Disjoint Cycles and Equitable Colorings in Graphs

H. Kierstead A. Kostochka T. Molla E. Yeager

yeager2@illinois.edu

21 April 2014

Corrádi-Hajnal

Corrádi-Hajnal Theorem

Corrádi-Hajnal, 1963

If G is a graph on n vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

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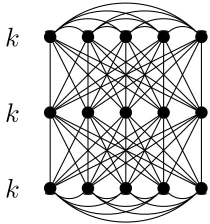
- Sharpness:



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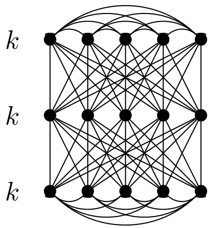
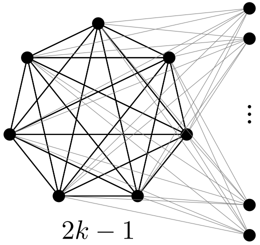
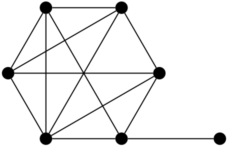




Dirac-Erdős, 1963



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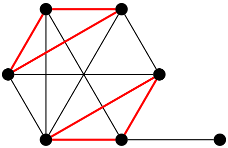

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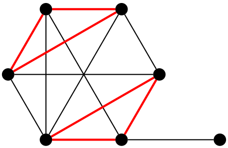

Let $V_{\geq c}$ be the number of vertices with degree at least c, etc.

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If $V_{\geq 2k} - V_{\leq 2k-2} \geq k^2 + 2k - 4$, $k \geq 3$, then G contains k disjoint cycles.

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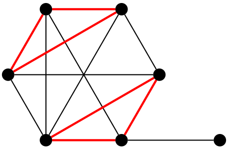

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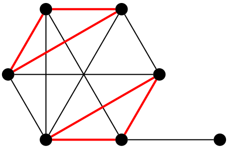

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- Unique result?
- "Probably not best possible"

Dirac-Erdős: (Lack of) Sharpness

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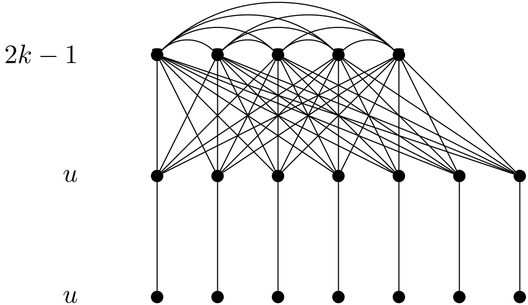
For $k \geq 3$, $V_{\geq 2k} - V_{\leq 2k-2} \geq 2k - 1$ does not guarantee k disjoint cycles.



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- Implies Corrádi-Hajnal

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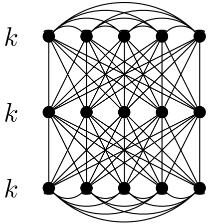

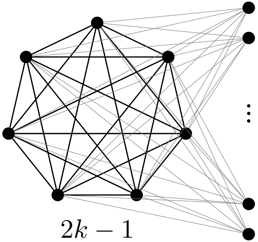

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Proof (Enomoto)

- Edge-maximal counterexample
  - $(k-1)$ disjoint cycles
  - Remaining graph at least 3 vertices
- Minimize number of vertices in cycles
- Maximize longest path in remainder

Kierstead-Kostochka-Y, 2014+

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$\alpha(G) \geq n - 2k + 1 \Rightarrow$ no k cycles

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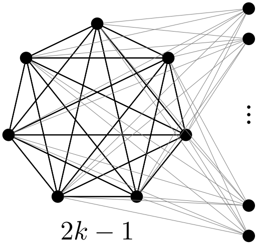

