Disjoint Cycles and Equitable Colorings

H. Kierstead A. Kostochka T. Molla E. Yeager

yeager2@illinois.edu

Louisiana State University
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Disjoint Cycles

Corrádi-Hajnal Theorem

Corrádi-Hajnal, 1963

If G is a graph on n vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

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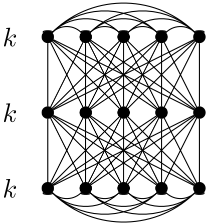

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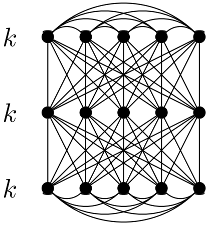

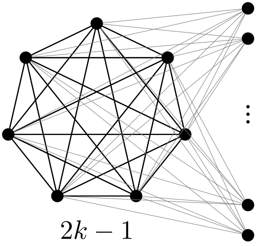

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Implies Corrádi-Hajnal

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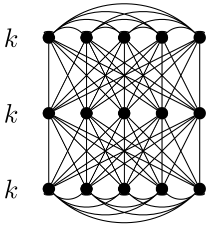

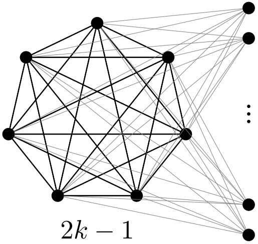

Kierstead-Kostochka-Y, 2014+

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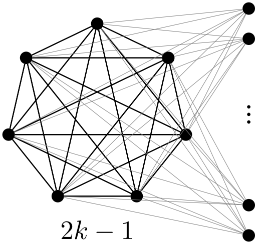

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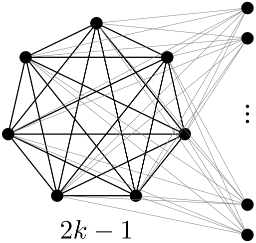

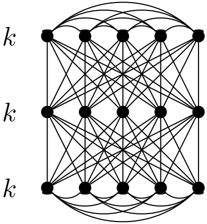

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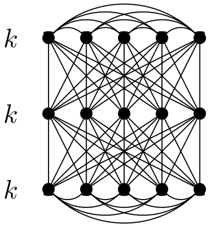
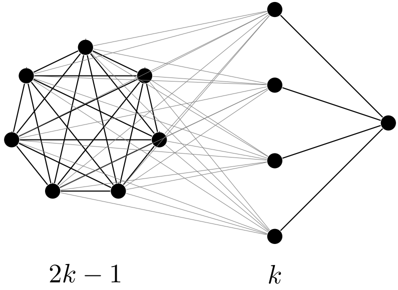
$n \geq 3k+1$





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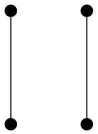

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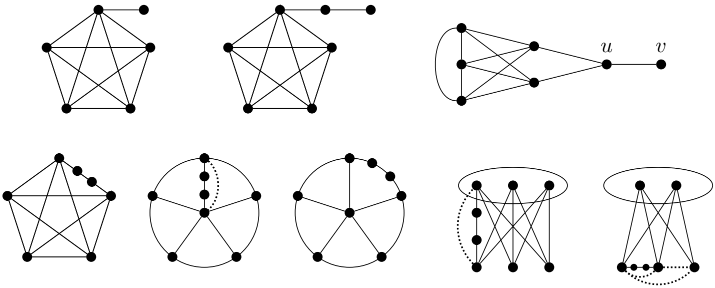

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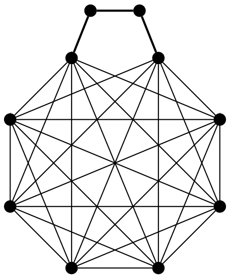

