Refinements of the Corrádi-Hajnal Theorem

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MIGHTY, September 2012

Corrádi-Hajnal Theorem

Theorem 1

[Corradi, Hajnal 1963] Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\delta(H) \geq 2k$. Then H contains k vertex-disjoint cycles.

Corrádi-Hajnal Theorem

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Corollary 2

Let $n = 3k$, and let H be an n-vertex graph with $\delta(H) \geq 2k$. Then H contains k vertex-disjoint triangles.

Refinements

Theorem 3

[Aigner, Brandt 1993]: Let H be an n-vertex graph with $\delta(H) \geq \frac{2n-1}{3}$. Then H contains each 2-factor.

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Definition

$\sigma_2(G) = \min_{xy \notin E(G)} \{d(x) + d(y)\}$

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Definition

$\sigma_2(G) = \min_{xy \notin E(G)}\{d(x) + d(y)\}$

Theorem 4

[Kostochka, Yu 2011]: *Let $n \geq 3$ and H be an n-vertex graph with $\sigma_2(H) \geq 4n/3 - 1$. Then H contains each 2-factor.*

Refinements

Theorem 5

[Fan, Kierstead 1996]: Let $n \geq 3$ and H be an n-vertex graph with $\delta(H) \geq \frac{2n-1}{3}$. Then H contains the square of the n-vertex path.

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Theorem 6

[Enomoto 1998; Wang 1999]: Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\sigma_2(H) \geq 4k - 1$. Then H contains k vertex-disjoint cycles.

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Theorem 5

[Fan, Kierstead 1996]: *Let $n \geq 3$ and H be an n-vertex graph with $\delta(H) \geq \frac{2n-1}{3}$. Then H contains the square of the n-vertex path.*

Theorem 6

[Enomoto 1998; Wang 1999]: *Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\sigma_2(H) \geq 4k - 1$. Then H contains k vertex-disjoint cycles.*

Theorem 7

[Kierstead, Kostochka, Y.]: *Let $k \geq 3$, $n \geq 3k + 1$, and let H be an n-vertex graph with $\delta(H) \geq 2k - 1$ and $\alpha(H) \leq n - 2k$. Then H contains k vertex-disjoint cycles.*

Proof Sketch: Theorem 7

Theorem (7)

[Kierstead, Kostochka, Y.]: Let $k \geq 3$, $n \geq 3k+1$, and let H be an n-vertex graph with $\delta(H) \geq 2k-1$ and $\alpha(H) \leq n-2k$. Then H contains k vertex-disjoint cycles.

Idea of Proof: Suppose G is an edge-maximal counterexample. Let $\mathcal{C}$ be a set of disjoint cycles in G such that:

- $|\mathcal{C}|$ is maximized,
- subject to the above, $\sum_{C \in \mathcal{C}} |C|$ is minimized, and
- subject to both other conditions, the length of a longest path in $G - \bigcup \mathcal{C}$ is maximized.

Proof of Therem 7

Goal (1)

$R := G - \mathcal{C}$ is a path

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Goal (2)

$|R| \geq 4$

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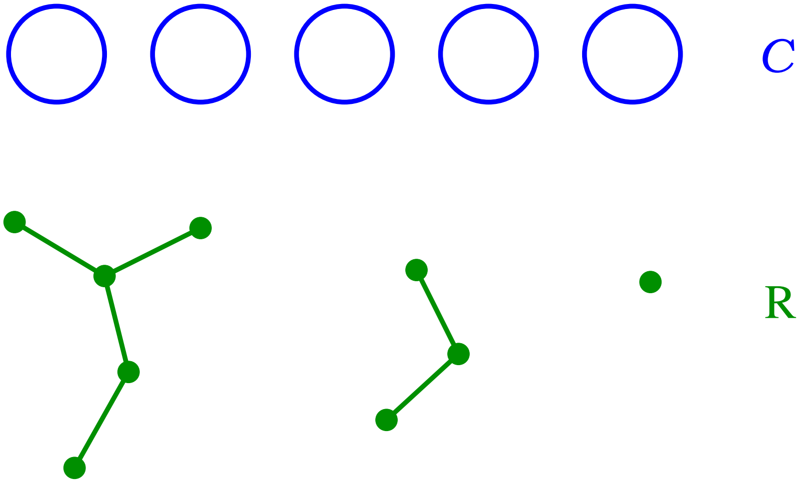
$|R| \geq 4$

Goal (3)

$|R| = 3$

Goal 1



Notice R is a forest. If R is not a path, it has at least three buds. Let a be an endpoint of a longest path P, and let c be a bud not on P.

Goal 1: R is a Path

Claim 1

Suppose R is not a path. $||\{a, c\}, C|| = 4$ for every $C \in \mathcal{C}$.

Claim 2

Suppose R is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves c in R, a and c share exactly the same two neighbors in C. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

R is a subdivided star.