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KKY, 2014$^+$

For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

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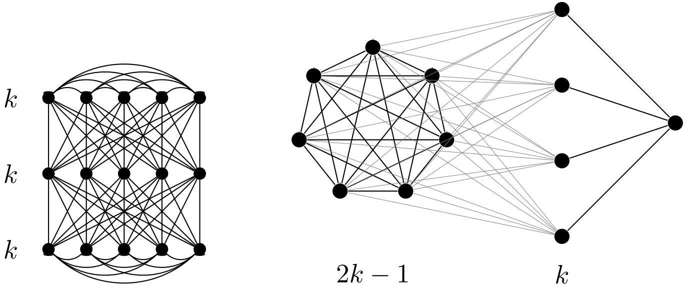

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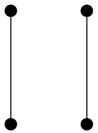

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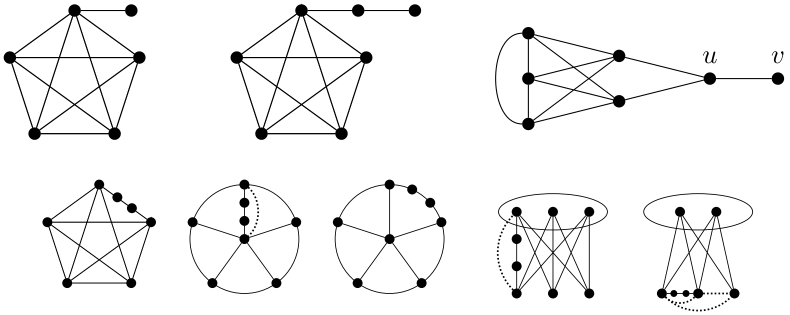

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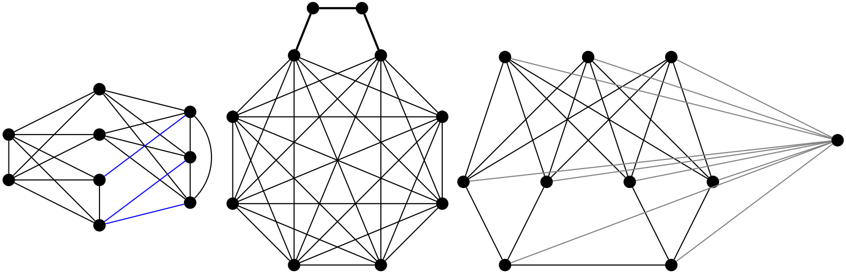

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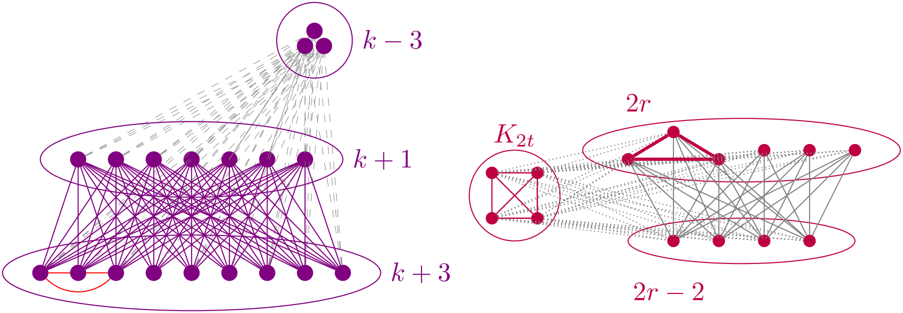
$$\sigma_2 = 4k - 4:$$



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Proof

(Like Enomoto)

- Let G be an edge-maximal counterexample.
- There exists a set of $(k-1)$ disjoint cycles.
- Choose the set of cycles with the least number of vertices, etc.

Dirac: $(2k-1)$-connected without k disjoint cycles

Dirac, 1963

What $(2k-1)$-connected graphs do not have k disjoint cycles?

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Answer to Dirac's Question

Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if

- $\alpha(G) \leq |G| - 2k$, and
- if k is odd and $H = K_{k,k} \subseteq \overline{G}$ then $\overline{G} - H$ is not k-equitable, and
- if $k = 2$ then G is not a wheel.