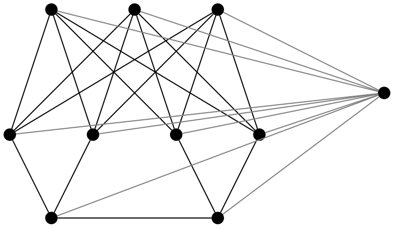

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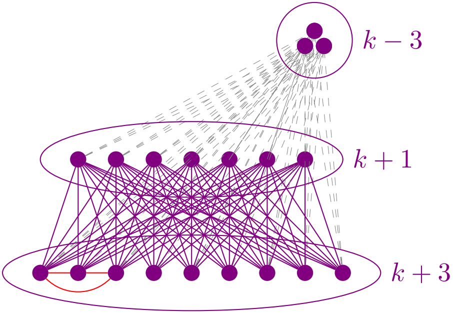
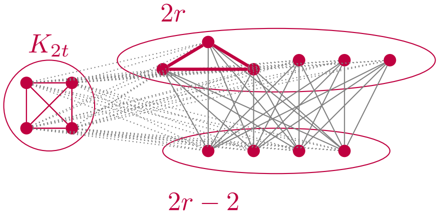
$$\sigma_2 = 4k-4:$$





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Proof

(Like Enomoto)

- Let G be an edge-maximal counterexample.
- There exists a set of $(k-1)$ disjoint cycles.
- Choose the set of cycles with the least number of vertices, etc.

Other Results

Egawa-Enomoto-Jendrol-Ota-Schiermeyer, 2007

Let G be a graph on n vertices. If $\alpha(G) \leq \sqrt{2n - 6k + 3} - 1$ then G contains k disjoint cycles.

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Let G be a graph on at least $3k$ vertices. If $|N(x) \cup N(y)| \geq 3k$ for every pair of nonadjacent vertices x, y then G contains k disjoint cycles.

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Finkel, 2008

Let G be a graph on at least $4k$ vertices. If $\delta(G) \geq 3k$ then G contains k disjoint chorded cycles.

Dirac's Question

Dirac: $(2k-1)$-connected without k disjoint cycles

Dirac, 1963

What $(2k-1)$-connected graphs do not have k disjoint cycles?

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Answer to Dirac's Question for Simple Graphs

Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if

- $\alpha(G) \leq |G| - 2k$, and
- if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$, and
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Further:

characterization for multigraphs

Multigraphs

Theorem (Extension of Corrádi-Hajnal to Multigraphs)

For $k \in \mathbb{Z}^+$, let G be a multigraph with simple degree at least $2k$. Then G has k disjoint cycles if and only if

$$|V(G)| \geq 3k - 2\ell - \alpha'$$

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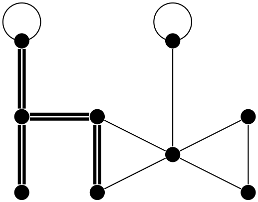

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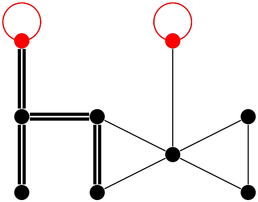

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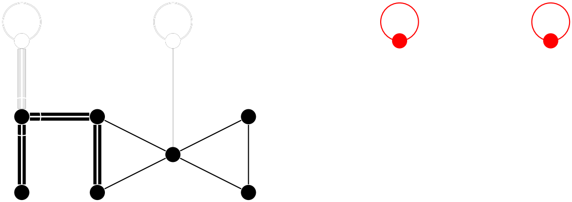

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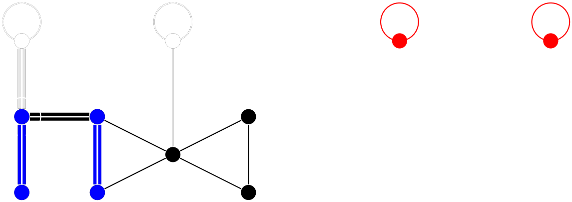

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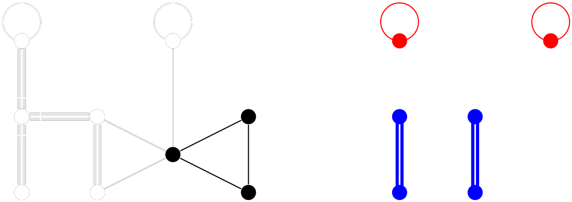

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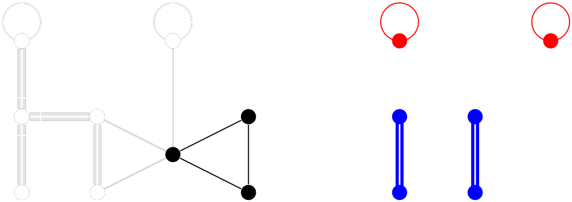

