Disjoint Cycles and A Question of Dirac

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Disjoint Cycles

Corrádi-Hajnal Theorem

Corrádi-Hajnal, 1963

If G is a graph on n vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

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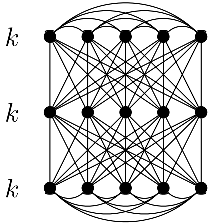

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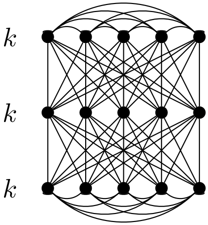

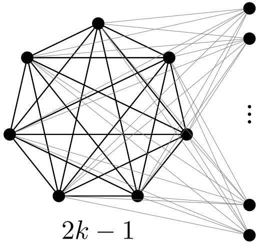

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Implies Corrádi-Hajnal

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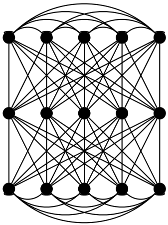

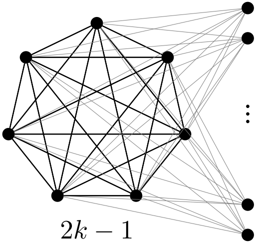

$2k-1$

Kierstead-Kostochka-Y, 2014+

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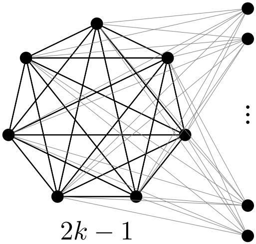

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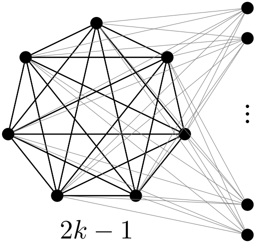

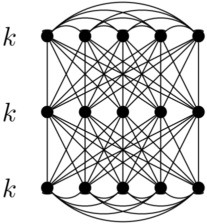

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KKY, 2014^+

For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

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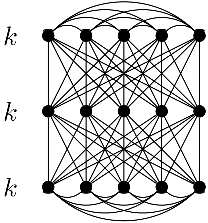

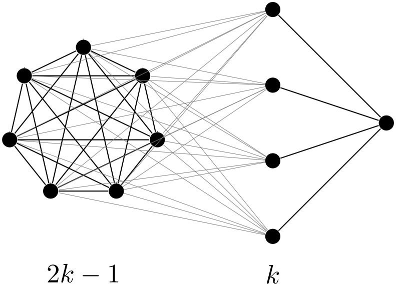

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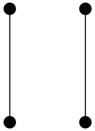

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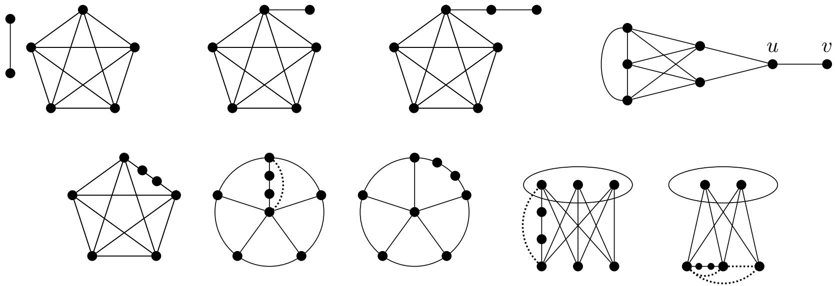

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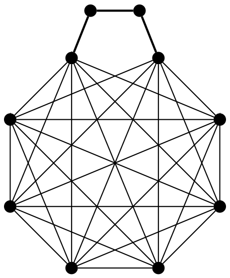