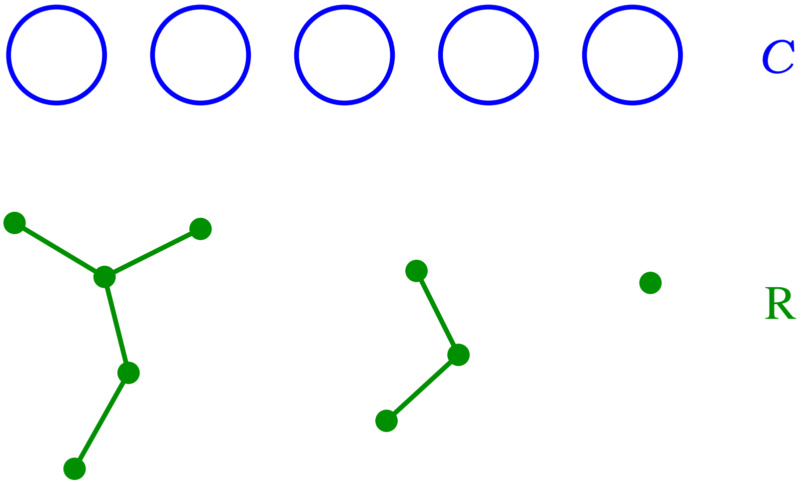
Claim 4

R is a path or a star.

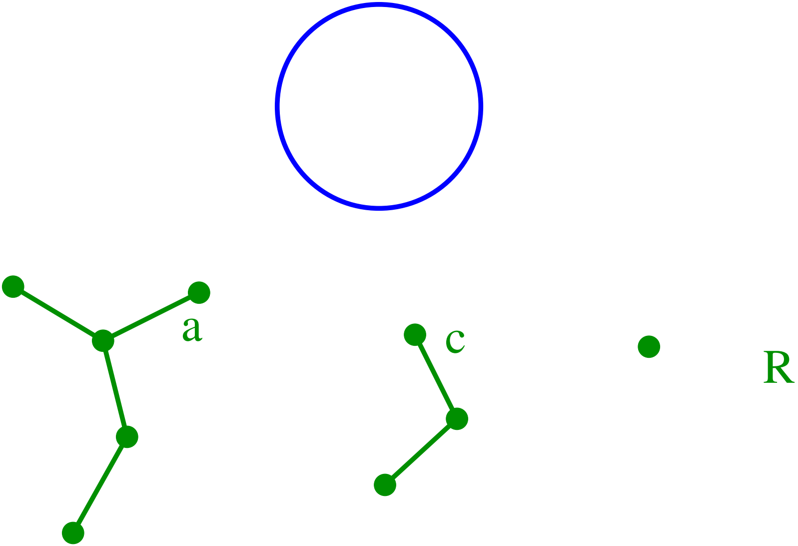
Claim 5

R is a path.

Claim 1



Claim 1



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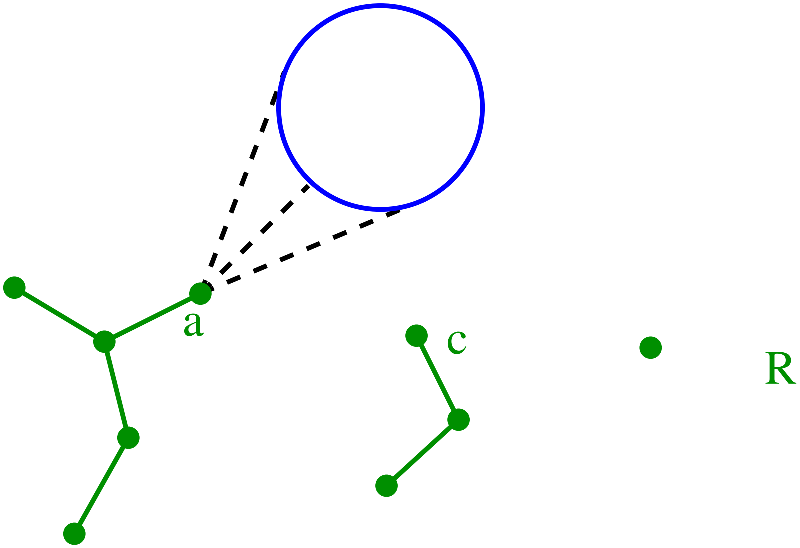

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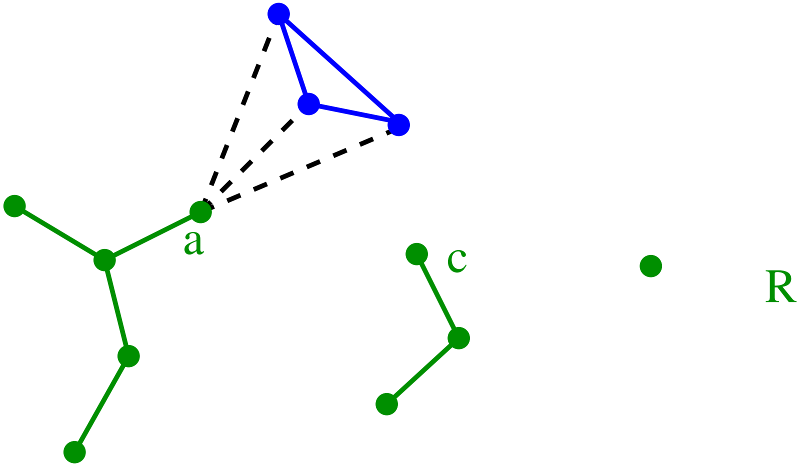