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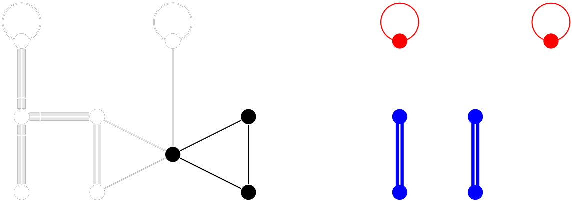

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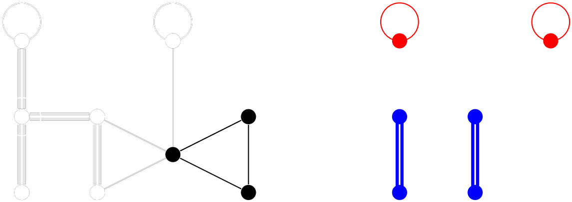

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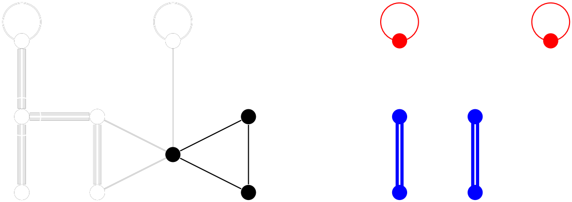

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Corollary

Let G be a multigraph with simple degree at least $2k - 1$ for some integer $k \geq 2$. Suppose G contains at least one loop. Then G has k disjoint cycles if and only if

$$|V(G)| \geq 3k - 2\ell - \alpha'.$$

$(2k-1)$-connected multigraphs with no k disjoint cycles

Answer to Dirac's Question *for multigraphs*:

Let $k \geq 2$ and $n \geq k$. Let G be an n-vertex graph with simple degree at least $2k-1$ and no loops. Let F be the simple graph induced by the strong edgs of G, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Then G does not contain k disjoint cycles if and only if one of the following holds:

- $n + \alpha' < 3k$;
- $|F| = 2\alpha'$ (i.e., F has a perfect matching) and either (i) k' is odd and $G - F = Y_{k',k'}$, or (ii) $k' = 2 < k$ and $G - F$ is a wheel with 5 spokes;
- G is extremal and either (i) some big set is not incident to any strong edge, or (ii) for some two distinct big sets I_j and $I_{j'}$, all strong edges intersecting $I_j \cup I_{j'}$ have a common vertex outside of $I_j \cup I_{j'}$;
- $n = 2\alpha' + 3k'$, k' is odd, and F has a superstar $S = \{v_0, \ldots, v_s\}$ with center v_0 such that either (i) $G - (F - S + v_0) = Y_{k'+1,k'}$, or (ii) $s = 2$, $v_1 v_2 \in E(G)$, $G - F = Y_{k'-1,k'}$ and G has no edges between $\{v_1, v_2\}$ and the set X_0 in $G - F$;
- $k = 2$ and G is a wheel, where some spokes could be strong edges;
- $k' = 2$, $|F| = 2\alpha' + 1 = n - 5$, and $G - F = C_5$.

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Idea of Proof

Suppose G is a multigraph with no loops, minimum simple degree at least $2k-1$, and $n = |G| \geq 3k - \alpha'$, but G does not contain k disjoint cycles.

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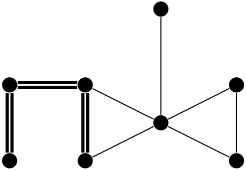

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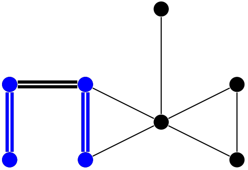

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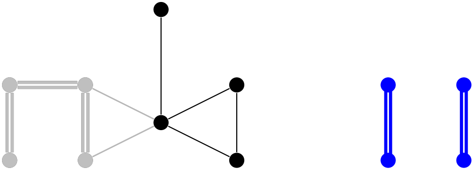