Coming Soon:

Dirac: $(2k-1)$-connected without k disjoint cycles

KKY, 2014+

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Coming Soon:

characterization for multigraphs

Faudree-Gould, 2005

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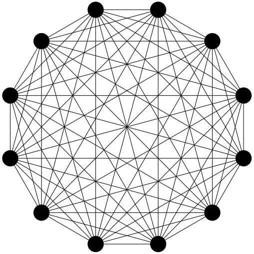
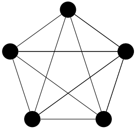


K_{3k-4}



K_5

Chorded Cycles

Finkel, 2008

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If G is a graph on $n \geq 4k$ vertices with $\delta(G) \geq 3k$, then G contains k disjoint chorded cycles.

Posed by Pósa, 1961

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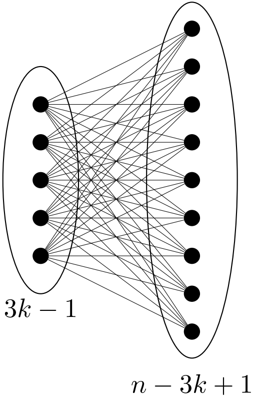

Chorded + Unchorded Cycles

Conjecture: Bialostocki-Finkel-Gyárfás, 2008

If G is a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then G contains $r + s$ cycles, s of them chorded.

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Let r and s be integers with $r + s \geq 1$, and let G be a graph on $n \geq 3r + 4s$ vertices. If $\boxed{\sigma_2(G) \geq 4r + 6s - 1}$, then G contains $r + s$ disjoint cycles, s of them chorded cycles.

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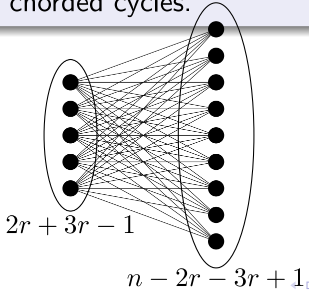

Neighborhood-Union Conditions

Qiao, 2012

Let r, s be nonnegative integers, and let G be a graph on at least $3r + 4s$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 3r + 4s + 1$. Then G contains $r + s$ disjoint cycles, s of them chorded.

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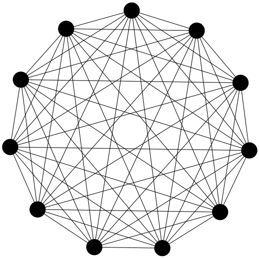
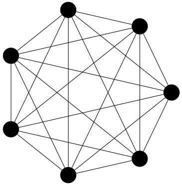
Sharpness (sort of):



K_{2s+3}



K_{2s-1}

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Gould-Hirohata-Horn, 2013

Let G be a graph on at least $4k$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 4k + 1$. Then G contains k disjoint chorded cycles.

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Let G be a graph on $n > 30k$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 2k + 1$. Then G contains k disjoint chorded cycles.

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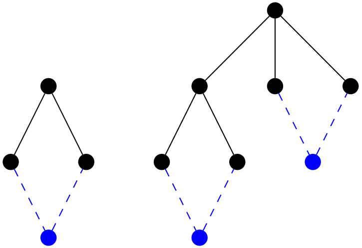

Gould-Hirohata-Horn, 2013

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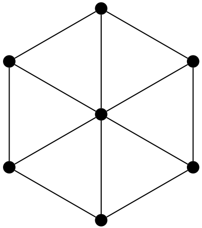
Let G be a graph on $n > 30k$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 2k + 1$. Then G contains k disjoint cycles.

Sharpness (of $|N(x) \cup N(y)| \geq 2k + 1$)

$k = 2$:



Multiply Chorded Cycles

We define $f(k)$ to be the number of chords in K_{k+1}, viewed as a cycle. That is, $f(k) = \frac{(k+1)(k-2)}{2}$.

Gould-Horn-Magnant, 2014

There exist s_0 and k_0 so that if $s \geq s_0$ and $k \geq k_0$, then there exists an $n_0 = n_0(s,k)$ so that if G has minimum degree at least sk and $|G| > n_0$, then G contains s disjoint cycles with at least $f(k)$ chords.

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Conjecture: $s_0 = k_0 = 1$, $n_0 = s(k+1)$

Multiply Chorded Cycles

Qiao-Zhang, 2010

Let G be a graph on $n \geq 4k$ vertices with $\delta(G) \geq \lfloor 7k/2 \rfloor$. Then G contains k disjoint, doubly chorded cycles.