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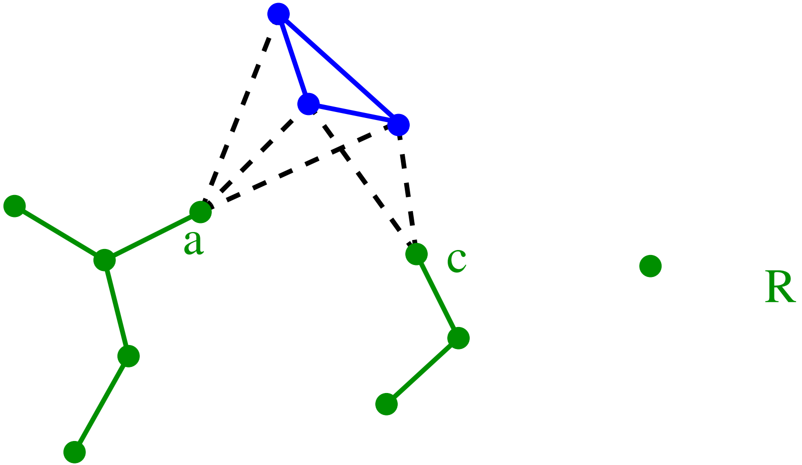

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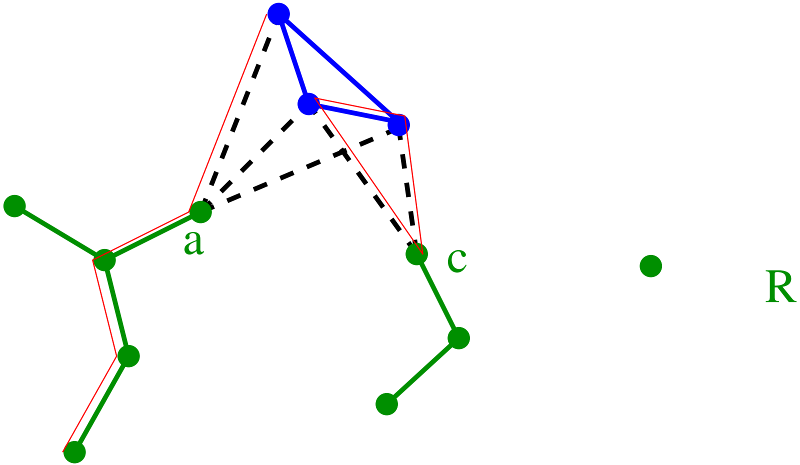

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We can now show $||\{a, c\}, C|| = 4$ by a counting argument, using the minimum degree of G. This proves Claim (1).

The same counting argument shows that a and c must have one neighbor in R, so R has no isolated vertices.

Goal 1: R is a Path

Claim 1

Suppose R is not a path. $||\{a, c\}, C|| = 4$ for every $C \in \mathcal{C}$.

Claim 2

Suppose R is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves c in R, a and c share exactly the same two neighbors in C. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

R is a subdivided star.

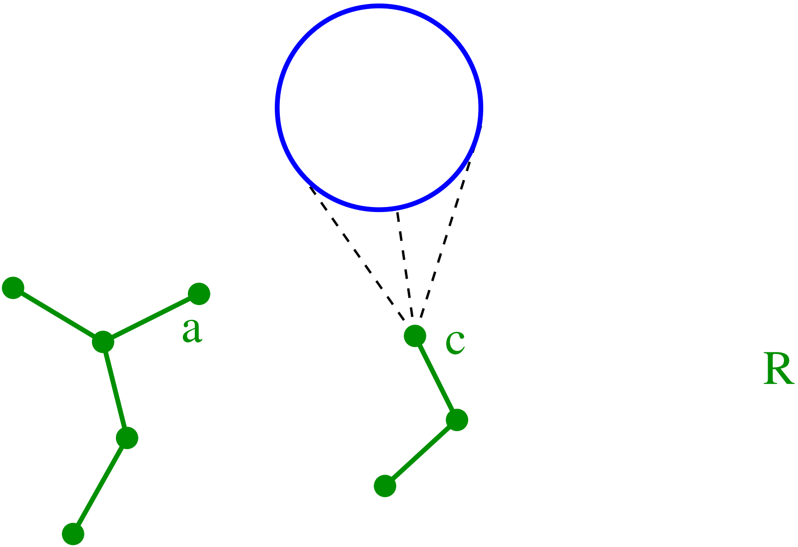
Claim 4

R is a path or a star.

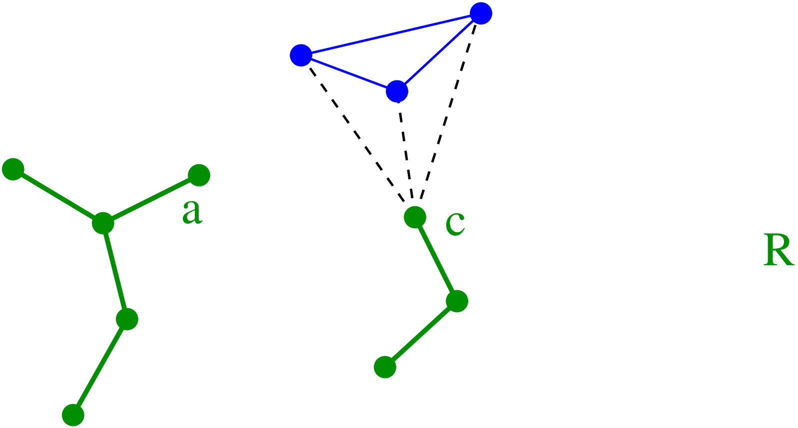
Claim 5

R is a path.