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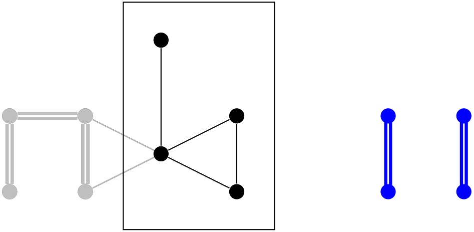

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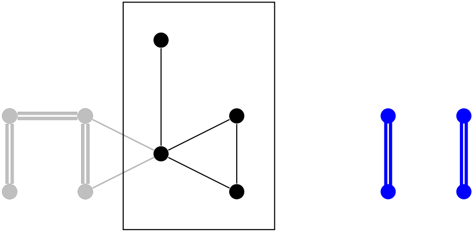

The remaining graph G' *does not* have $k' = k - \alpha'$ cycles, but $\delta(G') \geq (2k-1) - 2\alpha' = 2(k - \alpha') - 1 = 2k' - 1$.

$(2k-1)$-connected multigraphs with no k disjoint cycles

Answer to Dirac's Question *for multigraphs*:

Let $k \geq 2$ and $n \geq k$. Let G be an n-vertex graph with simple degree at least $2k-1$ and no loops. Let F be the simple graph induced by the strong edgs of G, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Then G does not contain k disjoint cycles if and only if one of the following holds:

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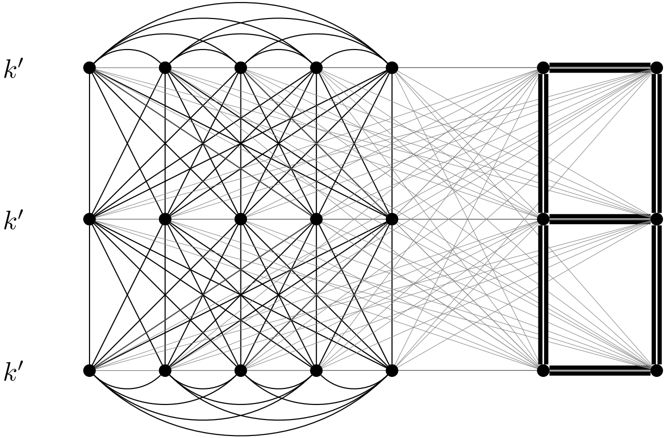
k' odd, F (strong edges) has a perfect matching

Example: $k = 8$, $\alpha' = 3$, $k' = 5$.



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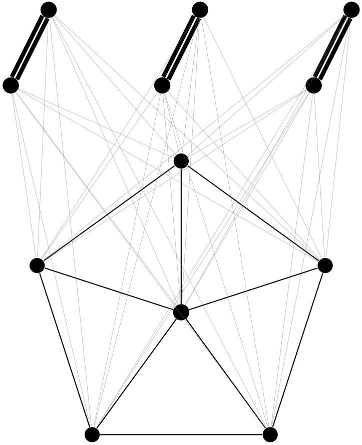
$k' = 2$, F (strong edges) has a perfect matching

Example: $k = 5$, $\alpha' = 3$, $k' = 2$



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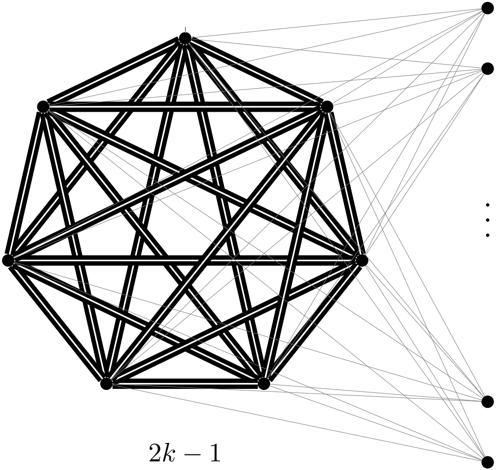
Big independent set, incident to no multiple edges



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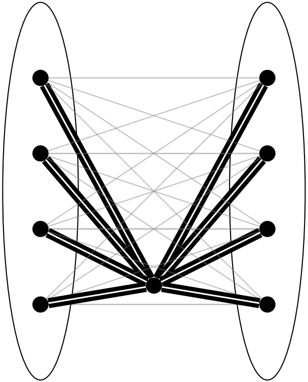
Two big independent sets; all incident strong edges share an endpoint