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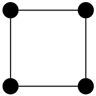
$$k = 1$$

$$\lfloor 7k/2 \rfloor = 3; \text{ use } C_4$$



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Sharp for small k:

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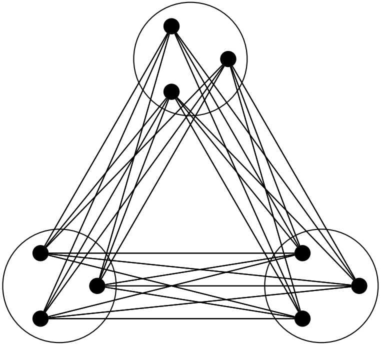
$\lfloor 7k/2 \rfloor = 7$; use $K_{3,3,3}$



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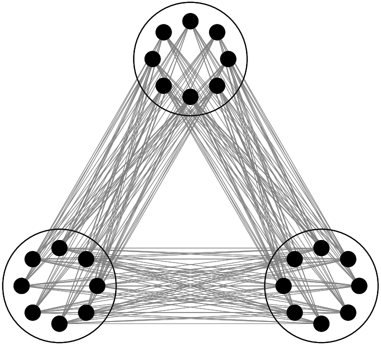
Sharp for small k:

$k = 5$

$\lfloor 7k/2 \rfloor = 17$; use $K_{8,8,8}$



Cycles with All Chords at One Vertex

Babu-Diwan, 2009

Theorem 1.1. Let $n_1, n_2, \ldots, n_k$ be integers, $n = \sum n_i$, $n_i \geq 3$, and let H_i be a cycle containing all possible chords incident to one vertex, or a tree, on n_i vertices. If G is a graph on at least n vertices with $\sigma_2(G) \geq 2(n-k)-1$, then G contains disjoint subgraphs isomorphic to $H_1, \ldots, H_k$.

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Sharpness?

Equitable Coloring

Equitable Coloring

Definition

An *equitable k-coloring* of a graph G is a proper coloring of $V(G)$ such that any two color classes differ in size by at most one.

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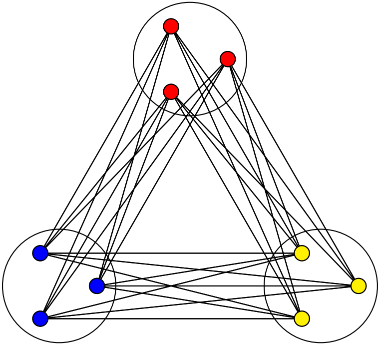

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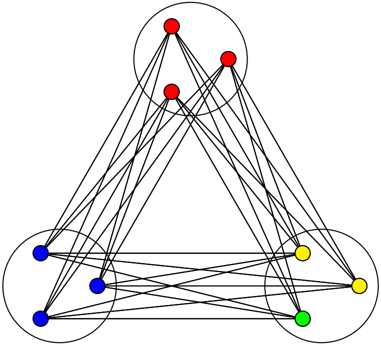

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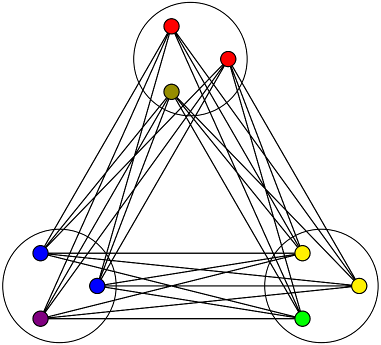

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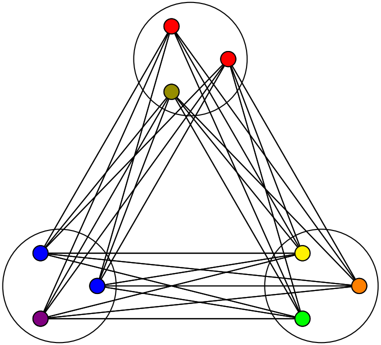

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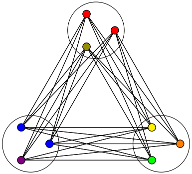

Application

Scheduling

- Vertices are tasks
- Edges are conflicts
- Colors are computers

Equitable Coloring and Cycles

$n = 3k$

If G has $n = 3k$ vertices and an equitable k-coloring, then $\overline{G}$ has k disjoint cycles (all triangles).