Claim 2



Claim 2



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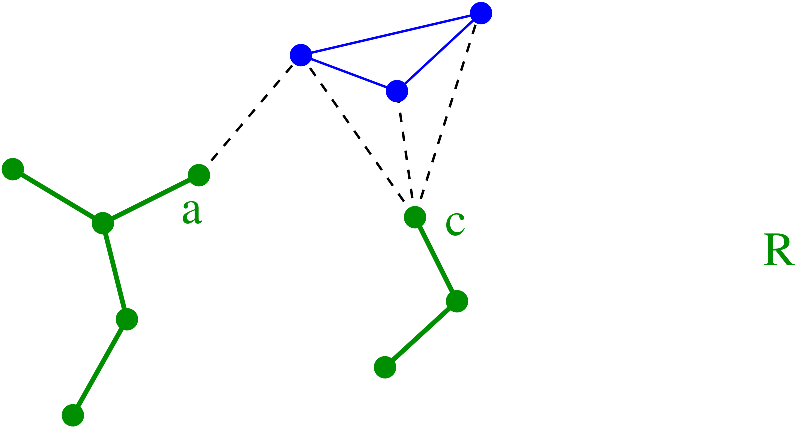

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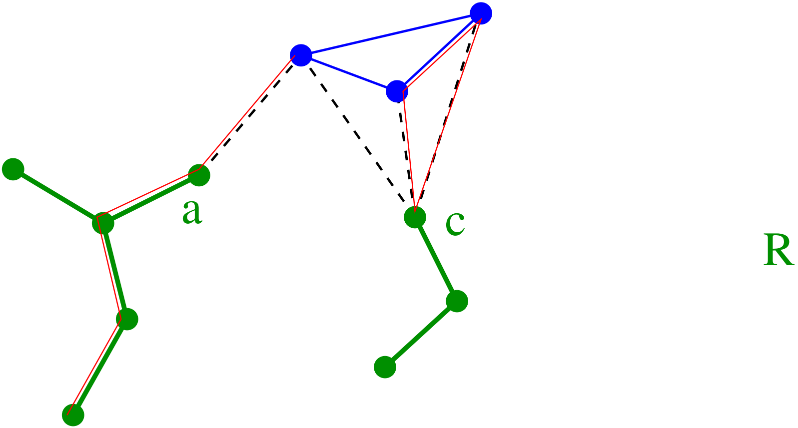
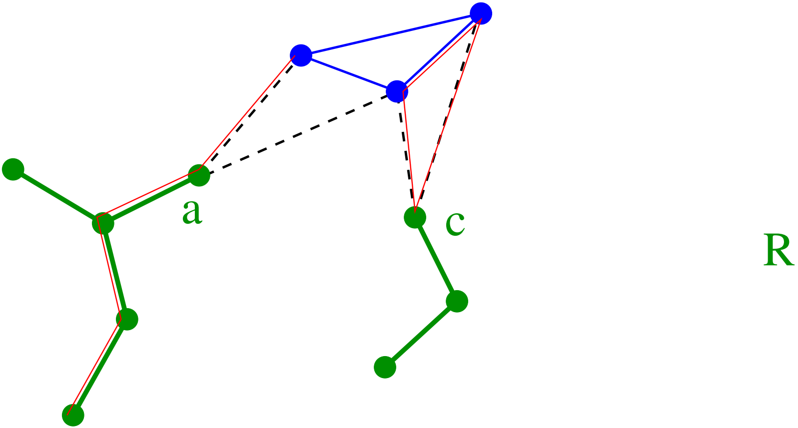


Claim 2

So we see that c can have at most 2 neighbors in any cycle $C \in \mathcal{C}$. By degree considerations, c must have precisely two neighbors in each cycle $C \in \mathcal{C}$. This tells us that a, as well, has precisely 2 neighbors to every cycle $C \in \mathcal{C}$.

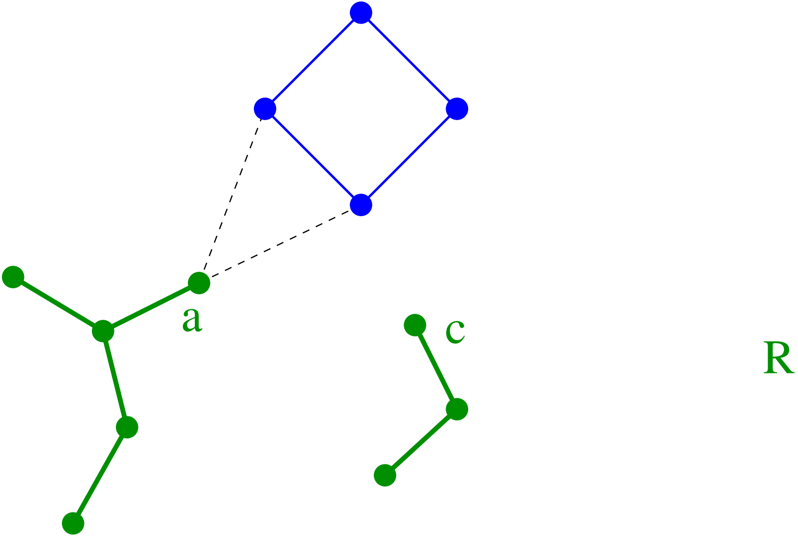
It remains only to show that no two leaves in R have different sets of neighbors, and if $|C| = 4$, the neighbors of our leaves are nonadjacent.