Example: $k = 3$, $\alpha' = 1$, $k' = 2$



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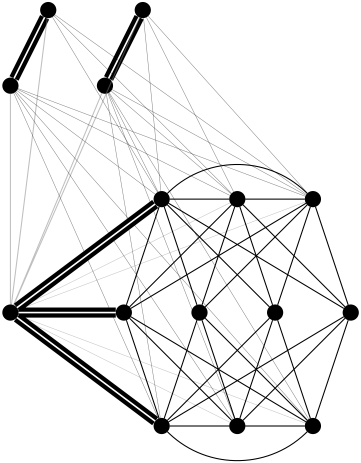
$n = 2\alpha' + 3k'$, k' odd; G has $Y_{k'+1,k'}$ subgraph

Example: $k = 6$, $\alpha' = 3$, $k' = 3$



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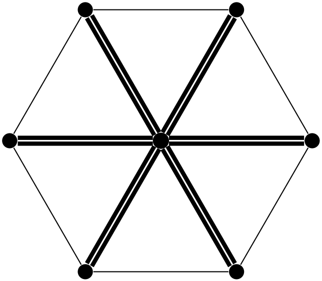
$k = 2$ and G is a wheel; spokes possibly strong



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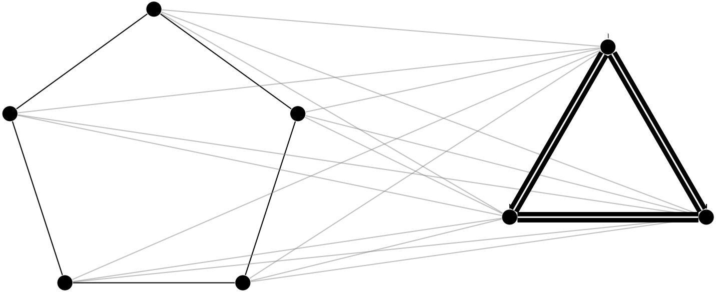

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Kierstead-Kostochka-Y, 2014+

KKY, 2014+

For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k+1$ and $\sigma_2(G) \geq 4k-3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n-2k$.

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An *equitable k-coloring* of a graph G is a proper coloring of $V(G)$ such that any two color classes differ in size by at most one.

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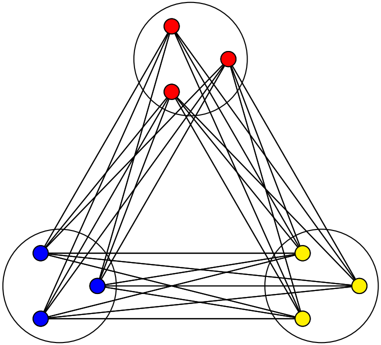

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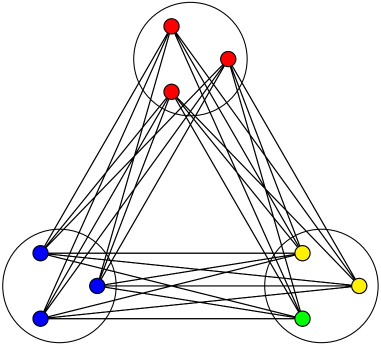

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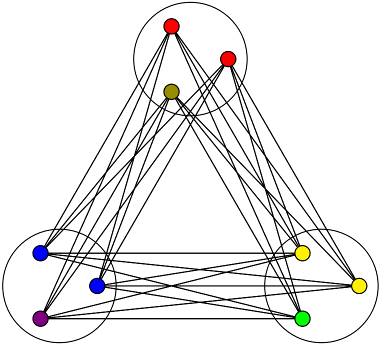

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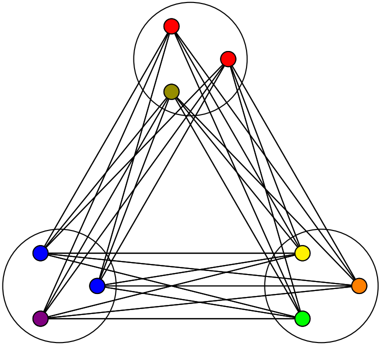

Equitable Coloring and Cycles

$n = 3k$

If G has $n = 3k$ vertices and an equitable k-coloring, then $\overline{G}$ has k disjoint cycles (all triangles).