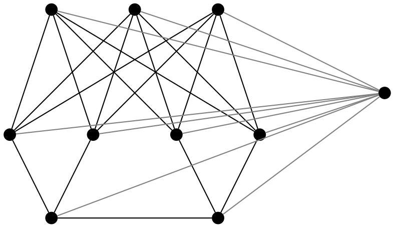

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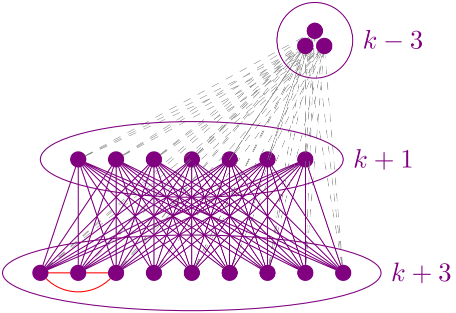
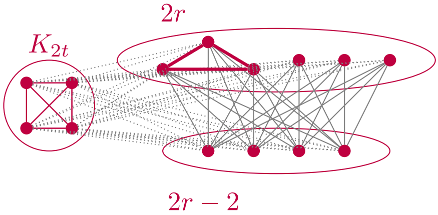
$$\sigma_2 = 4k - 4:$$





Dirac's Question

Dirac: $(2k-1)$-connected without k disjoint cycles

Dirac, 1963

What $(2k-1)$-connected graphs do not have k disjoint cycles?

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Answer to Dirac's Question for Simple Graphs

Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if

- $\alpha(G) \leq |G| - 2k$, and
- if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$, and
- if $k = 2$ then G is not a wheel.

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Further:

characterization for multigraphs

Multigraphs

Theorem (Extension of Corrádi-Hajnal to Multigraphs)

For $k \in \mathbb{Z}^+$, let G be a multigraph with simple degree at least $2k$. Then G has k disjoint cycles if and only if

$$|V(G)| \geq 3k - 2\ell - \alpha'$$

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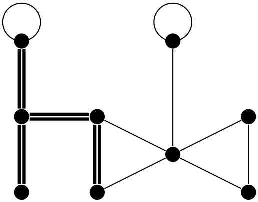

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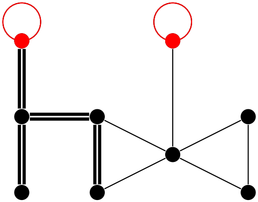

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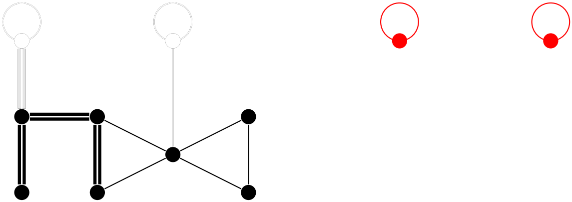

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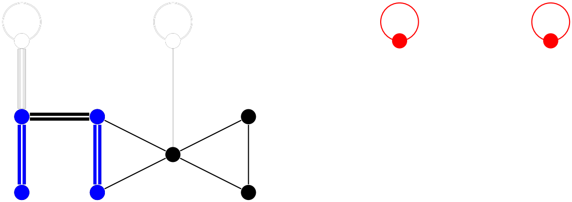

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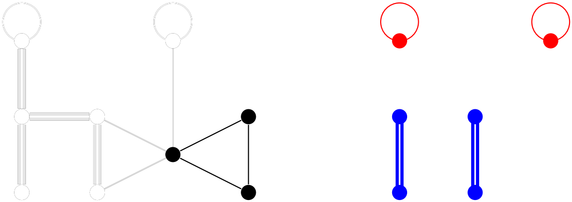

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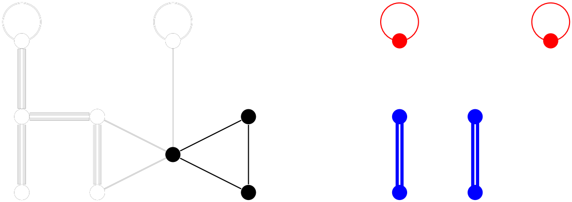