Claim 2



So if $|C| = 3$, then $N(a) \cap C = N(c) \cap C$, as desired.

Claim 2



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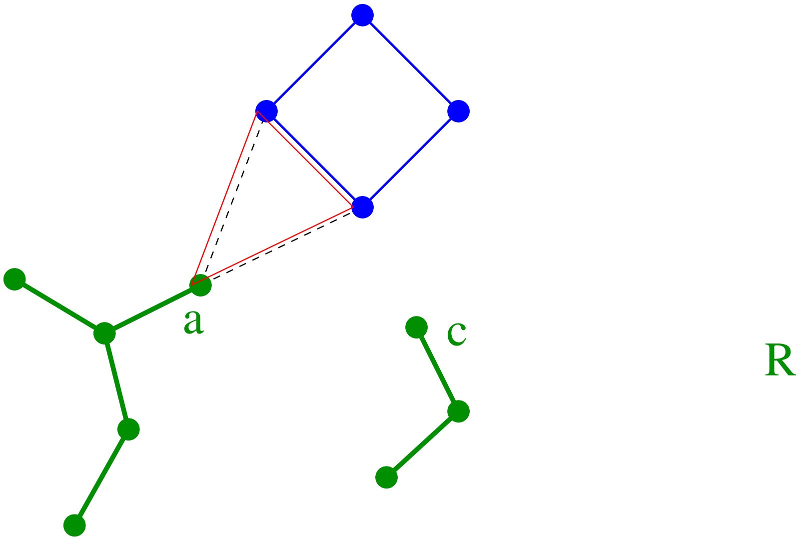

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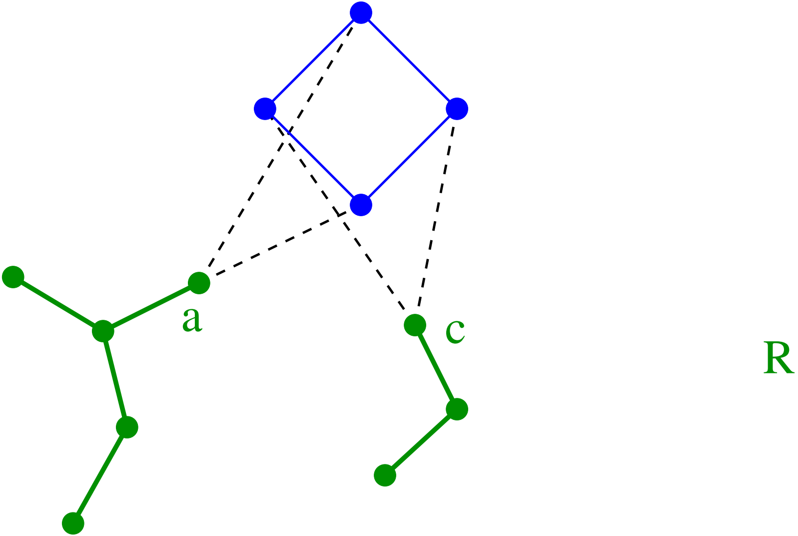

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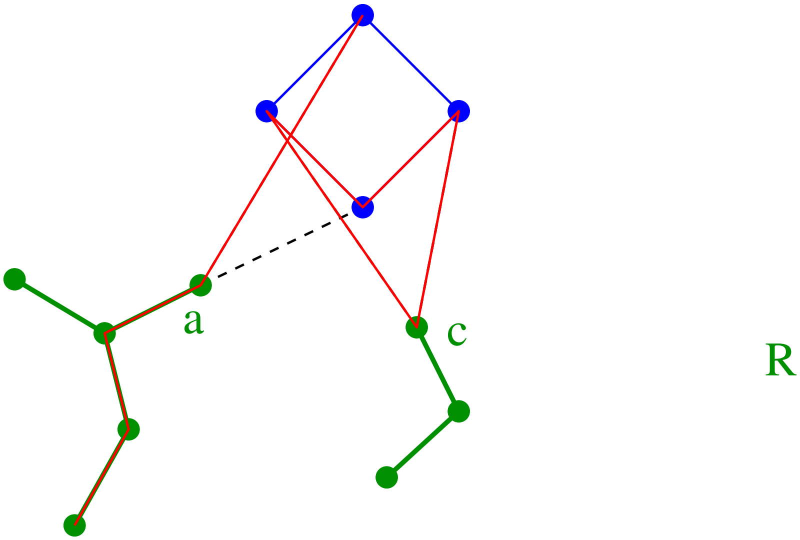

Claim 2

This proves Claim 2.

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Suppose R is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves c in R, a and c share exactly the same two neighbors in C. If $|C| = 4$, then those neighbors are nonadjacent.

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R is a subdivided star.