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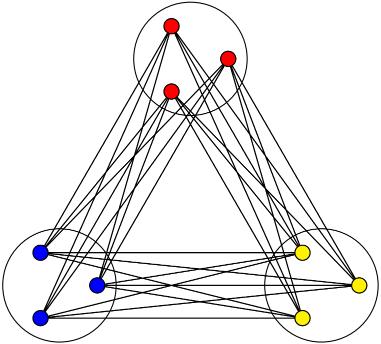

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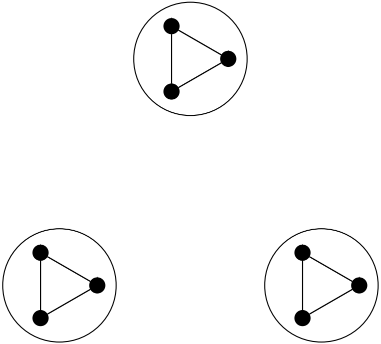

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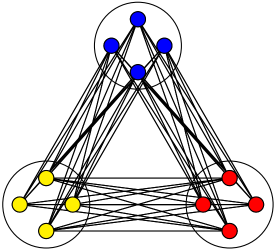

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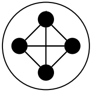

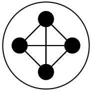

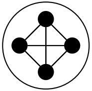

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independent sets $\leftrightarrow$ cliques

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Cycles, chorded cycles, cycles with $f(k)$ chords, etc: generalizations.

Equitable Coloring and Cycles

Kierstead-Kostochka, 2008

If G is a graph such that $d(x) + d(y) \leq 2k - 1$ for every edge xy, then G has an equitable k-coloring.

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Equivalent when $n = 3k$: 2(3k-1)-(2k-1)=4k-1

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Equivalent Statement for $n = 4k$

Let G be a graph on $4k$ vertices with $\Delta(G) \leq \lfloor k/2 \rfloor - 1$. Then G is equitably k-colorable.

Chen-Lih-Wu

Hajnal-Szemerédi, 1970

If $k \geq \Delta(G) + 1$, then G is equitably k-colorable.

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Chen-Lih-Wu Conjecture

A connected graph G is equitably $\Delta(G)$ colorable if G is different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$ for every $m \geq 1$.

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Chen-Lih-Wu Conjecture Re-stated

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then G is equitably k-colorable.

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CLW true if:

$\delta(G) \geq |G|/2$; $\Delta(G) \leq 4$; G planar with $\Delta(G) \geq 13$; G outerplanar, etc.

Ore Conditions

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Kierstead-Kostochka-Molla-Yeager, 2014+

If G is a $3k$-vertex graph such that for each edge xy, $d(x) + d(y) \leq 2k + 1$, then G is equitably k-colorable, or is one of several exceptions.

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Equivalent

If G is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles, or is one of several exceptions.

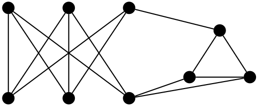
KKY, 2014+

For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

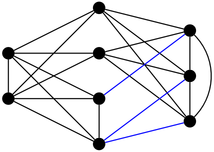
Exceptions

- $k = 3$

Equitable coloring:



Cycles:



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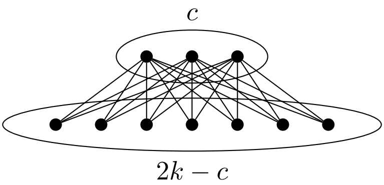

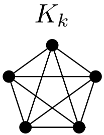

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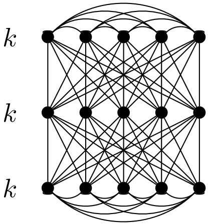

