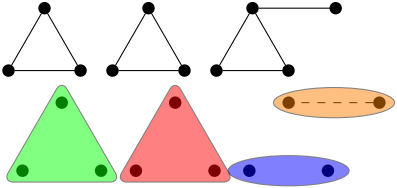

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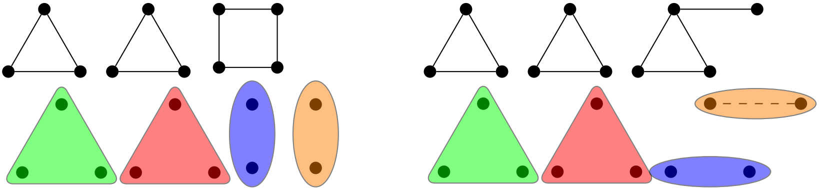

Conjecture (Chen-Lih-Wu)

If G is a $(k+1)$-colorable graph with $\Delta(G) \leq k+1$, then G has an equitable $(k+1)$-coloring or $k+1$ is odd and G contains $K_{k+1,k+1}$.

Future Directions

Conjecture (Chen-Lih-Wu)

If G is a k-colorable graph with $\Delta(G) \leq k$, then G has an equitable k-coloring or k is odd and G contains $K_{k,k}$.

Refining Chen-Lih-Wu for $n = 3k$:

Conjecture

Let G be a $3k$-vertex, k-colorable graph with $\overline{\sigma}(G) \leq 2k+1$. If G does not contain certain subgraphs, then G has an equitable k-coloring.

Ramsey Saturation

Definition

The saturation number of a family of graphs $\mathcal{H}$, denoted $sat(n; \mathcal{H})$, is the minimum number of edges over all graphs G on n vertices with the property that no member of $\mathcal{H}$ is a subgraph of G, but some member of $\mathcal{H}$ is a subgraph of $G + e$ for every edge $e \notin E(G)$.

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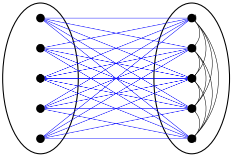

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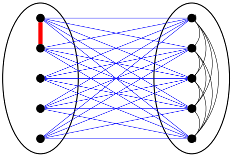

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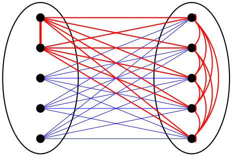

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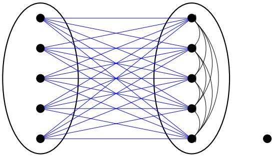

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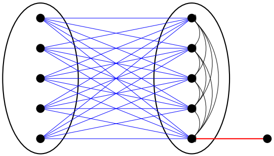

Ramsey Saturation Number

Definition

The Ramsey saturation number of $(H_1, \ldots, H_k)$, written $sat(n; \mathcal{R}(H_1, \ldots, H_k))$, is the minimum number of edges over all n-vertex graphs G such that an edge-coloring of G exists with no forbidden subgraphs, but G is edge-maximal with this property.

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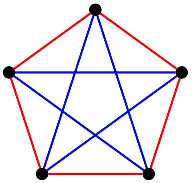
The Ramsey saturation number of $(H_1, \ldots, H_k)$*, written* $sat(n; \mathcal{R}(H_1, \ldots, H_k))$*, is the minimum number of edges over all n-vertex graphs G such that an edge-coloring of G exists with no forbidden subgraphs, but G is edge-maximal with this property.*

Example

$$sat(n; \mathcal{R}(K_3, K_3)) = \begin{cases} \binom{n}{2} & n \leq 5 \\ & \end{cases}$$



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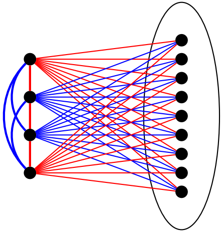

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Conjecture (Hanson, Toft)

$$sat(n; \mathcal{R}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r \end{cases}$$

where $r := r(k_1, \ldots, k_t)$.

Ramsey Saturation Number of Matchings

Theorem (FKY)

If $n > 3(m_1 + \cdots + m_t - t)$, *then*

$$sat(n; \mathcal{R}(m_1K_2, \ldots, m_tK_2)) = 3(m_1 + \cdots + m_t - t).$$

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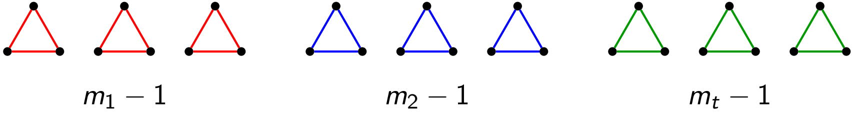

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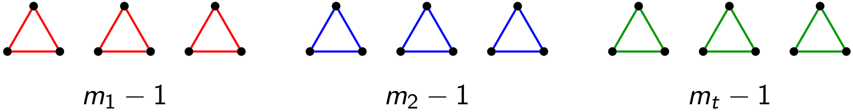

Trivial case: if $n < r(m_1K_2, \ldots, m_tK_2)$, then
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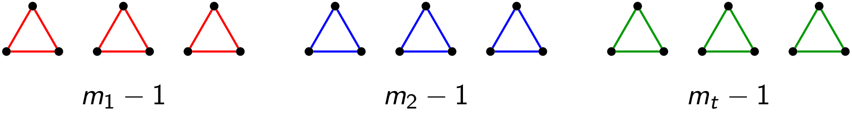

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$$r(m_1K_2, \ldots, m_tK_2) = \max\{m_1, \ldots, m_t\} + 1 + (m_1 + \cdots + m_t - t)$$

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- Matchings versus other kinds of graphs
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