Claim 4

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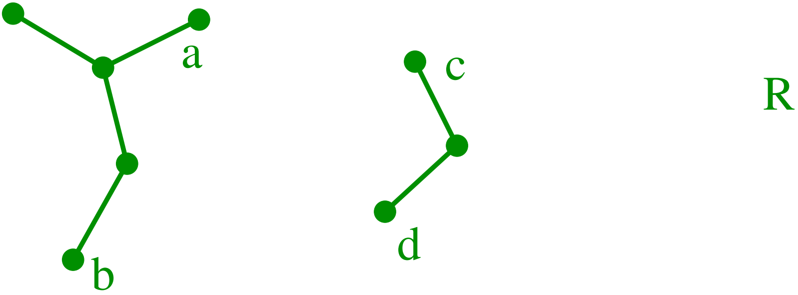
Claim 3

Suppose R is not a subdivided star. Then it has four leaves a, b, c, d such that the paths aRb and cRd exist and are disjoint.



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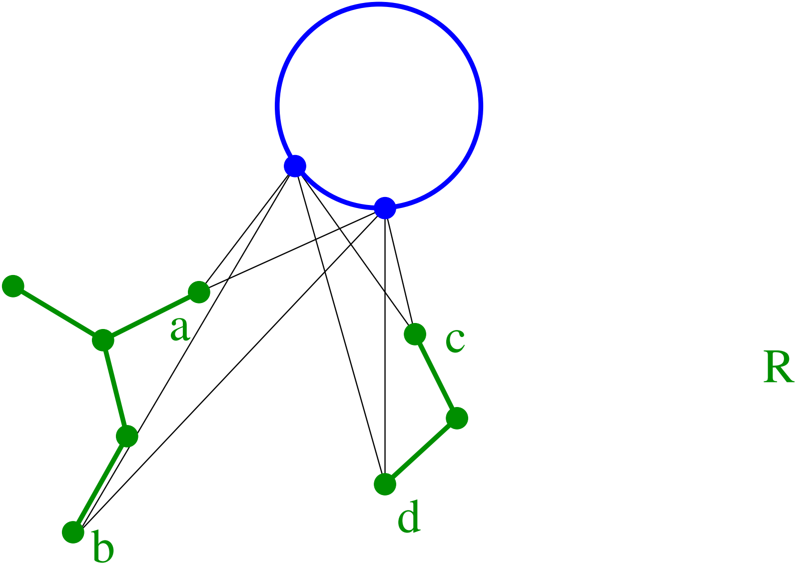

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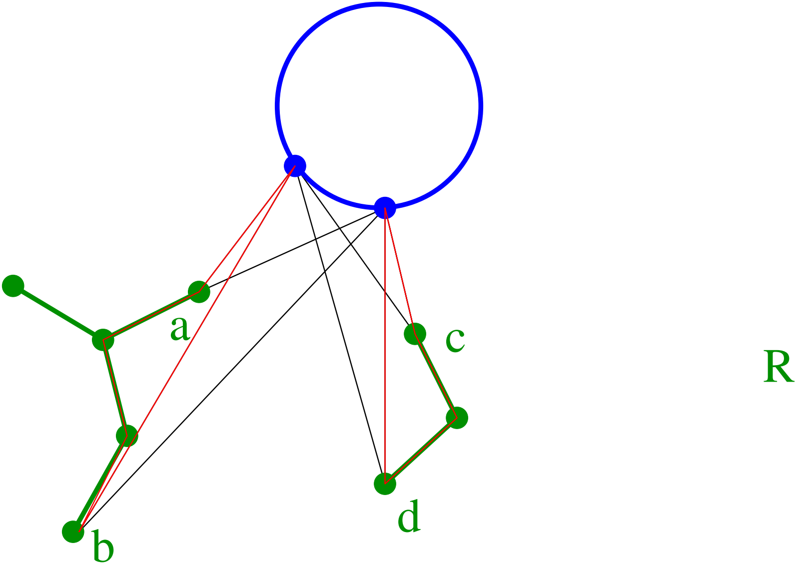

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R is a subdivided star.

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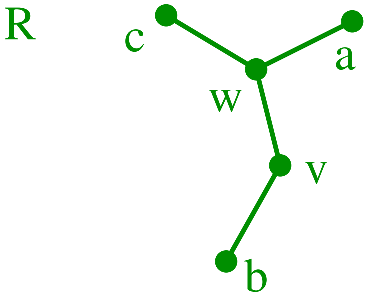
Claim 4

Suppose R is not a path or a star. We know it is a subdivided star, so there must be some unique vertex w with degree at least three. Since we assume it is not a star, there is also a vertex v of degree 2. Further, there exist leaves a, b, c so that vRb does not contain w and is disjoint rom aRc.