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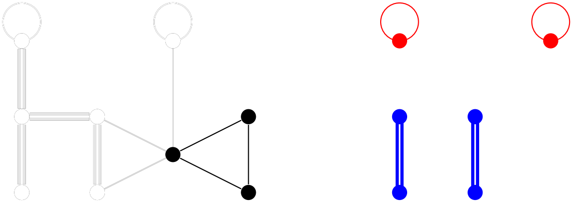
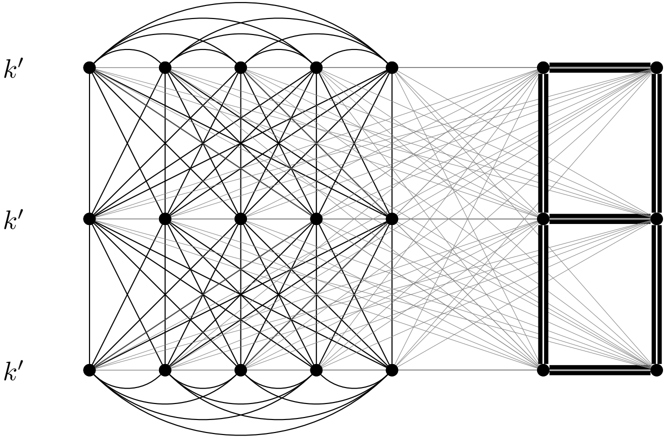


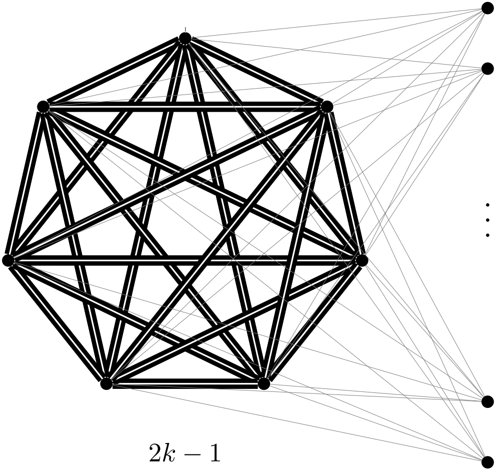
Remaining graph: min degree $\geq 2k = 2(k - \ell - \alpha') + \ell$

k' odd, F has a perfect matching

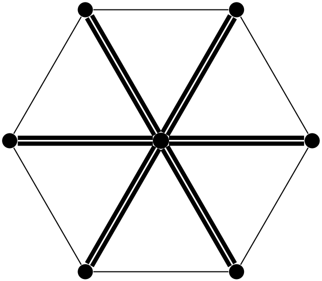
Example: $k = 8$, $\alpha' = 3$, $k' = 5$.



Big independent set, incident to no multiple edges

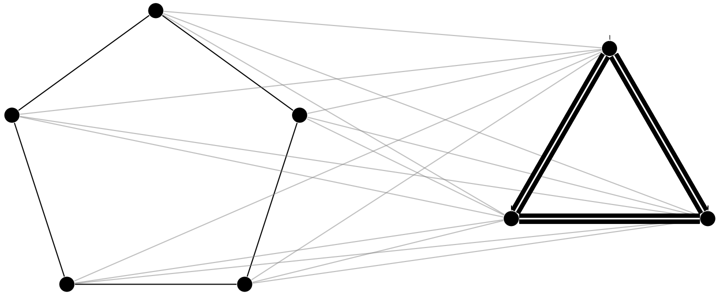


$k = 2$ and G is a wheel; spokes possibly strong



$k' = 2$; Remove 2-cyces remainder is a wheel

Example: $k = 3$, $\alpha' = 1$, $k' = 2$



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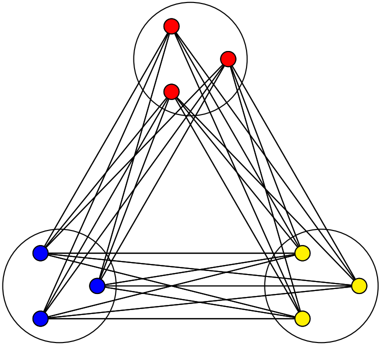