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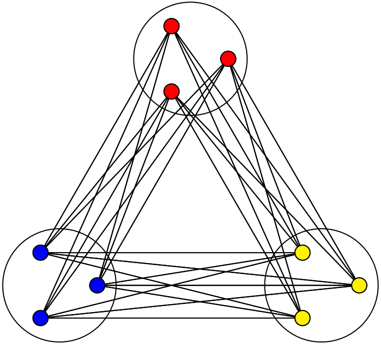

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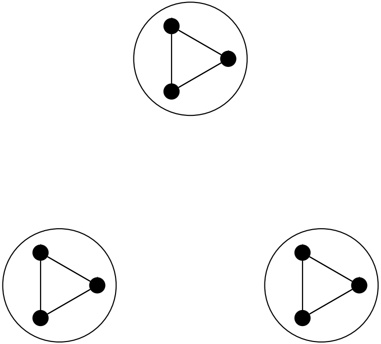

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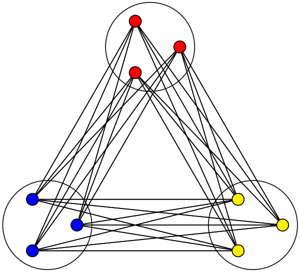

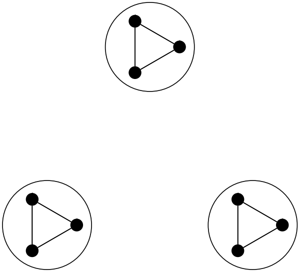

What's Really Going On

independent sets $\leftrightarrow$ cliques

Equitable Coloring and Cycles

Kierstead-Kostochka, 2008

If G is a graph such that $d(x) + d(y) \leq 2k - 1$ for every edge xy, then G has an equitable k-coloring.

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Enomoto 1998, Wang 1999

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$n = 3k$

Equivalent when $n = 3k$: 2(3k-1)-(2k-1)=4k-1

Chen-Lih-Wu

Hajnal-Szemerédi, 1970

If $k \geq \Delta(G) + 1$, then G is equitably k-colorable.

Chen-Lih-Wu

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If $k \geq \Delta(G) + 1$, then G is equitably k-colorable.

Chen-Lih-Wu Conjecture, 1994

A connected graph G is equitably $\Delta(G)$ colorable if G is different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$ for every $m \geq 1$.

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Chen-Lih-Wu Conjecture Re-stated

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then G is equitably k-colorable.

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Hajnal-Szemerédi, 1970

If $k \geq \Delta(G) + 1$, then G is equitably k-colorable.

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If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then G is equitably k-colorable.

CLW true if:

$\delta(G) \geq |G|/2$; $\Delta(G) \leq 4$; G planar with $\Delta(G) \geq 13$; G outerplanar, etc.

Ore Conditions

Kierstead-Kostochka, 2008

If G is a graph such that for each edge xy, $d(x) + d(y) \leq 2k - 1$, then G is equitably k-colorable.

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If G is a $3k$-vertex, k-colorable graph such that for each edge xy, $d(x) + d(y) \leq 2k + 1$, then G is equitably k-colorable, or is one of several exceptions.

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If G is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles, or is one of several exceptions.

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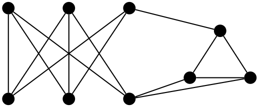
Kierstead-Kostochka-Yeager, 2014+

For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

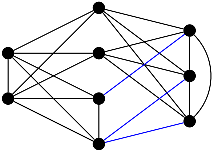
Exceptions

- $k = 3$

Equitable coloring:



Cycles:



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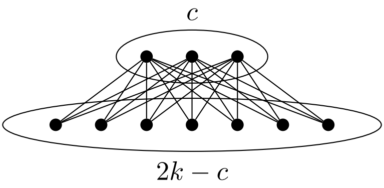
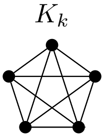
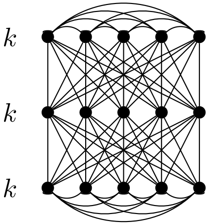
- *Equitable coloring:*