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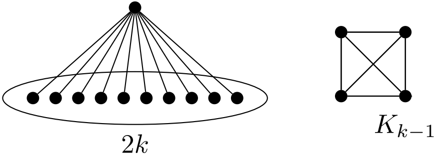

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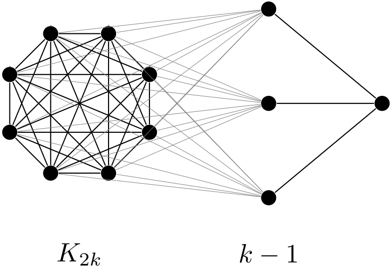
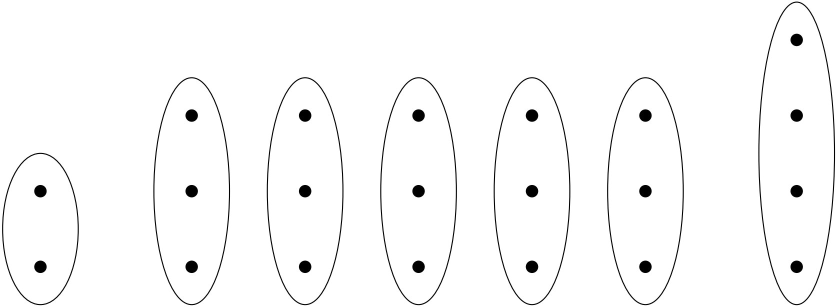


Proof of KKMY 2014+



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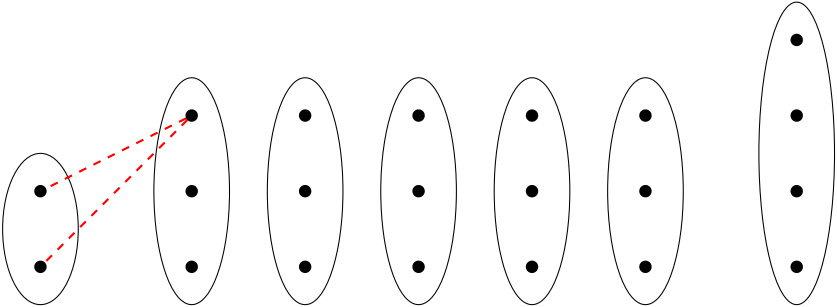

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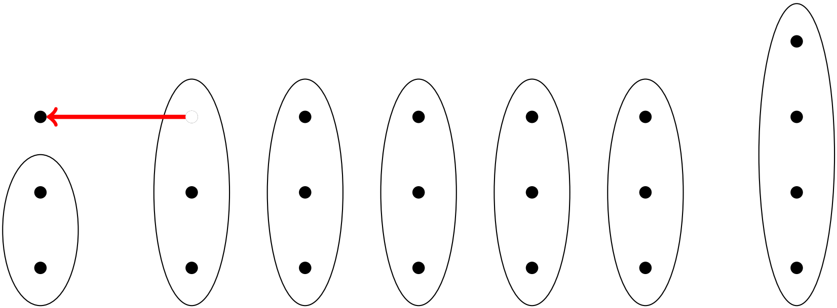

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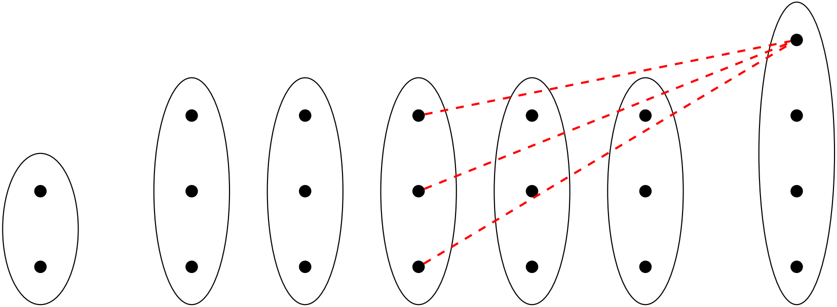

