Disjoint Cycles and Equitable Coloring

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Research in Graph Theory

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Disjoint Cycles

Corrádi-Hajnal Theorem

Corrádi-Hajnal, 1963

If G is a graph on n vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

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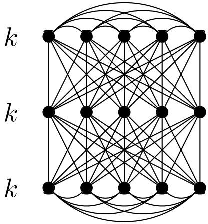

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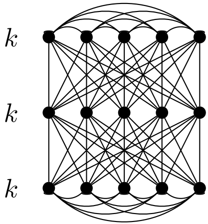

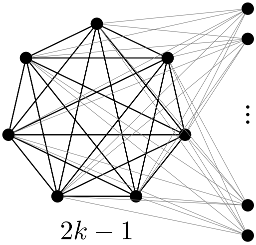

Enomoto, Wang

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$$\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$$

That is, low vertices form a clique.

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If G is a graph on n vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k disjoint cycles.

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Implies Corrádi-Hajnal

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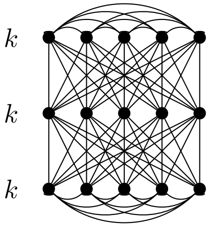

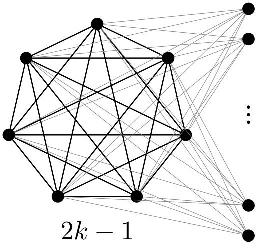

Kierstead-Kostochka-Y, 2015+

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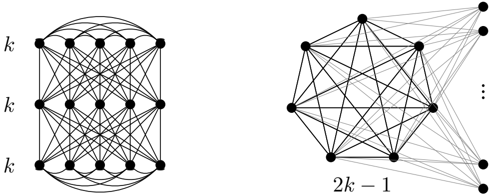

Kierstead-Kostochka-Yeager, 2015+

For $k \geq 4$, if G is a graph on n vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then G contains k disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

Dirac's Question

Dirac: $(2k-1)$-connected without k disjoint cycles

Dirac, 1963

What $(2k-1)$-connected graphs do not have k disjoint cycles?

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Answer to Dirac's Question for Simple Graphs (Kierstead-Kostochka-Yeager, 2015+)

Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if

- $\alpha(G) \leq |G| - 2k$, and
- if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$, and
- if $k = 2$ then G is not a wheel.

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Dirac, 1963

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Further:

Characterization for *multigraphs*
Kierstead-Kostochka-Yeager **Combinatorica**, to appear.

The Case $n = 3k$

A Missing Detail

Enomoto 1998, Wang 1999

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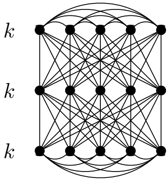

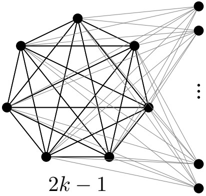

Kierstead-Kostochka-Yeager, 2015$^+$

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Kierstead-Kostochka-Molla-Yeager, 2015+

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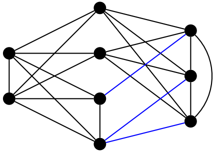
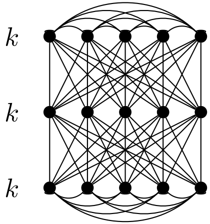
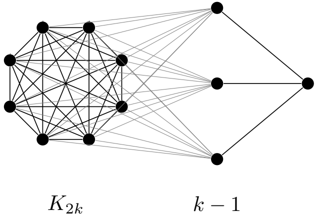
If G is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k-3$, then G **contains** k **disjoint cycles**, or is one of several exceptions, or $\overline{G}$ is not k-colorable.







Chen-Lih-Wu Conjecture

Hajnal-Szemerédi, 1970

If $k \geq \Delta(G) + 1$, then G is equitably k-colorable.

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If $\chi(G), \Delta(G) \leq k$, and if k is odd $K_{k,k} \not\subseteq G$, then G is equitably k-colorable.

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Kierstead-Kostochka-Molla-Yeager, 2015+

If G is a k-colorable $3k$-vertex graph such that for each edge xy, $d(x) + d(y) \leq 2k + 1$, then G is equitably k-colorable, or is one of several exceptions.

Thanks for Listening!