Cycles:



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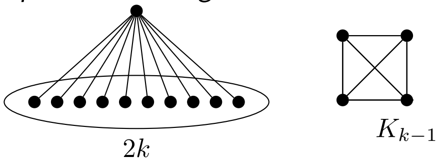

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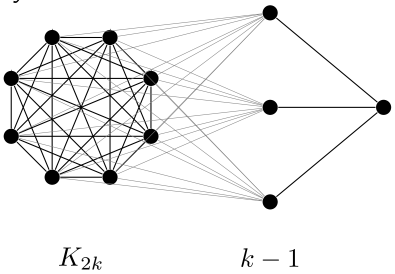
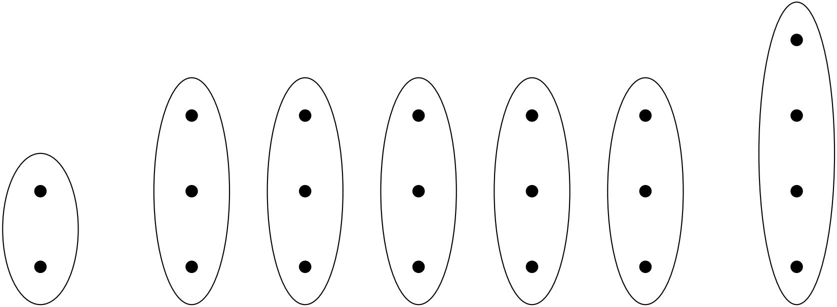


Proof of KKMY 2014+



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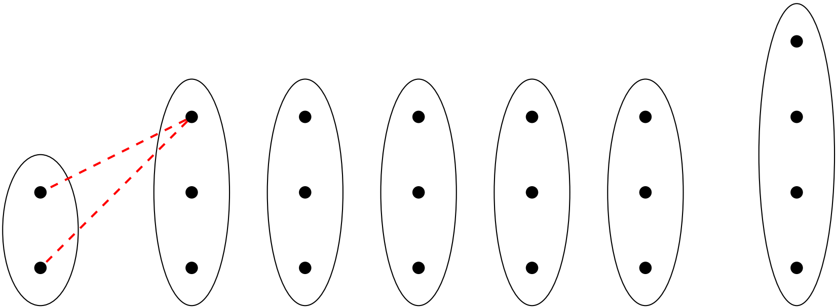
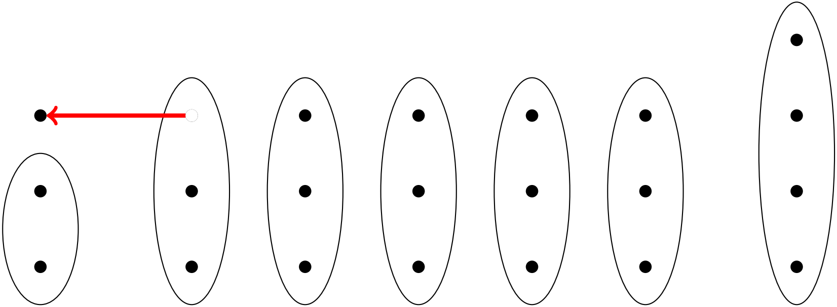


Proof of KKMY 2014+



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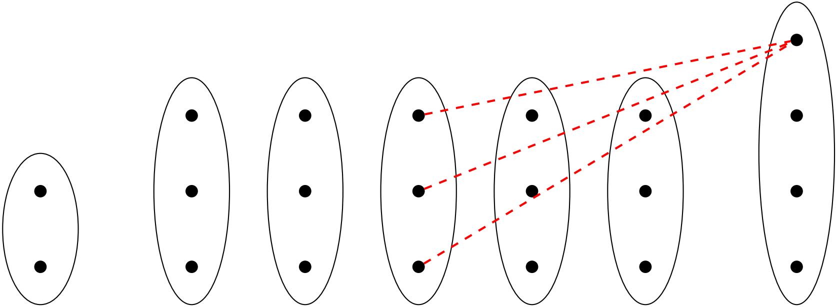
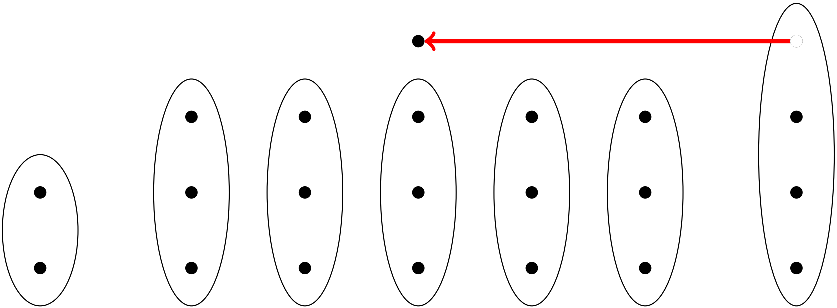