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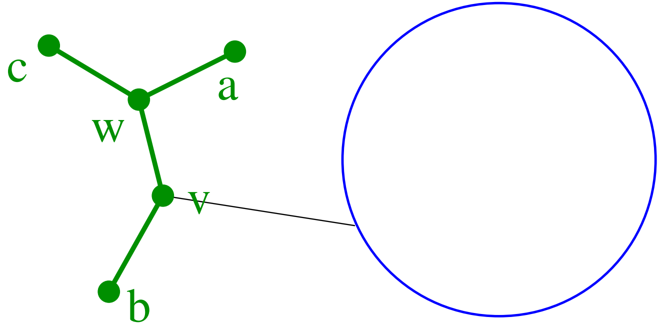

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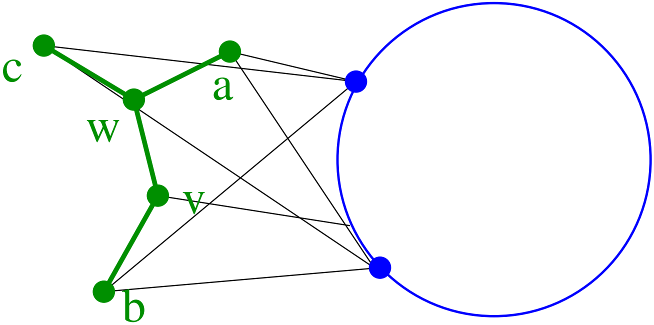

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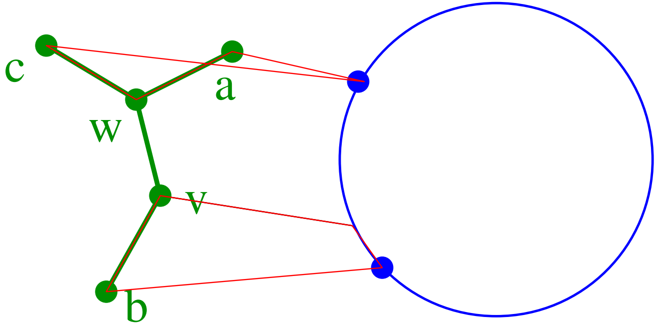

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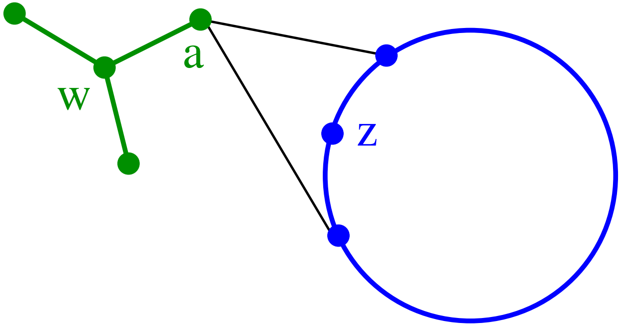
Suppose R is not a path. R has precisely one vertex w of degree at least 3.

Let z be an arbitrary vertex in $\mathcal{C} - N(a)$.



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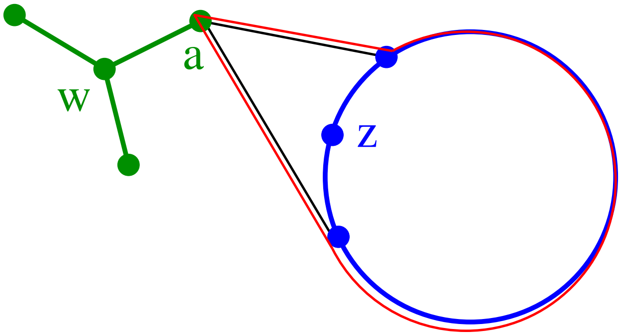

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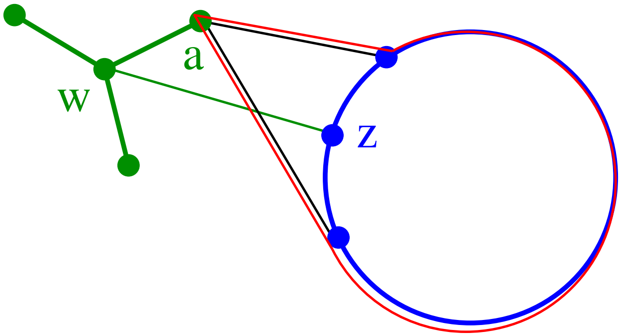

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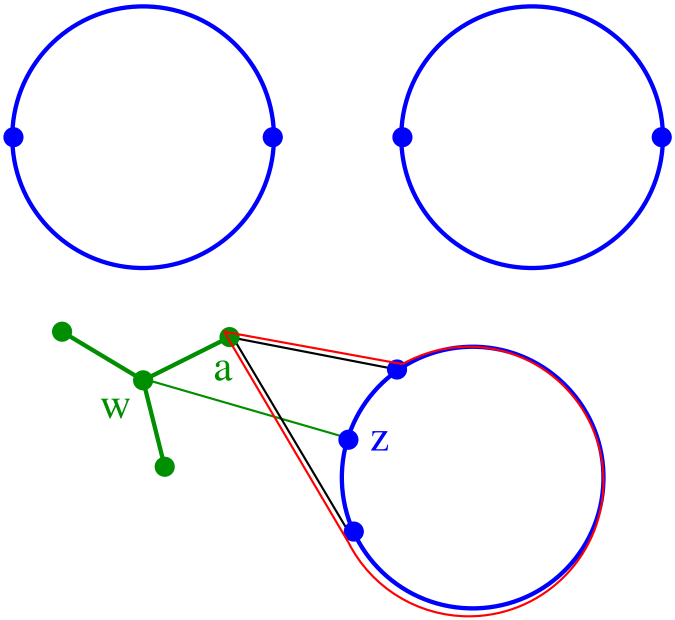

