Disjoint Cycles and Equitable Colorings in Graphs

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Disjoint Cycles

Corrádi-Hajnal Theorem

Corrádi-Hajnal, 1963

If G is a graph on n vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

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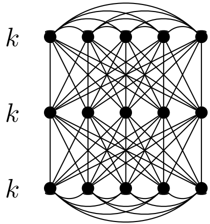

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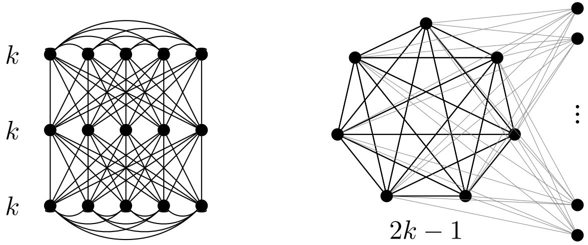

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Implies Corrádi-Hajnal

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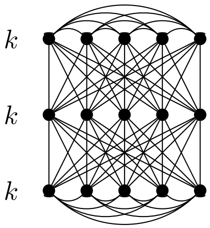

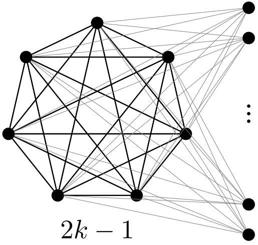

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Proof (Enomoto)

- Edge-maximal counterexample
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- Minimize number of vertices in cycles
- Maximize longest path in remainder

Kierstead-Kostochka-Y, 2014+

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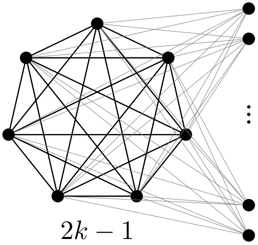

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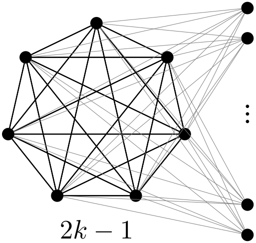

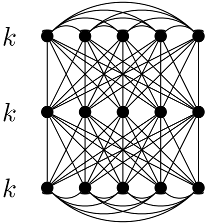

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KKY, 2014^+

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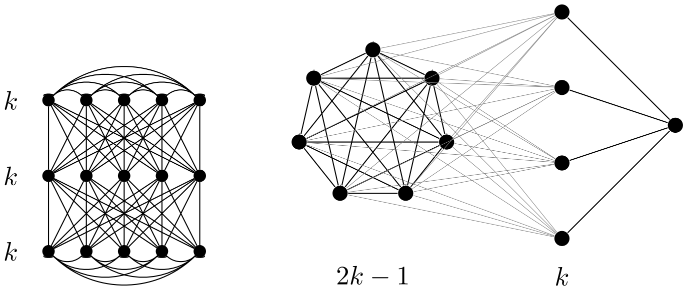

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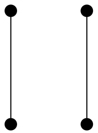

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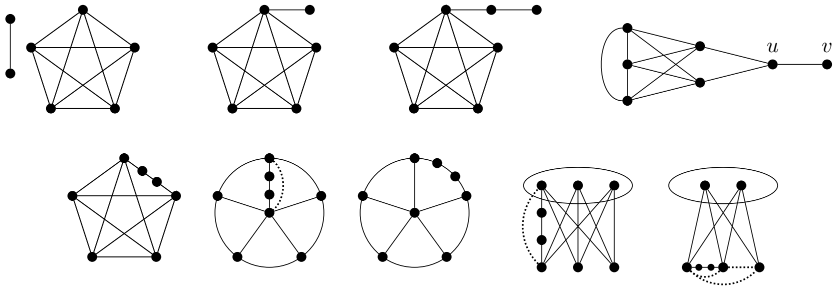

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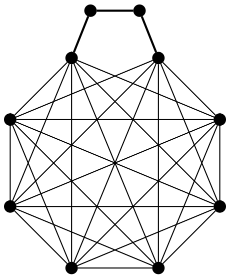

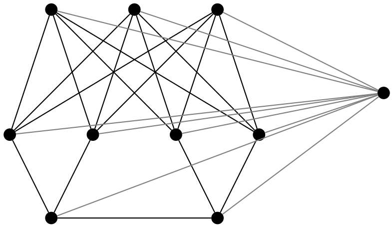

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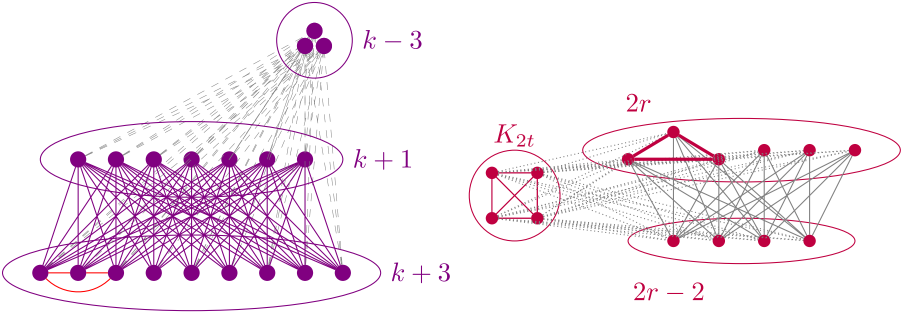
$$\sigma_2 = 4k - 4:$$



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Proof

(Like Enomoto)

- Let G be an edge-maximal counterexample.
- There exists a set of $(k - 1)$ disjoint cycles.
- Choose the set of cycles with the least number of vertices, etc.