Proof of KKMY 2014+



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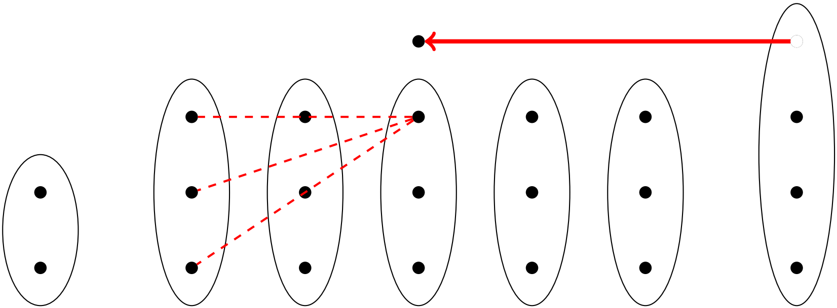

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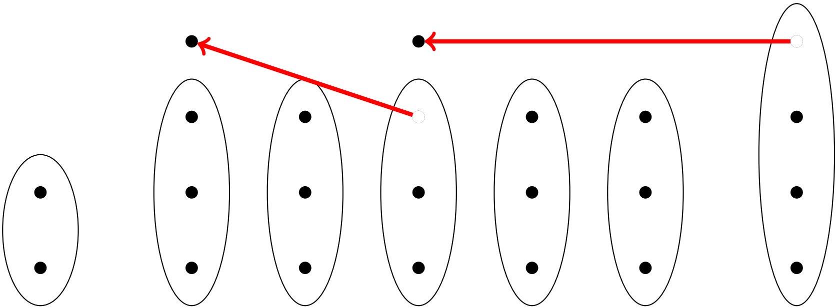
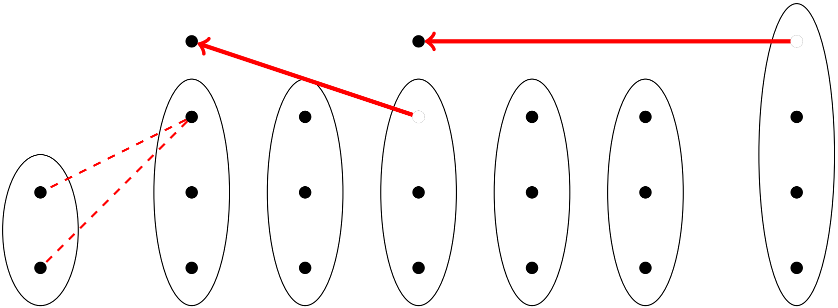


Proof of KKMY 2014+



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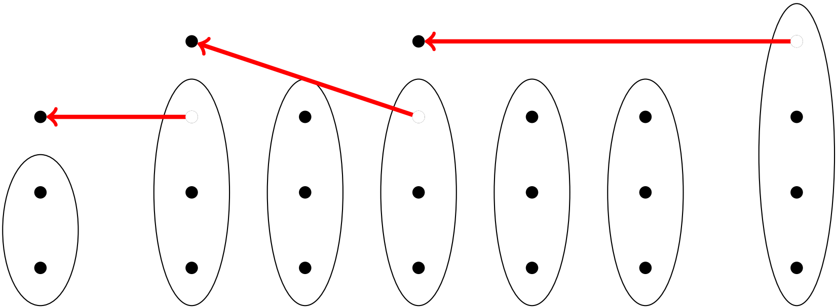

Thanks for Listening!