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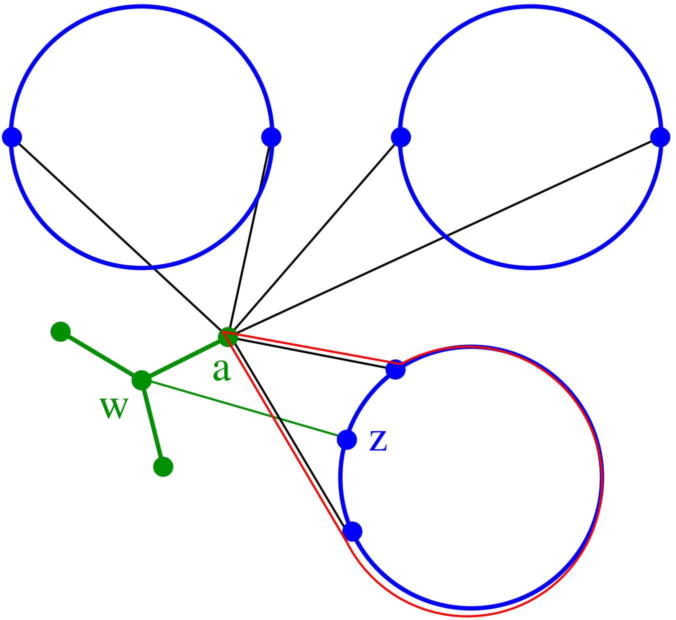

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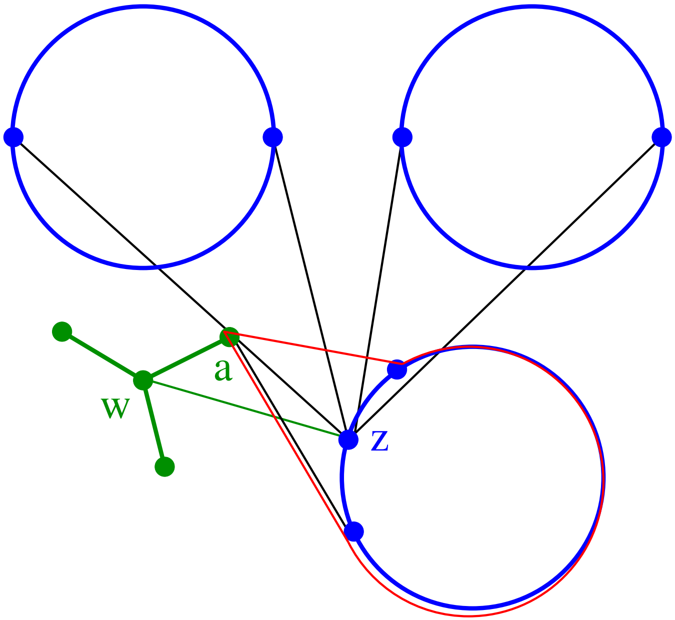

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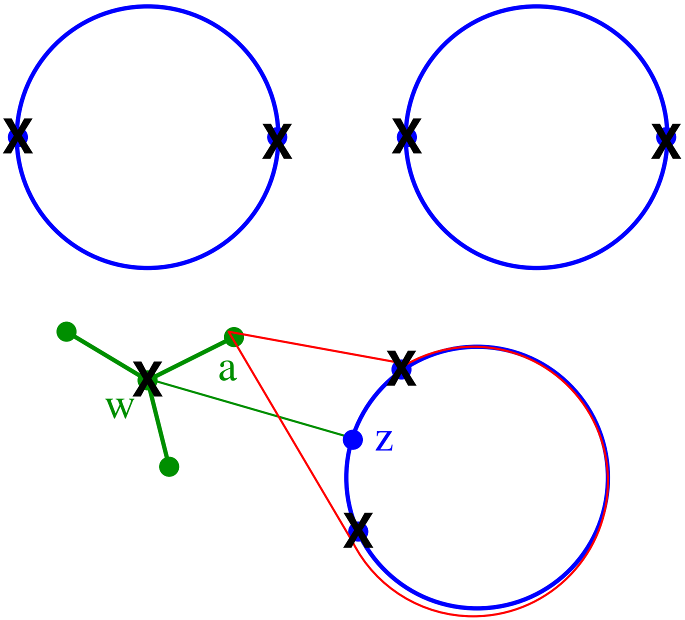

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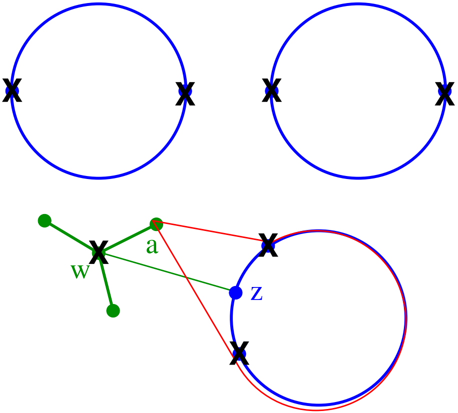

The independent set has size:

$$|V(G)| - 2(k-1) - 1 = n - 2k + 1$$

but we assumed $\alpha(G) \leq n - 2k$, a contradiction. This proves Claim 5, also Goal 1, that R is a path.

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Thank you for listening!