Dirac: $(2k-1)$-connected without k disjoint cycles

Dirac, 1963

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Answer to Dirac's Question

Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if

- $\alpha(G) \leq |G| - 2k$, and
- if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$, and
- if $k = 2$ then G is not a wheel.

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Further:

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Further:

characterization for multigraphs

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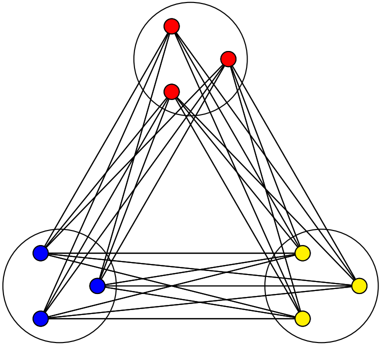

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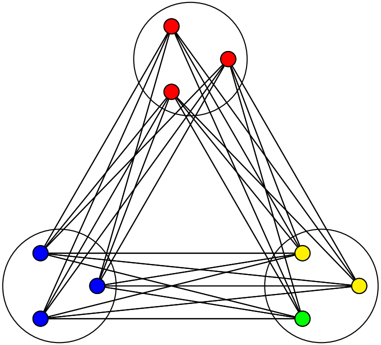

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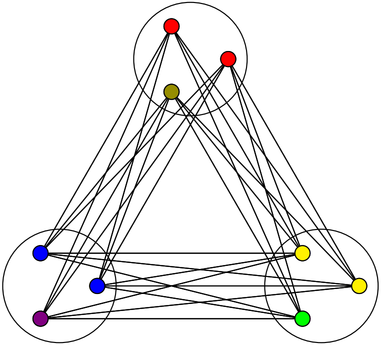

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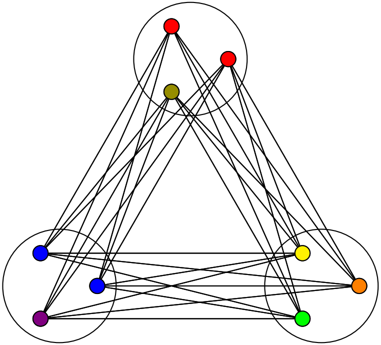

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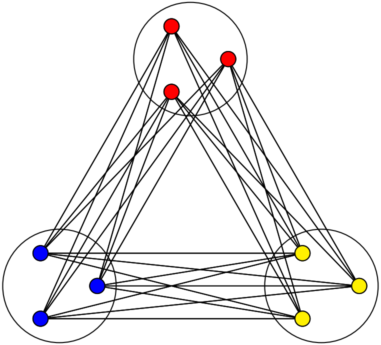

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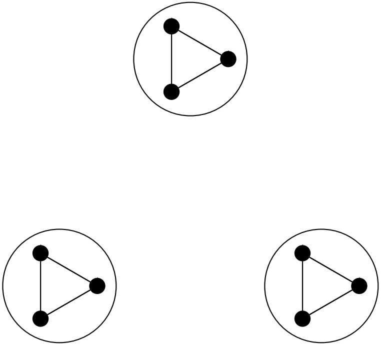

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What's Really Going On

independent sets $\leftrightarrow$ cliques

Chen-Lih-Wu

Hajnal-Szemerédi, 1970

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Chen-Lih-Wu Conjecture

If $\chi(G), \Delta(G) \leq k$, and if $K_{k,k} \not\subseteq G$ when k is odd, then G is equitably k-colorable.

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CLW true if:

$\delta(G) \geq |G|/2$; $\Delta(G) \leq 4$; G planar with $\Delta(G) \geq 13$; G outerplanar, etc.

Ore Conditions

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If G is a k-colorable $3k$-vertex graph such that for each edge xy, $d(x) + d(y) \leq 2k + 1$, then G is equitably k-colorable, or is one of several exceptions.

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If G is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles, or is one of several exceptions, or $\overline{G}$ is not k-colorable.

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Exceptions

- $k = 3$

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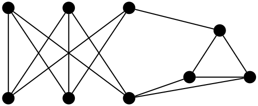

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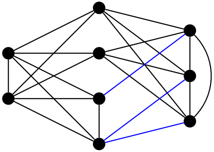

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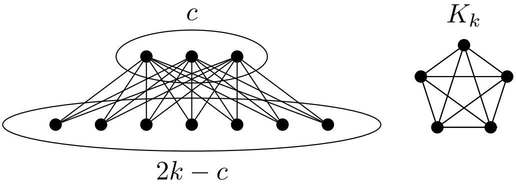

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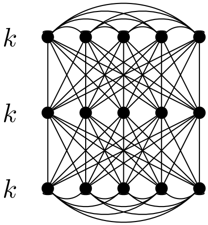

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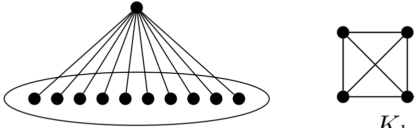

$2k$ K_{k-1}

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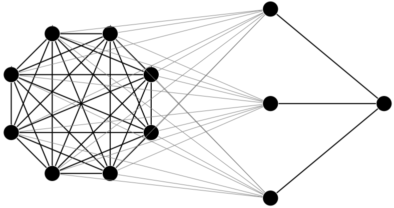

K_{2k} $k-1$

Thanks for Listening!