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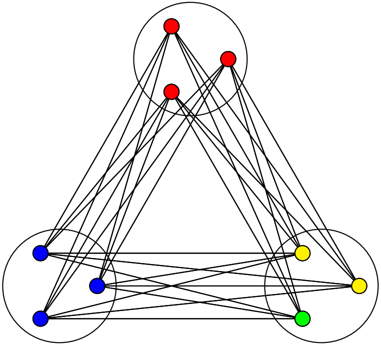

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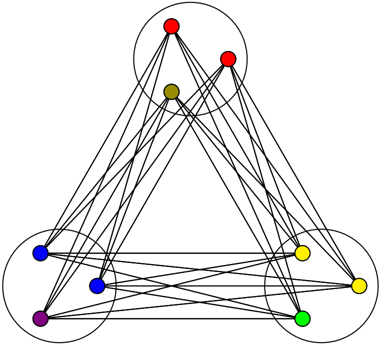

Equitable Coloring and Cycles

$n = 3k$

If G has $n = 3k$ vertices, then G has an equitable k-coloring if and only if $\overline{G}$ has k disjoint cycles (all triangles).

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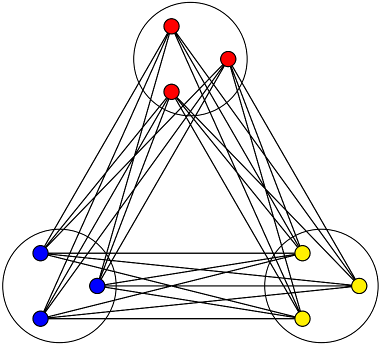

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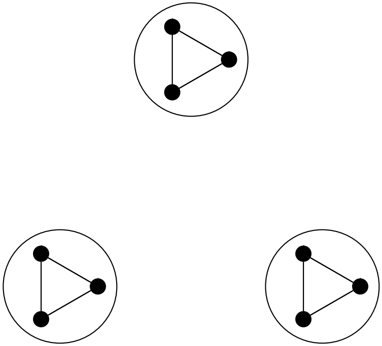

Chen-Lih-Wu

Hajnal-Szemerédi, 1970

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Chen-Lih-Wu Conjecture

If $\chi(G), \Delta(G) \leq k$, and if k is odd $K_{k,k} \not\subseteq G$, then G is equitably k-colorable.

Ore Conditions

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If G is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles, or is one of several exceptions, or $\overline{G}$ is not k-colorable.

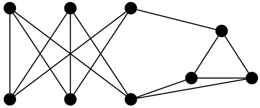
KKY, 2014+

For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

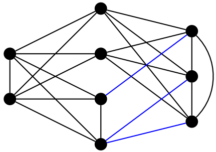
Exceptions

- $k = 3$

Equitable coloring:

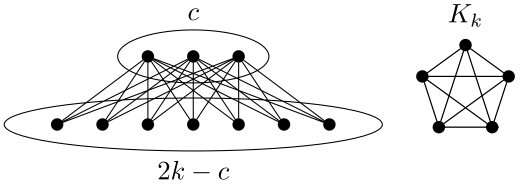


Cycles:

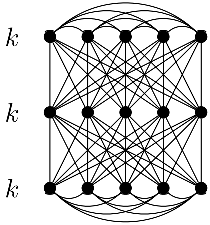


Exceptions

- *Equitable coloring:*

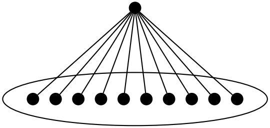


Cycles:

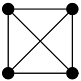


Exceptions

- *Equitable coloring:*

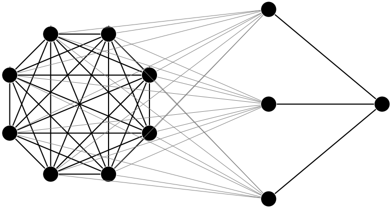


$2k$



K_{k-1}

Cycles:



K_{2k} $\qquad$ $k-1$

Thanks for Listening!