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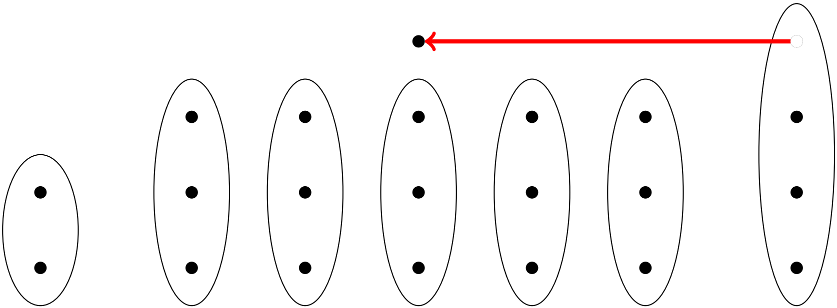

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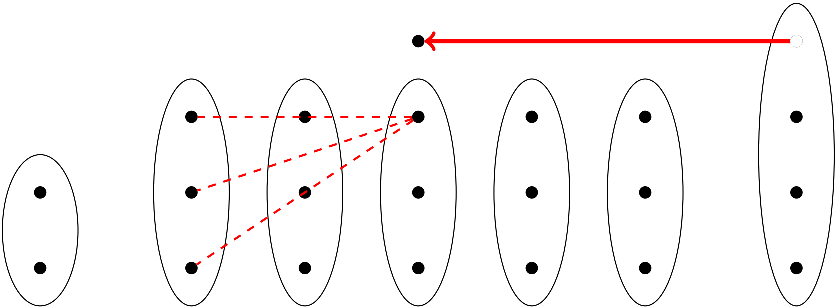

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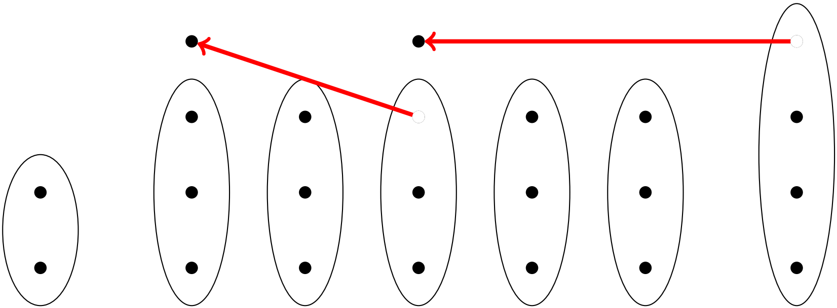

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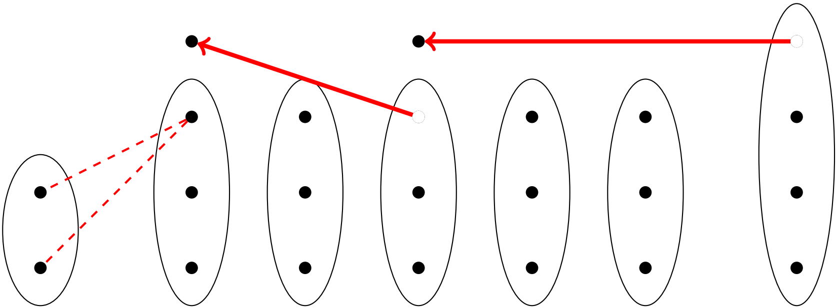

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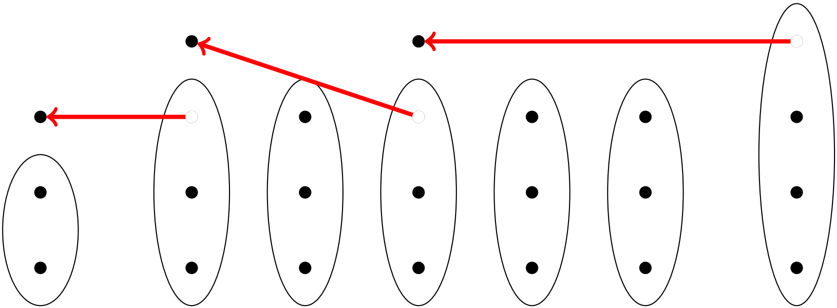

Thanks for Listening!